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Robust adaptive control of nonlinear systems with unknown time delays☆

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Abstract

In this paper, robust adaptive control is presented for a class of parametric-strict-feedback nonlinear systems with unknown time delays. Using appropriate Lyapunov–Krasovskii functionals, the uncertainties of unknown time delays are compensated for. Controller singularity problems are solved by employing practical robust control and regrouping unknown parameters. By using differentiable approximation, backstepping design can be carried out for a class of nonlinear systems in strict-feedback form. It is proved that the proposed systematic backstepping design method is able to guarantee global uniform ultimate boundedness of all the signals in the closed-loop system and the tracking error is proven to converge to a small neighborhood of the origin. Simulation results are provided to show the effectiveness of the proposed approach.

Keywords: Robust adaptive control; Time-delay systems; Differentiable control

1. Introduction

In recent years, there have been tremendous efforts in adaptive control of certain class of nonlinear systems. Adaptive control has proven its great capability in compensating for linearly parameterized uncertainties. To obtain global stability, some restrictions have to be made to system nonlinearities such as matching conditions (Taylor et al., 1989), extended matching conditions (Kanellakopoulos et al., 1991), or growth conditions (Sastry & Isidori, 1989). To overcome these restrictions, a recursive and systematic backstepping design was developed in Kanellakopoulos et al. (1991). The overparametrization problem was then removed in Krstic et al. (1992) by introducing the concept of tuning function. Several adaptive approaches for nonlinear systems with triangular structures have been proposed in Seto et al. (1994) and Ge et al. (2000). Robust adaptive backstepping control has been studied for certain class of nonlinear systems whose uncertainties are not only from parametric ones but also from unknown nonlinear functions in Polycarpou and Ioannou (1996) and Pan and Basar (1998), among others. For systems that are feedback linearizable, the certainty equivalent control is usually taken the form $u(t) = [-\hat{f}(x) + v(t)]/\hat{g}(x)$, where $\hat{f}(x)$ and $\hat{g}(x)$ are estimates of $f(x)$ and $g(x)$. In this case, the assumption of $\hat{g}(x) \neq 0$ should be made to avoid the singularity problem (Yesildirek & Lewis, 1995). When $g(x)$ is referred as to virtual control coefficient with known signs, several schemes have been developed to avoid singularity problems (Wang, 1994; Spooner & Passino, 1996; Ge et al., 2002).

Practically, systems with time delays are frequently encountered (e.g., process control). Time-delayed linear systems have been intensively investigated (Kolmanovskii et al., 1999). However, the useful tools such as linear matrix inequalities (LMIs) are hard to apply to nonlinear systems with time delays. Lyapunov design has been proven to be an effective tool in controller design for nonlinear systems. One major difficulty lies in the control of time-delayed nonlinear systems is that the delays are usually not perfectly known.

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One way to ensure stability robustness with respect to this uncertainty is to employ stability criteria valid for any nonnegative value of the delays, i.e., delay-independent results. A class of quadratic Lyapunov–Krasovskii functionals (Hale, 1977) has been used earlier as checking criteria for time-delay systems' stability. The unknown time delays are the main issue to be dealt with for the extension of backstepping design to such kinds of systems. In Nguang (2000), stabilizing controller design based on the Lyapunov–Krasovskii functionals was proposed for a class of nonlinear time-delay systems with a so-called "triangular structure". However, few attempts have been made towards systems with unknown parameters or unknown nonlinear functions. In Ge et al. (2003, 2004), practical backstepping design was studied for a class of nonlinear time-delay systems in strict-feedback form by solving the problem of differentiation of the intermediate control functions at the discontinuous points in a "practical sense", i.e., setting finite values at these points, though the intermediate control functions remain not smooth at all.

Motivated by previous works on the nonlinear systems with both unknown time delays and uncertainties from unknown parameters and nonlinear functions, we present in this paper a practical robust adaptive controller for a class of unknown nonlinear systems in a parametric-strict-feedback form by employing practical yet differentiable control. Using appropriate Lyapunov–Krasovskii functionals in the Lyapunov function candidate, the uncertainties from unknown time delays are removed such that the design of the stabilizing control law is free from these uncertainties. In this way, the iterative backstepping design procedure can be carried out directly. A novel smooth approximation is introduced to solve the differentiability problem for the intermediate control functions. Time-varying control gains rather than fixed gains are chosen to guarantee the boundedness of all the signals in closed-loop system. The global uniform ultimate boundedness (GUUB) of the signals in the closed-loop system is achieved and the output of the systems is proven to converge to a small neighborhood of the desired trajectory.

To the best of our knowledge, there is little work dealing with such a kind of systems in the literature at present stage. The proposed method expands the class of nonlinear systems that can be handled using adaptive control techniques. The main contributions of the paper lie in: (i) the employment of robust adaptive backstepping controller design for a class of unknown nonlinear time-delay systems in parametric-strict-feedback form, in which the unknown time delays are compensated for by using appropriate Lyapunov–Krasovskii functionals; (ii) the introduction of differentiable practical control in solving the controller singularity problem so that it can be carried out in backstepping design and guarantee the tracking error being confined in a compact domain of attraction; and (iii) the elegant re-grouping of unknown parameters, by which the controller singularity problem is effectively avoided, and the lumping of unknown parameter vectors as scalars, by which the number of parameters being estimated as well as the order and complexity of the controllers, are dramatically reduced.

2. Problem formulation and preliminaries

Consider a class of single-input-single-output (SISO) nonlinear time-delay systems

$$\begin{aligned} \dot{x}_i(t) &= g_i x_{i+1}(t) + f_i(\bar{x}_i(t)) \\ &\quad + h_i(\bar{x}_i(t-\tau_i)), \quad 1 \leqslant i \leqslant n-1, \\ \dot{x}_n(t) &= g_n u(t) + f_n(x(t)) + h_i(x(t-\tau_n)), \\ y(t) &= x_1(t), \end{aligned} \tag{1}$$

where $\bar{x}_i = [x_1, x_2, \ldots, x_i]^{\mathrm{T}}$, $x = [x_1, x_2, \ldots, x_n]^{\mathrm{T}} \in R^n$, $u \in R$, $y \in R$ are the state variables, system input and output, respectively, $f_i(\cdot)$ and $h_i(\cdot)$ are unknown smooth functions, g_i are unknown constants, and τ_i are unknown time delays of the states, $i = 1, \ldots, n$. The control objective is to design an adaptive controller for system (1) such that the output $y(t)$ follows a desired reference signal $y_d(t)$, while all signals in the closed-loop system are bounded. Define the desired trajectory $\bar{x}_{d(i+1)} = [y_d, \dot{y}_d, \ldots, y_d^{(i)}]^{\mathrm{T}}$, $i = 1, \ldots, n-1$, which is a vector of y_d up to its ith time derivative $y_d^{(i)}$. We have the following assumptions for the system functions, unknown time delays and reference signals.

Assumption 1. The signs of g_i are known, and there exist constants $g_{\max} \geqslant g_{\min} > 0$ such that $g_{\min} \leqslant |g_i| \leqslant g_{\max}$.

The above assumption implies that unknown constants g_i are strictly either positive or negative. Without losing generality, we shall only consider the case when $g_i > 0$. It should be emphasized that the bounds $g_{\min}$ and $g_{\max}$ are only required for analytical purposes, their true values are not necessarily known since they are not used for controller design.

