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Adaptive NN control of uncertain nonlinear pure-feedback systems☆

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Abstract

This paper is concerned with the control of nonlinear pure-feedback systems with unknown nonlinear functions. This problem is considered difficult to be dealt with in the control literature, mainly because that the triangular structure of pure-feedback systems has no affine appearance of the variables to be used as virtual controls. To overcome this difficulty, implicit function theorem is firstly exploited to assert the existence of the continuous desired virtual controls. NN approximators are then used to approximate the continuous desired virtual controls and desired practical control. With mild assumptions on the partial derivatives of the unknown functions, the developed adaptive NN control schemes achieve semi-global uniform ultimate boundedness of all the signals in the closed-loop. The control performance of the closed-loop system is guaranteed by suitably choosing the design parameters. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Adaptive neural control; Uncertain pure-feedback system; Backstepping

1. Introduction

In the past decade, interest in adaptive control of nonlinear systems has been ever increasing, and many significant developments have been achieved. As a breakthrough in nonlinear control area, adaptive backstepping was introduced to achieve global stability and asymptotic tracking for a large class of nonlinear systems in the parametric strict-feedback form by Kanellakopoulos, Kokotovic, and Morse (1991). Later, the overparametrization problem was successfully eliminated in Krstiċ, Kanellakopoulos, and Kokotovic (1992) through the tuning function method. In an effort to extend the backstepping idea to larger classes of nonlinear systems, Kanellakopoulos et al. (1991) studied the adaptive control problem of parametric pure-feedback systems and obtained regionally stable results; Seto, Annaswamy, and Baillieul (1994) proposed several adaptive approaches for nonlinear systems with a triangular structure. To accommodate uncertainties, robust adaptive backstepping control has been studied for nonlinear strict-feedback systems with time-varying disturbances and static or dynamic uncertainties in Freeman and Kokotoviċ (1996), Yao and Tomizuka (1997), Jiang and Praly (1998) and Pan and Basar (1998) (to name just a few).

On the other hand, adaptive neural control schemes have been found to be particularly useful for the control of highly uncertain, nonlinear and complex systems (see Lewis, Jagannathan, & Yeildirek, 1999; Ge, Hang, Lee, & Zhang, 2001 and the references therein). In the earlier NN control schemes, optimization techniques were mainly used to derive parameter adaptation laws with little analytical results for stability and performance. To overcome these problems, some elegant adaptive NN control approaches have been proposed based on Lyapunov's stability theory (Narendra & Parthasarathy, 1990; Polycarpou & Ioannou, 1992; Sanner & Slotine, 1992; Rovithakis & Christodoulou, 1994; Chen & Khalil, 1995; Yesidirek & Lewis, 1995; Spooner & Passino, 1996). However, one limitation of these schemes is that they can only be applied to nonlinear systems where certain types of matching conditions are required to be satisfied.

Using the idea of adaptive backstepping design (Krstiċ, Kanellakopoulos, & Kokotoviċ, 1995), several neural-based adaptive controllers (Polycarpou &

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Mears, 1998; Ge et al., 2001) have been investigated for some classes of nonlinear systems in the following strict-feedback form without the requirement of matching conditions

$$\dot{x}_i = f_i(\bar{x}_i) + g_i(\bar{x}_i)x_{i+1}, \quad 1 \leqslant i \leqslant n-1,$$
$$\dot{x}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n)u, \quad n \geqslant 2, \tag{1}$$
$$y = x_1,$$

where $\bar{x}_i = [x_1, \ldots, x_i]^{\mathrm{T}} \in R^i$, $i = 1, \ldots, n$, $u \in R$, $y \in R$ are state variables, system input and output, respectively; $f_i(\cdot)$ and $g_i(\cdot)$, $i = 1, \ldots, n$ are unknown smooth functions. In Polycarpou and Mears (1998), an indirect adaptive NN control scheme was presented for system (1) with the affine terms $g_i(\bar{x}_i) = 1$, $i = 1, \ldots, n-1$, and $g_n(\bar{x}_n) = g$ being an unknown constant. The unknown functions $f_i(\bar{x}_i)$, $i = 1, \ldots, n$ are firstly approximated on-line by neural networks, then a stabilizing controller is constructed based on the approximation. Through the introduction of novel integral Lyapunov functions, direct adaptive neural network control was proposed for system (1) (Ge et al., 2001), in which the possible controller singularity problem usually met in adaptive control is avoided without using projection.

While the nonlinear strict-feedback systems have been much investigated via backstepping design, only a few results are available in the literature for the control of nonlinear pure-feedback systems (Nam & Arapostations, 1988; Kanellakopoulos et al., 1991; Seto et al., 1994; Krstič et al., 1995). The pure-feedback system represents a more general class of triangular systems which have no affine appearance of the variables to be used as virtual controls. In practice, there are many systems falling into this category, such as mechanical systems (Ferrara & Giacomini, 2000), aircraft flight control system (Hunt & Meyer, 1997), biochemical process (Krstič et al., 1995), Duffing oscillator (Dong, Chen, & Chen, 1997), etc. As indicated in Krstič et al. (1995), it was quite restrictive to find the explicit virtual controls to stabilize the pure-feedback systems by using integrator backstepping. In Kanellakopoulos et al. (1991); Krstič et al. (1995), while excellent results are given for global stabilization of parametric strict-feedback systems, only local stability is achieved in a well defined region around origin for parametric pure-feedback systems. By imposing additional restrictions on the nonlinearities, global stability is obtained for a special case of the parametric pure-feedback systems in Seto et al. (1994). Note that in Kanellakopoulos et al. (1991); Seto et al. (1994); Krstič et al. (1995), the nonlinearities are known smooth functions, and the unknown parameters occur linearly.

In this paper, adaptive NN control schemes are proposed for the following uncertain nonlinear pure-feedback systems:

$$\Sigma_1: \begin{cases} \dot{x}_i = f_i(\bar{x}_i, x_{i+1}), \quad 1 \leqslant i \leqslant n-2, \\ \dot{x}_{n-1} = f_{n-1}(\bar{x}_{n-1}) + g_{n-1}(\bar{x}_{n-1})x_n, \\ \dot{x}_n = f_n(\bar{x}_n) + g_n(\bar{x}_{n-1})u, \quad n \geqslant 3, \\ y = x_1 \end{cases} \tag{2}$$

and

$$\Sigma_2: \begin{cases} \dot{x}_i = f_i(\bar{x}_i, x_{i+1}), \quad 1 \leqslant i \leqslant n-2, \\ \dot{x}_{n-1} = f_{n-1}(\bar{x}_{n-1}) + g_{n-1}(\bar{x}_{n-1})x_n, \\ \dot{x}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n)u, \quad n \geqslant 3, \\ y = x_1, \end{cases} \tag{3}$$

where $\bar{x}_i = [x_1, \ldots, x_i]^{\mathrm{T}} \in R^i$, $i = 1, \ldots, n$, $u \in R$, $y \in R$ are state variables, system input and output, respectively; $f_i(\bar{x}_i, x_{i+1})$ $(i = 1, \ldots, n-2)$, $f_j(\cdot)$ and $g_j(\cdot)$ $(j = n-1, n)$ are unknown smooth functions. Here the difference between Σ_1 and Σ_2 only lies in $g_n(\cdot)$, i.e., $g_n(\cdot) = g_n(\bar{x}_{n-1})$ in Σ_1, while $g_n(\cdot) = g_n(\bar{x}_n)$ in Σ_2.

Due to the difficulties for controlling non-affine systems, the systems considered in this paper are affine in control u. Moreover, they are also affine in x_n in the $\dot{x}_{n-1}$ equations. To the authors' knowledge, no effective method for this control problem exists in the literature at present stage. This is mainly due to the fact that it is very difficult to find virtual controls α_i in terms of $\bar{x}_i$ in backstepping design procedure. To overcome this difficulty, implicit function theorem is firstly exploited to assert the existence of the continuous desired virtual controls $\alpha_i^*(\bar{x}_i)$ for $i = 1, \ldots, n-2$, then NN approximators are used to approximate the continuous desired virtual controls α_i^* and desired practical control u^* as in Ge, Hang, and Zhang (1999). With the help of NN approximation, there is no need to solve the implicit functions for the explicit virtual controls and the practical controller to cancel the unknown functions in backstepping design. The idea of integrator backstepping is still employed, i.e., some of the state variables are considered as "virtual controls", and intermediate control laws are designed in the constructive design procedures. With mild assumptions on the partial derivatives of the unknown functions $f_i(\bar{x}_i, x_{i+1})$ $(i = 1, \ldots, n-2)$ as well as on $g_j(\cdot)$ $(j = n-1, n)$, the developed adaptive NN control scheme achieves semi-global uniform ultimate boundedness of all the signals in the closed-loop. Moreover, the output of the system is proven to converge to a small neighborhood of the desired trajectory. The control performance of the closed-loop system is guaranteed by suitably choosing the design parameters.

2. Problem formulation and preliminaries

The control objective is to design direct adaptive NN controllers for the systems such that (i) all the signals in the closed-loop remain semi-globally uniformly ultimately bounded, and (ii) the output y follows a desired trajectory y_d generated from the following smooth, bounded reference model:

$$\dot{x}_{di} = f_{di}(x_d), \quad 1 \leqslant i \leqslant m, \qquad y_d = x_{d1}, \tag{4}$$

where $x_d = [x_{d1}, x_{d2}, \ldots, x_{dm}]^{\mathrm{T}} \in R^m$ are the states, $y_d \in R$ is the system output, $f_{di}(\cdot)$, $i = 1, 2, \ldots, m$ are known smooth nonlinear functions. Assume that x_d remain bounded, i.e., $x_d \in \Omega_d$, $\forall t \geqslant 0$.

