Pergamon

0005-1098(95)00129-8

Automatica, Vol. 32, No. 2, pp. 273–278, 1996

0005-1098/96 \$15.00 + 0.00

Brief Paper

Adaptive Controller Design for Flexible Joint Manipulators*

SHUZHI S. GE†

Key Words—Adaptive control; singular perturbations; flexible joint robots.

Abstract—A new adaptive controller for flexible joint robots is presented here based on singular perturbation theory and using only position and velocity feedback by modelling the 'motor tracking error' Ke_m as the fast variables instead of the joint elastic forces, where K represents the joint elasticity and e_m is the motor tracking error. The resulting boundary layer and quasi-steady-state systems are made exponentially stable; therefore the statements of Tychonov's theorem are valid for an infinite time interval.

1. *Introduction*

Since it was pointed out by Sweet and Good (1984) that joint flexibility should be taken into account in both modelling and control if high performance is required, considerable research work has been carried out on the control of flexible-joint robots. A very good survey was given by Spong (1990), who listed about 100 papers. Singular perturbation techniques were applied to flexible joint robots in Marino and Nicosia (1985) Spong (1990) and among others in the literature.

One of the first adaptive results for flexible joint robots was suggested by Spong (1989), based on singular perturbation theory (SPT) and only using position and velocity feedback. The method is simple compared with most of the proposed controllers in the literature. The idea is based on adding a simple correction term to the control law for rigid bodied robots to damp out the elastic oscillation at the joints. The control design is based on SPT by modelling the joint elastic forces as the fast variables and the link variables as the slow variables. Some later work (Hung, 1990; Spong, 1990; Chang and Daniel, 1992) was based on the resulting singular perturbation model. Even though most results are only valid for finite intervals, they can be easily extended to hold for an infinite interval, as the resulting slow-time dynamics coincides with the rigid robot model, for which exponentially stable controllers exist.

Another adaptive controller was presented by Ge (1993), based on SPT and also only using position and velocity feedback by modelling the 'motor tracking error' Ke_m as the fast variable instead of the joint elastic forces, where K represents the joint elasticity and e_m is the motor tracking error. As a consequence, the slow-time dynamics are also different from those given by Spong (1989), as are the resulting control laws. It is shown that the singular perturbation model for the same system is not unique. The new choice of fast variable provides a different insight into the controller design problem of flexible joint robots. The adaptive control law in Ge (1993) is only valid for a finite time interval. Since the resulting quasi-steady-state system has an extra parameter-error-related term compared with the system for a rigid bodied robot, the adaptive control law is extended here to an infinite-time-interval controller by taking the extra term into account.

This case is more complicated analytically, and more on-line computation is needed. Most importantly, it presents an alternative singular perturbation model for controller design. Both of these methods (i) are based on the assumption of weak elasticity, i.e. high stiffness at the joints, and (ii) do not need acceleration and jerk measurements, which is very desirable because these are difficult to measure.

2. *Dynamics of flexible joint robots*

The simplified dynamics of multi-axis flexible joint robots (Spong, 1987) are described by

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = K(\theta - q), \tag{1}$$

$$J\ddot{\theta} + K(\theta - q) = u, \tag{2}$$

where q and $\theta \in \mathbb{R}^n$ denote the angular displacements of the link and the motor shafts respectively, $D(q): \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is the symmetric positive-definite inertia matrix of the rigid links, $C(q, \dot{q})\dot{q}: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ represents the Coriolis and centrifugal forces, $G(q): \mathbb{R}^n \to \mathbb{R}^n$ is the gravitational force vector, $K \in \mathbb{R}^{n\times n}$ is a diagonal positive-definite matrix representing the spring constant matrix of n joints, $J \in \mathbb{R}^{n\times n}$ is the matrix of the moments of the inertia of the motors, and $u \in \mathbb{R}^n$ is the external input torque vector.

To facilitate controller design, we give the following two structural properties.

1. The matrix $\dot{D}(q) - 2C(q, \dot{q})$ is skew-symmetric if $C(q, \dot{q})$ is defined using the *Christoffel symbols*.

2. Dynamic parameter linearity of the rigid links enables us to write

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = Y(q, \dot{q}, \ddot{q})P,$$

where $P \in \mathbb{R}^r$ is the vector of robot parameters and Y is the corresponding matrix of known functions.

In this paper the following assumptions are made.

1. The desired link trajectory $q_d(t) \in \mathbb{C}^4$, and all time derivatives up to the fourth order are bounded and continuously differentiable (uniformly bounded in time t).

2. $q, \dot{q}$ and $\theta, \dot{\theta}$ are measurable for $t \in [0, \infty]$.

Therefore the desired trajectory θ_d and its time derivatives

* Received 16 February 1994; revised 13 June 1994; revised 6 September 1994; revised 17 May 1995; received in final form 11 July 1995. This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor P. Ioannou under the direction of Editor C. C. Hang. Corresponding author Dr Shuzhi Ge. Tel. +65 7726821; Fax +65 7791103; E-mail elegesz@leonis.nus.sg.

† Department of Electrical Engineering, National University of Singapore, 10 Kent Ridge Crescent, Singapore 0511.

up to second order are also uniformly bounded in t. They can be expressed as

$$\theta_d = K^{-1}Y(q_d, \dot{q}_d, \ddot{q}_d)P + q_d, \quad (3)$$

$$\dot{\theta}_d = K^{-1}\dot{Y}(q_d, \dot{q}_d, \ddot{q}_d)P + \dot{q}_d, \quad (4)$$

$$\ddot{\theta}_d = K^{-1}\ddot{Y}(q_d, \dot{q}_d, \ddot{q}_d)P + \ddot{q}_d. \quad (5)$$

