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Approximation-based control of nonlinear MIMO time-delay systems☆

S.S. Ge*, K.P. Tee

Department of Electrical and Computer Engineering, National University of Singapore, Singapore 117576, Singapore

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Abstract

Approximation-based control is presented for a class of multi-input multi-output (MIMO) nonlinear systems in block-triangular form with unknown state delays. Neural networks (NNs) are utilized to approximate and compensate for unknown functions in the system dynamics, including the unknown bounds of the functions of delayed states. The use of a separation technique removes the need for any assumption on the function of delayed states, and allows the handling of multiple delays in each function of delayed states. By combining the use of Lyapunov–Krasovskii functionals and adaptive NN backstepping, the proposed control guarantees that all closed-loop signals remain bounded, while the outputs converge to a neighborhood of the desired trajectories. Simulation results demonstrate the effectiveness of the proposed scheme.

Keywords: Adaptive neural control; Uncertain time-delay systems; Nonlinear MIMO system; Lyapunov–Krasovskii functional

1. Introduction

Time-delay systems constitute an active topic of research. From an academic viewpoint, time-delay systems are challenging since they involve infinite-dimensional *functional differential equations*, which are more difficult to handle than finite-dimensional *ordinary differential equations* (Dugard & Verriest, 1998; Hale, 1977; Kolmanovskii & Myshkis, 1992). On the practical front, it is worth noting that time delay is frequently encountered in models of engineering systems, natural phenomena, and biological systems (Kolmanovskii & Myshkis, 1992). Delay may occur in the feedback loop of a plant, either in the states, inputs or outputs. Of great concern is the effect of time delay on stability and asymptotic performance (Kolmanovskii, Niculescu, & Gu, 1999).

Some of the useful tools in robust stability analysis for time-delay systems are based on the Lyapunov's second method, the Lyapunov–Krasovskii theorem, and the Lyapunov–Razumikhin theorem. These have been applied to time-delay systems that are linear (Gu, Kharitonov, & Chen, 2003; Kharitonov & Melchor-Aguilar, 2003; Kolmanovskii & Richard, 1999; Sun, Hsieh, & Yang, 1997), as well those that are nonlinear (Dugard & Verriest, 1998; Jankovic, 2001; Xu & Liu, 1994).

Following its success in stability analysis, the utility of Lyapunov–Krasovskii functionals in control design for time-delay systems was subsequently explored. In Wu (2000), linear systems with nonlinear functions of delayed states were considered. In Nguang (2000), Lyapunov–Krasovskii functionals were used with backstepping to obtain a robust controller for a class of single-input single-output (SISO) nonlinear time-delay systems with known bounds on the functions of delayed states, but it was commented that the results could not be constructively obtained (Zhou, Feng, & Nguang, 2002). The need for knowledge of system nonlinearities is removed with the use of adaptive neural network (NN) control in Ge, Hong, and Lee (2003). A subsequent work extended the problem to the case of completely unknown virtual control coefficients, through the use of Nussbaum-type functions (Ge, Hong, & Lee, 2004). As the above-mentioned works are essentially based on robust approaches, restrictions have been imposed on the functions of delayed states to facilitate Lyapunov synthesis, which may limit the applicability of the approach to certain practical systems.

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* Corresponding author. Tel.: +65 65166821; fax: +65 67791103.
E-mail addresses: elegesz@nus.edu.sg (S.S. Ge), kptee@nus.edu.sg (K.P. Tee).

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In this paper, approximation-based control is presented for a class of multi-input multi-output (MIMO) nonlinear systems with unknown state delays. By combining the use of Lyapunov–Krasovskii functionals and adaptive NN backstepping, the proposed control guarantees that all closed-loop signals remain bounded, while output tracking is achieved. Using a separation technique (Lin & Qian, 2002), we are able to decompose the norm of a general function of delayed states into a series of positive bounding functions of each delayed state. This is obtained free of any restrictive assumptions on the function of delayed states, and can be seen as an improvement over (Ge et al., 2003, 2004; Wu, 2000), in which each function of delayed states carried a common delay, and were assumed to be bounded by special functions.

The extension of results from SISO to MIMO systems is generally non-trivial, due to state and input interconnections found in MIMO systems, which tend to make the analysis much more complex. To simplify analysis, we consider the block-triangular realization of the MIMO system, which allows for backstepping to be carried out in a nested fashion across the subsystems, thus avoiding the need for a decoupling matrix (Ge & Wang, 2004). The difficulty is increased when time delays are present, and there are currently only few results available in the literature directed at MIMO time-delay systems (see Chou & Cheng, 2002; Palmor & Halevi, 1983; Wang & Mau, 1997; Wu, 2002, which are all for linear systems).

With the use of NN approximation, the bounding functions are not required to be known, unlike in Wu (2000), Ge et al. (2003, 2004). We also show that, for the special case whereby the bounds are known, the information can be exploited to achieve better quantification of performance bounds. Furthermore, hyperbolic tangent functions are used to handle the singularity problem encountered in Lyapunov synthesis. The properties of these functions are exploited to show that the NNs approximate well-defined functions, and to analyze stability and performance.

2. Problem formulation and preliminaries

2.1. Plant dynamics

Consider the n-input n-output continuous-time MIMO nonlinear system in block-triangular form with unknown time delays described by

$$\begin{aligned}
\dot{x}_{j,i_j} &= f_{j,i_j}(\bar{x}_{j,i_j}) + g_{j,i_j}(\bar{x}_{j,i_j})x_{j,i_j+1} + h_{j,i_j}(\bar{x}_{\tau_{j,i_j}}),\\
\dot{x}_{j,m_j} &= f_{j,m_j}(X,\bar{u}_{j-1}) + g_{j,m_j}(X_{g_j})u_j + h_{j,m_j}(X_\tau),\\
y_j &= x_{j,1} \qquad (1)
\end{aligned}$$

for $j = 1, 2, \ldots, n$, $i_j = 1, 2, \ldots, m_j - 1$, where $x_j = [x_{j,1}, x_{j,2}, \ldots, x_{j,m_j}]^{\mathrm{T}} \in R^{m_j}$ are the delay-free state variables of the jth subsystem; $y = [y_1, \ldots, y_n]^{\mathrm{T}} \in R^n$ are the outputs; $\bar{u}_j := [u_1, \ldots, u_j]^{\mathrm{T}}$ are the inputs for the first j subsystems; $\bar{x}_{j,i_j} := [x_{j,1}, \ldots, x_{j,i_j}]^{\mathrm{T}} \in R^{i_j}$ is the vector of delay-free states for the first i_j differential equations of the jth subsystem; $f_{j,i_j}(\cdot)$, $g_{j,i_j}(\cdot)$ and $h_{j,i_j}(\cdot)$ are unknown smooth nonlinear functions; the vector $X = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \ldots, x_n^{\mathrm{T}}]^{\mathrm{T}}$ contains all delay-free states; $x_{\tau_{j,i_j}} := x_{j,i_j}(t-\tau_{j,i_j})$ denotes the delayed state; and the vectors X_{g_j}, X_τ, and $\bar{x}_{\tau_{j,i_j}}$ are defined by

$$\begin{aligned}
X_\tau := [&x_{1,1}(t-\tau_{1,1}), \ldots, x_{1,m_1}(t-\tau_{1,m_1}), \ldots,\\
&x_{j,1}(t-\tau_{j,1}), \ldots, x_{j,m_j}(t-\tau_{j,m_j}), \ldots,\\
&x_{n,1}(t-\tau_{n,1}), \ldots, x_{n,m_n}(t-\tau_{n,m_n})]^{\mathrm{T}},
\end{aligned}$$

$$\bar{x}_{\tau_{j,i_j}} := [x_{j,1}(t-\tau_{j,1}), x_{j,2}(t-\tau_{j,2}), \ldots, x_{j,i_j}(t-\tau_{j,i_j})]^{\mathrm{T}},$$

$$X_{g_j} := [x_1^{\mathrm{T}}, \ldots, x_j^{\mathrm{T}}, \bar{x}^{\mathrm{T}}_{j+1,m_{(j+1)}-1}, \ldots, \bar{x}^{\mathrm{T}}_{n,m_n-1}]^{\mathrm{T}},$$

where $\tau_{j,i_j} > 0$ is the constant unknown time delay. Note that $X_{gn} = X$ and $\bar{u}_0 := 0$.

The term h_{j,i_j} is a function of the previous i_j delayed states of the jth subsystem, while h_{j,m_j}, which appears in the last equation of each subsystem, is a function of the delayed states of *all* subsystems. For $t \in [-\tau_{j,i_j}, 0]$, we have $x_{j,i_j}(t) = \phi_{j,i_j}(t)$, $j = 1, 2, \ldots, n$, $i_j = 1, 2, \ldots, i_n$, where the initial function, $\phi_{j,i_j}(t)$, is smooth and bounded. Throughout this paper, for clarity in presentation, we omit the argument t in the delay-free states $x(t)$.

Remark 1. In plant (1), we deal with a system with interconnected states carrying multiple constant delays embedded in the function $h_j(x_1(t-\tau_1), \ldots, x_j(t-\tau_j))$. Each state x_i, $i = 1, 2, \ldots, j$ is assigned an independent delay τ_i. This is clearly different from the case in Ge et al. (2003, 2004), where a common delay τ_j was assigned to the argument states in each function of delayed states $h_j(x_1(t-\tau_j), \ldots, x_j(t-\tau_j))$. As such, the functions of delayed states considered in this paper are more general.

Remark 2. Practical examples of systems in the form of (1) include most recycling processes such as recycled reactors, recycled storage tanks, cold rolling mills, which inherit delays in their state equations (Niculescu, 2001), as well as telemanipulation systems with transmission delays.

The control objective is to ensure that all signals are bounded while tracking the desired trajectories y_{dj}, $j = 1, 2, \ldots, n$ such that the tracking errors converge to a small neighborhood of the origin, i.e., $\lim_{t\to\infty}|y_j(t) - y_{dj}(t)| \leqslant \delta$ for some $\delta > 0$.

Assumption 1. The desired trajectories y_{dj}, $j = 1, 2, \ldots, n$, and their time derivatives up to the nth order, are continuous and bounded.

Assumption 2. The signs of $g_{j,i_j}(\cdot)$, for $j = 1, 2, \ldots, n$, $i_j = 1, 2, \ldots, m_j$, are known, and there exist constants $g_{0_{j,i_j}}$ and known smooth functions $\bar{g}_{j,i_j}(\cdot)$ such that $0 < g_{0_{j,i_j}} \leqslant |g_{j,i_j}(\cdot)| \leqslant \bar{g}_{j,i_j}(\cdot) < \infty$. Without loss of generality, we further assume that all of the signs are positive.

Lemma 1 ((Lin & Qian, 2002)). *For any continuous function* $h(\xi_1, \ldots, \xi_n) : R^{m_1} \times \cdots \times R^{m_n} \to R$ *satisfying* $h(0, \ldots, 0) = 0$, *where* $\xi_j \in R^{m_j}$ $(j=1, 2, \ldots, n, m_j > 0)$, *there exist positive smooth functions* $\varrho_j(\xi_j) : R^{m_j} \to R$ $(j=1, 2, \ldots, n)$ *satisfying* $\varrho_j(0) = 0$ *such that*

$$|h(\xi_1, \ldots, \xi_n)| \leqslant \sum_{j=1}^{n} \varrho_j(\xi_j). \tag{2}$$

Throughout this paper, we use the notation $\varrho_c^{\mathrm{b}}(x_{\tau_c})$ to denote the bounding function of the delayed state x_{τ_c} belonging to the differential equation for x_{b}.

Lemma 2. *For any constant* $\eta > 0$ *and any variable* $p \in R$, $\lim_{p \to 0} \tanh^2(p/\eta)/p = 0$.