Lemma 1. *Assume that $f(x,u): R^n \times R \to R$ is continuously differentiable $\forall (x,u) \in R^n \times R$, and there exists a positive constant d such that $\partial f(x,u)/\partial u > d > 0$, $\forall (x,u) \in R^n \times R$. Then there exists a continuous (smooth) function $u^* = u(x)$ such that $f(x,u^*) = 0$.*

Proof. Firstly, it can be proven that for every $x \in R^n$, there exists a unique $u(x) \in R$ such that $f(x,u(x)) = 0$. Because the partial derivative $\partial f(x,u)/\partial u$ has a lower positive bound d, i.e., $\partial f(x,u)/\partial u > d > 0$, $\forall (x,u) \in R^n \times R$, if $f(x,0) = c \neq 0$, then by mean value theorem (Apostol, 1963),

$$\begin{aligned} f\left(x, \frac{|c|}{d}\right) &= f(x,0) + \left.\frac{\partial f(x,u)}{\partial u}\right|_{u=u_*^1} \left(\frac{|c|}{d} - 0\right) \\ &> f(x,0) + |c| \geqslant 0, \\ f\left(x, -\frac{|c|}{d}\right) &= f(x,0) + \left.\frac{\partial f(x,u)}{\partial u}\right|_{u=u_*^2} \left(-\frac{|c|}{d} - 0\right) \\ &< f(x,0) - |c| \leqslant 0, \end{aligned} \tag{5}$$

where $u_*^1 \in (0, |c|/d)$ and $u_*^2 \in (-|c|/d, 0)$. Therefore, by intermediate value theorem (Apostol, 1963), there exists $u \in (-|c|/d, |c|/d)$ such that $f(x,u) = 0$. If $f(x,0) = c = 0$, then $u = 0$ is the trivial solution satisfying $f(x,u) = 0$. Secondly, it can be shown that $u(x)$ is a continuous (smooth) function of x. Fix x_0, apply implicit function theorem (Apostol, 1963) to $f(x,u)$ around the point $(x_0, u(x_0))$. Because the positivity of partial derivative to u, it can be seen that there exists a continuous (smooth) function $u_1(x)$ around $x = x_0$, such that $f(x, u_1(x)) = 0$ around $x = x_0$. The strict increasing of $f(x,u)$ with respect to u leads to $u_1(x) = u(x)$ for every x. Therefore, $u(x)$ is continuous (smooth) in x around x_0. Since x_0 is arbitrary, it is clear that $u(x)$ is a continuous (smooth) function in x. □

Note that NN approximation is only guaranteed within some compact sets in the derivation of the adaptive neural controller. Accordingly, the stability results obtained in this work are semi-global in the sense that, as long as the input variables of the NNs remain within some compact sets, where the compact sets can be made as large as desired, there exists controller(s) with sufficiently large number of NN nodes such that all the signals in the closed-loop remain bounded.

In control engineering, radial basis function (RBF) neural networks (NNs) are usually used as a tool for modeling nonlinear functions because of their good capabilities in function approximation. They belong to a class of linearly parameterized networks. For comprehensive treatment of NN approximation, see (Ge et al., 2001). In this paper, the following RBF NN (Haykin, 1999) is used to approximate the continuous function $h(Z): R^q \to R$,

$$h_{nn}(Z) = W^{\mathrm{T}} S(Z), \tag{6}$$

where the input vector $Z \in \Omega \subset R^q$, weight vector $W = [w_1, w_2, \ldots, w_l]^{\mathrm{T}} \in R^l$, the NN node number $l > 1$; and $S(Z) = [s_1(Z), \ldots, s_l(Z)]^{\mathrm{T}}$, with $s_i(Z)$ being chosen as the commonly used Gaussian functions, which have the form

$$s_i(Z) = \exp\left[\frac{-(Z-\mu_i)^{\mathrm{T}}(Z-\mu_i)}{\eta_i^2}\right], \quad i = 1, 2, \ldots, l, \tag{7}$$

where $\mu_i = [\mu_{i1}, \mu_{i2}, \ldots, \mu_{iq}]^{\mathrm{T}}$ is the center of the receptive field and η_i is the width of the Gaussian function. It has been proven that network (6) can approximate any continuous function over a compact set $\Omega_Z \subset R^q$ to arbitrary any accuracy as

$$h(Z) = W^{*\mathrm{T}} S(Z) + \varepsilon, \quad \forall Z \in \Omega_Z, \tag{8}$$

where W^* is ideal constant weights, and ε is the approximation error.

Assumption 1. There exist ideal constant weights W^* such that $|\varepsilon| \leqslant \varepsilon^*$ with constant $\varepsilon^* > 0$ for all $Z \in \Omega_Z$.

The ideal weight vector W^* is an "artificial" quantity required for analytical purposes. W^* is defined as the value of W that minimizes $|\varepsilon|$ for all $Z \in \Omega_Z \subset R^q$, i.e.,

$$W^* \triangleq \arg\min_{W \in R^l} \left\{ \sup_{Z \in \Omega_Z} |h(Z) - W^{\mathrm{T}} S(Z)| \right\}. \tag{9}$$

In the following, let $\|\cdot\|$ denote the 2-norm, and $\lambda_{\max}(B)$ and $\lambda_{\min}(B)$ denote the largest and smallest eigenvalues of a square matrix B, respectively.

3. Direct adaptive NN control for Σ_1

For the control of pure-feedback systems Σ_1 and Σ_2, define $g_i(\bar{x}_i, x_{i+1}) := \partial f_i(\bar{x}_i, x_{i+1})/\partial x_{i+1}$, $i = 1, \ldots, n-2$, which are also unknown nonlinear functions.

Assumption 2. The signs of $g_i(\bar{x}_i, x_{i+1})$, $i = 1, \ldots, n-2$, $g_{n-1}(\bar{x}_{n-1})$ and $g_n(\bar{x}_{n-1})$ are known, and there exist constants $g_{i1} > g_{i0} > 0$ such that (i) $|g_i(\cdot)| > g_{i0} > 0$, $\forall \bar{x}_n \in R^n$, and (ii) $|g_i(\cdot)| \leqslant g_{i1} < \infty$, $\forall \bar{x}_n \in \Omega_{\bar{x}_n} \subset R^n$ where $\Omega_{\bar{x}_n}$ is a compact region, $i = 1, \ldots, n$.

The above assumption implies that partial derivatives $g_i(\cdot)$, $i=1,\ldots,n$ are strictly either positive or negative. Without losing generality, it is assumed that $g_i(\cdot) > g_{i0} > 0$, $\forall \bar{x}_i \in R^i$.

Accordingly, the derivatives of $g_i(\cdot)$, $i=1,\ldots,n$ are given by

$$\dot{g}_i(\bar{x}_i,x_{i+1}) = \sum_{k=1}^{i+1} \frac{\partial g_i(\bar{x}_i,x_{i+1})}{\partial x_k}\dot{x}_k = \sum_{k=1}^{i+1} \frac{\partial g_i(\bar{x}_i,x_{i+1})}{\partial x_k} f_k(\bar{x}_{k+1}), \quad i=1,\ldots,n-2,$$

$$\dot{g}_{n-1}(\bar{x}_{n-1}) = \sum_{k=1}^{n-1} \frac{\partial g_{n-1}(\bar{x}_{n-1})}{\partial x_k}\dot{x}_k = \sum_{k=1}^{n-2} \frac{\partial g_{n-1}(\bar{x}_{n-1})}{\partial x_k} f_k(\bar{x}_{k+1}) + \frac{\partial g_{n-1}(\bar{x}_{n-1})}{\partial x_{n-1}}[f_{n-1}(\bar{x}_{n-1})+g_{n-1}(\bar{x}_{n-1})x_n],$$

$$\dot{g}_n(\bar{x}_{n-1}) = \sum_{k=1}^{n-1} \frac{\partial g_n(\bar{x}_{n-1})}{\partial x_k}\dot{x}_k = \sum_{k=1}^{n-2} \frac{\partial g_n(\bar{x}_{n-1})}{\partial x_k} f_k(\bar{x}_{k+1}) + \frac{\partial g_n(\bar{x}_{n-1})}{\partial x_{n-1}}[f_{n-1}(\bar{x}_{n-1})+g_{n-1}(\bar{x}_{n-1})x_n]. \quad (10)$$

Clearly, they only depend on states $\bar{x}_n$. Because $f_i(\cdot)$, $i=1,\ldots,n-1$, $g_{n-1}(\cdot)$ and $g_n(\cdot)$ are assumed to be smooth functions, $\dot{g}_i(\cdot)$, $i=1,\ldots,n$ are therefore bounded within a compact set. Accordingly, the following assumption is made for $\dot{g}_i(\cdot)$, $i=1,\ldots,n$.

Assumption 3. There exist constants $g_{id} > 0$ such that $|\dot{g}_i(\cdot)| \leqslant g_{id}$, $\forall \bar{x}_n \in \Omega_{\bar{x}_n} \subset R^n$ where $\Omega_{\bar{x}_n}$ is a compact region, $i=1,\ldots,n$.

In this section, direct adaptive NN design is presented by using RBF NN. At each recursive step i, an intermediate desired feedback control α_i^* is first shown to exist which possesses some desired stabilizing properties, and then the ith-order subsystem is stabilized with respect to a Lyapunov function V_i by the design of a stabilizing function α_i, where an RBF neural network is employed to approximate the unknown part in intermediate desired feedback control α_i^*. The control law u is designed in the last step.