Let Λ be a positive-definite matrix, and define

$$e = q_d - q, \quad (6)$$

$$\dot{q}_r = \dot{q}_d + \Lambda e, \quad (7)$$

$$r := \dot{q}_r - \dot{q} = \dot{e} + \Lambda e. \quad (8)$$

Let $\hat{P}$ be the time-varying estimate of the robot parameter P, and $\hat{D}, \hat{C}$ and $\hat{G}$ be the estimated matrices corresponding to D, C and G, obtained by replacing P by $\hat{P}$. Then the parameter linearity of the dynamics enables us to write

$$D(q)(\ddot{q}_r + \lambda_c r) + C(q, \dot{q})\dot{q}_r + G(q) = \Psi_g(q, \dot{q}, \dot{q}_r, \ddot{q}_r + \lambda_c r)P, \quad (9)$$

$$\hat{D}(q)(\ddot{q}_r + \lambda_c r) + \hat{C}(q, \dot{q})\dot{q}_r + \hat{G}(q) = \Psi_g(q, \dot{q}, \dot{q}_r, \ddot{q}_r + \lambda_c r)\hat{P}, \quad (10)$$

where Ψ_g is the corresponding matrix of known functions, and λ_c is a positive constant that is further clarified in Section 4. Define $(\tilde{*}) = (*) - (\hat{*})$; then, we have

$$\tilde{D}(q)(\ddot{q}_r + \lambda_c r) + \tilde{C}(q, \dot{q})\dot{q}_r + \tilde{G}(q) = \Psi_g(q, \dot{q}, \dot{q}_r, \ddot{q}_r + \lambda_c r)\tilde{P}. \quad (11)$$

Lemma 2.1. Let $e(t) = h * r$, where $h = L^{-1}(H(s))$ and $H(s)$ is an $n \times n$ strictly proper exponentially stable transfer function. Then $r \in L_n^2 \Rightarrow e \in L_n^2 \cap L_n^\infty$, $\dot{e} \in L_n^2$, e is continuous and $e \to 0$ as $t \to \infty$. If, in addition, $r \to 0$ as $t \to \infty$ then $\dot{e} \to 0$ (Desoer and Vidyasagar, 1975).

3. *Hierarchical control method*

The controller design technique (Ge, 1993), taking Ke_m as the fast variable, introduced in this section, is based on singular perturbation theory.

By freezing the estimated parameter vector $\hat{P}$ at t, the estimated position, velocity and acceleration can be expressed, from (3)–(5), as

$$\hat{\theta}_d := K^{-1}Y(q_d, \dot{q}_d, \ddot{q}_d)\hat{P} + q_d, \quad (12)$$

$$\hat{\dot{\theta}}_d := K^{-1}\dot{Y}(q_d, \dot{q}_d, \ddot{q}_d)\hat{P} + \dot{q}_d, \quad (13)$$

$$\hat{\ddot{\theta}}_d := K^{-1}\ddot{Y}(q_d, \dot{q}_d, \ddot{q}_d)\hat{P} + \ddot{q}_d. \quad (14)$$

Note that $\hat{\dot{\theta}}$ and $\hat{\ddot{\theta}}$ are not the first and second time derivatives of $\hat{\theta}$. They only serve as the estimates of $\dot{\theta}_d$ and $\ddot{\theta}_d$, and are to be used in the controller defined later because $\dot{\theta}_d$ and $\ddot{\theta}_d$ are not available under the assumption that P is an unknown constant. The corresponding estimation errors

$$\tilde{\theta}_d = \theta_d - \hat{\theta}_d = K^{-1}Y(q_d, \dot{q}_d, \ddot{q}_d)\tilde{P}, \quad (15)$$

$$\tilde{\dot{\theta}}_d = \dot{\theta}_d - \hat{\dot{\theta}}_d = K^{-1}\dot{Y}(q_d, \dot{q}_d, \ddot{q}_d)\tilde{P}, \quad (16)$$

$$\tilde{\ddot{\theta}}_d = \ddot{\theta}_d - \hat{\ddot{\theta}}_d = K^{-1}\ddot{Y}(q_d, \dot{q}_d, \ddot{q}_d)\tilde{P} \quad (17)$$

are bounded if $\tilde{P}$ is bounded. The boundedness of $\tilde{P}$ can be guaranteed by choosing an appropriate slow-timescale control law. This will be shown later.

Consider the control law

$$u = u_g + J(\hat{\ddot{\theta}}_d + K_{2v}\hat{\dot{e}}_m + K_{2p}\hat{e}_m), \quad (18)$$

where $\hat{e}_m = \hat{\theta}_d - \theta$, $\hat{\dot{e}}_m = \hat{\dot{\theta}}_d - \dot{\theta}$, K_{2v} and K_{2p} are diagonal positive-definite matrices, and u_g is the slow-timescale control, which will be defined in Section 4.

By defining $e_m = \theta_d - \theta$ and $\dot{e}_m = \dot{\theta}_d - \dot{\theta}$, we have

$$\hat{e}_m = \hat{\theta}_d - \theta = e_m - \tilde{\theta}_d, \quad (19)$$

$$\hat{\dot{e}}_m = \hat{\dot{\theta}}_d - \dot{\theta} = \dot{e}_m - \tilde{\dot{\theta}}_d, \quad (20)$$

$$\hat{\ddot{\theta}}_d = \ddot{\theta}_d - \tilde{\ddot{\theta}}_d. \quad (21)$$

Substituting the above three equations into (18), we obtain

$$u = v_g + J(\ddot{\theta}_d + K_{2v}\dot{e}_m + K_{2p}e_m) + \alpha, \quad (22)$$

where

$$\begin{aligned}\alpha &= -J(\tilde{\ddot{\theta}}_d + K_{2v}\tilde{\dot{\theta}}_d + K_{2p}\tilde{\theta}_d) \\ &= -JK^{-1}(\ddot{Y}(q_d, \dot{q}_d, \ddot{q}_d) + K_{2v}\dot{Y}(q_d, \dot{q}_d, \ddot{q}_d) \\ &\quad + K_{2p}Y(q_d, \dot{q}_d, \ddot{q}_d))\tilde{P} \\ &= \beta_d(t)\tilde{P}. \end{aligned} \quad (23)$$

Even though $\hat{e}_m, \hat{\dot{e}}_m$ and $\hat{\ddot{\theta}}_d$ are used in the controller, it is $e_m = \theta_d - \theta$ and its first and second derivatives that are used in stability analysis. The term α consists of the known function matrix β_d and the parameter estimation error vector $\tilde{P}$, which is absorbed in the slow-timescale parameter adaptation.