2.2. Function approximation with NNs

In this paper, we employ radial basis function (RBF) NN to approximate unknown functions. Universal approximation results indicate that, given a desired level of accuracy ε, approximation to that level of accuracy can be guaranteed by making l sufficiently large (Sanner & Slotine, 1992). Thus the NN $W^{\mathrm{T}} S(Z)$ can approximate any continuous function $p_{nn}(Z)$ as

$$p_{nn}(Z) = W^{\mathrm{T}} S(Z) + \varepsilon(Z) \quad \forall Z \in \Omega_Z \subset R^q, \tag{3}$$

where the input vector $Z \in \Omega_Z \subset R^q$, weight vector $W \in R^l$, and basis function vector $S(Z) = [s_1(Z), s_2(Z), \ldots, s_l(Z)]^{\mathrm{T}} \in R^l$, with l being the number of NN nodes and $s_i(Z)$ chosen as the commonly used Gaussian functions, which have the form

$$s_i(Z) = \exp[-(Z - \mu_i)^{\mathrm{T}}(Z - \mu_i)/\phi_i^2], \quad i = 1, \ldots, l, \tag{4}$$

where $\mu_i = [\mu_{i1}, \mu_{i2}, \ldots, \mu_{iq}]^{\mathrm{T}}$ is the center of the receptive field and ϕ_i is the width of the Gaussian function, and $\varepsilon(Z)$ is the approximation error which is bounded over the compact set, i.e., $|\varepsilon(Z)| \leqslant \bar{\varepsilon}$, $\forall Z \in \Omega_Z$ where $\bar{\varepsilon} > 0$ is an unknown constant. For analytical purposes, we define the ideal weight vector W^* as the value of W that minimizes $|\varepsilon|$ for all $Z \in \Omega_Z \subset R^q$, i.e., $W^* := \arg\min_{W \in R^l}\{\sup_{Z \in \Omega_Z} |p_{nn}(Z) - W^{\mathrm{T}} S(Z)|\}$.

Due to the nature of NN approximation, only semiglobal stability results are obtained. In particular, for bounded initial conditions in Ω_0, there exists a control that ensures all closed-loop signals remain bounded in Ω, where $\Omega_0 \subset \Omega$, provided that the NN is constructed to cover Ω (Ge & Wang, 2004). For practical applications, for every set of initial conditions, we can design the NN to be "sufficiently" large, although the required size of l cannot be determined a priori. The interested reader is referred to Ge and Wang (2004) for more details.

Remark 3. Although RBF NNs are considered in this paper, they can be replaced by any linearly parameterized networks such as fuzzy systems, polynomial, splines and wavelet networks.

3. Adaptive NN control design

In this section, we will consider two cases, namely the general case where the bound, $\varrho_{j,i_j}(\cdot)$, of the function of delayed states, $h_{j,i_j}(\cdot)$, are unknown, and the special case where $\varrho_{j,i_j}(\cdot)$ are known. It will be shown in the performance analysis (Section 4) that utilizing information of $\varrho_{j,i_j}(\cdot)$, if available, may result in better performance.

3.1. General case: unknown bounds of functions of delayed states

Noting that each subsystem is in strict-feedback form, our control design adopts embedded backstepping. We choose the intermediate and practical control laws, α_{j,i_j} and u_j, respectively, as follows:

$$\alpha_{j,1} = \frac{1}{\bar{g}_{j,1}(x_{j,1})}[-\kappa_{j,1} z_{j,1} + \hat{W}_{j,1}^{\mathrm{T}} S(Z_{j,1})], \tag{5}$$

$$\alpha_{j,i_j} = \frac{1}{\bar{g}_{j,i_j}(\bar{x}_{j,i_j})}[-\bar{g}_{j,i_j-1}(\bar{x}_{j,i_j-1}) z_{j,i_j-1} - \kappa_{j,i_j} z_{j,i_j} + \hat{W}_{j,i_j}^{\mathrm{T}} S(Z_{j,i_j})], \tag{6}$$

$$u_j = \frac{1}{\bar{g}_{j,m_j}(X_{gj})}[-\bar{g}_{j,m_j-1}(\bar{x}_{j,m_j-1}) z_{j,m_j-1} - \kappa_{j,m_j} z_{j,m_j} + \hat{W}_{j,m_j}^{\mathrm{T}} S(Z_{j,m_j})] \tag{7}$$

for $j = 1, 2, \ldots, n$, $i_j = 2, 3, \ldots, m_j - 1$, where $\kappa_{j,i_j} > 0$ and $\kappa_{j,m_j} > 0$ are design parameters, $\hat{W}_{j,i_j}$ are the NN weights estimates, and $S(\cdot)$ is the basis function vector. The error of each step, z_{j,i_j}, is defined as

$$z_{j,1} = x_{j,1} - y_{dj},$$
$$z_{j,i_j} = x_{j,i_j} - \alpha_{j,i_j-1}, \quad i_j = 2, 3, \ldots, m_j \tag{8}$$

and the inputs to the NNs as

$$Z_{j,1} = [x_{j,1}, y_{dj}, \dot{y}_{dj}]^{\mathrm{T}}, \tag{9}$$

$$Z_{j,i_j} = \left[\bar{x}_{j,i_j}^{\mathrm{T}}, \alpha_{j,i_j-1}, \frac{\partial \alpha_{j,i_j-1}}{\partial \bar{x}_{j,i_j-1}}, \omega_{j,i_j-1}\right]^{\mathrm{T}}, \tag{10}$$

$$Z_{j,m_j} = \left[X^{\mathrm{T}}, \alpha_{j,m_j-1}, \frac{\partial \alpha_{j,m_j-1}}{\partial \bar{x}_{j,m_j-1}}, \omega_{j,m_j-1}, \bar{u}_{j-1}^{\mathrm{T}}\right]^{\mathrm{T}} \tag{11}$$

for $j = 1, 2, \ldots, n$, $i_j = 2, 3, \ldots, m_j - 1$ where

$$\omega_{j,i_j-1} := \sum_{k=0}^{i_j-1} \frac{\partial \alpha_{j,i_j-1}}{\partial y_{dj}^{(k)}} y_{dj}^{(k+1)} + \sum_{k=1}^{i_j-1} \frac{\partial \alpha_{j,i_j-1}}{\partial \hat{W}_{j,k}} \dot{\hat{W}}_{j,k} \tag{12}$$

for $i_j = 2, 3, \ldots, m_j$, with $y_{dj}^{(k)}$ denoting $\mathrm{d}^k/\mathrm{d}t^k(y_{dj})$. The NN weights adaptation law is given by

$$\dot{\hat{W}}_{j,i_j} = -\Gamma_{j,i_j}[S(Z_{j,i_j})z_{j,i_j} + \sigma_{j,i_j}(\hat{W}_{j,i_j} - W^0_{j,i_j})] \tag{13}$$

for $j = 1, 2, \ldots, n$, $i_j = 1, 2, \ldots, m_j$, where $\Gamma_{j,i_j} = \gamma_{j,i_j} I_l > 0$ is the adaptation gain, $\sigma_{j,i_j} > 0$ is the σ-modification parameter, and W^0_{j,i_j} is a design constant. The σ-modification term imposes a growth restriction on the estimation weight vector $\hat{W}_{j,i_j}$. The parameter W^0_{j,i_j} can be designed to diminish the residual set to which the error signals converge.

Step j, 1 ($j = 1, \ldots, n$): the first step for the jth subsystem. Consider the first equation of the jth subsystem. To avoid control singularity, we employ integral Lyapunov functions (Ge, Hang, & Zhang, 2000). In this paper, we define the integral function as

$$V_{z_{j,1}} = \int_0^{z_{j,1}} \beta g^{-1}_{\lambda_{j,1}}(\beta + y_{dj})\,\mathrm{d}\beta, \tag{14}$$

where $g^{-1}_{\lambda_{j,1}}(\cdot) := \bar{g}_{j,1}(\cdot)/g_{j,1}(\cdot)$. Hence, $V_{z_{j,1}}(z_{j,1})$ is smooth, continuous, radially unbounded, positive definite, and satisfies $V_{z_{j,1}}(0) = 0$. The derivative of $V_{z_{j,1}}$ is given by

$$\dot{V}_{z_{j,1}} = z_{j,1}\left[g^{-1}_{\lambda_{j,1}} f_{j,1} + \bar{g}_{j,1}(z_{j,2} + \alpha_{j,1}) - \dot{y}_{dj}\int_0^1 g^{-1}_{\lambda_{j,1}}\,\mathrm{d}\theta + g^{-1}_{\lambda_{j,1}} h_{j,1}(x_{\tau_{j,1}})\right]. \tag{15}$$

As a result of Lemma 1 and completion of squares , the following inequality is obtained:

$$\dot{V}_{z_{j,1}} \leqslant z_{j,1}\left[g^{-1}_{\lambda_{j,1}} f_{j,1} + \bar{g}_{j,1}(z_{j,2} + \alpha_{j,1}) - \dot{y}_{dj}\int_0^1 g^{-1}_{\lambda_{j,1}}\,\mathrm{d}\theta\right] + \frac{1}{2}z^2_{j,1} g^{-2}_{\lambda_{j,1}} + \frac{1}{2}(\varrho^{j,1}_{j,1}(x_{\tau_{j,1}}))^2. \tag{16}$$

The intermediate control $\alpha_{j,1}$ can be designed to render the RHS negative within a compact set. Although the terms containing $f_{j,1}(x_{j,1})$ and $g_{j,1}(x_{j,1})$ are unknown, NNs can be used to estimate them. However, a problem arises when trying to approximate $\varrho_{j,1}(\cdot)$, because the NN inputs contain the time delays, which are unknown. One way to circumvent this problem is to eliminate the terms with time delay, by augmenting the integral function with the Lyapunov–Krasovskii functional, defined by

$$V_{U_{j,1}} = \int_{t-\tau_{j,1}}^{t} \frac{1}{2}\left(\varrho^{j,1}_{j,1}(x_{j,1}(\tau))\right)^2 \mathrm{d}\tau. \tag{17}$$

Differentiating $V_{U_{j,1}}$ with respect to time, we obtain

$$\dot{V}_{U_{j,1}} = \tfrac{1}{2}(\varrho^{j,1}_{j,1}(x_{j,1}))^2 - \tfrac{1}{2}(\varrho^{j,1}_{j,1}(x_{\tau j,1}))^2. \tag{18}$$

Following the method in Ge et al. (2003), by adding $\dot{V}_{U_{j,1}}$ to $\dot{V}_{z_{j,1}}$ in (16), the term $\frac{1}{2}(\varrho^{j,1}_{j,1}(x_{\tau_{j,1}}))^2$ will be cancelled exactly, yielding the following which is independent of time delay:

$$\dot{V}_{z_{j,1}} + \dot{V}_{U_{j,1}} \leqslant z_{j,1}[F_{j,1}(Z_{j,1}) + \bar{g}_{j,1}(z_{j,2} + \alpha_{j,1})], \tag{19}$$

where

$$F_{j,1}(Z_{j,1}) := g^{-1}_{\lambda_{j,1}} f_{j,1} - \dot{y}_{dj}\int_0^1 g^{-1}_{\lambda_{j,1}}\,\mathrm{d}\theta + \frac{1}{2}z_{j,1} g^{-2}_{\lambda_{j,1}} + \frac{1}{2z_{j,1}}(\varrho^{j,1}_{j,1}(x_{j,1}))^2. \tag{20}$$

Remark 4. It is not feasible to use $F_{j,1}(Z_{j,1})$ in (20) directly to construct the intermediate control $\alpha_{j,1}$, not only because $f_{j,1}$, $g_{j,1}$ and $\varrho_{j,1}$ are unknown, but also that there is a problem of singularity due to the last term in (20), which approaches ∞ as $z_{j,1}$ approaches zero. The latter poses a problem for the use of adaptive NNs to approximate $F_{j,1}(Z_{j,1})$, since it requires the function to be continuous over the compact set of approximation, $\Omega_{Z_{j,1}}$, but $F_{j,1}(Z_{j,1})$ is not defined at $z_{j,1} = 0$.