Assumption 2. The unknown functions $f_i(\cdot)$ and $h_i(\cdot)$ can be expressed as

$$f_i(\bar{x}_i(t)) = \theta_{fi}^{\mathrm{T}} F_i(\bar{x}_i(t)) + \delta_{fi}(\bar{x}_i(t)),$$

$$h_i(\bar{x}_i(t)) = \theta_{hi}^{\mathrm{T}} H_i(\bar{x}_i(t)) + \delta_{hi}(\bar{x}_i(t)),$$

where $F_i(\cdot)$, $H_i(\cdot)$ are known smooth function vectors, $\theta_{fi} \in R^{n_i}$, $\theta_{hi} \in R^{m_i}$ are unknown constant parameter vectors, n_i, m_i are positive integers, $\delta_{fi}(\cdot)$, $\delta_{hi}(\cdot)$ are unknown smooth functions, which satisfy the so-called triangular bounds conditions

$$|\delta_{fi}(\bar{x}_i(t))| \leqslant c_{fi}\phi_i(\bar{x}_i(t)),$$

$$|\delta_{hi}(\bar{x}_i(t))| \leqslant c_{hi}\psi_i(\bar{x}_i(t)),$$

where c_{fi}, c_{hi} are constant parameters, which are not necessarily known, and $\phi_i(\cdot)$, $\psi_i(\cdot)$ are known nonnegative smooth functions.

Assumption 2 is rather weak as only a rough form of $f_i(\cdot)$ and $h_i(\cdot)$ need to be known.

Assumption 3. The size of the unknown time delays is bounded by a known constants, i.e., $\tau_i \leqslant \tau_{\max}$, $i = 1, \ldots, n$.

There are many physical processes which are governed by nonlinear differential equations of form (1). Examples are recycled reactors, recycled storage tanks and cold rolling mills (Malek-Zavarei, 1987). In general, most of the recycling processes inherit delays in their state equations. Compared with the systems in Nguang (2000), the system we consider in this paper is more general in the sense that the uncertainty is due to both parametric uncertainty and unknown nonlinear functions. These unknown functions might come from inaccurate modeling or model reduction.

To make the problem formulation precise, the system is presented again as follows:

$$\begin{aligned} \dot{x}_i(t) &= g_i x_{i+1}(t) + \theta_{fi}^{\mathrm{T}} F_i(\bar{x}_i(t)) + \delta_{fi}(\bar{x}_i(t)) \\ &\quad + \theta_{hi}^{\mathrm{T}} H_i(\bar{x}_i(t-\tau_i)) + \delta_{hi}(\bar{x}_i(t-\tau_i)), \\ &\quad 1 \leqslant i \leqslant n-1 \\ \dot{x}_n(t) &= g_n u(t) + \theta_{fn}^{\mathrm{T}} F_n(x(t)) + \delta_{fn}(x(t)) \\ &\quad + \theta_{hn}^{\mathrm{T}} H_n(x(t-\tau_n)) + \delta_{hn}(x(t-\tau_n)), \\ y(t) &= x_1(t). \end{aligned} \tag{2}$$

Assumption 4. The desired trajectory vectors $\bar{x}_{di} \in \Omega_{di} \subset R^i$, $i=2, \ldots, n$ are continuous and available with Ω_{di} known compact set.

Lemma 1. *The following inequality holds for any* $\varepsilon_1 > 0$ *and for any* $\eta \in R$

$$0 \leqslant |\eta| - \eta \tanh\left(\frac{\eta}{\varepsilon_1}\right) \leqslant k\varepsilon_1,$$

where k *is a constant that satisfies* $k = \mathrm{e}^{-(k+1)}$, *i.e.,* $k = 0.2785$.

Lemma 2. *Even function* $q_i(x) : R \to R$

$$q_i(x) = \begin{cases} 1, & |x| \geqslant \lambda_{ai} + \lambda_{bi}, \\ c_{qi} \int_{\lambda_{ai}}^{x} [(\frac{\lambda_{bi}}{2})^2 - (\sigma - \lambda_{ai} - \frac{\lambda_{bi}}{2})^2]^{n-i} d\sigma, & \lambda_{ai} < x < \lambda_{ai} + \lambda_{bi}, \\ c_{qi} \int_{x}^{-\lambda_{ai}} [(\frac{\lambda_{bi}}{2})^2 - (\sigma + \lambda_{ai} + \frac{\lambda_{bi}}{2})^2]^{n-i} d\sigma, & -(\lambda_{ai} + \lambda_{bi}) < x < -\lambda_{ai}, \\ 0, & |x| \leqslant \lambda_{ai}, \end{cases} \tag{3}$$

where $c_{qi} = \frac{[2(n-i)+1]!}{\lambda_{bi}^{2(n-i)+1}[(n-i)!]^2}$, $\lambda_{ai}, \lambda_{bi} > 0$ *and integer* $i \in R^+$, *is* $(n-i)$*th differentiable, i.e.,* $q_i(x) \in C^{n-i}$ *and bounded by* 1.

3. Adaptive controller design for first-order systems

To illustrate the design methodology clearly, let us consider the tracking problem of a first-order system first

$$\begin{aligned} \dot{x}_1(t) &= g_1 u(t) + \theta_{f1}^{\mathrm{T}} F_1(x_1(t)) + \delta_{f1}(x_1(t)) \\ &\quad + \theta_{h1}^{\mathrm{T}} H_1(x_1(t-\tau_1)) + \delta_{h1}(x_1(t-\tau_1)) \end{aligned} \tag{4}$$

with $u(t)$ being the control input. Define $z_1 = x_1 - y_d$, we have

$$\begin{aligned} \dot{z}_1(t) &= g_1 u(t) + \theta_{f1}^{\mathrm{T}} F_1(x_1(t)) + \delta_{f1}(x_1(t)) \\ &\quad + \theta_{h1}^{\mathrm{T}} H_1(x_1(t-\tau_1)) + \delta_{h1}(x_1(t-\tau_1)) \\ &\quad - \dot{y}_d(t). \end{aligned} \tag{5}$$

Consider the scalar function $V_{z_1}(t) = \frac{1}{2g_1} z_1^2(t)$, whose time derivative along (5) is

$$\begin{aligned} \dot{V}_{z_1}(t) &= z_1(t)\{u(t) + \frac{1}{g_1}[\theta_{f1}^{\mathrm{T}} F_1(x_1(t)) + \delta_{f1}(x_1(t)) \\ &\quad + \theta_{h1}^{\mathrm{T}} H_1(x_1(t-\tau_1)) + \delta_{h1}(x_1(t-\tau_1)) - \dot{y}_d(t)]\}. \end{aligned}$$

Since $\delta_{f1}(\cdot)$ and $\delta_{h1}(\cdot)$ are partially known according to Assumption 2, we have

$$\begin{aligned} \dot{V}_{z_1}(t) &\leqslant z_1(t)u(t) + \frac{1}{g_1}[z_1(t)\theta_{f1}^{\mathrm{T}} F_1(x_1(t)) \\ &\quad + |z_1(t)| c_{f1}\phi_1(x_1(t)) + z_1(t)\theta_{h1}^{\mathrm{T}} H_1(x_1(t-\tau_1)) \\ &\quad + |z_1(t)| c_{h1}\psi_1(x_1(t-\tau_1)) - z_1(t)\dot{y}_d(t)]. \end{aligned} \tag{6}$$

Remark 1. It can be seen from (6) that the design difficulties come from two system uncertainties: unknown parameters and unknown time delay τ_1. Although $H_1(\cdot)$ and $\psi_1(\cdot)$ are known, they are functions of delayed state $x_1(t-\tau_1)$, which is undetermined due to the unknown time delay τ_1. Thus, functions $H_1(x_1(t-\tau_1))$ and $\psi_1(x_1(t-\tau_1))$ cannot be used in the controller design. In addition, the unknown time delay τ_1 and unknown parameters θ_{h1}^{T} and c_{h1} are entangled together in a nonlinear fashion, which makes the problem even more complex to solve. Therefore, we have to convert these related terms into such a form that the uncertainties from τ_1, θ_{h1}^{T} and c_{h1} can be dealt with separately.