Define $z_1 := Ke_m$ and $z_0 = K(\theta_d - q)$; then $z := K(\theta - q) = z_0 - z_1$. Substituting (22) into (2) yields

$$J(\ddot{e}_m + K_{2v}\dot{e}_m + K_{2p}e_m) + Ke_m = z_0 - u_g - \alpha, \quad (24)$$

$$\ddot{z}_1 + K_{2v}\dot{z}_1 + (K_{2p} + KJ^{-1})z_1 = KJ^{-1}(z_0 - u_g - \alpha), \quad (25)$$

on noting that K_{2p}, K_{2v} and K are all diagonal matrices.

Under the assumptions of (i) high stiffness K as in Spong (1989), and (ii) high motor feedback gains K_{2p} and K_{2v}, we have

$$K_{2p} + KJ^{-1} = \frac{K_1}{\epsilon^2}, \quad (26)$$

$$K_{2v} = \frac{K_2}{\epsilon}, \quad (27)$$

where $K_1 > 0$ and $K_2 > 0$ are 'not large' and ϵ will be close to 0. Therefore (25) can be written as

$$\epsilon^2\ddot{z}_1 + \epsilon K_2\dot{z}_1 + K_1 z_1 = (K_1 - \epsilon^2 K_{2p})(z_0 - u_g - \alpha). \quad (28)$$

The complete system can now be written as

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = z_0 - z_1, \quad (29)$$

$$\epsilon^2\ddot{z}_1 + \epsilon K_2\dot{z}_1 + K_1 z_1 = (K_1 - \epsilon^2 K_{2p})(z_0 - u_g - \alpha), \quad (30)$$

which can easily be written in the standard singular perturbation form

$$\dot{x} = f(t, x, y, \epsilon), \quad x \in \mathbb{R}^{2n}, \quad (31)$$

$$\epsilon\dot{y} = g(t, x, y, \epsilon, u_g), \quad y \in \mathbb{R}^{2n}, \quad u_g \in \mathbb{R}^n, \quad (32)$$

with the choice of state variables

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \quad y = \begin{bmatrix} z_1 \\ \epsilon\dot{z}_1 \end{bmatrix}$$

and with appropriate definitions of f and g. Setting $\epsilon = 0$ in (30) leads to an explicit expression for $\bar{z}_1$:

$$\bar{z}_1 = \bar{z}_0 - u_g - \alpha, \quad (33)$$

where the overbars indicate that the variables are defined at $\epsilon = 0$. Clearly the singular perturbation model is different from that in Spong (1989). It can be seen that the singular perturbation model is not unique for the same system.

Substituting (33) into (29) at $\epsilon = 0$ yields the *quasi-steady-state* model

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \bar{z}_0 - \bar{z}_1 = u_g + \alpha. \quad (34)$$

Then a so-called *boundary layer system* can be written in terms of the variable $\eta = z_1 - \bar{z}_1$ by assuming that $\bar{z}_1$ is constant on a fast timescale $\tau = t/\epsilon$. Changing to this fast timescale τ, and substituting $z_1 = \eta + \bar{z}_1$ into the fast subsystem (30), with

$$\frac{d\bar{z}_1}{d\tau} = \frac{d^2\bar{z}_1}{d\tau^2} = 0 \quad (35)$$

and $\epsilon = 0$, we have

$$\frac{d^2\eta}{d\tau^2} + K_2\frac{d\eta}{d\tau} + K_1(\eta + \bar{z}_1) = K_1(\bar{z}_0 - u_g - \alpha). \quad (36)$$

Using the expression (33) for $\bar{z}_1$, the above equation reduces to the *boundary layer equation*

$$\frac{d^2\eta}{d\tau^2} + K_2\frac{d\eta}{d\tau} + K_1\eta = 0. \tag{37}$$

Tychonov's Theorem. Under some technical conditions on the smoothness of the functions f and g, if the quasi-steady system has a unique solution defined on an interval $t \in [0, t_1]$ and if the boundary layer system is exponentially uniformly stable in (t, q) then there exists ϵ^* such that, for all $\epsilon < \epsilon^*$,

$$z_1(t) = \bar{z}_1(t) + \eta(t/\epsilon) + O(\epsilon), \tag{38}$$

$$q(t) = \bar{q}(t) + O(\epsilon) \tag{39}$$

holds uniformly for $t \in [0, t_1]$.

However, the above statements are only valid for a finite time interval. In order for them to hold for an infinite time interval for sufficiently small ϵ, both the boundary layer system (37) and the quasi-steady-state system (34) have to be exponentially stable (Kahlil, 1992). The boundary layer system can easily be made exponentially stable for appropriate K_1 and K_2, which in turn determine the values of K_{2v} and K_{2p} by (26) and (27) for known K, J and some $\epsilon < \epsilon^*$. An exponentially convergent controller for (34) is introduced next.

4. *Exponentially stable control for rigid bodied robots*

Because of the presence of the extra term $\alpha = \beta(t)\tilde{P}$ in (34) compared with the case of a rigid bodied robots, available exponential controllers for rigid bodied robots have to be modified so that the closed-loop systems are exponentially stable. The method presented here can be taken as an extension of that introduced in Slotine and Li (1991) to accommodate the extra term $\alpha = \beta_d\tilde{P}$. Let $w(t)$ be the impulse response of a stable, proper filter such as the first-order filter $\lambda_f/(s + \lambda_f)$ (whose impulse response is $e^{-\lambda_f t}$). Then, convoluting both sides of (1), i.e.

$$z = Y(q, \dot{q}, \ddot{q})P, \tag{40}$$

with w, we have

$$\int_0^t w(t-s)z(s)\,ds = \int_0^t w(t-s)(D\ddot{q} + C\dot{q} + G)\,ds. \tag{41}$$

Using partial integration, the first term on the right-hand side of (41) can be rewritten as

$$\int_0^t w(t-s)D\ddot{q}\,ds$$

$$= w(t-s)D\dot{q}\,|_0^t - \int_0^t \frac{d}{ds}(wD)\dot{q}\,ds \tag{42}$$

$$= w(0)D(q)\dot{q} - w(t)D(q(0))\dot{q}(0) \tag{43}$$

$$- \int_0^t [w(t-s)\dot{D}\dot{q} - \dot{w}(t-s)D\dot{q}]\,ds. \tag{44}$$

This means that (41) can be written as

$$y(t) = W(q, \dot{q})P, \tag{45}$$

where y is the filtered elastic force z and W is the filtered version of Y. The filtered elastic force can be computed if the elastic torque z is measurable (or it can be computed from $K(\theta - q)$ for known K and measurable θ and q), and the matrix W can also be computed from the available measurements of q and $\dot{q}$.