To deal with the difficulty mentioned in Remark 4, we note that (20) can be rewritten as

$$F_{j,1}(Z_{j,1}) = F^*_{j,1}(Z_{j,1}) + \frac{1}{z_{j,1}}\left(1 - 2\tanh^2\left(\frac{z_{j,1}}{\eta_{j,1}}\right)\right)U_{j,1},$$

where the functions $U_{j,1}(x_{j,1})$ and $F^*_{j,1}(Z_{j,1})$ are defined by

$$U_{j,1}(x_{j,1}) := \tfrac{1}{2}(\varrho^{j,1}_{j,1}(x_{j,1}))^2, \tag{21}$$

$$F^*_{j,1}(Z_{j,1}) := g^{-1}_{\lambda_{j,1}}(x_{j,1}) f_{j,1}(x_{j,1}) - \dot{y}_{dj}\int_0^1 g^{-1}_{\lambda_{j,1}}(x_{j,1})\,\mathrm{d}\theta + \frac{1}{2}z_{j,1} g^{-2}_{\lambda_{j,1}}(x_{j,1}) + \frac{2}{z_{j,1}}\tanh^2\left(\frac{z_{j,1}}{\eta_{j,1}}\right)U_{j,1}(x_{j,1}) \tag{22}$$

with $\eta_{j,1}$ as a positive constant. From Lemma 2, we know that the last term of (22) is defined even at $z_{j,1} = 0$. Thus, the unknown function $F^*_{j,1}(Z_{j,1})$ is well-defined in $\Omega_{Z_{j,1}}$, and can be approximated by the NN. The residual term $(F_{j,1} - F^*_{j,1})$ will be analyzed later.

Remark 5. The function $\tanh^2(z/\eta)$ is used in (22) due to the desirable property that $\tanh^2(z/\eta)/z$ is defined at $z = 0$, and thus can be approximated by NN. The constant η is a design parameter that can be adjusted to obtain better performance, as will be seen in Section 4. This approach is different from that of Ge et al. (2003), where discontinuous even functions were used, and from (Ge et al., 2004), where piecewise functions were constructed.

Therefore, we use NN to approximate the unknown terms, given by

$$F^*(Z_{j,1}) = -W^{*\mathrm{T}}_{j,1} S(Z_{j,1}) - \varepsilon_{j,1}(Z_{j,1}), \tag{23}$$

where $\varepsilon_{j,1}(Z_{j,1})$ is the approximation error satisfying $|\varepsilon_{j,1}(Z_{j,1})| \leqslant \bar{\varepsilon}_{j,1}$, with $\bar{\varepsilon}_{j,1}$ as a positive constant.

To analyze the stability in the presence of errors in estimation, we append a quadratic term of the weights estimation error to obtain a new Lyapunov function candidate

$$V_{j,1} = V_{z_{j,1}} + V_{U_{j,1}} + \tfrac{1}{2}\tilde{W}_{j,1}^{\mathrm{T}} \Gamma_{j,1}^{-1} \tilde{W}_{j,1}. \tag{24}$$

Throughout this paper, we denote by $\tilde{W}_{j,i_j} = \hat{W}_{j,i_j} - W^*_{j,i_j}$ the weights estimation error. Differentiating $V_{j,1}$ and substituting (5), (13), (19), (21), we obtain, using completion of squares, the following inequality:

$$\begin{aligned}\dot{V}_{j,1} \leqslant & -\left(\kappa_{j,1} - \frac{1}{4\lambda}\right) z_{j,1}^2 + \bar{g}_{j,1} z_{j,1} z_{j,2} + \lambda \varepsilon_{j,1}^2 \\ & - \frac{\sigma_{j,1}}{2}\|\tilde{W}_{j,1}\|^2 + \frac{\sigma_{j,1}}{2}\|W^*_{j,1} - W^0_{j,1}\|^2 \\ & + \left(1 - 2\tanh^2\left(\frac{z_{j,1}}{\eta_{j,1}}\right)\right) U_{j,1},\end{aligned} \tag{25}$$

where $\kappa_{j,1} > 1/2\lambda$. The $\bar{g}_{j,1} z_{j,1} z_{j,2}$ term will be cancelled in the next step, while the last term will be analyzed in the final step.

Step j, i_j $(j = 1, \ldots, n)$, $(i_j = 2, \ldots, m_j - 1)$: the i_jth step for the *j*th subsystem. Consider the i_jth equation of the *j*th subsystem. Define the integral function by

$$V_{z_{j,i_j}} = \int_0^{z_{j,i_j}} \beta g_{\lambda_{j,i_j}}^{-1}(\bar{x}_{j,i_j-1}, \beta + \alpha_{j,i_j-1})\, \mathrm{d}\beta. \tag{26}$$

The derivative of $V_{z_{j,i_j}}$ is given by

$$\begin{aligned}\dot{V}_{z_{j,i_j}} \leqslant & z_{j,i_j}\left[g_{\lambda_{j,i_j}}^{-1} f_{j,i_j} + \bar{g}_{j,i_j}(z_{j,i_j+1} + \alpha_{j,i_j}) \right. \\ & \left. - z_{j,i_j} \int_0^1 \theta \frac{\partial g_{\lambda_{j,i_j}}^{-1}}{\partial \bar{x}_{j,i_j-1}}\, \mathrm{d}\theta \dot{\bar{x}}_{j,i_j-1} - \dot{\alpha}_{j,i_j-1}\int_0^1 g_{\lambda_{j,i_j}}^{-1}\, \mathrm{d}\theta \right] \\ & + |z_{j,i_j}||g_{\lambda_{j,i_j}}^{-1}||h_{j,i_j}(\bar{x}_{\tau_{j,i_j}})|,\end{aligned} \tag{27}$$

where $\varrho_{j,k}^{j,i_j}(0) = 0$. Note that $\dot{\alpha}_{j,i_j-1}$ can be written as

$$\begin{aligned}\dot{\alpha}_{j,i_j-1} = & \sum_{k=1}^{i_j-1} \frac{\partial \alpha_{j,i_j-1}}{\partial x_{j,k}} [(f_{j,k} + g_{j,k} x_{j,k+1} + h_{j,k}(\bar{x}_{\tau_{j,k}})] \\ & + \omega_{j,i_j-1},\end{aligned} \tag{28}$$

where ω_{j,i_j-1} is defined in (12).

Remark 6. If the derivative signals $\dot{\bar{x}}_{j,i_j-1}$ are available, then $\dot{\alpha}_{j,i_j-1}$ is computable from (28), and the design of α_{j,i_j} is straightforward by utilizing these signals, similar to the procedure in Ge et al. (2003). However, when $\dot{\bar{x}}_{j,i_j-1}$ is not available, the control design is more involved by using the state equations, as shown in the subsequent steps.

Substituting (28) into (27) leads to the following:

$$\begin{aligned}\dot{V}_{z_{j,i_j}} \leqslant & z_{j,i_j}\left[g_{\lambda_{j,i_j}}^{-1} f_{j,i_j} + \bar{g}_{j,i_j}(z_{j,i_j+1} + \alpha_{j,i_j}) \right. \\ & - z_{j,i_j} \sum_{k=1}^{i_j-1} (f_{j,k} + g_{j,k} x_{j,k+1}) \int_0^1 \theta \frac{\partial g_{\lambda_{j,i_j}}^{-1}}{\partial x_{j,k}}\, \mathrm{d}\theta \\ & - \left(\sum_{k=1}^{i_j-1} \frac{\partial \alpha_{j,i_j-1}}{\partial x_{j,k}} (f_{j,k} + g_{j,k} x_{j,k+1}) + \omega_{j,i_j-1} \right) \\ & \left. \times \int_0^1 g_{\lambda_{j,i_j}}^{-1}\, \mathrm{d}\theta \right] + |z_{j,i_j}||g_{\lambda_{j,i_j}}^{-1}||h_{j,i_j}(\bar{x}_{\tau_{j,i_j}})| \\ & + |z_{j,i_j}| \left| \int_0^1 g_{\lambda_{j,i_j}}^{-1}\, \mathrm{d}\theta \right| \sum_{k=1}^{i_j-1} \left| \frac{\partial \alpha_{j,i_j-1}}{\partial x_{j,k}} \right| |h_{j,k}(\bar{x}_{\tau_{j,k}})| \\ & + |z_{j,i_j}|^2 \sum_{k=1}^{i_j-1} |h_{j,k}(\bar{x}_{\tau_{j,k}})| \left| \int_0^1 \theta \frac{\partial g_{\lambda_{j,i_j}}^{-1}}{\partial x_{j,k}}\, \mathrm{d}\theta \right|.\end{aligned} \tag{29}$$

Using Lemmas 1 and completion of squares on the last three terms of (29), the inequality can be rewritten as

$$\begin{aligned}\dot{V}_{z_{j,i_j}} \leqslant & z_{j,i_j}\left[g_{\lambda_{j,i_j}}^{-1} f_{j,i_j} + \bar{g}_{j,i_j}(z_{j,i_j+1} + \alpha_{j,i_j}) \right. \\ & - \sum_{k=1}^{i_j-1} (f_{j,k} + g_{j,k} x_{j,k+1}) \int_0^1 \theta \frac{\partial g_{\lambda_{j,i_j}}^{-1}}{\partial x_{j,k}}\, \mathrm{d}\theta \\ & - \left(\sum_{k=1}^{i_j-1} \frac{\partial \alpha_{j,i_j-1}}{\partial x_{j,k}} (f_{j,k} + g_{j,k} x_{j,k+1}) + \omega_{j,i_j-1} \right) \\ & \left. \times \int_0^1 g_{\lambda_{j,i_j}}^{-1}\, \mathrm{d}\theta \right] + \frac{i_j}{2} \sum_{k=1}^{i_j} (\varrho_{j,k}^{j,i_j}(x_{\tau_{j,k}}))^2 \\ & + \frac{i_j(i_j-1)}{2} \sum_{i=1}^{i_j-1} \sum_{k=1}^{i} (\varrho_{j,k}^{j,i}(x_{\tau_{j,k}}))^2 \\ & + \frac{1}{2} z_{j,i_j}^2 \left(g_{\lambda_{j,i_j}}^{-2} + \left| \int_0^1 g_{\lambda_{j,i_j}}^{-1}\, \mathrm{d}\theta \right|^2 \left\| \frac{\partial \alpha_{j,i_j-1}}{\partial \bar{x}_{j,i_j-1}} \right\|^2 \right. \\ & \left. + z_{j,i_j}^2 \left\| \int_0^1 \theta \frac{\partial g_{\lambda_{j,i_j}}^{-1}}{\partial \bar{x}_{j,i_j-1}}\, \mathrm{d}\theta \right\|^2 \right).\end{aligned} \tag{30}$$

Consider the Lyapunov–Krasovskii functional

$$V_{U_{j,i_j}} = \frac{i_j(i_j-1)}{2}\sum_{i=1}^{i_j-1}\sum_{k=1}^{i}\int_{t-\tau_{j,k}}^{t}(\varrho_{j,k}^{j,i}(x_{j,k}(\tau)))^2\,\mathrm{d}\tau + \frac{i_j}{2}\sum_{k=1}^{i_j}\int_{t-\tau_{j,k}}^{t}(\varrho_{j,k}^{j,i_j}(x_{j,k}(\tau)))^2\,\mathrm{d}\tau. \tag{31}$$

