Pergamon

PII: S0005-1098(98)00031-4

Automatica, Vol. 34, No. 7, pp. 881-888, 1998

Printed in Great Britain
0005-1098/98 $19.00 + 0.00

Brief Paper

A Quasi-Tracking Approach for Finite-Time Control of a Mass-Beam System*

G. ZHU† and S. S. GE†

Key Words—Flexible arm; terminal sliding mode control.

Abstract—This paper presents an approach for finite-time control of a distributed-parameter system: a flexible beam connected to a translational mass. A quasi-tracking variable is introduced, and a terminal sliding mode controller is developed to achieve finite-time convergence of the quasi-tracking variable to a properly pre-defined trajectory. This subsequently provides a set of boundary conditions to explicitly solve the corresponding boundary-value problem and prove finite-time regulation. End-point tracking control is also investigated. It is shown that the error between the end point of the beam and the pre-defined trajectory, though does not decay to zero, is always bounded by a small time-varying bound. A method to reduce the error bound is also given. The approach does not invoke any model truncation procedure, hence the problem of observation/control spillovers, which exists in the conventional truncated-model-based controller design methods, is essentially avoided. Satisfactory simulation results are provided to demonstrate the effectiveness of the presented approach. © 1998 Elsevier Science Ltd. All rights reserved.

1. *Introduction*

During the past two decades or so, modelling and control of mechanical systems with built-in distributed flexibility have been receiving increasing attention, due to the requirement for high accuracy and efficiency arising from modern industrial manufacturing and aerospace applications. Some well-known examples of such systems include flexible link robots and flexible structures in space, etc. These systems, whose dynamics are governed by partial differential equations (PDEs), are of distributed-parameter type and hence possess an infinite dimensionality.

Generally, a mechanical system with distributed flexibility is approximately modelled as a finite-dimensional system by modal analysis or finite element approach. Based on a model with finite dimensions, some well-developed controller design approaches are then applicable (Yeung and Chen, 1989; Young, 1993; Nathan and Singh, 1989), and among others. The most well-known problem existing in the truncated-model-based controllers is the control/observation spillovers, which may worsen the performance or even destroy the stability of the control system. In recent years, an alternative controller design approach has been investigated. The method directly utilizes the PDEs of the system and avoids the undesirable model truncation. Some design examples can be found in Luo (1993), Luo *et al.* (1995) and Ge *et al.* (1996).

In normal industrial applications, exponential stability or asymptotic stability are regarded good enough, but in some special conditions such as in astronaut assistance, finite-time control of robotic systems is an essential requirement. Finite-time rigid robot control has been investigated (Venkataraman and Gulati, 1992, 1993; Man *et al.*, 1994) using the *terminal sliding mode* technique. Although conventional sliding mode control has been employed (Young, 1993; Yeung and Chen, 1989; Nathan and Singh, 1989) for flexible robots and structures, their controllers are based on finite-dimensional truncated models. Currently, there is no general theory on solving PDEs with discontinuous terms. This is one of the difficulties in applying sliding mode control to distributed-parameter systems, as discussed in Hung *et al.* (1993).

In this paper, we present a quasi-tracking approach, incorporated with the terminal sliding mode technique, to achieve finite-time control of a translational mass–beam system, which is a distributed-parameter system. The system actually represents a class of SCARA/Cartesian flexible robots, which can be found widely used in automatic manufacturing assembly and some other applications (Luo *et al.*, 1995). Firstly, a quasi-tracking variable is introduced. A PDEs-based terminal sliding mode controller is then developed to achieve finite-time convergence of the quasi-tracking variable to a properly pre-defined trajectory. The approach is named "quasi-tracking" because the convergence of the quasi-tracking variable does not imply the tracking convergence of either mass motion or end-point motion. In the regulation case, by imposing some constraints to the pre-defined trajectory, it can be shown that the finite-time convergence of the quasi-tracking behaviour provides a set of boundary conditions, under which the corresponding boundary-value problem of the system has only trivial solution. In other words, the eigen-functions of the flexible system are all zero. This is the case only when the system is at the equilibrium position, therefore finite-time regulation is achieved. In the end-point tracking case, some of the constraints on the pre-defined trajectory are removed. It is shown that the error between the end point of the beam and the pre-defined trajectory, though does not decay to zero, is always bounded by a time-varying bound. The error bound can be arbitrarily reduced, of course, at the price of unboundedly increasing the control effort.