Step 1: Define $z_1 = x_1 - x_{d1}$. Its derivative is

$$\dot{z}_1 = f_1(x_1,x_2) - \dot{x}_{d1}. \quad (11)$$

From Assumption 2, we know that $\partial f_1(x_1,x_2)/\partial x_2 > g_{10} > 0$ for all $(x_1,x_2) \in R^2$. Define v_1 as

$$v_1 = -\dot{x}_{d1} + k_1 z_1, \quad (12)$$

where $k_1 > 0$ is a constant. It is clear that v_1 is a function of x_1 and x_d. Considering the fact that $\partial v_1/\partial x_2 = 0$, the following inequality holds

$$\frac{\partial[f_1(x_1,x_2)+v_1]}{\partial x_2} > g_{10} > 0.$$

According to Lemma 1, by viewing x_2 as a virtual control input, for every value of x_1 and v_1, there exists a smooth ideal control input $x_2 = \alpha_1^*(x_1,v_1)$ such that

$$f_1(x_1,\alpha_1^*) + v_1 = 0. \quad (13)$$

Using mean value theorem (Apostol, 1963), there exists λ_1 $(0 < \lambda_1 < 1)$ such that

$$f_1(x_1,x_2) = f_1(x_1,\alpha_1^*) + g_{1\lambda_1}(x_2 - \alpha_1^*), \quad (14)$$

where $g_{1\lambda_1} := g_1(x_1,x_{2\lambda_1})$ with $x_{2\lambda_1} = \lambda_1 x_2 + (1-\lambda_1)\alpha_1^*$. Note that Assumption 2 on $g_1(x_1,x_2)$ is still valid for $g_{1\lambda_1}$. Since $g_{1\lambda_1}$ is a function of x_1, x_2 and α_1^*, from (12) and (13), it can be seen that α_1^* is a function of x_1 and x_d, the derivation of $\dot{g}_{1\lambda_1}$ is

$$\dot{g}_{1\lambda_1} = \sum_{k=1}^{2} \frac{\partial g_{1\lambda_1}}{\partial x_k}\dot{x}_k + \sum_{k=1}^{m} \frac{\partial g_{1\lambda_1}}{\partial x_{dk}}\dot{x}_{dk} = \sum_{k=1}^{2} \frac{\partial g_{1\lambda_1}}{\partial x_k} f_k(\bar{x}_{k+1}) + \sum_{k=1}^{m} \frac{\partial g_{1\lambda_1}}{\partial x_{dk}} f_{dk}(x_d).$$

Since x_d is assumed to be bounded, similar to Assumption 3, it is reasonable to assume that $|\dot{g}_{1\lambda_1}|$ is bounded by some constant within some compact region. For uniformity of presentation, we assume that $|\dot{g}_{1\lambda_1}(\cdot)| \leqslant g_{1d}$, $\forall \bar{x}_n \in \Omega_{\bar{x}_n}$ and $x_d \in \Omega_d$, where g_{1d} is the same as in Assumption 3 for $|\dot{g}_1|$. In other words, Assumption 3 on $\dot{g}_1(x_1,x_2)$ is valid for $\dot{g}_{1\lambda_1}$.

Combining (11)–(14) yields

$$\dot{z}_1 = -k_1 z_1 + g_{1\lambda_1}(x_2 - \alpha_1^*). \quad (15)$$

By employing an RBF neural network $W_1^{\mathrm{T}} S_1(Z_1)$ to approximate $\alpha_1^*(x_1,v_1)$, where $Z_1 = [x_1,\dot{x}_{d1},z_1]^{\mathrm{T}} \in \Omega_1 \subset R^3$, α_1^* can be expressed as

$$\alpha_1^* = W_1^{*\mathrm{T}} S_1(Z_1) + \varepsilon_1, \quad (16)$$

where W_1^* denotes the ideal constant weights, and $|\varepsilon_1| \leqslant \varepsilon_1^*$ is the approximation error with constant $\varepsilon_1^* > 0$.

Let $\hat{W}_1$ be the estimate of W_1^*. Define $z_2 = x_2 - \alpha_1$ and let

$$\alpha_1 = -c_1 z_1 + \hat{W}_1^{\mathrm{T}} S_1(Z_1), \quad (17)$$

where c_1 is a positive constant to be specified later. Then, the dynamics of z_1 is governed by

$$\dot{z}_1 = -k_1 z_1 + g_{1\lambda_1}(z_2 + \alpha_1 - \alpha_1^*) = -k_1 z_1 + g_{1\lambda_1}[z_2 - c_1 z_1 + \tilde{W}_1^{\mathrm{T}} S_1(Z_1) - \varepsilon_1], \tag{18}$$

where $\tilde{W}_1 = \hat{W}_1 - W_1^*$. Through out this paper, define $(\tilde{\cdot}) = (\hat{\cdot}) - (\cdot)^*$.

Consider the Lyapunov function candidate

$$V_1 = \frac{1}{2g_{1\lambda_1}} z_1^2 + \frac{1}{2}\tilde{W}_1^{\mathrm{T}} \Gamma_1^{-1} \tilde{W}_1. \tag{19}$$

The derivative of V_1 is

$$\dot{V}_1 = \frac{z_1 \dot{z}_1}{g_{1\lambda_1}} - \frac{\dot{g}_{1\lambda_1} z_1^2}{2g_{1\lambda_1}^2} + \tilde{W}_1^{\mathrm{T}} \Gamma_1^{-1} \dot{\tilde{W}}_1 = -\frac{k_1}{g_{1\lambda_1}} z_1^2 + z_1 z_2 - c_1 z_1^2 - \frac{\dot{g}_{1\lambda_1}}{2g_{1\lambda_1}^2} z_1^2 - z_1 \varepsilon_1 + \tilde{W}_1^{\mathrm{T}} S_1(Z_1) z_1 + \tilde{W}_1^{\mathrm{T}} \Gamma_1^{-1} \dot{\hat{W}}_1. \tag{20}$$

Consider the following adaptation law

$$\dot{\hat{W}}_1 = \dot{\tilde{W}}_1 = \Gamma_1[-S_1(Z_1) z_1 - \sigma_1 \hat{W}_1], \tag{21}$$

where $\sigma_1 > 0$ is a small constant. Let $c_1 = c_{10} + c_{11}$, with c_{10} and $c_{11} > 0$. Then, Eq. (20) becomes

$$\dot{V}_1 = -\frac{k_1}{g_{1\lambda_1}} z_1^2 + z_1 z_2 - \left(c_{10} + \frac{\dot{g}_{1\lambda_1}}{2g_{1\lambda_1}^2} \right) z_1^2 - c_{11} z_1^2 - z_1 \varepsilon_1 - \sigma_1 \tilde{W}_1^{\mathrm{T}} \hat{W}_1. \tag{22}$$

By completion of squares, the following inequalities hold

$$-\sigma_1 \tilde{W}_1^{\mathrm{T}} \hat{W}_1 = -\sigma_1 \tilde{W}_1^{\mathrm{T}} (\tilde{W}_1 + W_1^*) \leqslant -\sigma_1 \|\tilde{W}_1\|^2 + \sigma_1 \|\tilde{W}_1\| \|W_1^*\| \leqslant -\frac{\sigma_1 \|\tilde{W}_1\|^2}{2} + \frac{\sigma_1 \|W_1^*\|^2}{2},$$

$$-c_{11} z_1^2 - z_1 \varepsilon_1 \leqslant \frac{\varepsilon_1^2}{4c_{11}} \leqslant \frac{\varepsilon_1^{*2}}{4c_{11}}. \tag{23}$$

Because $-(c_{10} + (\dot{g}_{1\lambda_1}/2g_{1\lambda_1}^2)) z_1^2 \leqslant -(c_{10} - (g_{1d}/2g_{10}^2)) z_1^2$, by choosing c_{10} large enough such that $c_{10}^* := c_{10} - (g_{1d}/2g_{10}^2) > 0$, the derivative of V_1 satisfies

$$\dot{V}_1 < -\frac{k_1}{g_{1\lambda_1}} z_1^2 + z_1 z_2 - c_{10}^* z_1^2 - \frac{\sigma_1 \|\tilde{W}_1\|^2}{2} + \frac{\sigma_1 \|W_1^*\|^2}{2} + \frac{\varepsilon_1^{*2}}{4c_{11}}. \tag{24}$$

Step 2: The derivative of $z_2 = x_2 - \alpha_1$ is

$$\dot{z}_2 = f_2(\bar{x}_2, x_3) - \dot{\alpha}_1. \tag{25}$$

From Assumption 2, we know that $\partial f_2(\bar{x}_2, x_3)/\partial x_3 > g_{20} > 0$ for all $\bar{x}_3 \in R^3$. Define

$$v_2 = -\dot{\alpha}_1 + k_2 z_2, \tag{26}$$

where $k_2 > 0$ is a constant. Considering the fact that $\partial v_2/\partial x_3 = 0$, the following inequality holds

$$\frac{\partial [f_2(\bar{x}_2, x_3) + v_2]}{\partial x_3} > g_{20} > 0.$$

According to Lemma 1, by viewing x_3 as a virtual control input, for every value of $\bar{x}_2$ and v_2, there exists a smooth ideal control input $x_3 = \alpha_2^*(\bar{x}_2, v_2)$ such that

$$f_2(\bar{x}_2, \alpha_2^*) + v_2 = 0. \tag{27}$$

Using mean value theorem (Apostol, 1963), there exists λ_2 $(0 < \lambda_2 < 1)$ such that

$$f_2(\bar{x}_2, x_3) = f_2(\bar{x}_2, \alpha_2^*) + g_{2\lambda_2}(x_3 - \alpha_2^*), \tag{28}$$

where $g_{2\lambda_2} := g_2(\bar{x}_2, x_{3\lambda_2})$ with $x_{3\lambda_2} = \lambda_2 x_3 + (1 - \lambda_2)\alpha_2^*$. Note that Assumption 2 on $g_2(\bar{x}_2, x_3)$ is still valid for $g_{2\lambda_2}$. Since $g_{2\lambda_2}$ is a function of x_1, x_2, x_3 and α_2^*, and from (26) and (27), it can be seen that α_2^* is a function of $\bar{x}_2$, z_2 and $\dot{\alpha}_1$, i.e., a function of x_1, x_2, x_d and $\hat{W}_1$, the derivation of $\dot{g}_{2\lambda_2}$ is

$$\dot{g}_{2\lambda_2} = \sum_{k=1}^{3} \frac{\partial g_{2\lambda_2}}{\partial x_k} \dot{x}_k + \sum_{k=1}^{m} \frac{\partial g_{2\lambda_2}}{\partial x_{dk}} \dot{x}_{dk} + \frac{\partial g_{2\lambda_2}}{\partial \hat{W}_1} \dot{\hat{W}}_1 = \sum_{k=1}^{2} \frac{\partial g_{1\lambda_1}}{\partial x_k} f_k(\bar{x}_{k+1}) + \sum_{k=1}^{m} \frac{\partial g_{1\lambda_1}}{\partial x_{dk}} f_{dk}(x_d) + \frac{\partial g_{2\lambda_2}}{\partial \hat{W}_1} [\Gamma_1(-S_1(Z_1) z_1 - \sigma_1 \hat{W}_1)].$$