Summing $\dot{V}_{z_{j,i_j}}$ and $\dot{V}_{U_{j,i_j}}$, the terms containing the unknown time delays are eliminated, yielding

$$\dot{V}_{z_{j,i_j}} + \dot{V}_{U_{j,i_j}} \leqslant z_{j,i_j}[F_{j,i_j}^*(Z_{j,i_j}) + \bar{g}_{j,i_j}(z_{j,i_j+1} + \alpha_{j,i_j})] + \left[1 - 2\tanh^2\left(\frac{z_{j,i_j}}{\eta_{j,i_j}}\right)\right]U_{j,i_j}, \tag{32}$$

where η_{j,i_j} is a positive constant, $F_{j,i_j}^*(Z_{j,i_j})$ is defined as:

$$\begin{aligned}F_{j,i_j}^* :=\ & g_{\lambda_{j,i_j}}^{-1} f_{j,i_j} - \sum_{k=1}^{i_j-1}(f_{j,k} + g_{j,k}x_{j,k+1})\int_0^1 \theta\frac{\partial g_{\lambda_{j,i_j}}^{-1}}{\partial x_{j,k}}\,\mathrm{d}\theta \\ & - \left(\sum_{k=1}^{i_j-1}\frac{\partial\alpha_{j,i_j-1}}{\partial x_{j,k}}(f_{j,k} + g_{j,k}x_{j,k+1}) + \omega_{j,i_j-1}\right)\int_0^1 g_{\lambda_{j,i_j}}^{-1}\,\mathrm{d}\theta \\ & + \frac{1}{2}z_{j,i_j}\left(g_{\lambda_{j,i_j}}^{-2} + \left|\int_0^1 g_{\lambda_{j,i_j}}^{-1}\,\mathrm{d}\theta\right|^2\left\|\frac{\partial\alpha_{j,i_j-1}}{\partial\bar{x}_{j,i_j-1}}\right\|^2 + z_{j,i_j}^2\left\|\int_0^1\theta\frac{\partial g_{\lambda_{j,i_j}}^{-1}}{\partial\bar{x}_{j,i_j-1}}\,\mathrm{d}\theta\right\|^2\right) \\ & + \frac{2}{z_{j,i_j}}\tanh^2\left(\frac{z_{j,i_j}}{\eta_{j,i_j}}\right)U_{j,i_j},\end{aligned} \tag{33}$$

and

$$U_{j,i_j}(\bar{x}_{j,i_j}) := \frac{i_j(i_j-1)}{2}\sum_{i=1}^{i_j-1}\sum_{k=1}^{i}(\varrho_{j,k}^{j,i}(x_{j,k}))^2 + \frac{i_j}{2}\sum_{k=1}^{i_j}(\varrho_{j,k}^{j,i_j}(x_{j,k}))^2. \tag{34}$$

From Lemma 2, we know that the last term of (33) is defined even at $z_{j,i_j}=0$. Thus, the unknown function $F_{j,i_j}^*(Z_{j,i_j})$ is well-defined in $\Omega_{Z_{j,i_j}}$, and can be approximated by the NN with input vector Z_{j,i_j} given by (10).

The NN output $\hat{W}_{j,i_j}^{\mathrm{T}}S(Z_{j,i_j})$ approximates $W_{j,i_j}^{*\mathrm{T}}S(Z_{j,i_j})$, which is defined in the following:

$$W_{j,i_j}^{*\mathrm{T}}S(Z_{j,i_j}) = -F_{j,i_j}^*(Z_{j,i_j}) - \varepsilon_{j,i_j}(Z_{j,i_j}) \tag{35}$$

with $\varepsilon_{j,i_j}(Z_{j,i_j})$ as the bounded approximation error satisfying $|\varepsilon_{j,i_j}(Z_{j,i_j})| \leqslant \bar{\varepsilon}_{j,i_j}$, where $\bar{\varepsilon}_{j,i_j}$ is a positive constant. Choosing the Lyapunov function candidate as

$$V_{j,i_j} = V_{j,i_j-1} + V_{z_{j,i_j}} + V_{U_{j,i_j}} + \tfrac{1}{2}\tilde{W}_{j,i_j}^{\mathrm{T}}\Gamma_{j,i_j}^{-1}\tilde{W}_{j,i_j} \tag{36}$$

with adaptation law (13) and completion of squares, we have that

$$\begin{aligned}\dot{V}_{j,i_j} \leqslant\ & -\sum_{k=1}^{i_j}\left(\kappa_{j,k} - \frac{1}{4\lambda}\right)z_{j,k}^2 + \bar{g}_{j,i_j}z_{j,i_j}z_{j,i_j+1} \\ & - \sum_{k=1}^{i_j}\frac{\sigma_{j,k}}{2}\|\tilde{W}_{j,k}\|^2 + \sum_{k=1}^{i_j}\left(\lambda\bar{\varepsilon}_{j,k}^2 + \frac{\sigma_{j,k}}{2}\|W_{j,k}^* - W_{j,k}^0\|^2\right) + \sum_{k=1}^{i_j}\left(1 - 2\tanh^2\left(\frac{z_{j,k}}{\eta_{j,k}}\right)\right)U_{j,k},\end{aligned} \tag{37}$$

where $\kappa_{j,i_j} > 1/2\lambda$, and the parameters λ, $\sigma_{j,k}$ and $W_{j,k}^0$ can be designed to make the positive residual term small. The $\bar{g}_{j,i_j}z_{j,i_j}z_{j,i_j+1}$ terms will be cancelled in the subsequent step, while the last two terms in (37) will be analyzed in the final step.

Remark 7. For clarity, we summarize the idea of handling the function of delayed states, $h_{j,i_j}(\bar{x}_{\tau_{j,i_j}})$, which contains multiple unknown time delays $\tau_{j,1},\ldots,\tau_{j,i_j}$. First, with Lemma 1, $h_{j,i_j}(\bar{x}_{\tau_{j,i_j}})$ is decomposed into positive bounding functions $\varrho_{j,i_j}^{j,i_j}(x_{\tau_{j,i_j}})$ in terms of each delayed state $x_{\tau_{j,i_j}}$. Since the delays are unknown, $\varrho_{j,i_j}^{j,i_j}(x_{\tau_{j,i_j}})$ cannot be approximated directly. However, by careful design of Lyapunov–Krasovskii functionals $V_{U_{j,i_j}}$, the bounding functions $\varrho_{j,i_j}^{j,i_j}(x_{\tau_{j,i_j}})$ in terms of delayed states are eliminated when $\dot{V}_{U_{j,i_j}}$ and $\dot{V}_{z_{j,i_j}}$ are summed.Thus, the remaining bounding functions $\varrho_{j,i_j}^{j,i_j}(x_{j,i_j})$ are in terms of delay-free states only, and are approximated by NNs.

Step j, m_j $(j=1,\ldots,n)$: the last step for the jth subsystem. Consider the last equation of subsystem Σ_j, where the control input u_j appears. Let $X_{j,m_j}^{\mathrm{c}} \subset X_{gj}$ be the complement of x_{j,m_j} in X_{gj}, such that $X_{j,m_j}^{\mathrm{c}} \cup x_{j,m_j} = X_{gj}$ and $X_{j,m_j}^{\mathrm{c}} \cap x_{j,m_j} = \emptyset$. Define the integral function by

$$V_{z_{j,m_j}} = \int_0^{z_{j,m_j}}\beta g_{\lambda_{j,m_j}}^{-1}(X_{j,m_j}^{\mathrm{c}}, \beta + \alpha_{j,m_j-1})\,\mathrm{d}\beta. \tag{38}$$

In view of the interconnections between the different subsystems in the last equation, and according to Lemma 1, we consider the following Lyapunov–Krasovskii functional, which has a form slightly different from that of the previous $m_j - 1$

equations:

$$V_{U_{j,m_j}} = \frac{1}{2}\sum_{q=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{m_i} m_q \int_{t-\tau_{i,k}}^{t} (\varrho_{i,k}^{j,m_j}(x_{\tau_{i,k}}))^2\,\mathrm{d}\tau + \frac{m_j(m_j-1)}{4}\sum_{p=1}^{m_j-1}\sum_{k=1}^{p}\int_{t-\tau_{j,k}}^{t}\left(\varrho_{j,k}^{j,p}(x_{\tau_{j,k}})\right)^2\mathrm{d}\tau + \frac{1}{4}\sum_{q=1}^{n}\sum_{i=1}^{n}\sum_{p=1}^{m_i-1}\sum_{k=1}^{p} m_q(m_q-1) \times \int_{t-\tau_{i,k}}^{t}(\varrho_{i,k}^{i,p}(x_{\tau_{i,k}}))^2\,\mathrm{d}\tau + \frac{j-1}{2}\sum_{q=1}^{n}\sum_{i=1}^{j-1}\sum_{p=1}^{n}\sum_{k=1}^{m_p} m_q \times \int_{t-\tau_{p,k}}^{t}(\varrho_{p,k}^{i,m_i}(x_{\tau_{p,k}}))^2\,\mathrm{d}\tau. \tag{39}$$

Summing $\dot{V}_{z_{j,m_j}}$ and $\dot{V}_{U_{j,m_j}}$ eliminates the delayed states from the analysis, yielding

$$\dot{V}_{z_{j,m_j}} + \dot{V}_{U_{j,m_j}} \leqslant z_{j,m_j}(F_{j,m_j}^{*}(Z_{j,m_j}) + \bar{g}_{j,m_j}u_j) + \left(1 - 2\tanh^2\left(\frac{z_{j,m_j}}{\eta_{j,m_j}}\right)\right)U_{j,m_j}, \tag{40}$$

where the functions $U_{j,m_j}(X)$ and $F_{j,m_j}^{*}(Z_{j,m_j})$ are defined as:

$$U_{j,m_j}(X) := \frac{1}{2}\sum_{q=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{m_i} m_q(\varrho_{i,k}^{j,m_j}(x_{i,k}))^2 + \frac{1}{4}\sum_{q=1}^{n}\sum_{i=1}^{n}\sum_{p=1}^{m_i-1}\sum_{k=1}^{p} m_q(m_q-1)(\varrho_{i,k}^{i,p}(x_{i,k}))^2 + \frac{m_j(m_j-1)}{4}\sum_{p=1}^{m_j-1}\sum_{k=1}^{p}(\varrho_{j,k}^{j,p}(x_{j,k}))^2 + \frac{j-1}{2}\sum_{q=1}^{n}\sum_{i=1}^{j-1}\sum_{p=1}^{n}\sum_{k=1}^{m_p} m_q(\varrho_{p,k}^{i,m_i}(x_{p,k}))^2, \tag{41}$$

$$\begin{aligned}F_{j,m_j}^{*}(Z_{j,m_j}) :=\ & g_{\lambda_{j,m_j}}^{-1}(X_{gj})f_{j,m_j}(X,\bar{u}_{j-1}) + \frac{2}{z_{j,m_j}}\tanh^2\left(\frac{z_{j,m_j}}{\eta_{j,m_j}}\right)U_{j,m_j}(X) \\ & - z_{j,m_j}\sum_{i=1}^{n}\sum_{k=1}^{m_i-1}\int_0^1 \theta\frac{\partial g_{\lambda_{j,m_j}}^{-1}(X_{gj})}{\partial x_{i,k}}\,\mathrm{d}\theta \times (f_{i,k}(\bar{x}_{i,k}) + g_{i,k}(\bar{x}_{i,k})x_{i,k+1}) \\ & - z_{j,m_j}\sum_{i=1}^{j-1}\int_0^1 \theta\frac{\partial g_{\lambda_{j,m_j}}^{-1}(X_{gj})}{\partial x_{i,m_i}}\,\mathrm{d}\theta \times (f_{i,m_i}(\bar{x}_{i,m_i}) + g_{i,m_i}(\bar{x}_{i,m_i})u_i) \\ & - \left(\sum_{k=1}^{m_j-1}\frac{\partial\alpha_{j,m_j-1}}{\partial x_{j,k}}((f_{j,k}(\bar{x}_{j,k}) + g_{j,k}(\bar{x}_{j,k})x_{j,k+1})) + \omega_{j,m_j-1}\right) \times \int_0^1 g_{\lambda_{j,m_j}}^{-1}(X_{gj})\,\mathrm{d}\theta \\ & + \frac{z_{j,m_j}}{2}\left(g_{\lambda_{j,m_j}}^{-2}(X_{gj}) + \left|\int_0^1 g_{\lambda_{j,m_j}}^{-1}(X_{gj})\,\mathrm{d}\theta\right|^2\left\|\frac{\partial\alpha_{j,m_j-1}}{\partial\bar{x}_{j,m_j-1}}\right\|^2 + z_{j,m_j}^2\left\|\int_0^1\theta\frac{\partial g_{\lambda_{j,m_j}}^{-1}(X_{gj})}{\partial X_{gj}}\,\mathrm{d}\theta\right\|^2\right).\end{aligned} \tag{42}$$