2. *Dynamic equations of the system*

The translational mass-beam system is demonstrated in Fig. 1. System motion is restricted in the horizontal plane and the effect of gravity is neglected. In Fig. 1, frame X–Y is the fixed inertia frame, frame x–y is the local reference frame moving with the mass, and system parameters and variables are defined as: L, the length of the beam; EI, the uniform flexural rigidity of the beam; ρ, the uniform mass per unit length of the beam; M_b, the translational mass; $f(t)$, the control force applied to the mass; $d(t)$, the position of the mass; and $y(x, t)$, the elastic deflection measured from the undeformed beam.

The deflection of the flexible beam is assumed to be small, and $p(x, t) := d(t) + y(x, t)$ is used to represent the position of a point

*Received 10 September 1996; revised 22 July 1997; received in final form 5 January 1998. This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor H. Logemann under the direction of Editor Roberto Tempo. *Corresponding author* Shuzhi S. Ge. Tel. +65 772 68 21; Fax +65 779 11 03; E-mail elegss@ee.nus.sg.

†Centre for Intelligent Control, Department of Electrical Engineering, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260.

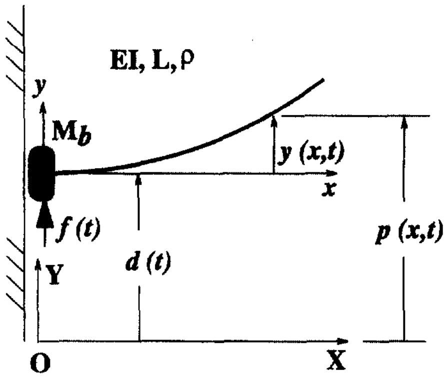

Fig. 1. The translational mass–beam system.

on the beam. We assume the longitude and torsion deformations of the beam are neglected. The total kinetic energy of the system is then given by

$$E_k = \frac{1}{2} M_b \dot{d}^2(t) + \frac{\rho}{2} \int_0^L \dot{p}^2(x, t)\, dx \tag{1}$$

and the total potential energy E_p is

$$E_p = \frac{EI}{2} \int_0^L [y''(x, t)]^2 dx, \tag{2}$$

where the dots and primes denote the derivatives with respect to time and space variable x, respectively. Substituting equations (1) and (2) into the extended Hamilton's Principle

$$\int_{t_0}^{t_f} \delta(E_k - E_p + f(t) d(t))\, dt = 0, \tag{3}$$

we arrive at, by recalling the global variable $p(x, t)$, the following dynamic equations and boundary conditions:

$$M_b \ddot{d}(t) = f(t) + EIy'''(0, t), \tag{4}$$

$$\rho \ddot{p}(x, t) = -EIp''''(x, t), \tag{5}$$

$$p(0, t) = d(t), \qquad p'(0, t) = 0, \tag{6}$$

$$p''(L, t) = 0, \qquad p'''(L, t) = 0. \tag{7}$$

It is noted that equation (5) is now in the same form as the Euler–Bernoulli beam equation, and the first boundary condition in equation (6) is not homogeneous. To solve equation (5), we need to construct another boundary condition, which can be achieved by using the terminal sliding mode technique as shown next.