Using Young's Inequality (Hardy, Littlewood, & Polya, 1952), we have

$$\begin{aligned} \dot{V}_{z_1}(t) &\leqslant z_1(t)u(t) + \frac{1}{g_1}[z_1(t)\theta_{f1}^{\mathrm{T}} F_1(x_1(t)) \\ &\quad + |z_1(t)| c_{f1}\phi_1(x_1(t)) \\ &\quad + \tfrac{1}{2} z_1^2(t)\theta_{h1}^{\mathrm{T}}\theta_{h1} \\ &\quad + \tfrac{1}{2} H_1^{\mathrm{T}}(x_1(t-\tau_1)) H_1(x_1(t-\tau_1)) \\ &\quad + \tfrac{1}{2} z_1^2(t) c_{h1}^2 + \tfrac{1}{2}\psi_1^2(x_1(t-\tau_1)) - z_1(t)\dot{y}_d(t)] \end{aligned} \tag{7}$$

where θ_{h1} and $H_1(x_1(t-\tau_1))$, and c_{h1} and $\psi_1(x_1(t-\tau_1))$ are separated, respectively. In fact, parameter vector θ_{h1} and function vector $H_1(x_1(t-\tau_1))$ have been lumped as scalars by applying Young's Inequality, for which they can be dealt with separately as detailed later.

To overcome the design difficulties from the unknown time delay τ_1, the following Lyapunov–Krasovskii functional can be considered

$$V_{U_1}(t) = \frac{1}{2g_1}\int_{t-\tau_1}^{t} U_1(x_1(\tau))\,\mathrm{d}\tau, \tag{8}$$

where $U_1(\cdot)$ is a positive definite function chosen as

$$U_1(x_1(t)) = H_1^{\mathrm{T}}(x_1(t))H_1(x_1(t)) + \psi_1^2(x_1(t)). \tag{9}$$

The time derivative of $V_{U_1}(t)$ is

$$\begin{aligned}\dot{V}_{U_1}(t) = \frac{1}{2g_1}[&H_1^{\mathrm{T}}(x_1)H_1(x_1) + \psi_1^2(x_1)\\ &- H_1^{\mathrm{T}}(x_1(t-\tau_1))H_1(x_1(t-\tau_1))\\ &- \psi_1^2(x_1(t-\tau_1))]\end{aligned}$$

which can be used to cancel the time-delay terms on the right-hand side of (7) and thus eliminate the design difficulty from the unknown time delay τ_1 without introducing any uncertainties to the system. For notation conciseness, we will omit the time variable after time-delay terms have been eliminated. Accordingly, we obtain

$$\dot{V}_{z_1} + \dot{V}_{U_1} \leqslant z_1(u + \theta_1^{\mathrm{T}}F_{\theta 1}) + \theta_{10}|z_1|\phi_{10}, \tag{10}$$

where θ_{10} is an unknown constant, θ_1 is an unknown constant vector, $\phi_{10}(\cdot)$ is a known function, and $F_{\theta 1}(\cdot)$ is a known function vector defined below

$$\theta_{10} := \frac{c_{f1}}{g_1}, \quad \theta_1 := \left[\frac{\theta_{f1}^{\mathrm{T}}}{g_1}, \frac{\theta_{h1}^{\mathrm{T}}\theta_{h1} + c_{h1}^2}{g_1}, \frac{1}{g_1}\right]^{\mathrm{T}} \in R^{n_1+2},$$

$$\phi_{10} := \phi_1,$$

$$F_{\theta 1} = \left[F_1^{\mathrm{T}}, \frac{1}{2}z_1, \frac{1}{2z_1}(H_1^{\mathrm{T}}H_1 + \psi_1^2) - \dot{y}_d\right]^{\mathrm{T}} \in R^{n_1+2}.$$

Note that the design of $u(t)$ is free from unknown time delay τ_1 at present stage. To stabilize $z_1(t)$, the following desired certainty equivalent control (Astrom & Wittenmark, 1995) under the assumption of exact knowledge could be proposed as

$$u^* = -k_1 z_1 - \theta_1^{\mathrm{T}}F_{\theta 1} - \beta_1(z_1), \tag{11}$$

where $k_1 > 0$ and $\beta_1(z_1) = \mathrm{sgn}(z_1)\theta_{10}\phi_{10}$.

Remark 2. The introduction of θ_1 has two advantages. Firstly, we only need to estimate $\frac{1}{g_1}$ rather than g_1 such that the possible controller singularity due to $\hat{g}_1 = 0$ is avoided. Secondly, after applying Young's inequality, unknown constant vector $\theta_{h1} \in R^{m_1}$ is lumped as a scalar $\theta_{h1}^{\mathrm{T}}\theta_{h1}$. By doing so, the number of parameters being estimated is dramatically reduced, which greatly reduces the order and complexity of the controller. However, it may result in a reduced fineness of the results.

However, controller singularity may occur since the proposed desired control (11) is not well defined at $z_1 = 0$. Therefore, care must be taken to guarantee the boundedness of the control. It is noted that the controller singularity takes place at the point $z_1 = 0$, where the control objective is supposed to be achieved. From a practical point of view, once the system reaches its origin, no control action should be taken for less power consumption. As $z_1 = 0$ is hard to detect owing to the existence of measurement noises, it is more practical to relax our control objective of convergence to a bounded region rather than the origin. Next, let us show that certain bounded region is a domain of attraction in the sense that all z_1 will enter into this region and will stay within thereafter. In the case that the parameters are unknown, we propose the practical robust adaptive control law to guarantee the systems stability as detailed in Lemma 3.

Lemma 3. *For the first-order system* (4), *if the practical robust control law is chosen as*

$$u = \begin{cases} -k_1(t)z_1 - \hat{\theta}_1^{\mathrm{T}}F_{\theta 1} - \beta_1(z_1, \hat{\theta}_{10}), & |z_1| \geqslant \lambda_{a1}, \\ 0, & |z_1| < \lambda_{a1}, \end{cases} \tag{12}$$

$$\beta_1(z_1, \hat{\theta}_{10}) = \mathrm{sgn}(z_1)\hat{\theta}_{10}\phi_{10}, \tag{13}$$

where $\hat{\theta}_{10}$ *and* $\hat{\theta}_1$ *are the estimates of* θ_{10} *and* θ_1, *respectively*, $k_1(t) \geqslant k^* > 0$ *with* k^* *being any positive constant, and the parameters are updated by*

$$\dot{\hat{\theta}}_{10} = \gamma_1|z_1|\phi_{10}, \tag{14}$$

$$\dot{\hat{\theta}}_1 = \Gamma_1 F_{\theta 1} z_1, \tag{15}$$

with $\gamma_1 > 0$ *and* $\Gamma_1 = \Gamma_1^{\mathrm{T}} > 0$, *then for any finite initial conditions* $x_1(0)$, $\hat{\theta}_{10}(0)$ *and* $\hat{\theta}_1(0)$, *all signals in the closed-loop system are bounded, and the tracking error* $z_1(t)$ *will finally stay in a compact set defined by* $\Omega_{z_1} = \{z_1 \in R \,|\, |z_1| \leqslant \lambda_{a1}\}$.