Similar to Step 1, it is reasonable to assume that $|\dot{g}_{2\lambda_2}|$ is bounded by some constant within some compact region. For uniformity of presentation, we assume that $|\dot{g}_{2\lambda_2}(\cdot)| \leqslant g_{2d}$, $\forall \bar{x}_n \in \Omega_{\bar{x}_n}$, $x_d \in \Omega_d$ and $\hat{W}_1 \in \Omega_{W_1}$, where g_{2d} is the same as in Assumption 3 for $|\dot{g}_2|$. In other words, Assumption 3 on $\dot{g}_2(\bar{x}_2, x_3)$ is valid for $\dot{g}_{2\lambda_2}$. Subsequently, such an assumption is also made for $\dot{g}_{i\lambda_i}$ in the following steps.

Combining (25)–(28) yields

$$\dot{z}_2 = -k_2 z_2 + g_{2\lambda_2}(x_3 - \alpha_2^*). \tag{29}$$

Since α_1 is a function of x_1, x_d and $\hat{W}_1$, $\dot{\alpha}_1$ is given by

$$\dot{\alpha}_1 = \frac{\partial \alpha_1}{\partial x_1} \dot{x}_1 + \frac{\partial \alpha_1}{\partial x_d} \dot{x}_d + \frac{\partial \alpha_1}{\partial \hat{W}_1} \dot{\hat{W}}_1 = \frac{\partial \alpha_1}{\partial x_1} f_1(\bar{x}_2) + \phi_1, \tag{30}$$

where

$$\phi_1 = \frac{\partial \alpha_1}{\partial x_d}\dot{x}_d + \frac{\partial \alpha_1}{\partial \hat{W}_1}[\Gamma_1(-S_1(Z_1)z_1 - \sigma_1 \hat{W}_1)] \tag{31}$$

is computable.

By employing an RBF neural network $W_2^{\mathrm{T}} S_2(Z_2)$ to approximate $\alpha_2^*(\bar{x}_2, v_2)$, where $Z_2 = [\bar{x}_2^{\mathrm{T}}, \partial\alpha_1/\partial x_1, \phi_1, z_2]^{\mathrm{T}} \in \Omega_2 \subset R^5$, α_2^* can be expressed as

$$\alpha_2^* = W_2^{*\mathrm{T}} S_2(Z_2) + \varepsilon_2, \tag{32}$$

where W_2^* denotes the ideal constant weights, and $|\varepsilon_2| \leqslant \varepsilon_2^*$ is the approximation error with constant $\varepsilon_2^* > 0$.

Remark 1. From the definitions of v_2 in (26), and $\dot{\alpha}_1$ in (30), it can be seen that $\alpha_2^*(\bar{x}_2, v_2)$ in (28) is a function of $\bar{x}_2$, x_d, z_2 and $\hat{W}_1$. However, neural network weights $\hat{W}_1$ are not recommended to be used as inputs to the NN controller under construction. This is because of the curse of dimensionality of RBF NN which may result in the number of NN inputs to be too large for $\hat{W}_2^{\mathrm{T}} S_2(Z_2)$. By defining the intermediate variables $\partial\alpha_1/\partial x_1$ and ϕ_1 which are computable, the NN approximation $\hat{W}_2^{\mathrm{T}} S_2(Z_2)$ for $\alpha_2^*(\bar{x}_2, v_2)$ can be computed by using the minimal number of NN inputs $Z_2 = [\bar{x}_2^{\mathrm{T}}, \partial\alpha_1/\partial x_1, \phi_1, z_2]^{\mathrm{T}}$. Note also that the variable vector x_d does not appear in Z_2, because it has been combined into the intermediate variable ϕ_1.

Since W_2^* is unknown, α_2^* cannot be realized in practice. Let $\hat{W}_2$ be the estimate of W_2^*. Define $z_3 = x_3 - \alpha_2$ and let

$$\alpha_2 = -z_1 - c_2 z_2 + \hat{W}_2^{\mathrm{T}} S_2(Z_2), \tag{33}$$

where c_2 is a positive constant to be specified later. Then, the dynamics of z_2 is governed by

$$\begin{aligned}\dot{z}_2 &= -k_2 z_2 + g_{2\lambda_2}(z_3 + \alpha_2 - \alpha_2^*) \\ &= -k_2 z_2 + g_{2\lambda_2}[z_3 - z_1 - c_2 z_2 + \tilde{W}_2^{\mathrm{T}} S_2(Z_2) - \varepsilon_2]. \end{aligned}\tag{34}$$

Consider the Lyapunov function candidate

$$V_2 = V_1 + \frac{1}{2g_{2\lambda_2}} z_2^2 + \frac{1}{2}\tilde{W}_2^{\mathrm{T}} \Gamma_2^{-1} \tilde{W}_2. \tag{35}$$

The derivative of V_2 is

$$\begin{aligned}\dot{V}_2 &= \dot{V}_1 + \frac{z_2 \dot{z}_2}{g_{2\lambda_2}} - \frac{\dot{g}_{2\lambda} z_2^2}{2g_{2\lambda_2}^2} + \tilde{W}_2^{\mathrm{T}} \Gamma_2^{-1} \dot{\tilde{W}}_2, \\ &= \dot{V}_1 - \frac{k_2}{g_{2\lambda_2}} z_2^2 - z_1 z_2 + z_2 z_3 - c_2 z_2^2 - \frac{\dot{g}_{2\lambda}}{2g_{2\lambda_2}^2} z_2^2 - z_2 \varepsilon_2 \\ &\quad + \tilde{W}_2^{\mathrm{T}} S_2(Z_2) z_2 + \tilde{W}_2^{\mathrm{T}} \Gamma_2^{-1} \dot{\tilde{W}}_2. \end{aligned}\tag{36}$$

Consider the following adaptation law

$$\dot{\hat{W}}_2 = \dot{\tilde{W}}_2 = \Gamma_2[-S_2(Z_2)z_2 - \sigma_2 \hat{W}_2], \tag{37}$$

where $\sigma_2 > 0$ is a small constant. Let $c_2 = c_{20} + c_{21}$, where c_{20} and $c_{21} > 0$. By using (24), (36) and (37), and with some completion of squares and straightforward derivation similar to those employed in Step 1, the derivative of V_2 becomes

$$\begin{aligned}\dot{V}_2 &< z_2 z_3 - \sum_{j=1}^{2} \frac{k_j}{g_{j\lambda}} z_j^2 - \sum_{j=1}^{2} c_{j0}^* z_j^2 - \sum_{j=1}^{2} \frac{\sigma_j \|\tilde{W}_j\|^2}{2} \\ &\quad + \sum_{j=1}^{2} \frac{\sigma_j \|W_j^*\|^2}{2} + \sum_{j=1}^{2} \frac{\varepsilon_j^{*2}}{4c_{j1}}, \end{aligned}\tag{38}$$

where $c_{20}^* := c_{20} - (g_{2d}/2g_{20}^2) > 0$.