3. *Quasi-tracking approach*

Firstly, we introduce a quasi-tracking variable

$$\eta = d(t) + \lambda p(L, t), \tag{8}$$

where λ is a scalar to be determined later. The variable η will be controlled to track signal $(1 + \lambda) r(t)$, with $r(t)$ being a pre-defined trajectory satisfying some constraints for regulation and end-point tracking control design. The quasi-tracking error can be defined as

$$w = \eta - (1 + \lambda) r(t), \tag{9}$$

$$= (1 + \lambda)\, [d(t) - r(t)] + \lambda y(L, t), \tag{10}$$

$$= [d(t) - r(t)] + \lambda [p(L, t) - r(t)], \tag{11}$$

From equations (10) and (11), one observes that, due to the existence of the end-point deflection $y(L, t)$, $w \to 0$ implies neither $[d(t) - r(t)] \to 0$ nor $[p(L, t) - r(t)] \to 0$, i.e. neither the mass nor the end point of the beam will converge to $r(t)$. This is the reason that "quasi-tracking" is used to describe this approach. In the following two sub-sections, we shall show that by incorporation with the terminal sliding mode technique, finite-time regulation and bounded end-point tracking can be achieved.

3.1. *Finite-time regulation.* The control objective is to regulate the system to the equilibrium position $d(t) = 0$, $y(x, t) = 0$. We make the following assumption:

A1. At the initial operation moment $t = 0$, the system is such that $d(0) = d_0 \neq 0$ and $\dot{d}(0) = y(x, 0) = \dot{y}(x, 0) = 0$ hold, i.e. initially the mass stops at $d(0) = d_0$, and the beam is at rest.

Moreover, we assume the pre-defined trajectory $r(t)$ satisfies the following constraints:

C1. $\ddot{r}(t)$ is bounded;
C2. $\dot{r}(0) = 0$ and $r(0) = d_0$; and
C3. $r(t) \equiv 0$ for $t > T$ for a given $T > 0$.

Assumption A1 covers a large class of initial conditions of the system, and constraints C1–C3 are all easily achievable. The reasons we impose the constraints on $r(t)$ will become clear later in Remarks 1–3.

Corresponding to the quasi-tracking error w, we define the sliding variable

$$s = \dot{w} + k w^{q_1/q_2}, \tag{12}$$

where $k > 0$ is a control parameter, and

$$q_1, q_2 = 2j + 1, \quad j = 1, 2, \ldots, \tag{13}$$

$$q_1 < q_2 < 2q_1. \tag{14}$$

Selecting the Lyapunov function to be $V = M_b s^2/2$ and using equation (4) with the controller $f(t)$ being defined as

$$f(t) = -\operatorname{sgn}(s)\left[b_1 |y'''(0, t)| + b_2 \left|\lambda \ddot{p}(L, t) - (1 + \lambda) \ddot{r}(t) + k \frac{q_1}{q_2} w^{q_1/q_2 - 1} \dot{w}\right| + \varepsilon \right], \tag{15}$$

where $b_1 \geq EI$, $b_2 \geq M_b$, and $\varepsilon > 0$, we have

$$\dot{V} = s[EIa_1 + M_b a_2] - |s|[b_1 |a_1| + b_2 |a_2| + \varepsilon],$$

where

$$a_1 = y'''(0, t), \quad a_2 = \lambda \ddot{p}(L, t) - (1 + \lambda) \ddot{r}(t) + k \frac{q_1}{q_2} w^{q_1/q_2 - 1} \dot{w}.$$

Since $sEIa_1 \leq |s| b_1 |a_1|$ and $sM_b a_2 \leq |s| b_2 |a_2|$, we have $\dot{V} \leq -\varepsilon |s|$, which implies that the system motion will reach the sliding mode $s = 0$ in a finite time smaller than $M_b |s(0)|/\varepsilon$ (Slotine and Li, 1991), and then remain on the sliding mode. Moreover, from equation (12), it is easy to show (Man *et al.*, 1994; Venkataraman and Gulati, 1992, 1993) that the quasi-tracking error w converges along $s = 0$ to zero in a finite time which is given by

$$T_w = \frac{q_2}{k(q_2 - q_1)} w^{1 - q_1/q_2}(t_s),$$

where t_s is the finite time instant at which s reaches zero. In the literature (Man *et al.*, 1994; Venkataraman and Gulati, 1992, 1993), such an $s = 0$ is named as *terminal sliding mode.* In the terminal sliding mode $s = 0$, substitution of $\dot{w} = -k w^{q_1/q_2}$ into the controller (15) yields

$$f(t) = -\operatorname{sgn}(s)\left[b_1 |y'''(0, t)| + b_2 \left|\lambda \ddot{p}(L, t) - (1 + \lambda) \ddot{r}(t) - \frac{q_1}{q_2} k^2 w^{2q_1/q_2 - 1}\right| + \varepsilon \right]. \tag{16}$$