Define $\hat{y} = W(q, \dot{q})\hat{P}$ and $\tilde{y} = y - \hat{y}$. An exponentially stable controller for the system (34), namely

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = u_g + \beta_d\tilde{P}, \tag{46}$$

is summarised in the following theorem.

Theorem 4.1. The quasi-steady-state system (46) is exponentially stable (i.e. there is exponential convergence of $e, \dot{e}$ and $\tilde{P}$) under the following control law and parameter adaptation algorithms:

$$u_g = \hat{D}(q)\ddot{q}_r + \hat{C}(q, \dot{q})\dot{q}_r + G(q) + K_D r, \tag{47}$$

$$K_D = \lambda_c\hat{D}, \tag{48}$$

$$\dot{\hat{P}} = \Gamma(\Psi^T r + \gamma_1 W^T\tilde{y}), \tag{49}$$

$$\Psi = \Psi_g - \beta_d, \tag{50}$$

$$\frac{d}{dt}\Gamma^{-1}(t) = -\lambda(t)\Gamma^{-1} + \gamma_2 W^T W, \tag{51}$$

with $2\gamma_1 > \gamma_2 > 0$ and the variable forgetting factor λ defined as

$$\lambda(t) = \lambda_0(1 - \|\Gamma\|/k_0),$$

where k_0 and λ_0 are two positive constants specifying the upper bound of the gain matrix norm and the maximum forgetting rate.

Proof. See Appendix A.

Note that (49) and (51) are more general than that given by Slotine and Li (1991). By taking $\gamma_1 = \gamma_2 = 1$, we have the special case introduced by Slotine and Li.

Theorem 4.2. From (51) and (52), one can show that $\forall t \ge 0$, $\lambda(t) \ge 0$, and $\Gamma(t) \le k_0 I$, where I is a dimensionally compatible unit matrix. In addition, if W is persistently exciting then there exists $\lambda_1 > 0$, $\forall t \ge 0$, $\lambda(t) \ge \lambda_1$, and Γ will be upper and lower bounded uniformly, i.e.

$$k_2 I \le \Gamma(t) \le k_1 i \tag{52}$$

where $0 < k_2 < k_1 < k_0$.

Proof. See Appendix B.

We have shown that the quasi-steady-state system (34) is exponentially stable under the control law and parameter adaptation algorithms of Theorem 4.1. For closed-loop stability, persistently excitation of the trajectory is not required, as can be seen from Theorems 4.1 and 4.2. In this case, $e, \dot{e}, \Gamma$ and $\hat{P}$ are all bounded, and $\Gamma^{-1} \ge I/k_0$. When the trajectory is persistently excited, the additional numerical properties of Γ stated in Theorem 4.2 does not affect the overall stability of the system.

Since the boundary layer system (37) can easily be made exponentially stable for appropriate K_1 and K_2, which in turn determine the values of K_{2v} and K_{2p} by (36) and (39) the statements of Tychonov's theorem are valid for an infinite time interval for sufficiently small ϵ.

5. *Simulation tests*

The robot used for simulation is a two-link planar manipulator. The equivalent motor inertia J is a unit matrix to make it have the same size as the link inertia (a common situation). The two-link robot arm is modelled by

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = K(\theta - q), \tag{53}$$

$$J\ddot{\theta} + K(\theta - q) = u. \tag{54}$$

The functions $D(q)$, $C(q, \dot{q})$ and $G(q)$ are expressed in terms of the elements of the parameter vector $P = [p_1\ p_2\ p_3\ p_4\ p_5]^T$ as follows:

$$D(q) = \begin{bmatrix} p_1 + p_2 + 2p_3\cos q_2 & p_2 + p_3\cos q_2 \\ p_2 + p_3\cos q_2 & p_2 \end{bmatrix},$$

$$C(q, \dot{q}) = \begin{bmatrix} -p_3\dot{q}_2\sin q_2 & -p_3(\dot{q}_1 + \dot{q}_2)\sin q_2 \\ p_3\dot{q}_1\sin q_2 & 0 \end{bmatrix},$$

$$G(q) = \begin{bmatrix} [p_4\cos q_1 + p_5\cos(q_1 + q_2)]g \\ p_5\cos(q_1 + q_2)\,g \end{bmatrix},$$

where the gravitational acceleration $g \in \mathbb{R}$ is assumed to be 9.81. The numerical values for the parameter vector P, and the known K and J are shown in Table 1.