From Lemma 2, we know that the second term of (42) is defined even at $z_{j,m_j}=0$. Thus, the unknown function $F_{j,m_j}^{*}(Z_{j,m_j})$ is well-defined in $\Omega_{Z_{j,m_j}}$, and can be approximated by NN with input vector Z_{j,m_j} defined in (11). The NN output $\hat{W}_{j,m_j}^{\mathrm{T}}S(Z_{j,m_j})$ approximates $W_{j,m_j}^{*\mathrm{T}}S(Z_{j,m_j})$, which is defined by

$$W_{j,m_j}^{*\mathrm{T}}S(Z_{j,m_j}) = -F_{j,m_j}^{*}(Z_{j,m_j}) - \varepsilon_{j,m_j}(Z_{j,m_j}) \tag{43}$$

with $\varepsilon_{j,m_j}(Z_{j,m_j})$ as the approximation error satisfying $|\varepsilon_{j,m_j}(Z_{j,m_j})| \leqslant \bar{\varepsilon}_{j,m_j}$, where $\bar{\varepsilon}_{j,m_j}$ is a positive constant.

We consider the Lyapunov function candidate as

$$V_{j,m_j} = V_{j-1,m_{(j-1)}} + V_{j,m_j-1} + V_{z_{j,m_j}} + V_{U_{j,m_j}} + \tfrac{1}{2}\tilde{W}_{j,m_j}^{\mathrm{T}}\Gamma_{j,m_j}^{-1}\tilde{W}_{j,m_j}. \tag{44}$$

The derivative of V_{j,m_j} along the closed-loop trajectories can be shown to satisfy the inequality:

$$\dot{V}_{j,m_j} \leqslant -\sum_{i=1}^{j}\sum_{k=1}^{m_i}\left[\left(\kappa_{i,k} - \frac{1}{4\lambda}\right)z_{i,k}^2 + \frac{\sigma_{i,k}}{2}\|\tilde{W}_{i,k}\|^2\right] + \sum_{i=1}^{j}\sum_{k=1}^{m_i}\left(\lambda\bar{\varepsilon}_{i,k}^2 + \frac{\sigma_{i,k}}{2}\|W_{i,k}^{*} - W_{i,k}^{0}\|^2\right) + \sum_{i=1}^{j}\sum_{k=1}^{m_i}\left(1 - 2\tanh^2\left(\frac{z_{i,k}}{\eta_{i,k}}\right)\right)U_{i,k}, \tag{45}$$

where $\kappa_{j,m_j} > 1/2\lambda$.

After Step n, m_n, the last step for the nth subsystem, it can be shown that the derivative of V_{n,m_n} along the closed-loop trajectories satisfies the following inequality:

$$\dot{V}_{n,m_n} \leqslant -\sum_{j=1}^{n}\sum_{k=1}^{m_j}\left(\bar{\kappa}_{j,k}z_{j,k}^2 + \frac{\sigma_{j,k}}{2}\|\tilde{W}_{j,k}\|^2\right) + C + \sum_{j=1}^{n}\sum_{k=1}^{m_j}\left[1 - 2\tanh^2\left(\frac{z_{j,k}}{\eta_{j,k}}\right)\right]U_{j,k}, \tag{46}$$

where $\bar{\kappa}_{j,k} := \kappa_{j,k} - 1/4\lambda > 0$, and the constant C is defined by

$$C = \sum_{j=1}^{n}\sum_{k=1}^{m_j}\left(\lambda\bar{\varepsilon}_{j,k}^2 + \frac{\sigma_{j,k}}{2}\|W_{j,k}^* - W_{j,k}^0\|^2\right). \tag{47}$$

Lemma 3. *Consider the set* $\Omega_{c_{z_{j,i_j}}}$ *defined by* $\Omega_{c_{z_{j,i_j}}} := \{z_{j,i_j} \,|\, |z_{j,i_j}| < 0.8814\eta_{j,i_j}\}$ for $j=1,2,\ldots,n$, $i_j=1,2,\ldots,m_j$. *Then, for any* $z_{j,i_j} \notin \Omega_{c_{z_{j,i_j}}}$, *the inequality* $[1-2\tanh^2(\frac{z_{j,i_j}}{\eta_{j,i_j}})] \leqslant 0$ *is satisfied.*

Theorem 1. *For system* (1), *under Assumptions* 1 *and* 2, *control law* (7) *and the NN adaptation law* (13), *all closed-loop trajectories remain bounded.*

Proof. From (46), it can be seen that the first term is negative definite and that the second term is a positive constant. However, the last term may be positive or negative, depending on the size of $z_{j,k}$. For analysis of stability, three cases need to be considered:

Case 1: $z_{j,i_j} \in \Omega_{c_{z_{j,i_j}}}$ $\forall j = 1, 2, \ldots, n$, $i_j = 1, 2, \ldots, m_j$.

From the weight adaptation law (13), we know that $\hat{W}_{j,i_j}$ is bounded, since z_{j,i_j} is bounded by construction. As W_{j,i_j}^* is a constant, the weights estimation error $\tilde{W}_{j,i_j}$ must also be bounded. From Assumption 1, we have that $y_{dj}, y_{dj}^{(1)}, \ldots, y_{dj}^{(m_j)}$ are bounded. Thus it implies the boundedness of $x_{j,1}$, and in turn the boundedness of $\alpha_{j,1}$ in (5). Following this chain of reasoning, the boundedness of x_{j,i_j}, α_{j,i_j}, and u_j can be deduced. As such, all closed-loop signals are bounded.

Case 2: $z_{j,i_j} \notin \Omega_{c_{z_{j,i_j}}}$ $\forall j = 1, 2, \ldots, n$, $i_j = 1, 2, \ldots, m_j$.

From Lemma 3 and the fact that $U_{j,k} \geqslant 0$, we know that

$$\sum_{j=1}^{n}\sum_{k=1}^{m_j}\left[1 - 2\tanh^2\left(\frac{z_{j,k}}{\eta_{j,k}}\right)\right]U_{j,k} \leqslant 0. \tag{48}$$

Therefore, (46) can be rewritten as

$$\dot{V}_{n,m_n} \leqslant -\sum_{j=1}^{n}\sum_{k=1}^{m_j}\left(\bar{\kappa}_{j,k}z_{j,k}^2 + \frac{\sigma_{j,k}}{2}\|\tilde{W}_{j,k}\|^2\right) + C. \tag{49}$$

Since $\bar{\kappa}_{j,k} > 0$, we know that $z_{j,k}$ and $\tilde{W}_{j,k}$ are bounded, for $j = 1, \ldots, n$, $k = 1, \ldots, m_j$. Together with the boundedness of y_{d_j} in Assumption 1, it can be shown that $x_{j,k}$, and hence all closed-loop trajectories, are bounded.

Case 3: $z_{j,i_j} \in \Omega_{c_{z_{j,i_j}}}$ and $z_{k,i_k} \notin \Omega_{c_{z_{k,i_k}}}$ for $(j, i_j) \neq (k, i_k)$.

Denote by Σ_K the subsystem consisting of $z_{k,i_k} \notin \Omega_{c_{z_{k,i_k}}}$, and Σ_J the subsystem consisting of $z_{j,i_j} \in \Omega_{c_{z_{j,i_j}}}$. Define Lyapunov function candidate for Σ_K as follows:

$$V_{\Sigma_K} := \sum_{(j,k)\in\Sigma_K}\left\{V_{z_{j,k}} + V_{U_{j,k}} + \frac{1}{2}\tilde{W}_{j,k}^{\mathrm{T}}\Gamma_{j,k}^{-1}\tilde{W}_{j,k}\right\}. \tag{50}$$

It can be shown that the derivative is given by

$$\dot{V}_{\Sigma_K} \leqslant \sum_{(j,k)\in\Sigma_K}\left\{-\bar{\kappa}_{j,k}z_{j,k}^2 - \frac{\sigma_{j,k}}{2}\|\tilde{W}_{j,k}\|^2 + C_{j,k} + \left[1 - 2\tanh^2\left(\frac{z_{j,k}}{\eta_{j,k}}\right)\right]U_{j,k}\right\} + \sum_{(j,k)\in\Sigma_K} z_{j,k}(\bar{g}_{j,k}z_{j,k+1} - \bar{g}_{j,k-1}z_{j,k-1}), \tag{51}$$

where $z_{j,m_j+1} := 0$, $z_{j,0} := 0$, and

$$C_{j,k} := \lambda\bar{\varepsilon}_{j,k}^2 + \frac{\sigma_{j,k}}{2}\|W_{j,k}^* - W_{j,k}^0\|^2. \tag{52}$$

The second summation term of (51) can be expressed as

$$\begin{aligned}\sum_{(j,k)\in\Sigma_K} z_{j,k}[\bar{g}_{j,k}z_{j,k+1} - \bar{g}_{j,k-1}z_{j,k-1}] &= \sum_{\substack{(j,k)\in\Sigma_K,\\(j,k+1)\in\Sigma_J}} \bar{g}_{j,k}z_{j,k}z_{j,k+1} - \sum_{\substack{(j,k)\in\Sigma_K,\\(j,k-1)\in\Sigma_J}} \bar{g}_{j,k-1}z_{j,k-1}z_{j,k} \\ &\quad + \sum_{\substack{(j,k)\in\Sigma_K,\\(j,k+1)\in\Sigma_K}} \bar{g}_{j,k}z_{j,k}z_{j,k+1} - \sum_{\substack{(j,k)\in\Sigma_K,\\(j,k-1)\in\Sigma_K}} \bar{g}_{j,k-1}z_{j,k-1}z_{j,k},\end{aligned}$$

where it can be verified that the last two terms cancel out during backstepping. From Assumption 2, we have that $\bar{g}_{j,k}(\cdot) \leqslant G < \infty$ for all j, k, where G is an unknown positive constant. Hence, we obtain that

$$\begin{aligned}\sum_{(j,k)\in\Sigma_K} z_{j,k}[\bar{g}_{j,k}z_{j,k+1} - \bar{g}_{j,k-1}z_{j,k-1}] &\leqslant \sum_{\substack{(j,k)\in\Sigma_K,\\(j,k+1)\in\Sigma_J}}\left(\frac{1}{8\lambda}z_{j,k}^2 + 2\lambda G^2 z_{j,k+1}^2\right) + \sum_{\substack{(j,k)\in\Sigma_K,\\(j,k-1)\in\Sigma_J}}\left(\frac{1}{8\lambda}z_{j,k}^2 + 2\lambda G^2 z_{j,k-1}^2\right) \\ &\leqslant \sum_{(j,k)\in\Sigma_K}\frac{1}{4\lambda}z_{j,k}^2 + \sum_{(j,k)\in\Sigma_J} 2\lambda G^2(0.8814\eta_{j,k})^2.\end{aligned} \tag{53}$$

By Lemma 3 and the fact that $U_{j,k} \geqslant 0$, it can be obtained that $\sum_{(j,k)\in\Sigma_K}[1 - 2\tanh^2(z_{j,k}/\eta_{j,k})]U_{j,k} \leqslant 0$, which implies that the term can be dropped from the right-hand side of (51).