Proof. To show Ω_{z_1} to be a domain of attraction, we first find a Lyapunov function candidate $V_1(t) > 0$ such that $\dot{V}_1(t) \leqslant 0$, $\forall z_1 \notin \Omega_{z_1}$. For $|z_1| \geqslant \lambda_{a1}$, let us consider the following Lyapunov function candidate:

$$V_1(t) = V_{z_1}(t) + V_{U_1}(t) + \tfrac{1}{2}\gamma_1^{-1}\tilde{\theta}_{10}^2(t) + \tfrac{1}{2}\tilde{\theta}_1^{\mathrm{T}}(t)\Gamma_1^{-1}\tilde{\theta}_1(t),$$

where $\tilde{(\cdot)} = \hat{(\cdot)} - (\cdot)$. The time derivative of $V_1(t)$ along (10) is

$$\begin{aligned}\dot{V}_1(t) \leqslant{}& z_1(u + \theta_1^{\mathrm{T}}F_{\theta 1}) + \theta_{10}|z_1|\phi_{10} + \gamma_1^{-1}\tilde{\theta}_{10}\dot{\hat{\theta}}_{10}\\ &+ \tilde{\theta}_1^{\mathrm{T}}\Gamma_1^{-1}\dot{\hat{\theta}}_1.\end{aligned} \tag{16}$$

Substituting (12)–(15) into (16), we obtain $\dot{V}_1 \leqslant -k_1(t)z_1^2 \leqslant -k^* z_1^2 \leqslant 0$. Hence, $V_1(t)$ is a Lyapunov function and $z_1(t)$, $x_1(t)$, $\hat{\theta}_{10}(t)$, $\hat{\theta}_1(t)$ are bounded. In addition, the domain Ω_{z_1} is attractive in the sense that z_1 will be driven onto Ω_{z_1} in a finite time, and then after stay within. For $|z_1| < \lambda_{a1}$, since $z_1 = x_1 - x_d$, $\dot{\hat{\theta}}_{10} = 0$ and $\dot{\hat{\theta}}_1 = 0$, x_1 is bounded, $\hat{\theta}_{10}$ and $\hat{\theta}_1$ are kept unchanged in bounded values. We can readily conclude that the tracking error$|z_1(t)| \leqslant \lambda_{a1}$ while all the other closed-loop signals are bounded. □

The key point of the proposed design lies in two aspects. Firstly, the Lyapunov–Krasovskii functional is utilized such that the design difficulties from unknown time delay has been removed. Secondly, the practical control scheme including a deadzone has employed to avoid possible controller singularity. It is well known in Utkin (1978) and Slotine & Li (1991) that the above discontinuous control scheme should be avoided as it will cause chattering phenomena and excite high-frequency unmodeled dynamics. Furthermore, we would like to extend the methodology described in this section from first-order systems to more general nth-order systems. To achieve this objective, the iterative backstepping design can be used, which requires the differentiation of the control u and the control component β_1 at each step. Therefore, appropriate smooth control functions shall be used, and at the same time the controller should guarantee the boundedness of all the signals in the closed-loop and z_1 will still stay in certain domain of attraction.

4. Adaptive controller design for nth-order systems

In this section, the adaptive design will be extended to nth-order systems (2) and the stability results of the closed-loop system are presented. Note that the extension requires the smoothness of intermediate control functions to certain degree, which is not straightforward but very much involved. In the recursive backstepping design, the computation of the intermediate control function $\alpha_i(t)$ in each step requires that of $\dot{\alpha}_{i-1}(t), \ddot{\alpha}_{i-2}(t), \ldots, \alpha_1^{(i-1)}$. As a result, α_i need to be at least $(n-i)$th differentiable. On the other hand, the unknown time delay terms of all the previous subsystems will appear in Step i, which have to be compensated for one by one.

The backstepping design procedure contains n steps. At each step, an intermediate control function $\alpha_i(t)$ shall be developed using an appropriate Lyapunov function $V_i(t)$. The design of both the control laws and the adaptive laws are based on the following change of coordinates: $z_1 = x_1 - y_d$, $z_i = x_i - \alpha_{i-1}$, $i = 2, \ldots, n$. For conciseness of notation, let us define the scalar function $V_{z_i}(t)$, the Lyapunov–Krasovskii functional $V_{U_i}(t)$, and the Lyapunov function candidate $V_i(t)$ as

$$V_{z_i}(t) = \frac{1}{2g_i} z_i^2(t), \tag{17}$$

$$V_{U_i}(t) = \frac{1}{2g_i} \sum_{j=1}^{i} \int_{t-\tau_j}^{t} U_j(\bar{x}_j(\tau))\,\mathrm{d}\tau,$$

$$U_j(\bar{x}_j(t)) = H_j^{\mathrm{T}}(\bar{x}_j(t)) H_j(\bar{x}_j(t)) + \psi_j^2(\bar{x}_j(t)), \tag{18}$$

$$V_i(t) = V_{z_i}(t) + V_{U_i}(t) + \tfrac{1}{2}\gamma_i^{-1}\tilde{\theta}_{i0}^2(t) + \tfrac{1}{2}\tilde{\theta}_i^{\mathrm{T}}(t)\Gamma_i^{-1}\tilde{\theta}_i(t), \tag{19}$$

where $\tilde{\theta}_{i0} = \hat{\theta}_{i0} - \theta_{i0}$ and $\tilde{\theta}_i = \hat{\theta}_i - \theta_i$ with $\hat{\theta}_{i0}$ and $\hat{\theta}_i$ being the estimates of θ_{i0} and θ_i, respectively, constant $\gamma_i > 0$, matrix $\Gamma_i = \Gamma_i^{-1} > 0$, and unknown parameters θ_{i0} and θ_i are defined as

$$\theta_{i0} := \max\{c_{f1}, \ldots, c_{fi}\},$$

$$\theta_i := \left[\frac{\theta_{fi}^{\mathrm{T}}}{g_i}, \frac{\theta_{hi}^{\mathrm{T}}\theta_{hi} + c_{hi}^2}{g_i}, \frac{g_{i-1}}{g_i}, \frac{g_{i-1}}{g_i}\theta_{i-1}^{\mathrm{T}}\right]^{\mathrm{T}} \in R^{\bar{n}_i}. \tag{20}$$

Let us consider the following robust adaptive control law:

$$\alpha_i = q_i(z_i)[-k_i(t)z_i - \hat{\theta}_i^{\mathrm{T}} F_{\theta i} - \beta_i], \tag{21}$$

$$k_i(t) = k_{i0} + \frac{1}{z_i^2} \sum_{j=1}^{i} \int_{t-\tau_{\max}}^{t} U_j(\bar{x}_j(\tau))\,\mathrm{d}\tau, \tag{22}$$

$$\dot{\hat{\theta}}_{i0} = q_i(z_i)\gamma_i(z_i\xi_i - \sigma_{i0}\hat{\theta}_{i0}), \tag{23}$$

$$\dot{\hat{\theta}}_i = q_i(z_i)\Gamma_i(F_{\theta i} z_i - \sigma_i \hat{\theta}_i), \tag{24}$$

where $\beta_i = \hat{\theta}_{i0}\xi_i$, $\xi_i = \phi_{i0}\tanh(\frac{z_i\phi_{i0}}{\varepsilon_i})$, $k_{i0} > 0$ is a design constant, $\varepsilon_i > 0$ is a small constant, constant $\gamma_i > 0$, matrix $\Gamma_i = \Gamma_i^{-1} > 0$, $\sigma_{i0}, \sigma_i > 0$ are small constants to introduce the σ–modification for the closed-loop system, and known function $\phi_{i0}(\cdot)$ and known function vector $F_{\theta i}(\cdot)$ are defined, respectively, as