5.1. *Trajectory planning.* The desired trajectory for each axis is expressed as a Hermite polynomial of the fifth degree

Table 1. Parameters used for simulation

Parameter	Value
$k_1 = k_2$	$400\,\mathrm{N\,m^{-1}}$
$J_1 = J_2$	$1.0\,\mathrm{kg\,m^2}$
$[p_1\ \ p_2\ \ p_3\ \ p_4\ \ p_5]$	$[1.66\ \ 0.42\ \ 0.63\ \ 3.75\ \ 1.25]\,\mathrm{kg\,m^2}$

in t with continuous bounded position, velocity and acceleration. The general expression for the desired position trajectory is

$$q_d(t, t_d) = q_0 + \left(6.0\frac{t^5}{t_d^5} - 15\frac{t^4}{t_d^4} + 10.0\frac{t^3}{t_d^3}\right)(q_f - q_0), \tag{55}$$

where q_0 and q_f are the arm initial and final positions, and t_d represents the time at which the desired arm trajectory reaches the desired final position. In the simulation tests the following values were chosen:

$$t_d = 1.0\,\mathrm{s},\quad q_d(0) = [-1.0\ \ -1.0]^T\,\mathrm{rad},\quad q_d(t_d) = [0.0\ \ 1.0]^T\,\mathrm{rad}.$$

Assume the manipulator is initially at rest with $q(0) = [-0.98\ \ -1.01]^T$, $\dot{q}(0) = [0.0\ \ 0.0]^T$ and $\theta(0) = [-0.98\ \ -1.01]^T$, $\dot{\theta}(0) = [0.0\ \ 0.0]^T$. For the controller gains, $K_{2p} = \mathrm{diag}\,[60.0]$, $K_{2v} = \mathrm{diag}\,[60.0]$, while $\lambda_c = 1.0$, $\Lambda = \mathrm{diag}\,[5.0]$. For the parameter adaptation, $\gamma_1 = 5.0$, $\gamma_2 = 4.0$, $\lambda_0 = 1.0$, $\Gamma^{-1}(0) = \mathrm{diag}\,[50.0]$, $k_0 = 1.0$ and $\hat{P}(0) = [2.0\ 1.0\ 1.0\ 3.5\ 1.0]^T\,\mathrm{kg\,m^2}$. It is clear that $\hat{P}(0) \neq P$ and $\Gamma(0) < k_0 I$. A first-order filter $\lambda_f/(s + \lambda_f)$ with $\lambda_f = 3.0$ is used to compute W and y.

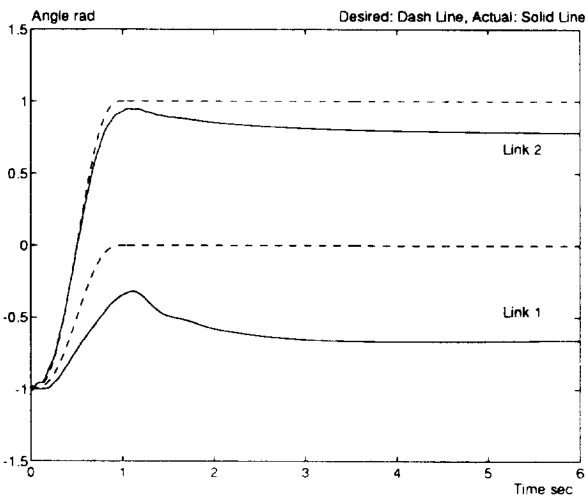

Fig. 1. Link trajectory tracking without parameter adaptation.

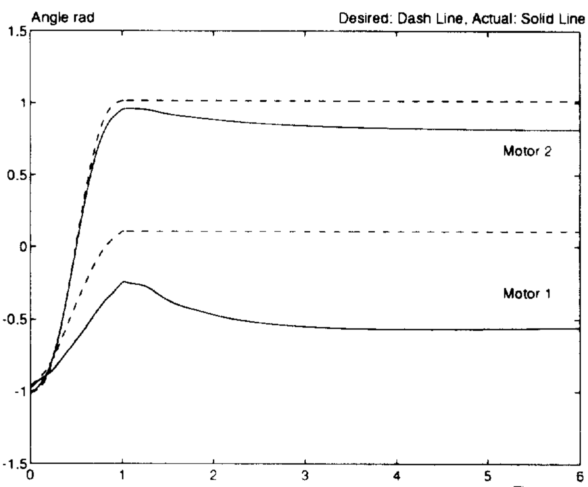

Fig. 2. Motor trajectory tracking without parameter adaptation.

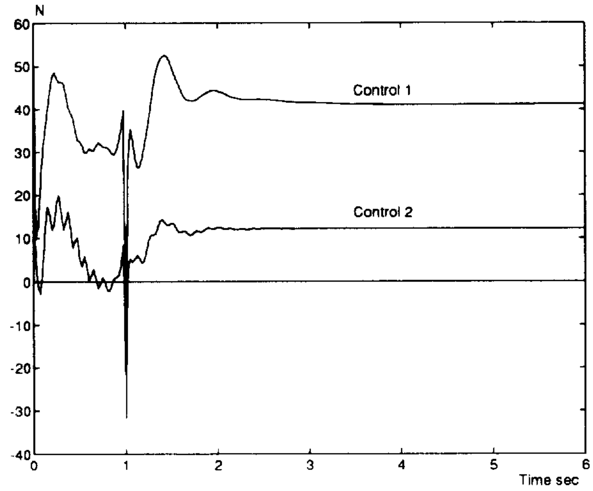

Fig. 3. Control signals without parameter adaptation.

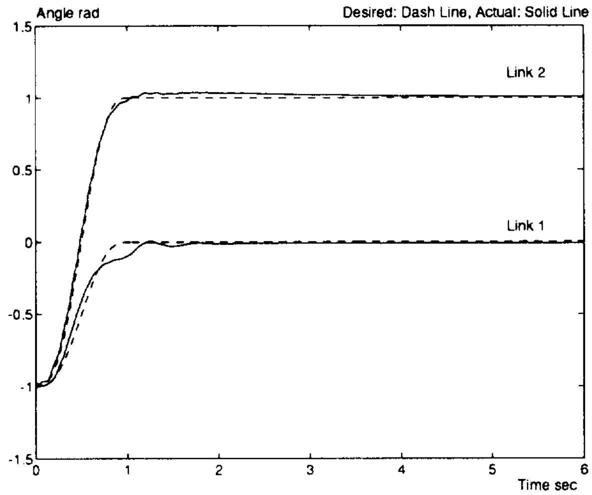

Fig. 4. Link trajectory tracking with parameter adaptation.

5.2. *Motor feedback control.* In general, motor feedback control is used in industry, partly because of its easy instrumentation. It has also been shown that, without motor feedback, link feedback control may lead to an unstable closed-loop system (Ge, 1993). By cutting out the slow-timescale control u_g in the controller (18), we have conventional motor feedback control. The responses of the links and the motors are shown in Figs 1 and 2 respectively, under the controls shown in Fig. 3. It can be seen that this control scheme has large tracking errors.