For the control system, we have the following remarks:

Remark 1. Due to assumption A1 and constraint C2, we actually have $s(0) = 0$ and thus $s \equiv 0$, i.e. the system is always kept on the terminal sliding mode. Consequently, *the controller given by* equation (16) *is used in the entire control process.* Invoking equation (14) and constraint C1, it is easy to check that $f(t)$ in equation (16) is always bounded. Moreover, since $s \equiv 0$, we have $t_s = 0$, and thus $T_w = 0$ because $w(0) = 0$ due to assumption A1 and constraint C2. Therefore, $w \equiv 0$ also holds.

Remark 2. Without assumption A1 and constraint C2, $s(0) = 0$ cannot be guaranteed. When $s(0) \neq 0$, controller (15) is to be used to drive the system to the terminal sliding mode $s = 0$ in a finite time smaller than $M_b|s(0)|/\varepsilon$, as shown above. However, during this finite-time interval, it is possible at some time instants that $w = 0$ though $s \neq 0$. If this happens, the term $w^{q_1/q_2 - 1}$ in equation (15) goes to infinity because $q_1/q_2 - 1 < 0$. Therefore, assumption A1 and constraints C1–C2 are all necessary to obtain a bounded control.

Remark 3. Due to constraint C3 and $w \equiv 0$ shown in *Remark* 1, it can be concluded that the quasi-tracking variable η, from equation (9), will converge to zero in finite time T.

Up to now, we have shown that $\eta = 0$ can be achieved in finite time T. If we can further show that the system stops at $d(t) = 0$, $y(x, t) = 0$ provided that $\eta = 0$, the regulation in finite time T is then proven. Indeed, this is guaranteed by Theorem 1 below.

Theorem 1. For the closed-loop system described by equations (4)–(7), the control (16), assumption A1, and constraints C1–C3 for trajectory $r(t)$ being satisfied, if the scalar λ is selected such that $1 < \lambda < 3.83$, regulation can be achieved in finite time T.

Proof. From the discussion above, we know that the quasi-tracking variable η converges to zero in finite time T. Note that $\eta = 0$ implies

$$\lambda p(L, t) = -d(t). \tag{17}$$

Combining equation (17) with the first boundary condition in equation (6) yields $p(0, t) = -\lambda p(L, t)$, which, together with the rest three boundary conditions, leads to the following set of four boundary conditions

$$p(0, t) = -\lambda p(L, t), \qquad p'(0, t) = 0, \tag{18}$$

$$p''(L, t) = 0, \qquad p'''(L, t) = 0. \tag{19}$$

Moreover, it is noted that equation (5) is a homogeneous equation. This allows us to invoke the variable separation method (Kreyszig, 1993) to solve equation (5) under equation (18) and equation (19). Consequently, $p(x, t)$ is represented by

$$p(x, t) = \Phi(x)Q(t). \tag{20}$$

Equation (5) can then be written as

$$\frac{\Phi''''}{\Phi}\frac{EI}{\rho} = -\frac{\ddot{Q}}{Q}. \tag{21}$$

Since the left-hand side of equation (21) is only space-dependent while the right-hand side is a purely time-varying function, it is obvious that both sides must equal a constant. If we denote the constant by K, we obtain two ordinary differential equations, namely,

$$\ddot{Q}(t) + KQ(t) = 0, \tag{22}$$

$$\Phi''''(x) = \frac{\rho}{EI}K\Phi(x). \tag{23}$$

From equations (18) and (19), we have

$$\Phi(0) = -\lambda\Phi(L), \qquad \Phi'(0) = 0, \tag{24}$$

$$\Phi''(L) = 0, \qquad \Phi'''(L) = 0. \tag{25}$$

Equation (23) and conditions (24) and (25) describe the corresponding boundary-value problem, whose solutions are known as the eigen-functions of the system. We now come to solve the boundary-value problem with different K's.