$$\phi_{i0} := \phi_i + \sum_{j=1}^{i-1} \left|\frac{\partial\alpha_{i-1}}{\partial x_j}\right| \phi_j,$$

$$F_{\theta i} := \Bigg[F_i^{\mathrm{T}}, \frac{1}{2}z_i, -\frac{\partial\alpha_{i-1}}{\partial x_{i-1}}x_i, -\frac{\partial\alpha_{i-1}}{\partial x_{i-1}}F_{i-1}^{\mathrm{T}}, \frac{1}{2}z_i\left(\frac{\partial\alpha_{i-1}}{\partial x_{i-1}}\right)^2, - \frac{\partial\alpha_{i-1}}{\partial x_{i-2}}x_{i-1}, -\frac{\partial\alpha_{i-1}}{\partial x_{i-2}}F_{i-2}^{\mathrm{T}}, \frac{1}{2}z_i\left(\frac{\partial\alpha_{i-1}}{\partial x_{i-2}}\right)^2, \ldots, - \frac{\partial\alpha_{i-1}}{\partial x_1}x_2, -\frac{\partial\alpha_{i-1}}{\partial x_1}F_1^{\mathrm{T}}, \frac{1}{2}z_i\left(\frac{\partial\alpha_{i-1}}{\partial x_1}\right)^2, \frac{1}{2z_i}\sum_{j=1}^{i} H_j^{\mathrm{T}}H_j + \psi_j^2 - \omega_{i-1}\Bigg]^{\mathrm{T}} \in R^{\bar{n}_i},$$

$$\bar{n}_i = \sum_{j=1}^{i} n_j + 2i$$

with $\omega_{i-1} = \frac{\partial\alpha_{i-1}}{\partial\bar{x}_{di}}\dot{\bar{x}}_{di} + \sum_{j=1}^{i-1}(\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_{j0}}\dot{\hat{\theta}}_{j0} + \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_j}\dot{\hat{\theta}}_j)$. Note that when $i = n$, $\alpha_i = u$. In addition, define the positive constants $c_i := \min\{\frac{3}{2}g_{\min}k_{i0}, 2g_{\min}, \sigma_{i0}\gamma_i, \frac{\sigma_i}{\lambda_{\max}(\Gamma_i^{-1})}\}$ and $\lambda_i := \frac{1}{2}\sigma_{i0}\theta_{i0}^2 + \frac{1}{2}\sigma_i\|\theta_i\|^2 + 0.2785\varepsilon_i\theta_{i0}$.

Step 1: Let us firstly consider the z_1-subsystem as

$$\dot{z}_1(t) = g_1(z_2(t) + \alpha_1(t)) + \theta_{f1}^{\mathrm{T}}F_1(x_1(t)) + \delta_{f1}(x_1(t)) + \theta_{h1}^{\mathrm{T}}H_1(x_1(t-\tau_1)) + \delta_{h1}(x_1(t-\tau_1)) - \dot{y}_d(t) \tag{25}$$

Following the same procedure as in Section 3 by choosing $V_{z_1}(t)$ in (17) and V_{U_1} in (18), and applying Assumption 2 and Young's inequality, we obtain

$$\dot{V}_{z_1} + \dot{V}_{U_1} \leqslant z_1z_2 + z_1(\alpha_1 + \theta_1^{\mathrm{T}}F_{\theta 1}) + \theta_{10}|z_1|\phi_{10}. \tag{26}$$

As stated in Section 3, the control objective now is to show that z_1 will converge to certain domain of attraction rather than the origin. At the same time, the control functions shall be smooth or at least differentiable to certain degree. Let us consider the smooth adaptive scheme (21)–(24) and the Lyapunov function candidate $V_1(t)$ given in (19).The time derivative of $V_1(t)$ along (21)–(24) when $|z_1| \geqslant \lambda_{a1} + \lambda_{b1}$ is

$$\dot{V}_1(t) \leqslant -k_{10}z_1^2 - \int_{t-\tau_{\max}}^{t} U_1(x_1(\tau))\,\mathrm{d}\tau + z_1 z_2 + \theta_{10}\left[|z_1|\phi_{10} - z_1\phi_{10}\tanh\left(\frac{z_1\phi_{10}}{\varepsilon_1}\right)\right] - \sigma_{10}\tilde{\theta}_{10}\hat{\theta}_{10} - \sigma_1\tilde{\theta}_1^{\mathrm{T}}\hat{\theta}_1.$$

Using the inequalities $-\frac{1}{4}k_{10}z_1^2 + z_1z_2 \leqslant \frac{1}{k_{10}}z_2^2$, $-\sigma_{10}\tilde{\theta}_{10}\hat{\theta}_{10} \leqslant -\frac{1}{2}\sigma_{10}\tilde{\theta}_{10}^2 + \frac{1}{2}\sigma_{10}\theta_{10}^2$, and $-\sigma_1\tilde{\theta}_1^{\mathrm{T}}\hat{\theta}_1 \leqslant -\frac{1}{2}\sigma_1\|\tilde{\theta}_1\|^2 + \frac{1}{2}\sigma_1\|\theta_1\|^2$, and applying Lemma 1, we have

$$\dot{V}_1(t) \leqslant -\frac{3}{4}k_{10}z_1^2 - \int_{t-\tau_{\max}}^{t} U_1(x_1(\tau))\,\mathrm{d}\tau + \frac{1}{k_{10}}z_2^2 - \frac{1}{2}\sigma_{10}\tilde{\theta}_{10}^2 - \frac{1}{2}\sigma_1\|\tilde{\theta}_1\|^2 + \lambda_1. \tag{27}$$

Since $\tau_1 \leqslant \tau_{\max}$ according to Assumption 3, the inequality $\int_{t-\tau_1}^{t} U_1(x_1(\tau))\,\mathrm{d}\tau \leqslant \int_{t-\tau_{\max}}^{t} U_1(x_1(\tau))\,\mathrm{d}\tau$ holds. Accordingly, (27) becomes

$$\dot{V}_1(t) \leqslant -c_1V_1(t) + \lambda_1 + \frac{1}{k_{10}}z_2^2. \tag{28}$$

Remark 3. If there is no extra term z_2^2 within inequality (28), we can conclude that for $|z_1| \geqslant \lambda_{a1} + \lambda_{b1}$, $V_1(t)$ is bounded, and thus z_1, $\hat{\theta}_{10}$ and $\hat{\theta}_1$ are bounded, and for $|z_1| < \lambda_{a1} + \lambda_{b1}$, the boundness of these signals directly follows similarly as in Lemma 3 and is independent of z_2. However, it may not be the case due to the presence of the extra term z_2^2. It is found that if z_2 can be regulated as bounded, say, $|z_2| \leqslant z_{2\max}$ with $z_{2\max}$ being finite, we have

$$\dot{V}_1(t) \leqslant -c_1V_1(t) + \bar{\lambda}_1$$

with $\bar{\lambda}_1 = \lambda_1 + \frac{1}{k_{10}}z_{2\max}^2$. The stability analysis for this case will be shown later.

Remark 4. Note that both the intermediate control function (21) and the updating laws (23), (24) are differentiable, which makes it possible to carry out the backstepping design in the next steps.

The regulation of z_2 will be shown in the next steps.