Substituting (53) into (51) yields:

$$\dot{V}_{\Sigma_K} \leqslant -\sum_{(j,k)\in\Sigma_K} \left(\bar{\kappa}_{j,k} z_{j,k}^2 + \frac{\sigma_{j,k}}{2}\|\tilde{W}_{j,k}\|^2\right) + C_{\Sigma_K}, \tag{54}$$

where

$$C_{\Sigma_K} := \sum_{(j,k)\in\Sigma_K} \left(\lambda\bar{\varepsilon}_{j,k}^2 + \frac{\sigma_{j,k}}{2}\|W_{j,k}^* - W_{j,k}^0\|^2\right) + \sum_{(j,k)\in\Sigma_J} 2\lambda G^2(0.8814\eta_{j,k})^2 \tag{55}$$

is a positive constant. Since $\bar{\kappa}_{j,k} > 0$, we can conclude the boundedness of closed-loop trajectories in Σ_K, using a similar reasoning as that in Case 2.

Subsequently, we show boundedness of closed-loop trajectories in the entire system. First, note that the error signal z_{j,i_j}, for $(j, i_j) \in \Sigma_J$, is bounded by construction, i.e., $z_{j,i_j} \in \Omega_{c_{z_{j,i_j}}}$. It follows that $\hat{W}_{j,i_j}$ is bounded, which implies that $\tilde{W}_{j,i_j}$ is bounded since W_{j,i_j}^* is a constant. Now, considering the entire system (1), from Assumption 1, we have that y_{dj}, $y_{dj}^{(1)}, \ldots, y_{dj}^{(m_j)}$ are bounded. Thus it implies the boundedness of $x_{j,1}$, and in turn the boundedness of $\alpha_{j,1}$ in (5). Following this chain of reasoning, the boundedness of x_{j,i_j}, α_{j,i_j}, and u_j can be deduced. Therefore, we can conclude that all closed-loop signals in the entire system are bounded under Case 3.

Therefore, for all three cases, we have shown that the closed-loop signals are bounded. This concludes the proof. □

3.2. Special case: known bounds of functions of delayed states

When knowledge of $\varrho_{j,i_j}(\cdot)$—the bounds of the functions of delayed states—is available, they can be stated explicitly in the control law, instead of being approximated by NNs. To this end, we introduce additional assumptions, which are required in the subsequent developments.

Assumption 3. The unknown time delays $\tau_{j,k}$ ($j=1, 2, \ldots, n$, $k = 1, 2, \ldots, m_j$) are bounded by a known constant $\tau_{\max}$.

Assumption 4. For $j = 1, \ldots, n$, $k = 1, \ldots, m_j$, the following inequalities hold $\forall t$:

$$\left|\int_{t-\tau_{max}}^{t} U_{j,k}(x(\tau))d\tau\right| \leqslant M_{j,k}(x(t)) \ (1)$$

$$\left|\int_{t-\tau_{max}}^{t} \frac{\partial U_{j,k}(x(\tau))}{\partial x_i} d\tau\right| \leqslant M_{d_{j,k}}(x(t)) \ (2)$$

$$|U_{j,k}(x(t)) - U_{j,k}(x(t-\tau_{\max}))| < \bar{U}(x(t))$$

where $M_{j,k}(\cdot)$, $M_{d_{j,k}}(\cdot)$, and $\bar{U}(\cdot)$ are continuous positive functions, and x_i is each element of state vector x.

Remark 8. Assumption 4 is valid for many physical systems that are free from finite escape phenomenon. For such systems, the boundedness of integration or difference over a finite time interval is reasonable.

The intermediate and actual controls are designed as:

$$\alpha_{j,1} = \frac{1}{\bar{g}_{j,1}}\left[-\kappa_{j,1} z_{j,1}\int_0^1 \theta\bar{g}_{j,1}\,\mathrm{d}\theta - \kappa_{j,1}\frac{z_{j,1}\cosh(z_{j,1})}{1+z_{j,1}^2} \times \int_{t-\tau_{\max}}^{t} U_{j,1}\,\mathrm{d}\tau - \frac{1}{2\lambda} z_{j,1} + \hat{W}_{j,1}^{\mathrm{T}} S(Z_{j,1})\right], \tag{56}$$

$$\alpha_{j,i_j} = \frac{1}{\bar{g}_{j,i_j}}\left[-\bar{g}_{j,i_j-1} z_{j,i_j-1} - \kappa_{j,i_j} z_{j,i_j}\int_0^1 \theta\bar{g}_{j,i_j}\,\mathrm{d}\theta - \kappa_{j,i_j}\frac{z_{j,i_j}\cosh(z_{j,i_j})}{1+z_{j,i_j}^2}\int_{t-\tau_{\max}}^{t} U_{j,i_j}\,\mathrm{d}\tau - \frac{1}{2\lambda} z_{j,i_j} + \hat{W}_{j,i_j}^{\mathrm{T}} S(Z_{j,i_j})\right], \tag{57}$$

$$u_j = \frac{1}{\bar{g}_{j,m_j}}\left[-\bar{g}_{j,m_j-1} z_{j,m_j-1} - \kappa_{j,m_j} z_{j,m_j}\int_0^1 \theta\bar{g}_{j,m_j}\,\mathrm{d}\theta - \kappa_{j,m_j}\frac{z_{j,m_j}\cosh(z_{j,m_j})}{1+z_{j,m_j}^2}\int_{t-\tau_{\max}}^{t} U_{j,m_j}\,\mathrm{d}\tau - \frac{1}{2\lambda} z_{j,m_j} + \hat{W}_{j,m_j}^{\mathrm{T}} S(Z_{j,m_j})\right] \tag{58}$$

for $j = 1, 2, \ldots, n$, $i_j = 2, 3, \ldots, m_j - 1$, where the functions $U_{j,1}(\cdot)$, $U_{j,i_j}(\cdot)$, and $U_{j,m_j}(\cdot)$ are defined in (21), (34), and (41), respectively; and the ideal NN is defined by

$$W_{j,i_j}^{*\mathrm{T}} S(Z_{j,i_j}) = -F_{j,i_j}^*(Z_{j,i_j}) - \varepsilon_{j,i_j}(Z_{j,i_j}) \tag{59}$$

for $j = 1, 2, \ldots, n$, $i_j = 1, 2, \ldots, m_j$.

Remark 9. The virtual controls $\alpha_{j,k}$, for $j = 1, \ldots, n$, $k = 1, \ldots, m_j - 1$, are delay-dependent functions, and the term $\dot{\alpha}_{j,k}$ reintroduces delayed states $\bar{x}_{j,k}(t - \tau_{max})$, which poses a problem for the design in subsequent steps. However, based on Assumption 4, it can be shown that the partial derivatives of $\alpha_{j,k}$ are bounded by delay-independent functions, which then can be approximated by *NN* in backstepping without problem.

Remark 10. The choice of function $z\cosh(z)/(1+z^2)$ in (56)–(58), which was introduced in Ge et al. (2004), is not unique. In general, any function $\psi(z)$, for which $z\psi(z)$ is well-defined, and which satisfies the property

$$0 \leqslant z\psi(c) \leqslant z\psi(z) \quad \text{for } c \leqslant |z|, \tag{60}$$

where c is a positive constant, is a viable choice.

Theorem 2. *For system* (1), *under Assumptions* 1–3, *control law* (58), *NN adaptation law* (13), *and known bounding functions* $\varrho_{j,k}(\cdot)$, ($j = 1, \ldots, n$, $k = 1, \ldots, m_j$). *all closed-loop trajectories remain bounded.*

Proof. Consider the final step, with Lyapunov function candidate (44), control law (58), and adaptation law (13). Following the same procedure outlined for the proof of the general case, and noting the following properties (Ge et al., 2003):

$$\frac{z_{j,k}^2}{2} \leqslant V_{z_{j,k}} \leqslant \frac{z_{j,k}^2}{g_{0_{j,k}}} \int_0^1 \theta \bar{g}_{j,k}\,d\theta \tag{61}$$

$$\int_{t-\tau_{j,k}}^{t} [\varrho_{j,k}^i(x_{j,k}(\tau))]^2\,d\tau \leqslant \int_{t-\tau_{\max}}^{t} [\varrho_{j,k}^i(x_{j,k}(\tau))]^2\,d\tau \tag{62}$$

for $\tau_{j,k} \leqslant \tau_{\max}$, it can be shown that

$$\dot{V}_{n,m_n} \leqslant -\sum_{j=1}^{n}\sum_{k=1}^{m_j}\left[\frac{\sigma_{j,k}}{2\lambda_{\max}(\Gamma_{j,k}^{-1})}\tilde{W}_{j,k}^{\mathrm{T}}\Gamma_{j,k}^{-1}\tilde{W}_{j,k} + \kappa_{j,k}\left(g_{0_{j,k}}V_{z_{j,k}} + \frac{z_{j,k}^2\cosh(z_{j,k})}{1+z_{j,k}^2}V_{U_{j,k}}\right)\right] + C + \sum_{j=1}^{n}\sum_{k=1}^{m_j}\left[1-2\tanh^2\left(\frac{z_{j,k}}{\eta_{j,k}}\right)\right]U_{j,k}, \tag{63}$$

where $\lambda_{\max}(\Gamma_{j,k}^{-1})$ denotes the maximal eigenvalue of $\Gamma_{j,k}^{-1}$. To analyze the stability of the closed-loop system from (63), we need to consider three cases, similar to the analysis performed at the end of Section 3.1.

Case 1: $z_{j,i_j} \in \Omega_{c_{z_{j,i_j}}}$ $\forall j = 1, 2, \ldots, n,\ i_j = 1, 2, \ldots, m_j$.

All closed-loop signals are bounded, following the same analysis outlined in Case 1 of Theorem 1.

Case 2: $z_{j,i_j} \notin \Omega_{c_{z_{j,i_j}}}$ $\forall j = 1, 2, \ldots, n,\ i_j = 1, 2, \ldots, m_j$.

Due to Lemma 3, the last term of (63) can be removed from the analysis. Furthermore, we know that $|z_{j,k}| \geqslant 0.8814\eta_{j,k}$, which implies that

$$\frac{z_{j,k}^2\cosh(z_{j,k})}{1+z_{j,k}^2} \geqslant \frac{(0.8814\eta_{j,k})^2\cosh(0.8814\eta_{j,k})}{1+(0.8814\eta_{j,k})^2}. \tag{64}$$

As a result, it can be shown that

$$\dot{V}_{n,m_n} \leqslant -\rho V_{n,m_n} + C, \tag{65}$$

where C is defined in (47) and ρ as follows:

$$\rho := \min_{j,k}\left[\min\left(\kappa_{j,k}g_{0_{j,k}}, \frac{\sigma_{j,k}}{\lambda_{\max}(\Gamma_{j,k}^{-1})}, \frac{\kappa_{j,k}(0.8814\eta_{j,k})^2\cosh(0.8814\eta_{j,k})}{1+(0.8814\eta_{j,k})^2}\right)\right] > 0. \tag{66}$$

From (65), it is straightforward to show that all closed-loop trajectories are bounded (Ge & Wang, 2004).