When $K = 0$, the solution to equation (23) possesses the general form of

$$\Phi(x) = C_1x^3 + C_2x^2 + C_3x + C_4,$$

Substituting it into equations (24) and (25) yields $C_1 = C_2 = C_3 = C_4 = 0$ provided that

$$\lambda \neq -1. \tag{26}$$

Therefore, we have $\Phi(x) = 0$.

When $K < 0$, letting $K = -\omega^2$ with ω being a non-zero number, equation (23) can be rewritten as

$$\Phi''''(x) = -\left(\frac{\beta}{L}\right)^4\Phi(x), \tag{27}$$

where

$$\left(\frac{\beta}{L}\right)^4 = \frac{\rho}{EI}\omega^2. \tag{28}$$

The general solution to equation (27) is of the form

$$\Phi(x) = C_1e^{ax}\sin(ax) + C_2e^{ax}\cos(ax) + C_3e^{-ax}\sin(ax) + C_4e^{-ax}\cos(ax) \tag{29}$$

where $a := \sqrt{2}\beta/2L \neq 0$. By substituting equation (29) into equations (24) and (25), and letting $Z_1 = \sin(aL) + \cos(aL)$ and $Z_2 = \cos(aL) - \sin(aL)$, we obtain the following set of equations:

$$\begin{aligned} &\lambda e^{aL}\sin(aL)C_1 + (1 + \lambda e^{aL}\cos(aL))C_2 + \lambda e^{-aL}\sin(aL)C_3 \\ &\quad + (1 + \lambda e^{-aL}\cos(aL))C_4 = 0, \\ &C_1 + C_2 + C_3 - C_4 = 0, \\ &e^{aL}\cos(aL)C_1 - e^{aL}\sin(aL)C_2 - e^{-aL}\cos(aL)C_3 \\ &\quad + e^{-aL}\sin(aL)C_4 = 0 \\ &Z_2e^{aL}C_1 - Z_1e^{aL}C_2 + Z_1e^{-aL}C_3 + Z_2e^{-aL}C_4 = 0. \end{aligned} \tag{30}$$

from which we will solve for C_i $(i = 1, 2, 3, 4)$. The determinant of the coefficient matrix of equation (30), which is a function of a, is given by

$$\Delta = 4[\cosh^2(aL) + 2\lambda\cosh(aL)\cos(aL) + \cos^2(aL)] \tag{31}$$

From the basic knowledge of linear algebra, if $\Delta \neq 0$, then equation (30) has only trivial solution, i.e. $C_1 = C_2 = C_3 = C_4 = 0$. Subsequently, to achieve $\Phi(x) = 0$, we need to decide λ such that $\Delta \neq 0$ holds for any a. Through some algebra derivations, it was found that there does not exist any constant λ for which $\Delta < 0$ always holds. Therefore, we consider the case of $\Delta > 0$. When $\cos(aL) > 0$, $\Delta > 0$ can be guaranteed for any a if there holds

$$\lambda > -\frac{\cosh^2(aL) + \cos^2(aL)}{2\cosh(aL)\cos(aL)}.$$

Because $\cos(aL) > 0$, we have

$$\min_a\left[\frac{\cosh^2(aL) + \cos^2(aL)}{2\cosh(aL)\cos(aL)}\right] = 1.$$

Thus, λ should be $\lambda > -1$. When $\cos(aL) < 0$, $\Delta > 0$ for any a provided that

$$\lambda < \frac{\cosh^2(aL) + \cos^2(aL)}{2\cosh(aL)|\cos(aL)|}.$$

With the help of MATLAB™ (MathWorks, Inc.), it can be found that

$$\min_a\left[\frac{\cosh^2(aL) + \cos^2(aL)}{2\cosh(aL)|\cos(aL)|}\right] = 3.83.$$