Step i ($2 \leqslant i \leqslant n-1$): Similar procedures are taken for each steps when $i = 2, \ldots, n-1$ as in Step 1. The time derivative of $z_i(t)$ is given by

$$\dot{z}_i(t) = g_i[z_{i+1}(t) + \alpha_i(t)] + \theta_{fi}^{\mathrm{T}}F_i(\bar{x}_i(t)) + \delta_{fi}(\bar{x}_i(t)) + \theta_{hi}^{\mathrm{T}}H_i(\bar{x}_i(t-\tau_i)) + \delta_{hi}(\bar{x}_i(t-\tau_i)) - \dot{\alpha}_{i-1}(t). \tag{29}$$

Since $\alpha_{i-1}(t)$ is a function of $\bar{x}_{i-1}$, $\bar{x}_{di}$, $\hat{\theta}_{10}, \ldots, \hat{\theta}_{i-1,0}$, $\hat{\theta}_1, \ldots, \hat{\theta}_{i-1}$, $\dot{\alpha}_{i-1}(t)$ can be expressed as

$$\dot{\alpha}_{i-1}(t) = \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\dot{x}_j + \omega_{i-1}.$$

Considering the scalar functions $V_{z_i}(t)$ in (17) and the functional $V_{U_i}(t)$ in (18) and noting Assumption, we have

$$\dot{V}_{z_i} + \dot{V}_{U_i} \leqslant z_i z_{i+1} + z_i(\alpha_i + \theta_i^{\mathrm{T}}F_{\theta i}) + \theta_{i0}|z_i|\phi_{i0}, \tag{30}$$

where unknown parameters θ_{i0} and θ_i, known functions $\phi_{i0}(\cdot)$ and $F_{\theta i}(\cdot)$ are defined before.

Similarly, consider the robust adaptive intermediate control law (21)–(24) and the Lyapunov function candidate $V_i(t)$ given in (19). For $|z_i| \geqslant \lambda_{ai} + \lambda_{bi}$, the time derivative of $V_i(t)$ is

$$\dot{V}_i(t) \leqslant -c_iV_i(t) + \lambda_i + \frac{1}{k_{i0}}z_{i+1}^2. \tag{31}$$

If z_{i+1} can be regulated as bounded, say, $|z_{i+1}| \leqslant z_{i+1,\max}$ with $z_{i+1,\max}$ being finite, from (31), we have that $\dot{V}_i(t) \leqslant -c_iV_i(t) + \bar{\lambda}_i$ with $\bar{\lambda}_i = \lambda_i + \frac{1}{k_{i0}}z_{i+1,\max}^2$. The stability analysis for this case will be shown later and the effect of z_{i+1} will be handled in the next steps.

For $|z_i| < \lambda_{ai} + \lambda_{bi}$, similarly as in Step 1, the following two cases are considered: (i) if $|z_{i-1}| < \lambda_{a,i-1} + \lambda_{b,i-1}$, the following two sub-cases shall be considered for $|z_{i-2}| \geqslant \lambda_{a,i-2} + \lambda_{b,i-2}$ and $|z_{i-2}| < \lambda_{a,i-2} + \lambda_{b,i-2}$, and the same procedure will apply backwards for z_{i-3} till z_1; and (ii) if $|z_{i-1}| \geqslant \lambda_{a,i-1} + \lambda_{b,i-1}$, we know from Step $(i-1)$ that $\dot{V}_{i-1}(t) \leqslant -c_{i-1}V_{i-1}(t) + \lambda_{i-1} + \frac{1}{k_{i-1,0}}z_i^2$, hence

$$\dot{V}_{i-1}(t) \leqslant -c_{i-1}V_{i-1}(t) + \bar{\lambda}_{i-1}$$

with $\bar{\lambda}_{i-1} = \lambda_{i-1} + \frac{1}{k_{i-1,0}}(\lambda_{ai} + \lambda_{bi})^2$. The stability results will be given later.

Step n: This is the final step, since the actual control u appears in the derivative of $z_n(t)$ as given in

$$\dot{z}_n(t) = g_nu(t) + \theta_{fn}^{\mathrm{T}}F_n(x(t)) + \delta_{fn}(x(t)) + \theta_{hn}^{\mathrm{T}}H_n(x(t-\tau_n)) + \delta_{hn}(x(t-\tau_n)) - \dot{\alpha}_{n-1}(t). \tag{32}$$

Since $\alpha_{n-1}(t)$ is a function of $\bar{x}_{n-1}$, $\bar{x}_{dn}$, $\hat{\theta}_{10}, \ldots, \hat{\theta}_{n-1,0}$, $\hat{\theta}_1, \ldots, \hat{\theta}_{n-1}$, $\dot{\alpha}_{n-1}(t)$ can be expressed as

$$\dot{\alpha}_{n-1}(t) = \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}\dot{x}_j + \omega_{n-1}(t).$$

Considering the scalar functions $V_{z_n}(t)$ in (17) and the functional V_{U_i} in (18), and applying Assumption 2 and the Young's Inequality, we have

$$\dot{V}_{z_n} + \dot{V}_{U_n} \leqslant z_n(u + \theta_n^{\mathrm{T}}F_{\theta n}) + \theta_{n0}|z_n|\phi_{n0}. \tag{33}$$

Similarly, consider the robust adaptive control law (21)–(24) and the Lyapunov function candidate $V_n(t)$ given in (19).

For $|z_n| \geqslant \lambda_{an} + \lambda_{bn}$, the final control $u(t)$ is invoked and the time derivative of $V(t)$ along (21)–(24) and (33) is

$$\dot{V}_n(t) \leqslant -c_n V_n(t) + \lambda_n. \tag{34}$$

It is known from (34) that $V_n(t)$ is bounded, hence z_n, $\hat{\theta}_{n0}$ and $\hat{\theta}_n$ are bounded.

For $|z_n| < \lambda_{an} + \lambda_{bn}$, two cases are considered: (i) if $|z_{n-1}| < \lambda_{a,n-1} + \lambda_{b,n-1}$, we shall similarly consider the two sub-cases, i.e., $|z_{n-2}| \geqslant \lambda_{a,n-2} + \lambda_{b,n-2}$ and $|z_{n-2}| < \lambda_{a,n-2} + \lambda_{b,n-2}$ and such analysis shall be carried out backwards till z_1; and (ii) $|z_{n-1}| \geqslant \lambda_{a,n-1} + \lambda_{b,n-1}$, we know from Step $(n-1)$ that $\dot{V}_{n-1}(t) \leqslant -c_{n-1}V_{n-1}(t) + \lambda_{n-1} + \frac{1}{k_{n-1,0}} z_n^2$, hence

$$\dot{V}_{n-1}(t) \leqslant -c_{n-1}V_{n-1}(t) + \bar{\lambda}_{n-1}$$

with $\bar{\lambda}_{n-1} = \lambda_{n-1} + \frac{1}{k_{n-1,0}}(\lambda_{an} + \lambda_{bn})^2$.

Theorem 1 shows the stability and control performance of the closed-loop adaptive system.

Theorem 1. *Consider the closed-loop system consisting of the plant* (2) *under Assumptions* 1–4. *If we apply the control laws* (21)–(24), *the following properties can be guaranteed under any finite initial conditions*: (i) z_i, $\hat{\theta}_{i0}$, $\hat{\theta}_i$ *and* x_i, $i = 1, \ldots, n$, *are globally uniformly ultimately bounded*; and (ii) *the signal* $z(t) = [z_1, \ldots, z_n]^{\mathrm{T}} \in R^n$ *will eventually converge to the compact set defined by*

$$\Omega_z := \{z \mid \|z\| \leqslant \mu\}$$

with $\mu = \max\{\sqrt{2g_{\max}\rho}, \sqrt{\sum_{j=1}^n (\lambda_{ai} + \lambda_{bi})^2}\}$ *and the compact set* Ω_z *can be made as small as desired by an appropriate choice of the design parameters*.