Step i ($3 \leqslant i \leqslant n-2$): The derivative of $z_i = x_i - \alpha_{i-1}$ is

$$\dot{z}_i = f_i(\bar{x}_i, x_{i+1}) - \dot{\alpha}_{i-1}. \tag{39}$$

From Assumption 2, we know that $\partial f_i(\bar{x}_i, x_{i+1})/\partial x_{i+1} > g_{i0} > 0$ for all $\bar{x}_{i+1} \in R^{i+1}$. Define

$$v_i = -\dot{\alpha}_{i-1} + k_i z_i, \tag{40}$$

where $k_i > 0$ is a constant. Considering the fact that $\partial v_i/\partial x_{i+1} = 0$, the following inequality holds

$$\frac{\partial[f_i(\bar{x}_i, x_{i+1}) + v_i]}{\partial x_{i+1}} > g_{i0} > 0.$$

According to Lemma 1, by viewing x_{i+1} as a virtual control input, for every value of $\bar{x}_i$ and v_i, there exists a smooth ideal control input $x_{i+1} = \alpha_i^*(\bar{x}_i, v_i)$ such that

$$f_i(\bar{x}_i, \alpha_i^*) + v_i = 0. \tag{41}$$

Using mean value theorem (Apostol, 1963), there exists λ_i ($0 < \lambda_i < 1$) such that

$$f_i(\bar{x}_i, x_{i+1}) = f_i(\bar{x}_i, \alpha_i^*) + g_{i\lambda_i}(x_{i+1} - \alpha_i^*), \tag{42}$$

where $g_{i\lambda_i} := g_i(\bar{x}_i, x_{(i+1)\lambda_i})$ with $x_{(i+1)\lambda_i} = \lambda_i x_{i+1} + (1-\lambda_i)\alpha_i^*$. Note that Assumptions 2 and 3 on $g_i(\bar{x}_i, x_{i+1})$ is still valid for $g_{i\lambda_i}$. Combining (39)–(42) yields

$$\dot{z}_i = -k_i z_i + g_{i\lambda_i}(x_{i+1} - \alpha_i^*). \tag{43}$$

Since α_{i-1} is a function of $\bar{x}_{i-1}, x_d$ and $\hat{W}_1, \ldots, \hat{W}_{i-1}$, $\dot{\alpha}_{i-1}$ is given by

$$\dot{\alpha}_{i-1} = \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k}(f_k(\bar{x}_{k+1})) + \phi_{i-1}, \tag{44}$$

where

$$\begin{aligned}\phi_{i-1} &= \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_d}\dot{x}_d \\ &\quad + \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{W}_k}[\Gamma_k(-S_k(Z_k)z_k - \sigma_k \hat{W}_k)] \end{aligned}\tag{45}$$

is computable.

By employing an RBF neural network $W_i^{\mathrm{T}} S_i(Z_i)$ to approximate $\alpha_i^*(\bar{x}_i, v_i)$, where $Z_i = [\bar{x}_i^{\mathrm{T}}, \partial\alpha_{i-1}/\partial x_1, \ldots, \partial\alpha_{i-1}/\partial x_{i-1}, \phi_{i-1}, z_i]^{\mathrm{T}} \in \Omega_i \subset R^{2i+1}$, α_i^* can be expressed as

$$\alpha_i^* = W_i^{*\mathrm{T}} S_i(Z_i) + \varepsilon_i, \tag{46}$$

where W_i^* denotes the ideal constant weights, and $|\varepsilon_i| \leqslant \varepsilon_i^*$ is the approximation error with constant $\varepsilon_i^* > 0$.

Since W_i^* is unknown, α_i^* cannot be realized in practice. Let $\hat{W}_i$ be the estimate of W_i^*. Define $z_{i+1} = x_{i+1} - \alpha_i$ and let

$$\alpha_i = -z_{i-1} - c_i z_i + \hat{W}_i^{\mathrm{T}} S_i(Z_i), \tag{47}$$

where c_i is a positive constant to be specified later. Then, Eq. (43) becomes

$$\begin{aligned} \dot{z}_i &= -k_i z_i + g_{i\lambda_i}(z_{i+1} + \alpha_i - \alpha_i^*) \\ &= -k_i z_i + g_{i\lambda_i}[z_{i+1} - z_{i-1} - c_i z_i + \tilde{W}_i^{\mathrm{T}} S_i(Z_i) - \varepsilon_i]. \end{aligned} \tag{48}$$

Consider the Lyapunov function candidate

$$V_i = V_{i-1} + \frac{1}{2g_{i\lambda_i}} z_i^2 + \frac{1}{2} \tilde{W}_i^{\mathrm{T}} \Gamma_i^{-1} \tilde{W}_i. \tag{49}$$

The derivative of V_i is

$$\begin{aligned} \dot{V}_i &= \dot{V}_{i-1} + \frac{z_i \dot{z}_i}{g_{i\lambda_i}} - \frac{\dot{g}_{i\lambda_i} z_i^2}{2g_{i\lambda_i}^2} + \tilde{W}_i^{\mathrm{T}} \Gamma_i^{-1} \dot{\hat{W}}_i \\ &= \dot{V}_{i-1} - \frac{k_i}{g_{i\lambda_i}} z_i^2 - z_{i-1} z_i + z_i z_{i+1} - c_i z_i^2 - \frac{\dot{g}_{i\lambda_i}}{2g_{i\lambda_i}^2} z_i^2 \\ &\quad - z_i \varepsilon_i + \tilde{W}_i^{\mathrm{T}} S_i(Z_i) z_i + \tilde{W}_i^{\mathrm{T}} \Gamma_i^{-1} \dot{\hat{W}}_i. \end{aligned} \tag{50}$$

Consider the following adaptation law

$$\dot{\hat{W}}_i = \dot{\tilde{W}}_i = \Gamma_i[-S_i(Z_i) z_i - \sigma_i \hat{W}_i], \tag{51}$$

where $\sigma_i > 0$ is a small constant. Let $c_i = c_{i0} + c_{i1}$, where c_{i0} and $c_{i1} > 0$. By using the equation for $\dot{V}_{i-1}$ corresponding to (38), the Eqs. (50) and (51), and with some completion of squares and straightforward derivation similar to those employed in Step 1, the derivative of V_i becomes

$$\begin{aligned} \dot{V}_i &< z_i z_{i+1} - \sum_{j=1}^{i} \frac{k_j}{g_{j\lambda}} z_j^2 - \sum_{j=1}^{i} c_{j0}^* z_j^2 - \sum_{j=1}^{i} \frac{\sigma_j \|\tilde{W}_j\|^2}{2} \\ &\quad + \sum_{j=1}^{i} \frac{\sigma_j \|W_j^*\|^2}{2} + \sum_{j=1}^{i} \frac{\varepsilon_j^{*2}}{4c_{j1}}, \end{aligned} \tag{52}$$

where $c_{i0}^* := c_{i0} - (g_{id}/2g_{i0}^2) > 0$.

Step $n-1$: The derivative of $z_{n-1} = x_{n-1} - \alpha_{n-2}$ is $\dot{z}_{n-1} = f_{n-1}(\bar{x}_{n-1}) + g_{n-1}(\bar{x}_{n-1}) x_n - \dot{\alpha}_{n-2}$. By viewing x_n as a virtual control to stabilize the $(z_1, \ldots, z_{n-1})$-subsystem, there exists a desired feedback control

$$\begin{aligned} \alpha_{n-1}^* &= -z_{n-2} - c_{n-1} z_{n-1} \\ &\quad - \frac{1}{g_{n-1}(\bar{x}_{n-1})}(f_{n-1}(\bar{x}_{n-1}) - \dot{\alpha}_{n-2}), \end{aligned} \tag{53}$$

where c_{n-1} is a positive constant to be specified later, $g_{n-1}(\bar{x}_{n-1})$ and $f_{n-1}(\bar{x}_{n-1})$ are unknown smooth functions of $\bar{x}_{n-1}$, and $\dot{\alpha}_{n-2}$ is a function of $\bar{x}_{n-2}$, x_d and $\hat{W}_1, \ldots, \hat{W}_{n-2}$. Therefore, $\dot{\alpha}_{n-2}$ can be expressed as

$$\dot{\alpha}_{n-2} = \sum_{k=1}^{n-2} \frac{\partial \alpha_{n-2}}{\partial x_k}(g_k(\bar{x}_k) x_{k+1} + f_k(\bar{x}_k)) + \phi_{n-2}, \tag{54}$$

where

$$\phi_{n-2} = \sum_{k=1}^{n-2} \frac{\partial \alpha_{n-2}}{\partial x_d} \dot{x}_d + \sum_{k=1}^{n-2} \frac{\partial \alpha_{n-2}}{\partial \hat{W}_k}[\Gamma_k(S_k(Z_k) z_k - \sigma_k \hat{W}_k)] \tag{55}$$

is computable.

By employing an RBF neural network $W_{n-1}^{\mathrm{T}} S_{n-1}(Z_{n-1})$ to approximate the following unknown part of α_{n-1}^* in (53)

$$-\frac{1}{g_{n-1}(\bar{x}_{n-1})}(f_{n-1}(\bar{x}_{n-1}) - \dot{\alpha}_{n-2}), \tag{56}$$

where $Z_{n-1} = [\bar{x}_{n-1}^{\mathrm{T}}, \partial\alpha_{n-2}/\partial x_1, \ldots, \partial\alpha_{n-2}/\partial x_{n-2}, \phi_{n-2}]^{\mathrm{T}} \in \Omega_{n-1} \subset R^{2(n-1)}$, α_{n-1}^* can be expressed as

$$\alpha_{n-1}^* = -z_{n-2} - c_{n-1} z_{n-1} + W_{n-1}^{*\mathrm{T}} S_{n-1}(Z_{n-1}) + \varepsilon_{n-1}, \tag{57}$$

where W_{n-1}^* denotes the ideal constant weights, and $|\varepsilon_{n-1}| \leqslant \varepsilon_{n-1}^*$ is the approximation error with constant $\varepsilon_{n-1}^* > 0$.