5.3. *Adaptive control.* By including the slow-timescale control u_g defined by (47) in the controller (18), we have the proposed adaptive control method, which blends the motor and the link feedbacks together. The responses of the links and the motors are used as shown in Figs 4 and 5 respectively. The adaptive controls and parameter variations are shown in Figs 6 and 7 respectively. It can be seen that the tracking errors are much smaller than that in the motor feedback control case. In fact, the static tracking errors can be eliminated because of the 'learning' mechanism. The estimated parameters are bounded, which guarantees that the estimated motor positions, velocities and accelerations are also bounded.

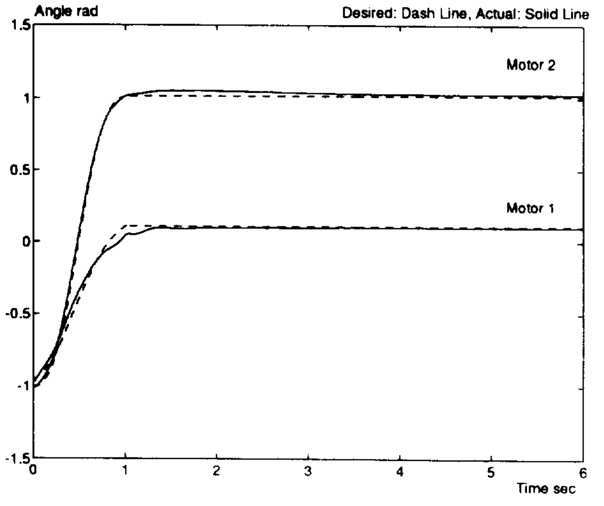

Fig. 5. Motor trajectory tracking with parameter adaptation.

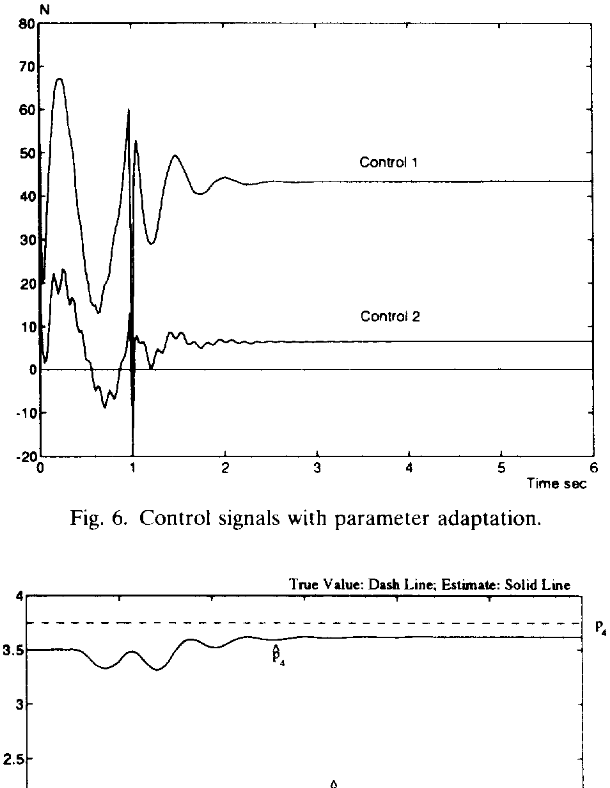

Fig. 6. Control signals with parameter adaptation.

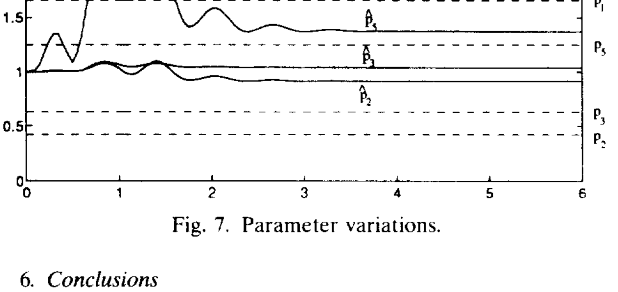

Fig. 7. Parameter variations.

6. *Conclusions*

A new adaptive controller has been presented based on SPT and using only position and velocity feedback by modelling the 'motor tracking error' Ke_m as the fast variable instead of the joint elastic forces. As a consequence, the slow-timescale dynamics and the resulting control laws are also different from those given in Spong (1989). As the boundary layer system and the quasi-steady-state system are exponentially stable, the statements of Tychonov's theorem are valid for an infinite time interval. Most importantly, an alternative singular perturbation model has been presented for controller design.

References

Chang, Y.-Z. and R. W. Daniel (1992). On the adaptive control of flexible joint robots, *Automatica*, **28**, 969–974.

Desoer, C. and M. Vidyasagar (1975). *Feedback Systems: Input-Output Properties.* Academic Press, New York.

Ge, Shuzhi (1993). Nonlinear adaptive control of robots. PhD thesis, Department of Mechanical Engineering, Imperial College of Science, Technology, and Medicine, London.

Hung, J. Y. (1989). Robust control of flexible joint robots. PhD thesis, Department of Electrical and Computer Science, University of Illinois at Urbana–Champaign.

Khalil, H. (1992). *Nonlinear Systems.* Macmillan, New York.

Marino, R. and S. Nicosia (1985). Singular perturbation techniques in the adaptive controls of elastic robots. In *Proc. 1st IFAC Symp. on Robot Control,* Barcelona, pp. 95–100.

Slotine, J.-J. E and W. Li (1991). *Applied Nonlinear Control.* Prentice-Hall, Englewood Cliffs, NJ.

Spong, M. W. (1987). Modeling and control of elastic joint robots. *J. Dyn. Syst. Meas. Control,* **109,** 310–319.

Spong, M. W. (1989). Adaptive control of flexible joint robots. *Syst. Control Lett.*, **13,** 15–21.