Proof. Consider the following Lyapunov function candidate:

$$V(t) = \sum_{i=1}^n \left[V_{z_i}(t) + V_{U_i}(t) + \frac{1}{2}\gamma_i^{-1}\tilde{\theta}_{i0}^2(t) + \frac{1}{2}\tilde{\theta}_i^{\mathrm{T}}(t)\Gamma_i^{-1}\tilde{\theta}_i(t) \right], \tag{35}$$

where $V_{z_i}(t)$, $V_{U_i}(t)$, $i = 1, \ldots, n$ are defined as before, and $\tilde{(\cdot)} = \hat{(\cdot)} - (\cdot)$. The following three cases are considered:

Case 1: All z_i's, $i = 1, \ldots, n$, are satisfying $|z_i| \geqslant \lambda_{ai} + \lambda_{bi}$. From the previous derivation, we have the following inequality for $|z_i| \geqslant \lambda_{ai} + \lambda_{bi}$, $i = 1, \ldots, n$

$$\dot{V}(t) \leqslant -cV(t) + \lambda,$$

where $c := \min\{c_1, \ldots, c_n\}$ and $\lambda := \sum_{i=1}^n \lambda_i$. Let $\rho := \lambda/c$, it follows that

$$0 \leqslant V(t) \leqslant [V(0) - \rho]\mathrm{e}^{-ct} + \rho \leqslant V(0) + \rho, \tag{36}$$

where the constant $V(0) = \sum_{i=1}^n [\frac{1}{2g_i} z_i^2(0) + \frac{1}{2}\gamma_i^{-1}\tilde{\theta}_{i0}^2(0) + \frac{1}{2}\tilde{\theta}_i^{\mathrm{T}}(0)\Gamma_i^{-1}\tilde{\theta}_i(0)]$.

Considering (35), we know that

$$\sum_{i=1}^n z_i^2 \leqslant 2g_{\max}[V(0) + \rho] \tag{37}$$

$$\sum_{i=1}^n \tilde{\theta}_{i0}^2 \leqslant 2\max\{\gamma_i\}[V(0) + \rho],$$

$$\sum_{i=1}^n \|\tilde{\theta}_i\|^2 \leqslant \frac{2[V(0) + \rho]}{\lambda_{\min}\{\Gamma_i^{-1}\}}. \tag{38}$$

It can be seen from (36)–(38) that $V(t)$ is bounded, hence z_i, $\hat{\theta}_{i0}$ and $\hat{\theta}_i$ are uniformly bounded for $|z_i| \geqslant \lambda_{ai} + \lambda_{bi}$, $i = 1, \ldots, n$. In addition, from (35) and (36), we have

$$\|z\| \leqslant \sqrt{2g_{\max}[(V(0) - \rho)\mathrm{e}^{-ct} + \rho]}$$

i.e., $\lim_{t\to\infty}\|z\| = \sqrt{2g_{\max}\rho}$. Since the above analysis is carried out for $|z_i| \geqslant \lambda_{ai} + \lambda_{bi}$, $i = 1, \ldots, n$, we have that $\lim_{t\to\infty}\|z\| = \max\{\sqrt{2g_{\max}\rho}, \sqrt{\sum_{j=1}^n (\lambda_{ai} + \lambda_{bi})^2}\}$.

Case 2: All z_i's, $i = 1, \ldots, n$, are satisfying $|z_i| < \lambda_{ai} + \lambda_{bi}$. In this case, $V_n(t)$ is bounded, hence z_i, x_i, $\hat{\theta}_{i0}$ and $\hat{\theta}_i$, $i = 1, \ldots, n$ are all bounded. In addition, $\|z\| \leqslant \sqrt{\sum_{j=1}^n (\lambda_{ai} + \lambda_{bi})^2}$.

Case 3: Some z_i's are satisfying $|z_i| \geqslant \lambda_{ai} + \lambda_{bi}$, while some z_j's are satisfying $|z_j| < \lambda_{aj} + \lambda_{bj}$. For $|z_i| \geqslant \lambda_{ai} + \lambda_{bi}$, the control effort α_i will render $\dot{V}_i \leqslant -c_i V_i + \lambda_i + \frac{1}{k_{i0}} z_{i+1}^2$. If z_{i+1} is bounded, the boundedness of z_i can be guaranteed. Otherwise, the control effort α_{i+1} will be invoked, which yields $\dot{V}_{i+1} \leqslant -c_{i+1}V_{i+1} + \lambda_{i+1} + z_{i+2}^2$. Similarly, regulation of z_{i+2} will be left to the next steps till the final step where z_n will be regulated as bounded. Therefore, those z_i's will be regulated as bounded finally. For those $|z_j| < \lambda_{aj} + \lambda_{bj}$, their boundedness has already obtained.

Therefore, we can conclude from Cases 1–3 that all the closed-loop signals are GUUB and there does exist a compact set Ω_z such that z will eventually converge to. This completes the proof. □

Remark 5. Theorem 1 shows that the system tracking error converges to a domain of attraction defined by compact set Ω_z rather than the origin. This is due to the introduction of the practical control, the smooth β_i control component and the σ-modification for the parameter adaptation. Even though the size of the compact set is unknown due to the unknown parameters $g_{\min}$, $g_{\max}$, θ_{i0} and θ_i, $i = 1, \ldots, n$, it is possible to make it as small as possible by appropriately choosing the design parameters. However, parameters such as λ_{ai} or λ_{bi} cannot be made zero to avoid possible control singularity and computational singularity. Therefore, in practical applications, the design parameters should be adjusted carefully for achieving suitable transient performance and control action.

Remark 6. The unknown parameters have been rearranged into a newly defined vector in each step of the iterative backstepping design. By doing so, on one hand, unknown vectors $\theta_{hi}, i=1,\ldots,n$ have been lumped as scalars, which reduces the number of parameters to be estimated in each step and finally reduces the order of the controller dramatically. On the other hand, we only need to estimate $\frac{1}{g_i}$ rather than g_i such that possible controller singularities due to $\hat{g}_i=0$ have been avoided.

5. Simulation studies

To illustrate the proposed robust adaptive control algorithms, we consider the following second-order plant:

$$\dot{x}_1(t)=g_1x_2(t)+\theta_{f1}x_1^2(t)+\delta_{f1}(x_1(t)),$$

$$\dot{x}_2(t)=g_1u(t)+\theta_{h2}x_2(t-\tau_2)+\delta_{h2}(x(t-\tau_2)),$$

$$y(t)=x_1(t),$$

where g_1, g_2 are unknown virtual control coefficients, θ_{f1}, θ_{h2} are unknown parameters, and $\delta_{f1}(\cdot)$, $\delta_{h2}(\cdot)$ are unknown functions. For simulation purpose, we assume that $g_1=2$, $g_2=1$, $\theta_{f1}=0.1$, $\theta_{h2}=0.2$, and let $\delta_{f1}=0.6\sin(x_1)$, $\delta_{h2}=0.5(x_1^2+x_2^2)\sin(x_2)$. The bounds on $\delta_{f1}(\cdot)$ and $\delta_{h2}(\cdot)$ are $|\delta_{f1}(x_1)|\leqslant c_{f1}\phi_1(x_1)$, $|\delta_{h2}(x)|\leqslant c_{h2}\psi_2(x)$, where $c_{f1}=0.6$, $\phi_1(x_1)=1$, $c_{h2}=0.5$, $\psi_2(x)=x_1^2+x_2^2$. The unknown time delays are $\tau_1=0$, $\tau_2=3\,\text{s}$. The control objective is to track the desired reference signal $y_d(t)=0.5[\sin(t)+\sin(0.5t)]$. For the design of robust adaptive controller, let $z_1=x_1-y_d$, $z_2=x_2-\alpha_1$ and $\hat{\theta}_1,\hat{\theta}_2$ be the estimates of unknown parameter vectors $\theta_1=[\frac{\theta_{f1}}{g_1},\frac{1}{g_1}]^{\mathrm{T}}$, $\theta_2=[\frac{\theta_{h2}^2+c_{h2}^2}{g_2},\frac{g_1}{g_2},\frac{\theta_{f1}}{g_2},\frac{1}{g_2}]^{\mathrm{T}}$, respectively, we have