Since W_{n-1}^* is unknown, α_{n-1}^* cannot be realized in practice. Let $\hat{W}_{n-1}$ be the estimate of W_{n-1}^*. Define $z_n = x_n - \alpha_{n-1}$ and let

$$\alpha_{n-1} = -z_{n-2} - c_{n-1} z_{n-1} + \hat{W}_{n-1}^{\mathrm{T}} S_{n-1}(Z_{n-1}). \tag{58}$$

Then, the dynamics of z_{n-1} is governed by

$$\begin{aligned} \dot{z}_{n-1} &= f_{n-1}(\bar{x}_{n-1}) + g_{n-1}(\bar{x}_{n-1})(z_n + \alpha_{n-1}) - \dot{\alpha}_{n-2} \\ &= g_{n-1}[z_n - z_{n-2} - c_{n-1} z_{n-1} \\ &\quad + \tilde{W}_{n-1}^{\mathrm{T}} S_{n-1}(Z_{n-1}) - \varepsilon_{n-1}]. \end{aligned} \tag{59}$$

Consider the Lyapunov function candidate

$$V_{n-1} = V_{n-2} + \frac{1}{2g_{n-1}(\bar{x}_{n-1})} z_{n-1}^2 + \frac{1}{2} \tilde{W}_{n-1}^{\mathrm{T}} \Gamma_{n-1}^{-1} \tilde{W}_{n-1}. \tag{60}$$

The derivative of V_{n-1} is

$$\dot V_{n-1} = \dot V_{n-2} + \frac{z_{n-1}\dot z_{n-1}}{g_{n-1}} - \frac{\dot g_{n-1} z_{n-1}^2}{2g_{n-1}^2} + \tilde W_{n-1}^{\mathrm T}\Gamma_{n-1}^{-1}\dot{\hat W}_{n-1}$$
$$= \dot V_{n-2} - z_{n-2}z_{n-1} + z_{n-1}z_n - c_{n-1}z_{n-1}^2 - \frac{\dot g_{n-1}}{2g_{n-1}^2}z_{n-1}^2 - z_{n-1}\varepsilon_{n-1}$$
$$+ \tilde W_{n-1}^{\mathrm T}S_{n-1}(Z_{n-1})z_{n-1} + \tilde W_{n-1}^{\mathrm T}\Gamma_{n-1}^{-1}\dot{\hat W}_{n-1}. \quad (61)$$

Consider the following adaptation law

$$\dot{\hat W}_{n-1} = \dot{\tilde W}_{n-1} = \Gamma_{n-1}[-S_{n-1}(Z_{n-1})z_{n-1} - \sigma_{n-1}\hat W_{n-1}], \quad (62)$$

where $\sigma_{n-1} > 0$ is a small constant. Let $c_{n-1} = c_{(n-1)0} + c_{(n-1)1}$, where $c_{(n-1)0}$ and $c_{(n-1)1} > 0$. With some completion of squares and straightforward derivation similar to those employed in Step 1, the derivative of V_i becomes

$$\dot V_{n-1} < z_{n-1}z_n - \sum_{j=1}^{n-2}\frac{k_j}{g_{j\lambda}}z_j^2 - \sum_{k=1}^{n-1}c_{k0}^* z_k^2 + \sum_{k=1}^{n-1}\frac{\varepsilon_k^{*2}}{4c_{k1}}$$
$$-\sum_{k=1}^{n-1}\frac{\sigma_k\|\tilde W_k\|^2}{2} + \sum_{k=1}^{n-1}\frac{\sigma_k\|W_k^*\|^2}{2}, \quad (63)$$

where

$$c_{(n-1)0}^* := c_{(n-1)0} - \frac{g_{(n-1)d}}{2g_{(n-1)0}^2} > 0. \quad (64)$$

Step n: This is the final step. The derivative of $z_n = x_n - \alpha_{n-1}$ is

$$\dot z_n = f_n(\bar x_n) + g_n(\bar x_{n-1})u - \dot\alpha_{n-1}. \quad (65)$$

To stabilize the whole system $(z_1,\ldots,z_n)$, there exists a desired feedback control

$$u^* = -z_{n-1} - c_n z_n - \frac{1}{g_n}(f_n - \dot\alpha_{n-1}), \quad (66)$$

where c_n is a positive constant to be specified later.

Since α_{n-1} is a function of $\bar x_{n-1}$, x_d and $\hat W_1,\ldots,\hat W_{n-1}$, $\dot\alpha_{n-1}$ is given by

$$\dot\alpha_{n-1} = \sum_{k=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_k}f_k(\bar x_{k+1}) + \phi_{n-1}, \quad (67)$$

where

$$\phi_{n-1} = \sum_{k=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_d}\dot x_d + \sum_{k=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat W_k}[\Gamma_k(-S_k(Z_k)z_k - \sigma_k\hat W_k)] \quad (68)$$

is computable.

By employing an RBF neural network $W_n^{\mathrm T}S_n(Z_n)$ to approximate the following unknown part of u^* in (66)

$$-\frac{1}{g_n(\bar x_{n-1})}(f_n(\bar x_n) - \dot\alpha_{n-1}), \quad (69)$$

where $Z_n = [\bar x_n^{\mathrm T}, \partial\alpha_{n-1}/\partial x_1,\ldots,\partial\alpha_{n-1}/\partial x_{n-1}, \phi_{n-1}]^{\mathrm T} \in \Omega_n \subset R^{2n}$, u^* can be expressed

$$u^* = -z_{n-1} - c_n z_n + W_n^{*\mathrm T}S_n(Z_n) + \varepsilon_n, \quad (70)$$

where W_n^* denotes the ideal constant weights, and $|\varepsilon_n| \leqslant \varepsilon_n^*$ is the approximation error with constant $\varepsilon_n^* > 0$.

Since W_n^* is unknown, u^* cannot be realized in practice. Let $\hat W_n$ be the estimate of W_n^*, and

$$u = -z_{n-1} - c_n z_n + \hat W_n^{\mathrm T}S_n(Z_n), \quad (71)$$

where c_n is a positive constant to be specified later. Then, Eq. (65) becomes

$$\dot z_n = -k_n z_n + g_n[-z_{n-1} - c_n z_n + \tilde W_n^{\mathrm T}S_n(Z_n) - \varepsilon_n]. \quad (72)$$

Consider the Lyapunov function candidate

$$V_n = V_{n-1} + \frac{1}{2g_n}z_n^2 + \frac{1}{2}\tilde W_n^{\mathrm T}\Gamma_n^{-1}\tilde W_n. \quad (73)$$

The derivative of V_n is

$$\dot V_n = \dot V_{n-1} + \frac{z_n\dot z_n}{g_n} - \frac{\dot g_n z_n^2}{2g_n^2} + \tilde W_n^{\mathrm T}\Gamma_n^{-1}\dot{\hat W}_n$$
$$= \dot V_{n-1} - z_{n-1}z_n - c_n z_n^2 - \frac{\dot g_n}{2g_n^2}z_n^2 - z_n\varepsilon_n$$
$$+ \tilde W_n^{\mathrm T}S_n(Z_n)z_n + \tilde W_n^{\mathrm T}\Gamma_n^{-1}\dot{\hat W}_n. \quad (74)$$

Consider the following adaptation law

$$\dot{\hat W}_n = \dot{\tilde W}_n = \Gamma_n[-S_n(Z_n)z_n - \sigma_n\hat W_n], \quad (75)$$

where $\sigma_n > 0$ is a small constant. Let $c_n = c_{n0} + c_{n1}$, where c_{n0} and $c_{n1} > 0$. With the completion of squares and straightforward derivation similar to those employed in Step 1, the derivative of V_n satisfies the following inequality

$$\dot V_n < -\sum_{j=1}^{n-2}\frac{k_j}{g_{j\lambda}}z_j^2 - \sum_{j=1}^{n}c_{j0}^* z_j^2 - \sum_{j=1}^{n}\frac{\sigma_j\|\tilde W_j\|^2}{2}$$
$$+\sum_{j=1}^{n}\frac{\sigma_j\|W_j^*\|^2}{2} + \sum_{j=1}^{n}\frac{\varepsilon_j^{*2}}{4c_{j1}}$$
$$< -\sum_{j=1}^{n}c_{j0}^* z_j^2 - \sum_{j=1}^{n}\frac{\sigma_j\|\tilde W_j\|^2}{2}$$
$$+\sum_{j=1}^{n}\frac{\sigma_j\|W_j^*\|^2}{2} + \sum_{j=1}^{n}\frac{\varepsilon_j^{*2}}{4c_{j1}}, \quad (76)$$

where $c_{n0}^* := c_{n0} - g_{nd}/2g_{n0}^2 > 0$.

Theorem 1. *Consider the closed-loop system consisting of the plant* (2), *the reference model* (4), *the controller* (71) *and the NN weight updating laws* (21), (37), (51) *and* (75). *Assume there exists sufficiently large compact sets* $\Omega_i \in R^{2i+1}$, $i=1,\dots,n-2$ *and* $\Omega_i \in R^{2i}$, $i=n-1,n$ *such that* $Z_i \in \Omega_i$ *for all* $t \geqslant 0$. *Then, all signals in the closed-loop system remain bounded, and the output tracking error* $y(t)-y_d(t)$ *converges to a small neighborhood around zero by appropriately choosing design parameters.*

Proof. Let

$$\delta := \sum_{k=1}^{n} \frac{\sigma_k \|W_k^*\|^2}{2} + \sum_{k=1}^{n} \frac{\varepsilon_k^{*2}}{4c_{k1}}.$$

If c_{k0}^* is chosen such that $c_{k0}^* \geqslant \gamma/2g_{k0}$, i.e., $c_{k0} > \gamma/2g_{k0} + g_{kd}/2g_{k0}^2$, $k=1,\dots,n$, where γ is a positive constant, and σ_k and Γ_k are chosen such that $\sigma_k \geqslant \gamma\lambda_{\max}\{\Gamma_k^{-1}\}$, $k=1,\dots,n$, then from (76) the following inequality holds

$$\begin{aligned} \dot{V}_n &< -\sum_{k=1}^{n} c_{k0}^* z_k^2 - \sum_{k=1}^{n} \frac{\sigma_k \|\tilde{W}_k\|^2}{2} + \delta \\ &\leqslant -\sum_{k=1}^{n} \frac{\gamma}{2g_{k0}} z_k^2 - \sum_{k=1}^{n} \frac{\gamma \tilde{W}_k^{\mathrm{T}} \Gamma_k^{-1} \tilde{W}_k}{2} + \delta \\ &\leqslant -\gamma \left[\sum_{k=1}^{n} \frac{1}{2g_k} z_k^2 + \sum_{k=1}^{n} \frac{\tilde{W}_k^{\mathrm{T}} \Gamma_k^{-1} \tilde{W}_k}{2} \right] + \delta \\ &\leqslant -\gamma V_n + \delta. \end{aligned} \tag{77}$$

Let $\rho := \delta/\gamma > 0$, then (73) satisfies

$$0 \leqslant V_n(t) < \rho + (V_n(0) - \rho)\exp(-\gamma t). \tag{78}$$

Therefore z_i and $\hat{W}_i$ are uniformly ultimately bounded. Since $z_1 = x_1 - x_{d1}$, $z_i = x_i - \alpha_{i-1}$ and x_d is bounded, it can be seen that x_i is bounded. Using (71), it is concluded that control u is also bounded. Thus, all the signals in the closed-loop system remain bounded.