Case 3: $z_{j,i_j} \in \Omega_{c_{z_{j,i_j}}}$ and $z_{k,i_k} \notin \Omega_{c_{z_{k,i_k}}}$ for $(j, i_j) \neq (k, i_k)$.

Denote by Σ_K the subsystem consisting of $z_{k,i_k} \notin \Omega_{c_{z_{k,i_k}}}$, and Σ_J the subsystem consisting of $z_{j,i_j} \in \Omega_{c_{z_{j,i_j}}}$. Define Lyapunov function candidate V_{Σ_K} according to (50). By similar reasoning as that in Case 3 of Theorem 1, we can show that

$$\dot{V}_{\Sigma_K} \leqslant -\rho_K V_{\Sigma_K} + C_{\Sigma_K}, \tag{67}$$

where $\rho_K := \rho|_{(j,k)\in\Sigma_K}$ and C_{Σ_K} is defined in (55). Therefore, we can conclude the boundedness of trajectories in Σ_K. As a result, we have that $z_{j,i_j} \in \Omega_{c_{z_{j,i_j}}}$ and $z_{k,i_k} \notin \Omega_{c_{z_{k,i_k}}}$, for $(j, i_j) \neq (k, i_k)$, are all bounded, from which we can conclude, via signal chasing, that all closed-loop signals in the entire system remain bounded.

It is clear that, for all three cases, the closed-loop signals are bounded. This concludes the proof. □

4. Performance analysis

The adaptive NN controller designed in the preceding section has been shown to ensure boundedness of trajectories in the closed-loop system. In this section, we quantify the convergence properties of the system under the following theorems. We will present and compare between the two cases of known and unknown bounds on the delay functions of delayed states.

For ease of notation, define $\eta := [\eta_{1,1}, \ldots, \eta_{j,i_j}, \ldots, \eta_{n,m_n}]^{\mathrm{T}}$, $z(t) = [z_{1,1}, \ldots, z_{j,i_j}, \ldots, z_{n,m_n}]^{\mathrm{T}}$, $\tilde{W} = [\tilde{W}_{1,1}, \ldots, \tilde{W}_{j,i_j}, \ldots, \tilde{W}_{n,m_n}]^{\mathrm{T}}$, $\Gamma := \mathrm{blockdiag}\{\Gamma_{1,1}, \ldots, \Gamma_{j,i_j}, \ldots, \Gamma_{n,m_n}\}$, $\eta_{\Sigma_J} := \{\eta_{j,k}\}|_{j,k\in\Sigma_J}$, $W_{\Sigma_J}^* := \{W_{j,k}^*\}|_{(j,k)\in\Sigma_J}$, $\tilde{W}_{\Sigma_J} := \{\tilde{W}_{j,k}\}|_{(j,k)\in\Sigma_J}$, $\rho_K := \rho|_{(j,k)\in\Sigma_K}$, $l_{\Sigma_J} := \sum_{(j,k)\in\Sigma_J} l_{j,k}$, and $l := \sum_{j=1}^{n}\sum_{k=1}^{m_j} l_{j,k}$.

4.1. General case: unknown bounds of functions of delayed states

Theorem 3. *Consider the closed-loop system consisting of the plant* (1) *under Assumptions* 1 *and* 2, *control law* (7) *and adaptation law* (13). *Given that the initial conditions are bounded, the error signal $z(t)$ will remain, in the mean square sense, within a compact set defined by*

$$\Omega_z := \{z \in R^{m_1+\cdots+m_n} \,|\, z_{\mathrm{ms}} \leqslant \mu_z\} \tag{68}$$

with $z_{\mathrm{ms}} := (1/t)\int_0^t \|z(\tau)\|^2\,d\tau$, *and* μ_z *defined by*

$$\mu_z := \max\left\{(0.8814)^2\|\eta\|^2, \frac{1/t(V_{n,m_n}|_{t=0}) + C}{\min_{j,k}(\kappa_{j,k}) - 1/4\lambda}, \frac{1/t(V_{\Sigma_K}|_{t=0}) + C_{\Sigma_K}}{\min_{(j,k)\in\Sigma_K}(\kappa_{j,k}) - 1/2\lambda} + (0.8814)^2\|\eta_{\Sigma_J}\|^2\right\} \tag{69}$$

Proof. From inequality (46), it is implied that

$$\dot{V}_{n,m_n} \leqslant -\left(\min_{j,k}(\kappa_{j,k}) - \frac{1}{4\lambda}\right)\sum_{j=1}^{n}\sum_{k=1}^{m_n} z_{j,k}^2 + C + \sum_{j=1}^{n}\sum_{k=1}^{m_j}\left[1-2\tanh^2\left(\frac{z_{j,k}}{\eta_{j,k}}\right)\right]U_{j,k}. \tag{70}$$

Case 1: $z_{j,i_j} \in \Omega_{c_{z_{j,i_j}}}$ $\forall j = 1, 2, \ldots, n,\ i_j = 1, 2, \ldots, m_j$.

For $|z_{j,i_j}| < 0.8814\eta_{j,i_j}$, it is clear that $\|z\|^2 < (0.8814)^2\|\eta\|^2$, which implies that

$$z_{\mathrm{ms}} < (0.8814)^2\|\eta\|^2. \tag{71}$$

Case 2: $z_{j,i_j} \notin \Omega_{c_{z_{j,i_j}}}$ $\forall j = 1, 2, \ldots, n,\ i_j = 1, 2, \ldots, m_j$.

For $z_{j,i_j} \geqslant 0.8814\eta_{j,i_j}$, the last summation term of (70) is negative, according to Lemma 3, and thus can be removed. Integrating both sides over the interval $[0, t]$, and dividing by t, we obtain

$$\frac{1}{t}(V_{n,m_n} - V_{n,m_n}|_{t=0}) \leqslant -\frac{1}{t}\left(\min_{j,k}(\kappa_{j,k}) - \frac{1}{4\lambda}\right) \times \int_0^t \sum_{j=1}^{n}\sum_{k=1}^{m_n} z_{j,k}^2\, \mathrm{d}\tau + C. \quad (72)$$

Hence, the error in the mean square z_{ms} is given by

$$z_{\mathrm{ms}} = \frac{1}{t}\int_0^t \|z\|^2\, \mathrm{d}\tau \leqslant \frac{1/t(V_{n,m_n}|_{t=0}) + C}{\min\limits_{j,k}(\kappa_{j,k}) - 1/4\lambda}. \quad (73)$$

Case 3: $z_{j,i_j} \in \Omega_{c_{z_{j,i_j}}}$ and $z_{k,i_k} \notin \Omega_{c_{z_{k,i_k}}}$ for $(j, i_j) \neq (k, i_k)$.

For the subsystem Σ_K consisting of $z_{k,i_k} \notin \Omega_{c_{z_{k,i_k}}}$, we obtain, from Lemma 3 and (53), that

$$\dot{V}_{\Sigma_K} \leqslant -\left(\min_{(j,k)\in\Sigma_K}(\kappa_{j,k}) - \frac{1}{2\lambda}\right) \sum_{(j,k)\in\Sigma_K} z_{j,k}^2 + C_{\Sigma_K}, \quad (74)$$

where C_{Σ_K} is defined in (55), which yields

$$z_{\mathrm{ms}}|_{\Sigma_K} := \frac{1}{t}\int_0^t \sum_{(j,k)\in\Sigma_K} z_{j,k}^2\, \mathrm{d}\tau \leqslant \frac{1/t(V_{\Sigma_K}|_{t=0}) + C_{\Sigma_K}}{\min\limits_{(j,k)\in\Sigma_K}(\kappa_{j,k}) - 1/2\lambda}. \quad (75)$$

For the subsystem Σ_J consisting of $z_{j,i_j} \in \Omega_{c_{z_{j,i_j}}}$, we know that

$$z_{\mathrm{ms}}|_{\Sigma_J} := s - \frac{1}{t}\int_0^t \sum_{(j,k)\in\Sigma_J} z_{j,k}^2\, \mathrm{d}\tau < (0.8814)^2\|\eta_{\Sigma_J}\|^2. \quad (76)$$

Combining (75) and (76), the bound on the error in the mean square sense is as follows:

$$z_{\mathrm{ms}} \leqslant \frac{1/t(V_{\Sigma_K}|_{t=0}) + C_{\Sigma_K}}{\min\limits_{(j,k)\in\Sigma_K}(\kappa_{j,k}) - 1/2\lambda} + (0.8814)^2\|\eta_{\Sigma_J}\|^2. \quad (77)$$

Therefore, based on (71), (73), and (77) for the three cases, it is straightforward to obtain that $z_{\mathrm{ms}} \leqslant \mu_z$. □

Corollary 1. *The closed-loop error signal $z(t)$ eventually converges to the set $\Omega_{zs} := \{z | z_{\mathrm{ms}} \leqslant \mu_{zs}\}$, where*

$$\mu_{zs} := \max\left\{(0.8814)^2\|\eta\|^2, \frac{C}{\min_{j,k}(\kappa_{j,k}) - 1/4\lambda}, \frac{C_{\Sigma_K}}{\min_{(j,k)\in\Sigma_K}(\kappa_{j,k}) - 1/2\lambda} + (0.8814)^2\|\eta_{\Sigma_J}\|^2\right\}. \quad (78)$$

Proof. The proof is straightforward by taking the limit of μ_z in (69) as $t \to \infty$ to obtain μ_{zs} in (78). □

Theorem 4. *For the same closed-loop system as that described in Theorem 3, the NN weights error $\tilde{W}$ remain, in the mean square sense, within a compact set defined by*

$$\Omega_W := \{\tilde{W} \in R^l | \tilde{W}_{\mathrm{ms}} \leqslant \mu_W\} \quad (79)$$

with $\tilde{W}_{\mathrm{ms}} := 1/t \int_0^t \|\tilde{W}(\tau)\|^2\, \mathrm{d}\tau$, *and*

$$\mu_W := \max\left\{3\left[\|\tilde{W}|_{t=0}\|^2 + l\left(\frac{0.8814\|\eta\|}{\min_{j,k}(\sigma_{j,k})}\right)^2 + \|W^*\|^2\right], \frac{2}{\min_{j,k}(\sigma_{j,k})}\left(\frac{V_{n,m_n}|_{t=0}}{t} + C\right), \frac{2}{\min_{(j,k)\in\Sigma_K}(\sigma_{j,k})}\left(\frac{V_{\Sigma_K}|_{t=0}}{t} + C_{\Sigma_K}\right) + 3\left[\left\|\tilde{W}_{\Sigma_J}|_{t=0}\right\|^2 + l_{\Sigma_J}\left(\frac{0.8814\|\eta_{\Sigma_J}\|}{\min_{(j,k)\in\Sigma_J}(\sigma_{j,k})}\right)^2 + \|W_{\Sigma_J}^*\|^2\right]\right\}. \quad (80)$$

Proof. The method of proof is similar to that of Theorem 3 and is omitted. □

From the above analysis, it can be seen that the bounds of the mean square signal z_{ms} is proportional to the initial conditions $V_{m,m_n}|_{t=0}$, and inversely proportional to time t. This means that the worst possible performance of the signals, as represented by the bounds, will decrease as time increases, such that the signal z always remain within the compact set Ω_z, and eventually converge to the steady state compact set Ω_{zs}, which can be seen to depend only on the design parameters and not the initial conditions. Therefore, the transient and steady state tracking errors can be made smaller by appropriate choice of parameters $\eta_{j,k}$, $l_{j,k}$, λ, $W_{j,k}^0$, $\Gamma_{j,k}$, $\sigma_{j,k}$ and $\kappa_{j,k}$.