$$\alpha_1(t)=q_1(z_1)[-k_1z_1-\hat{\theta}_1^{\mathrm{T}}F_{\theta 1}-\beta_1],$$

$$u(t)=\begin{cases}-k_2z_2-\hat{\theta}_2^{\mathrm{T}}F_{\theta 2}-\beta_2, & |z_2|\geqslant c_{z_2},\\ 0, & |z_2|<c_{z_2},\end{cases}$$

$$\beta_i=\hat{\theta}_{i0}\xi_i,\ \xi_i=\phi_{i0}\tanh\left(\frac{z_i\phi_{i0}}{\varepsilon_i}\right),$$

$$\dot{\hat{\theta}}_{i0}=\gamma_i(z_i\xi_i-\sigma_{i0}\hat{\theta}_{i0}),\quad \dot{\hat{\theta}}_i=\Gamma_i(F_{\theta i}z_i-\sigma_i\hat{\theta}_i),\quad i=1,2,$$

where $k_i(t)$ is given in (22), $i=1,2$. The following design parameters are adopted in the simulation: $[x_1(0),x_2(0)]^{\mathrm{T}}=[0.1,0.1]^{\mathrm{T}}$, $\gamma_1=\gamma_2=1$, $\Gamma_1=\Gamma_2=\mathrm{diag}\{1\}$, $\sigma_{10}=\sigma_{20}=\sigma_1=\sigma_2=0.05$, $\theta_{10}^0=\theta_{20}^0=0$, $\theta_1^0=\theta_2^0=0$, $k_{10}=k_{20}=0.8$, $\varepsilon_1=\varepsilon_2=0.1$, and $c_{z_1}=c_{z_2}=1.0\mathrm{e}^{-3}$.

From Fig. 1, it was seen that satisfactory transient tracking performance was obtained after 10 s of adaptation periods. Figs. 2 and 3 show the boundedness of the control input and the estimates of the parameters in the control loop. Among the design parameters, the choices of c_{z_i} are critical for achieving good control performance. Through extensive

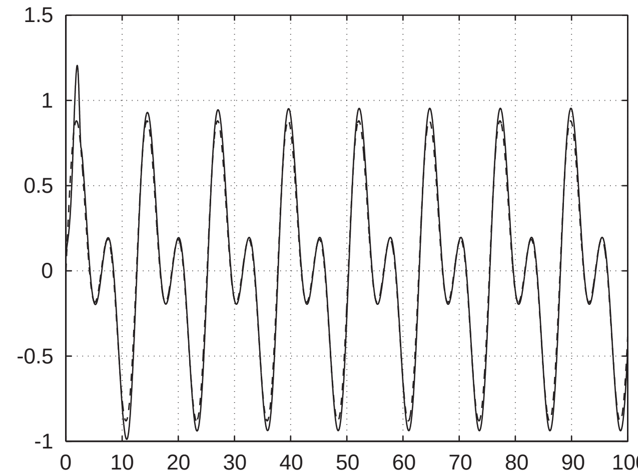

Fig. 1. Output $y(t)$ (solid), and reference y_d (dashed).

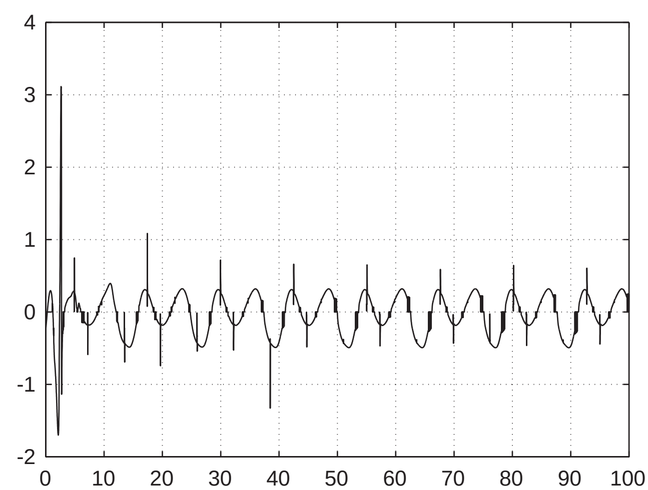

Fig. 2. Control input $u(t)$.

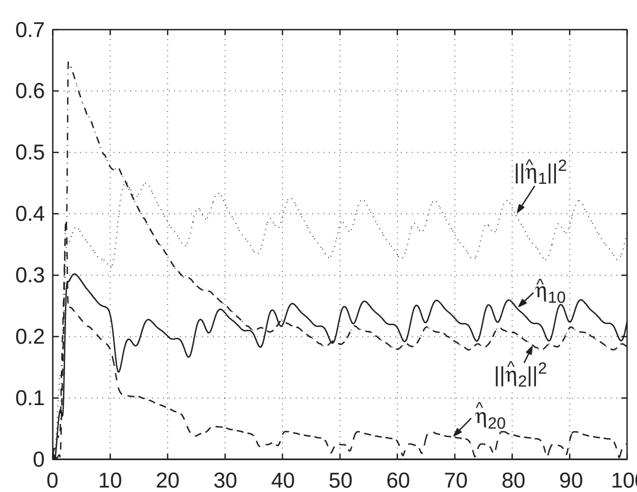

Fig. 3. Parameter estimates: $\hat{\theta}_{10}$ (solid), $\hat{\theta}_{20}$ (dashed), $\|\hat{\theta}_1\|^2$ (dotted), $\|\hat{\theta}_2\|^2$ (dash-dotted).

simulation study, it was found that c_{z_i} should not be chosen as too small. From analytical point of view, it is found that the known functions $F_{\theta i}$ which are used for on-line parameters tuning contain possibly singular terms. The robust design is then carried out to make sure those terms to be bounded. Although c_{z_i} can be chosen arbitrarily small theoretically, it is not the case in real implementation due to the limited actuator tolerance and computational capacity.

6. Conclusion

A robust adaptive control has been addressed for a class of parametric-strict-feedback nonlinear systems with varying unknown time delays. The uncertainty from unknown time delays has been compensated for through the use of appropriate Lyapunov–Krasovskii functionals. The controller has been made to be free from singularity problem by employing practical robust control and regrouping unknown parameters. Backstepping design has been carried out for a class of nonlinear systems in strict feedback form by using differentiable approximation. The proposed systematic backstepping design method has been proved to be able to guarantee global uniformly ultimately boundedness of closed-loop signals. In addition, the output of the system has been proven to converge to an arbitrarily small neighborhood of the origin. Simulation results have been provided to show the effectiveness of the proposed approach.

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