Combining Eq. (73) and inequality (78) yields

$$\begin{aligned} \sum_{k=1}^{n} \frac{1}{2g_k} z_k^2 &< \rho + (V_n(0) - \rho)\exp(-\gamma t) \\ &< \rho + V_n(0)\exp(-\gamma t). \end{aligned} \tag{79}$$

Let $g^* = \max_{1 \leqslant i \leqslant n}\{g_{i1}\}$. Then, the following inequality holds

$$\frac{1}{2g^*} \sum_{k=1}^{n} z_k^2 \leqslant \sum_{k=1}^{n} \frac{1}{2g_k} z_k^2 < \rho + V_n(0)\exp(-\gamma t) \tag{80}$$

that is,

$$\sum_{k=1}^{n} z_k^2 < 2g^*\rho + 2g^* V_n(0)\exp(-\gamma t), \tag{81}$$

which implies that given $\mu > \sqrt{2g^*\rho}$, there exists T such that for all $t \geqslant T$, the tracking error satisfies

$$|z_1(t)| = |x_1(t) - x_{d1}(t)| = |y(t) - y_d(t)| < \mu, \tag{82}$$

where μ is the size of a small residual set which depends on the NN approximation error ε_i and controller parameters c_i, σ_i and Γ_i. It is easily seen that the increase in the control gain c_i, adaptive gain Γ_i and NN node number l_i will result in a better tracking performance. □

Remark 2. In general, implicit functions of the desired virtual controls are very difficult to be solved, even in the case when the system nonlinearities $f_i(\bar{x}_i, x_{i+1})$ $(i = 1,\dots,n-2)$ are known functions. With the help of NN, there is no need to solve the implicit functions for the explicit virtual controls and the practical controller to cancel the unknown functions in each backstepping design procedure.

Remark 3. The adaptive NN controller (71) with adaptation laws (21), (37), (51) and (75) are highly structural, and independent of the complexities of the system nonlinearities. Such a structural property is particularly suitable for parallel processing and hardware implementation in practical applications. Simulation studies have been conducted to verify the effectiveness of the scheme. It is omitted here for conciseness of presentation.

4. Direct adaptive NN control for Σ_2

In this section, the design procedure is very similar to that of Section 3, except that integral Lyapunov function is employed in controller design to avoid the possible singularity problem caused by $g_n(\bar{x}_n)$ in the last equation of Σ_2.

For system Σ_2, all the assumptions on $g_i(\cdot)$, $i = 1, \cdot, n-1$ are the same. The following assumption is made for $g_n(\bar{x}_n)$.

Assumption 4. The sign of $g_n(\bar{x}_n)$ is known, and there exist a constant $g_{n0} > 0$ and a known smooth function $\bar{g}_n(\bar{x}_n)$ such that $\bar{g}_n(\bar{x}_n) \geqslant |g_n(\bar{x}_n)| \geqslant g_{n0}, \forall \bar{x}_n \in R^n$. Without losing generality, it is assumed that $\bar{g}_n(\bar{x}_n) \geqslant g_n(\bar{x}_n) \geqslant g_{n0}$, $\forall \bar{x}_n \in R^n$.

The direct adaptive controller is given by

$$u = \frac{1}{\bar{g}_n(\bar{x}_n)}[-z_{n-1} - c_n z_n + \hat{W}_n^{\mathrm{T}} S_n(Z_n)], \tag{83}$$

where

$$z_1 = x_1 - x_{dj}, \quad z_{i+1} = x_{i+1} - \alpha_i, \ \ 1 \leqslant i \leqslant n-1,$$

$$\alpha_1 = -c_1 z_1 + \hat{W}_1^{\mathrm{T}} S_1(Z_1),$$

$$\alpha_i = -z_{i-1} - c_i z_i + \hat{W}_i^{\mathrm{T}} S_i(Z_i),$$

$$Z_1 = [x_1, \dot{x}_{d1}, z_1]^{\mathrm{T}} \in \Omega_1 \subset R^3,$$

$$Z_i = \left[\bar{x}_i^{\mathrm{T}}, \frac{\partial \alpha_{i-1}}{\partial x_1}, \ldots, \frac{\partial \alpha_{i-1}}{\partial x_{i-1}}, \phi_{i-1}, z_i\right]^{\mathrm{T}} \in \Omega_i \subset R^{2i+1},$$
$$2 \leqslant i \leqslant n-2,$$

$$Z_{n-1} = \left[\bar{x}_{n-1}^{\mathrm{T}}, \frac{\partial \alpha_{n-2}}{\partial x_1}, \ldots, \frac{\partial \alpha_{n-2}}{\partial x_{n-2}}, \phi_{n-2}\right]^{\mathrm{T}} \in \Omega_{n-1} \subset R^{2(n-1)},$$

$$Z_n = \left[\bar{x}_n^{\mathrm{T}}, \alpha_{n-1}, \frac{\partial \alpha_{n-1}}{\partial x_1}, \ldots, \frac{\partial \alpha_{n-1}}{\partial x_{n-1}}, \phi_{n-1}\right]^{\mathrm{T}} \in \Omega_n \subset R^{2n+1},$$

$$\phi_{i-1} = \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} f_k(\bar{x}_k) + \frac{\partial \alpha_{i-1}}{\partial x_d}\dot{x}_d + \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{W}_k}\dot{\hat{W}}_k,$$
$$2 \leqslant i \leqslant n, \tag{84}$$

where $c_i > 0$ are design constants, RBF NNs $W_i^{\mathrm{T}} S_i(Z_i)$ are used to approximate the unknown functions in the controller design, with $\hat{W}_i$ being the estimates to W_i^*, and the adaptation laws being given by

$$\dot{\hat{W}}_i = \Gamma_i[-S_i(Z_i)z_i - \sigma_i \hat{W}_i], \tag{85}$$

where $\Gamma_i = \Gamma_i^{\mathrm{T}} > 0$, and $\sigma_i > 0$, $i = 1, \ldots, n$ are positive constant design parameters.

Theorem 2. *Consider the closed-loop system consisting of the plant* (3), *the reference model* (4), *the controller* (83) *and the NN weight updating laws* (85). *Assume there exists sufficiently large compact sets Ω_i such that $Z_i \in \Omega_i$ for all $t \geqslant 0$. Then, all signals in the closed-loop system remain bounded, and the output tracking error $y(t) - y_d(t)$ converges to a small neighborhood around zero by appropriately choosing design parameters.*

Proof. From the first to the $(n-1)$th step, the proof is carried out along the same lines as that of Theorem 1, thus it is omitted here. In the last step, integral-type Lyapunov function is employed in controller design to avoid the possible singularity problem caused by $g_n(\bar{x}_n)$ in case the quadratic Lyapunov function candidate is chosen.

Step n: The derivative of $z_n = x_n - \alpha_{n-1}$ is

$$\dot{z}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n)u - \dot{\alpha}_{n-1}. \tag{86}$$

Define $\beta_n(\bar{x}_n) = \bar{g}_n(\bar{x}_n)/g_n(\bar{x}_n)$, and a smooth scalar function

$$V_{z_n} = \int_0^{z_n} \sigma \beta_n(\bar{x}_{n-1}, \sigma + \alpha_{n-1})\,\mathrm{d}\sigma$$
$$= z_n^2 \int_0^1 \theta \beta_n(\bar{x}_{n-1}, \theta z_n + \alpha_{n-1})\,\mathrm{d}\theta. \tag{87}$$

Noting that $1 \leqslant \beta_n(\bar{x}_{n-1}, \theta z_n + \alpha_{n-1}) \leqslant \bar{g}_n(\bar{x}_{n-1}, \theta z_n + \alpha_{n-1})/g_{n0}$ (Assumption 4), the following inequality holds

$$\frac{z_n^2}{2} \leqslant V_{z_n} \leqslant \frac{z_n^2}{g_{n0}} \int_0^1 \theta \bar{g}_n(\bar{x}_{n-1}, \theta z_n + \alpha_{n-1})\,\mathrm{d}\theta. \tag{88}$$

Therefore, V_{z_n} is positive definite and radically unbounded with respective to z_n. Choose

$$V_n = V_{n-1} + \int_0^{z_n} \sigma \beta_n(\bar{x}_{n-1}, \sigma + \alpha_{n-1})\,\mathrm{d}\sigma + \frac{1}{2}\tilde{W}_n^{\mathrm{T}} \Gamma_n^{-1} \tilde{W}_n \tag{89}$$

as a Lyapunov function candidate. Its time derivative becomes

$$\dot{V}_n = \dot{V}_{n-1} + z_n \beta_n(\bar{x}_n)\dot{z}_n + \tilde{W}_n^{\mathrm{T}} \Gamma_n^{-1} \dot{\hat{W}}_n + \int_0^{z_n} \sigma \left[\frac{\partial \beta_n(\bar{x}_{n-1}, \sigma + \alpha_{n-1})}{\partial \bar{x}_{n-1}} \dot{\bar{x}}_{n-1} + \frac{\partial \beta_n(\bar{x}_{n-1}, \sigma + \alpha_{n-1})}{\partial \alpha_{n-1}} \dot{\alpha}_{n-1}\right] \mathrm{d}\sigma. \tag{90}$$

Since α_{n-1} is a function of $\bar{x}_{n-1}$, x_d and $\hat{W}_1, \ldots, \hat{W}_{n-1}$, $\dot{\alpha}_{n-1}$ is given by

$$\dot{\alpha}_{n-1} = \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_k} f_k(\bar{x}_{k+1}) + \phi_{n-1}, \tag{91}$$

where

$$\phi_{n-1} = \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_d}\dot{x}_d + \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{W}_k}[\Gamma_k(-S_k(Z_k)z_k - \sigma_k \hat{W}_k)] \tag{92}$$

is computable.