Spong, M. W. (1990). Control of flexible joint robots: a survey. Report UILU-ENG-90-2203, DC-116, Coordinated Science Laboratory, University of Illinois at Urbana–Champaign.

Sweet, L. M. and M. C. Good (1984). Re-definition of the robot motion control problem: effects of plant dynamics, drive system constraints, and user requirements. In *Proc. 23rd IEEE Conf. on Decision and Control,* Las Vegas, NV, pp. 724–731.

Appendix A—Exponential convergence of e, $\dot e$ and $\tilde P$

Choose the positive function V of time as

$$V = \tfrac{1}{2}r^T D(q) r + \tfrac{1}{2}\tilde{P}^T \Gamma^{-1} P. \tag{A.1}$$

Differentiating along the system dynamics, we have

$$\begin{aligned}\dot V &= r^T D(q)\dot r + \tfrac12 r^T \dot D r + \tilde P^T \Gamma^{-1}\dot{\tilde P} + \tfrac12 \tilde P^T \dot\Gamma^{-1}\tilde P\\ &= r^T[D(q)\ddot q_r - D(q)\ddot q] + \tfrac12 r^T \dot D r - \tfrac12 r^T\\ &\quad \times[\dot D - 2C(q,\dot q)]r + \tilde P^T\Gamma^{-1}\dot{\tilde P} + \tfrac12\tilde P^T\dot\Gamma^{-1}\tilde P\\ &= r^T[D(q)\ddot q_r + C(q,\dot q)r - D(q)\ddot q] + \tilde P^T\Gamma^{-1}\dot{\tilde P} + \tfrac12\tilde P^T\dot\Gamma^{-1}\tilde P,\end{aligned}$$

where the skew-symmetric property of $\dot D - 2C(q, \dot q)$ is used. Substituting (46) into the above equation leads to

$$\begin{aligned}\dot V &= r^T[D(q)\ddot q_r + C(q,\dot q)\dot q_r + G(q) - u_g - \beta_d \tilde P]\\ &\quad + \tilde P^T\Gamma^{-1}\dot{\tilde P} + \tfrac12\tilde P^T\dot\Gamma^{-1}\tilde P.\end{aligned}$$

Substituting the definition of u_g, (47) and (48), we have

$$\begin{aligned}\dot V &= r^T[\tilde D(q)\ddot q_r + \tilde C(q,\dot q)\dot q_r + \tilde G(q) - \lambda_c \hat D(q) r - \beta_d\tilde P]\\ &\quad + \tilde P^T\Gamma^{-1}\dot{\tilde P} + \tfrac12\tilde P^T\dot\Gamma^{-1}\tilde P\\ &= r^T[\tilde D(q)\ddot q_r + \tilde C(q,\dot q)\dot q_r + \tilde G(q) - \lambda_c D(q) r + \lambda_c\tilde D(q) r - \beta_d\tilde P]\\ &\quad + \tilde P^T\Gamma^{-1}\dot{\tilde P} + \tfrac12\tilde P^T\dot\Gamma^{-1}\tilde P\\ &= -\lambda_c r^T D(q) r + r^T[\tilde D(q)(\ddot q_r + \lambda_c r) + \tilde C(q,\dot q)\dot q_r + \tilde G(q) - \beta_d\tilde P]\\ &\quad + \tilde P^T\Gamma^{-1}\dot{\tilde P} + \tfrac12\tilde P^T\dot\Gamma^{-1}\tilde P.\end{aligned}$$

From (11), we obtain

$$\begin{aligned}\dot V &= -\lambda_c r^T D(q) r + r^T(\Psi_g\tilde P - \beta_d\tilde P)\\ &\quad + \tilde P^T\Gamma^{-1}\dot{\tilde P} + \tfrac12\tilde P^T\dot\Gamma^{-1}\tilde P.\end{aligned} \tag{A.2}$$

As $\Psi = \Psi_g - \beta_d$ and $\dot{\tilde P} = -\dot{\hat P}$ (for constant P), we have

$$\begin{aligned}\dot V &= -\lambda_c r^T D(q) r + r^T\Psi\tilde P + \tilde P^T\Gamma^{-1}\dot{\tilde P} + \tfrac12\tilde P^T\dot\Gamma^{-1}\tilde P\\ &= -\lambda_c r^T D(q) r + r^T\Psi\tilde P - \tilde P^T\Gamma^{-1}\dot{\hat P} + \tfrac12\tilde P^T\dot\Gamma^{-1}\tilde P.\end{aligned} \tag{A.3}$$

Since $\tilde y = W\tilde P$, (49) can be written as

$$\dot{\hat P} = \Gamma(\Psi^T r + \gamma_1 W^T W\tilde P). \tag{A.4}$$

Substituting the definitions of $\dot{\Gamma}^{-1}$, (51), and $\dot{\hat{P}}$ into (A.3) we have

$$\begin{aligned}\dot{V} &= -\lambda_c r^T D(q) r - \gamma_1 \tilde{P}^T W^T W \tilde{P} \\ &\quad -\tfrac{1}{2}\lambda \tilde{P}^T \Gamma^{-1} \tilde{P} + \tfrac{1}{2}\gamma_2 \tilde{P}^T W^T W \tilde{P} \\ &= -\lambda_c r^T D(q) r - \tfrac{1}{2}\lambda \tilde{P}^T \Gamma^{-1} \tilde{P} \\ &\quad -(\gamma_1 - \tfrac{1}{2}\gamma_2)\tilde{P}^T W^T W \tilde{P}. \qquad (A.5)\end{aligned}$$

If $\gamma_1 > \frac{1}{2}\gamma_2$, we have

$$\begin{aligned}\dot{V} &\le -\lambda_c r^T D(q) r - \tfrac{1}{2}\lambda \tilde{P}^T \Gamma^{-1} \tilde{P} \\ &\le -\lambda_m V \le 0, \qquad (A.6)\end{aligned}$$

where $\lambda_m = \min(2\lambda_c, \lambda)$. Therefore $\dot{V} + \lambda_m V \le 0 \Rightarrow V \le V(0)e^{-\lambda_m t}$. This in turn implies the exponential convergence of r and $\tilde{P}$ to zero. The exponential convergence of e and $\dot{e}$ to zero follows as a result of exponential convergence of r, according to Lemma 2.1.