4.2. Special case: known bounds of functions of delayed states

It has been reported that characterizing transient performance in the mean square sense does not eliminate the possibility of the undesirable "bursting" phenomenon (Hsu & Costa, 1987), where large-magnitude fluctuations occur for finite intervals of time in a cyclic manner, due to unmodeled dynamics or other factors not accounted for in the design.

By exploiting knowledge of the bounds of the functions of delayed states, the following results show that the *exponential form* Lyapunov inequality (65) can be obtained, which not only ensures boundedness of all closed-loop signals, but also allows better quantification of the transient and asymptotic bounds of the error signals. The explicit quantification of the error bounds helps to reduce unexpected intermittent degradation of performance, such as "bursting". However, the controller complexity has increased considerably, and the practitioner needs to assess the trade-off between the gain in performance and the increased difficulty in implementation.

Theorem 5. *Consider the closed-loop system consisting of the plant* (1) *under Assumptions* 1–3, *control law* (58) *and adaptation law* (13). *Furthermore, the bounds on the functions of delayed states,* $\varrho(\cdot)$, *are known. Given that the initial conditions are bounded, the error signal* $z(t)$ *remains within the compact set defined by*

$$\Pi_z := \{z \in R^{m_1+\cdots+m_n} \mid \|z\| \leqslant v_z\} \tag{81}$$

with

$$v_z := \max\left\{0.8814\|\eta\|, \sqrt{2\left(V_{n,m_n}|_{t=0} + \frac{C}{\rho}\right)}, \sqrt{2\left(V_{\Sigma_K}|_{t=0} + \frac{C_{\Sigma_K}}{\rho_K}\right)} + 0.8814\|\eta_{\Sigma_J}\|\right\}. \tag{82}$$

Corollary 2. *The closed-loop error signal* $z(t)$ *eventually converges to the set* $\Pi_{zs} := \{z \mid \|z\| \leqslant v_{zs}\}$ *where*

$$v_{zs} := \max\left\{\sqrt{\frac{2C}{\rho}}, \sqrt{\frac{2C_{\Sigma_K}}{\rho_K}} + 0.8814\|\eta_{\Sigma_J}\|, 0.8814\|\eta\|\right\}. \tag{83}$$

Theorem 6. *For the same closed-loop system as that described in Theorem* 5, *the NN weights error* $\tilde{W}$ *remain within a compact set defined by*

$$\Pi_W := \{\tilde{W} \in R^l \mid \|\tilde{W}\| \leqslant v_W\} \tag{84}$$

where

$$v_W := \max\left\{\|\tilde{W}|_{t=0}\| + \frac{0.8814\|\eta\|\sqrt{l}}{\min_{j,k}(\sigma_{j,k})} + \|W^*\|, \sqrt{\frac{2(V_{n,m_n}|_{t=0} + C/\rho)}{\lambda_{\min}(\Gamma^{-1})}}, \sqrt{\frac{2(V_{\Sigma_K}|_{t=0} + C_{\Sigma_K}/\rho_K)}{\lambda_{\min}(\Gamma_{\Sigma_K}^{-1})}} + \left\|\tilde{W}_{\Sigma_J}|_{t=0}\right\| + \frac{0.8814\|\eta_{\Sigma_J}\|\sqrt{l_{\Sigma_J}}}{\min_{(j,k)\in\Sigma_J}(\sigma_{j,k})} + \|W^*_{\Sigma_J}\|\right\}. \tag{85}$$

The proofs of Theorems 5–6 and Corollary 2 are similar to their counterparts in Section 4.1, and are omitted for brevity.

5. Simulation

In this section, a simulation study is presented to verify the effectiveness of the adaptive NN control design for the general case where the bounds on the functions of delayed states are not known. Consider the following two-input, two-output system in block-triangular form

$$\dot{x}_{1,1} = -x_{1,1} + (1 + \sin^2(x_{1,1}))x_{1,2} + x^2_{\tau_{1,1}},$$

$$\dot{x}_{1,2} = x_{1,1}x_{1,2} + x_{2,1} + x_{2,2} + (1 + \sin^2(x_{1,1}))u_1 + x_{\tau_{1,2}},$$

$$\dot{x}_{2,1} = -x_{2,1} + x_{2,2} + x_{\tau_{2,1}},$$

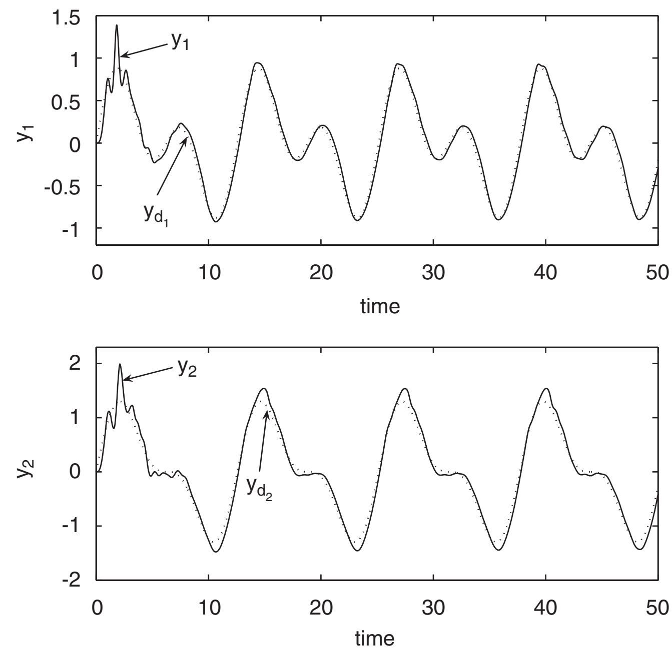

Fig. 1. Output tracking performance.

$$\dot{x}_{2,2} = (x_{1,2} + x_{2,1})x_{2,2} - x_{1,1}u_1 + (2 - \sin(x_{2,1}x_{2,2} - x_{1,1}))u_2 + x_{\tau_{1,1}}x_{\tau_{2,2}}, \tag{86}$$

where $x_{\tau_{j,i_j}} := x_{j,i_j}(t - \tau_{j,i_j})$, for $j = 1, 2$, $i_j = 1, 2$; and the time delays are $\tau_{1,1} = 2.0$, $\tau_{1,2} = 1.5$, $\tau_{2,1} = 0.5$, and $\tau_{2,2} = 1.0$. The desired trajectories to be tracked by the controller are given by $y_{d1} = 0.5(\sin(t) + \sin(0.5t))$ and $y_{d2} = 0.5\sin(t) + \sin(0.5t)$.

We consider the centers for $S(Z_{1,1})$ and $S(Z_{2,1})$ to be evenly spaced in a regular lattice in R^3, and that for $S(Z_{1,2})$ and $S(Z_{2,2})$ to be evenly spaced in regular lattices in R^7 and R^8, respectively. Employing three nodes for each input dimension, we end up with $3^3 = 27$ nodes for networks $\hat{W}^{\mathrm{T}}_{1,1}S(Z_{1,1})$ and $\hat{W}^{\mathrm{T}}_{2,1}S(Z_{2,1})$, $3^7 = 2187$ nodes for $\hat{W}^{\mathrm{T}}_{1,2}S(Z_{1,2})$ and $3^8 = 6561$ nodes for $\hat{W}^{\mathrm{T}}_{2,2}S(Z_{2,2})$. The design parameters are chosen as $\kappa_{j,i_j} = 1.0$, $\Gamma_{j,i_j} = \mathrm{diag}\{1.0\}$, $\sigma_{1,1} = 0.03$, $\sigma_{1,1} = 0.01$, $\sigma_{1,1} = 0.03$, $\sigma_{1,1} = 0.01$, and $W^0_{j,i_j} = 0.1$, while the initial conditions are $x_{j,i_j}(0) = 0.0$, $\hat{W}_{j,i_j}(0) = 0.0$, and $\phi_{j,i_j}(t) = 0.0$ for $-\tau_{j,i_j} \leqslant t < 0$, where $j = 1, 2$ and $i_j = 1, 2$.

As observed in Fig. 1, the tracking performance is fairly good. At the same time, the boundedness of the control signals u_1 and u_2 and the NN weights can be seen in Fig. 2. In fact, the tracking performance can be improved as desired by choosing larger values for $\kappa_{1,1}$ and $\kappa_{2,1}$, or by increasing the number of NN nodes.

6. Conclusion

This paper has proposed adaptive NN control for a class of block-triangular MIMO nonlinear systems with interconnected states carrying multiple constant delays. It has been shown that the tracking errors remain bounded within a neighborhood of the origin. At the same time, all other signals in the closed loop

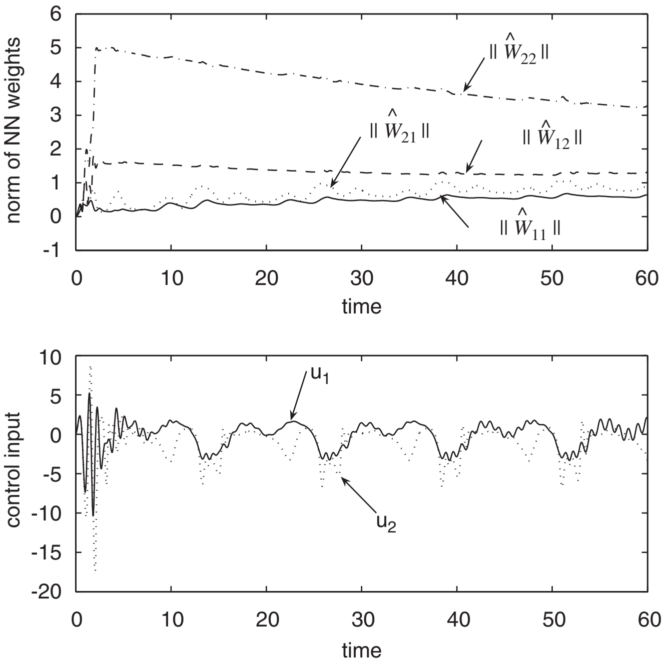

Fig. 2. Control signals and norms of NN weights.

are semi-globally uniformly ultimately bounded. For the special case whereby the bounds on the functions of delayed states are known, we show that this information can be exploited to obtain better quantification of performance bounds.

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Shuzhi Sam Ge (IEEE Fellow) is a Professor at Department of Electrical and Computer Engineering, the National University of Singapore. He received his B.Sc. degree from Beijing University of Aeronautics and Astronautics (BUAA), and the Ph.D. degree and the Diploma of Imperial College (DIC) from Imperial College of Science, Technology and Medicine, University of London. He has (co)-authored three books: Adaptive Neural Network Control of Robotic Manipulators (World Scientific, 1998), Stable Adaptive Neural Network Control (Kluwer, 2001) and Switched Linear Systems: Control and Design (Springer, 2005), and over 300 international journal and conference papers. He has been serving as Associate Editors for IEEE Transactions on Automatic Control, IEEE Transactions on Control Systems Technology, IEEE Transactions on Neural Networks, Automatica, and Editor for International Journal of Control, Automation and Systems. Dr. Ge is the recipient of the 1999 National Technology Award, 2001 University Young Research Award, 2002 Temasek Young Investigator Award, Singapore, and 2004 Outstanding Overseas Young Researcher Award from National Science Foundation, China. His current research interests include robotics, adaptive control, hybrid systems, and system development.

Keng Peng Tee received the B.Eng degree and the M.Eng degree from the National University of Singapore, in 2001 and 2003, respectively, both in mechanical engineering. He is currently pursuing the Ph.D. degree at the Department of Electrical and Computer Engineering, National University of Singapore, with the support of the A*STAR Graduate Scholarship. His current research interests include intelligent and adaptive control with applications in mechatronic systems.