Using the fact that

$$\int_0^{z_n} \sigma \frac{\partial \beta_n(\bar{x}_{n-1}, \sigma + \alpha_{n-1})}{\partial \alpha_{n-1}} \dot{\alpha}_{n-1}\,\mathrm{d}\sigma = \dot{\alpha}_{n-1} \int_0^{z_n} \sigma \frac{\partial \beta_n(\bar{x}_{n-1}, \sigma + \alpha_{n-1})}{\partial \sigma}\,\mathrm{d}\sigma = \dot{\alpha}_{n-1}\left[z_n \beta_n(\bar{x}_n) - \int_0^{z_n} \beta_n(\bar{x}_{n-1}, \sigma + \alpha_{n-1})\,\mathrm{d}\sigma\right],$$

we obtain

$$\dot{V}_n = \dot{V}_{n-1} + z_n[\bar{g}_n(\bar{x}_n)u + h_n(Z_n)] + \tilde{W}_n^{\mathrm{T}} \Gamma_n^{-1} \dot{\hat{W}}_n, \tag{93}$$

where

$$h_n(Z_n) = \beta_n(\bar{x}_n) f_n(\bar{x}_n) + z_n \int_0^1 \theta \frac{\partial \beta_n(\bar{x}_{n-1}, \theta z_n + \alpha_{n-1})}{\partial \bar{x}_{n-1}} \dot{\bar{x}}_{n-1}\,\mathrm{d}\theta - \dot{\alpha}_{n-1} \int_0^1 \beta_n(\bar{x}_{n-1}, \theta z_n + \alpha_{n-1})\,\mathrm{d}\theta \tag{94}$$

with

$$Z_n = \left[\bar{x}_n^{\mathrm{T}}, \alpha_{n-1}, \frac{\partial \alpha_{n-1}}{\partial x_1}, \frac{\partial \alpha_{n-1}}{\partial x_2}, \ldots, \frac{\partial \alpha_{n-1}}{\partial x_{n-1}}, \phi_{n-1}\right]^{\mathrm{T}} \in \Omega_n \subset R^{2n+1}.$$

To stabilize the whole system $(z_1, \ldots, z_n)$, there exists a desired feedback control

$$u^* = \frac{1}{\bar{g}_n(\bar{x}_n)}[-z_{n-1} - c_n z_n - h_n(Z_n)], \qquad (95)$$

where c_n is a positive constant to be specified later.

By employing an RBF neural network $W_n^{\mathrm{T}} S_n(Z_n)$ to approximate $-h_n(Z_n)$, u^* can be expressed

$$u^* = \frac{1}{\bar{g}_n(\bar{x}_n)}[-z_{n-1} - c_n z_n + W_n^{*\mathrm{T}} S_n(Z_n) + \varepsilon_n], \qquad (96)$$

where W_n^* denotes the ideal constant weights, and $|\varepsilon_n| \leqslant \varepsilon_n^*$ is the approximation error with constant $\varepsilon_n^* > 0$.

Since W_n^* is unknown, u^* cannot be realized in practice. Let $\hat{W}_n$ be the estimate of W_n^*, and the controller be chosen as

$$u = \frac{1}{\bar{g}_n(\bar{x}_n)}[-z_{n-1} - c_n z_n + \hat{W}_n^{\mathrm{T}} S_n(Z_n)]. \qquad (97)$$

The derivative of V_n becomes

$$\dot{V}_n = \dot{V}_{n-1} - z_{n-1} z_n - c_n z_n^2 - z_n \varepsilon_n + \tilde{W}_n^{\mathrm{T}} S_n(Z_n) z_n + \tilde{W}_n^{\mathrm{T}} \Gamma_n^{-1} \dot{\tilde{W}}_n. \qquad (98)$$

Consider the following adaptation law

$$\dot{\hat{W}}_n = \dot{\tilde{W}}_n = \Gamma_n[-S_n(Z_n) z_n - \sigma_n \hat{W}_n], \qquad (99)$$

where $\sigma_n > 0$ is a small constant. Let $c_n = c_{n0} + c_{n1}$, where $c_{n1} > 0$ and

$$c_{n0} = \frac{\gamma}{g_{n0}} \int_0^1 \theta \bar{g}_n(\bar{x}_{n-1}, \theta z_n + \alpha_{n-1})\, \mathrm{d}\theta > 0 \qquad (100)$$

with γ being a positive constant. Then, Eq. (98) becomes

$$\dot{V}_n = \dot{V}_{n-1} - z_{n-1} z_n - c_{n0} z_n^2 - c_{n1} z_n^2 - z_n \varepsilon_n - \sigma_n \tilde{W}_n^{\mathrm{T}} \hat{W}_n. \qquad (101)$$

By using the completion of squares and straightforward derivation similar to those employed in Section 3, the derivative of V_n satisfies

$$\begin{aligned} \dot{V}_n &< -\sum_{j=1}^{n-2} \frac{k_j}{g_{j\lambda}} z_j^2 - \sum_{j=1}^{n-1} c_{j0}^* z_j^2 - c_{n0} z_n^2 - \sum_{j=1}^{n} \frac{\sigma_j \|\tilde{W}_j\|^2}{2} \\ &\quad + \sum_{j=1}^{n} \frac{\sigma_j \|W_j^*\|^2}{2} + \sum_{j=1}^{n} \frac{\varepsilon_j^{*2}}{4c_{j1}} \\ &< -\sum_{j=1}^{n-1} c_{j0}^* z_j^2 - c_{n0} z_n^2 - \sum_{j=1}^{n} \frac{\sigma_j \|\tilde{W}_j\|^2}{2} \\ &\quad + \sum_{j=1}^{n} \frac{\sigma_j \|W_j^*\|^2}{2} + \sum_{j=1}^{n} \frac{\varepsilon_j^{*2}}{4c_{j1}}. \end{aligned} \qquad (102)$$

Let

$$\delta := \sum_{k=1}^{n} \frac{\sigma_k \|W_k^*\|^2}{2} + \sum_{k=1}^{n} \frac{\varepsilon_k^{*2}}{4c_{k1}}.$$

If c_{k0}^* is chosen such that $c_{k0}^* \geqslant \gamma/2g_{k0}$, i.e., $c_{k0} > \gamma/2g_{k0} + g_{kd}/2g_{k0}^2$, $k = 1, \ldots, n-1$, where γ is a positive constant, and σ_k and Γ_k are chosen such that $\sigma_k \geqslant \gamma \lambda_{\max}\{\Gamma_k^{-1}\}$, $k = 1, \ldots, n$, then from (102) the following inequality holds

$$\begin{aligned} \dot{V}_n &< -\sum_{k=1}^{n-1} c_{k0}^* z_k^2 - c_{n0} z_n^2 - \sum_{k=1}^{n} \frac{\sigma_k \|\tilde{W}_k\|^2}{2} + \delta \\ &\leqslant -\sum_{k=1}^{n-1} \frac{\gamma}{2g_{k0}} z_k^2 - \frac{\gamma z_n^2}{g_{n0}} \int_0^1 \theta \bar{g}_n(\bar{x}_{n-1}, \theta z_n + \alpha_{n-1})\, \mathrm{d}\theta \\ &\quad - \sum_{k=1}^{n} \frac{\gamma \tilde{W}_k^{\mathrm{T}} \Gamma_k^{-1} \tilde{W}_k}{2} + \delta \\ &\leqslant -\gamma V_n + \delta. \end{aligned} \qquad (103)$$

Let $\rho := \delta/\gamma > 0$, then (89) satisfies

$$0 \leqslant V_n(t) < \rho + (V_n(0) - \rho)\exp(-\gamma t). \qquad (104)$$

Therefore, following the same procedure in the proof of Theorem 1, it can be concluded that all the signals in the closed-loop system, including x_i, $\hat{W}_i$ $(i = 1, \ldots, n)$ and u remain bounded. Moreover, the output tracking error $y(t) - y_d(t)$ converges to a small neighborhood around zero by appropriately choosing design parameters. □

5. Conclusion

In this paper, direct adaptive NN control schemes are presented for nonlinear pure-feedback systems with unknown nonlinear functions. Implicit function theorem is firstly exploited to assert the existence of the continuous desired virtual controls. NN approximators are then used to approximate the continuous desired virtual controls and desired practical control. With mild assumptions on the partial derivatives of the unknown functions, the developed adaptive NN control scheme achieves semi-global uniform ultimate boundedness of all the signals in the closed-loop. Moreover, the output of the system is proven to converge to a small neighborhood of the desired trajectory. The control performance of the closed-loop system is guaranteed by suitably choosing the design parameters. Further research on this topic lies in the exploration of control schemes for pure-feedback system Σ_2 where the $(n-1)$th equation is non-affine in x_n, and/or the nth equation is non-affine in u.

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