Appendix B—Boundedness of Γ

The proof is exactly the same as that given in Slotine and Li (1991). For completeness, it is repeated here. Since the magnitude of the gain matrix Γ is an indicator of the excitation level of W, it is reasonable to correlate the forgetting factor variation with $\|\Gamma(t)\|$. The so-called bounded-gain forgetting technique is to choose

$$\lambda(t) = \lambda_0(1 - \|\Gamma\|/k_0). \qquad (B.1)$$

Substituting (B.1) into (51), i.e.

$$\frac{d}{dt}\Gamma^{-1}(t) = -\lambda(t)\Gamma^{-1} + \gamma_2 W^T W, \qquad (B.2)$$

we have

$$\dot{\Gamma}^{-1}(t) = -\lambda_0 \Gamma^{-1} + \frac{\lambda_0}{k_0}\|\Gamma\|\Gamma^{-1} + \gamma_2 W^T W. \qquad (B.3)$$

This leads to

$$\begin{aligned}\Gamma^{-1}(t) &= \Gamma^{-1}(0)e^{-\lambda_0 t} + \int_0^t e^{-\lambda_0(t-s)} \\ &\quad \times \left(\frac{\lambda_0}{k_0}\|\Gamma\|\Gamma^{-1} + \gamma_2 W^T W\right) ds. \qquad (B.4)\end{aligned}$$

Using the fact that $\|\Gamma\|\Gamma^{-1} \ge I$, which follows from the fact that

$$\|\Gamma\|\Gamma^{-1} - I = \Gamma^{-1/2}(\|\Gamma\| I - \Gamma)\Gamma^{-1/2} \ge 0, \qquad (B.5)$$

we have

$$\begin{aligned}&\int_0^t e^{-\lambda_0(t-s)}\frac{\lambda_0}{k_0}\|\Gamma\|\Gamma^{-1} ds \\ &\quad \ge I\int_0^t e^{-\lambda_0(t-s)}\frac{\lambda_0}{k_0} ds \ge I\frac{1}{k_0}\int_0^t e^{-\lambda_0(t-s)}\lambda_0 ds \\ &\quad \ge I\frac{1}{k_0}\int_0^t d(e^{-\lambda_0(t-s)}) \ge I\frac{1}{k_0}e^{-\lambda_0(t-s)}|_{s=0}^t \\ &\quad \ge I\frac{1}{k_0}(1 - e^{-\lambda_0 t}).\end{aligned}$$

Therefore (B.4) can be written as

$$\begin{aligned}\Gamma^{-1}(t) &\ge [\Gamma^{-1}(0) - k_0^{-1}I]e^{-\lambda_0 t} + \frac{1}{k_0}I \\ &\quad + \int_0^t e^{-\lambda_0(t-s)}\gamma_2 W^T W ds. \qquad (B.6)\end{aligned}$$

Note that $\Gamma(0) \le k_0 I$ guarantees the positive-definiteness of $\Gamma^{-1}(t)$, and therefore, for all $\gamma_2 > 0$, $\forall t \ge 0$,

$$\Gamma^{-1}(t) \ge \frac{1}{k_0}I, \qquad (B.7)$$

so that $\Gamma(t) \le k_0 I$. This in turn implies, from (B.1), that $\lambda(t) \ge 0$. Thus we have shown the boundedness of Γ and the non-negative nature of $\lambda(t)$.

If $W(t)$ is persistently exciting, i.e. if there exist positive constants T and α_1 such that $\forall t \ge 0$,

$$\int_t^{t+T} W^T W ds \ge \alpha_1 I, \qquad (B.8)$$

we can further show that $\lambda(t) \ge \lambda_1 > 0$. To do this, note that, from (B.6) and (B.8), for $t \ge \delta$,

$$\begin{aligned}\Gamma^{-1}(t) &\ge \left(\frac{1}{k_0} + \gamma_2 e^{-\lambda_0\delta}\alpha_1\right)I, \\ \Gamma(t) &\le \frac{k_0}{1 + k_0\alpha_1\gamma_2 e^{-\lambda_0\delta}}I.\end{aligned} \qquad (B.9)$$

This in turn leads to the uniform lower boundedness of the forgetting factor by a positive constant:

$$\lambda(t) = \frac{\lambda_0}{k_0}(k_0 - \|\Gamma\|) \ge \frac{\lambda_0 k_0 \alpha_1 \gamma_2 e^{-\lambda_0\delta}}{1 + k_0\alpha_1\gamma_2 e^{-\lambda_0\delta}} = \lambda_1. \qquad (B.10)$$

Under the persistently excitation condition, $\Gamma(t)$ can also be shown to be uniformly lower bounded by a constant positive-definite matrix—a property desirable for estimating time-varying parameters. From (B.2), the estimator gain can be solved explicitly:

$$\begin{aligned}\Gamma^{-1}(t) &= \Gamma^{-1}(0)\exp\left[-\int_0^t \lambda(s) ds\right] \\ &\quad + \gamma_2\int_0^t \exp\left[-\int_s^t \lambda(\sigma) dv\right] W^T(s)W(s) ds. \qquad (B.11)\end{aligned}$$

Therefore, based on (B.11) and (B.10), we obtain

$$\Gamma^{-1}(t) \le \Gamma^{-1}(0) + \gamma_2\int_0^t e^{-\lambda_1(t-s)}W^T(s)W(s) ds \qquad (B.12)$$

The second term on the right-hand side can be regarded as the output of the stable filter

$$\dot{M} + \lambda_1 M = W^T W. \qquad (B.13)$$

M is bounded if W is bounded. Thus, from (B.13) and (B.9), if W is persistently exciting and upper-bounded, Γ will be upper- and lower-bounded uniformly, i.e.

$$k_2 I \le \Gamma(t) \le k_1 I, \qquad (B.14)$$

where $0 < k_2 < k_1 < k_0$.