Therefore, when $K < 0$, the eigenfunctions of the system are all zero provided that λ satisfies

$$-1 < \lambda < 3.83. \quad (32)$$

The last choice for K is $K > 0$. Let $K = \omega^2$ with ω being a nonzero number. Recalling equation (28), equation (23) can be similarly rewritten as

$$\Phi''''(x) = \left(\frac{\beta}{L}\right)^4 \Phi(x). \quad (33)$$

The general solution to equation (33) is of the form

$$\Phi(x) = C_1 \cos\frac{\beta x}{L} + C_2 \cosh\frac{\beta x}{L} + C_3 \sin\frac{\beta x}{L} + C_4 \sinh\frac{\beta x}{L} \quad (34)$$

From equations (24) and (25), we have

$$\begin{aligned}(1 + \lambda\cos(\beta))C_1 + (1 + \lambda\cosh(\beta))C_2 + \lambda\sin(\beta)C_3 \\ + \lambda\sinh(\beta)C_4 = 0 \\ C_3 + C_4 = 0, \quad (35) \\ C_1\cos(\beta) - C_2\cosh(\beta) + C_3\sin(\beta) - C_4\sinh(\beta) = 0, \\ C_1\sin(\beta) + C_2\sinh(\beta) - C_3\cos(\beta) + C_4\cosh(\beta) = 0,\end{aligned}$$

from which we solve for C_i ($i = 1, 2, 3, 4$). Similarly, equation (35) has only trivial solution provided that its determinant of the coefficient matrix is not zero, i.e.,

$$\Delta = 2[\cos(\beta)\cosh(\beta) + \lambda\cos(\beta) + \lambda\cosh(\beta) + 1] \neq 0.$$

Note that Δ can be rewritten as

$$\Delta = 2[(1 + \cos(\beta))(1 + \cosh(\beta)) + (\lambda - 1)(\cos(\beta) + \cosh(\beta))] \quad (36)$$

Because $\cosh(\beta) \geq 1$ in which the equality sign holds only at $\beta = 0$, we can achieve $\Delta > 0$ for all β if $\lambda > 1$. This leads to $\Phi(x) = 0$ when $K > 0$. Recalling equations (26) and equations (32), one can see that to guarantee $\Phi(x) \equiv 0$, λ should satisfy the following inequality:

$$1 < \lambda < 3.83, \quad (37)$$

which is the same as the condition given in Theorem 1. Hence, if inequality (37) holds, then $\eta = 0$ implies that the eigenfunctions of the system are all identically zeros. This is the case only if the system stops at the equilibrium position, i.e. $p(x, t) = 0$. Therefore, regulation is achieved in finite time T. □

3.2. *Bounded end-point tracking.* Now, we investigate the end-point tracking of the system with the controller given in equation (16). In this case, we remove constraint C3, but constraints C1 and C2 are retained due to the reasons stated in Remarks 1 and 2. Moreover, inequality (37) needs no longer to be satisfied for tracking control, because $p(x, t) = 0$ is not required. However, we shall assume that $\lambda \neq 0$ and -1 hold, because (i) $\lambda = 0$ corresponds to the pure mass motion control, which is of no interest here; and (ii) $\lambda = -1$ leads to $s = -\dot{y}(L, t) - ky^{q_1/q_2}(L, t)$ and $f(t) \equiv 0$, and is thus making no sense.

Let us go back to check the quasi-tracking error w given in equations (9)–(11), and keep in mind that $w \equiv 0$ still holds. Firstly, we consider equation (10). Since equation (16) is a stable controller, the end-point of the beam must be vibrating around zero and $y(L, t)$ is bounded. Therefore, the tracking error between the mass motion $d(t)$ and the trajectory $r(t)$ is also bounded. Then, from equation (11), we can conclude that the end-point tracking error $p(L, t) - r(t)$ is bounded as well. Indeed, we have Theorem 2 below.

Theorem 2. For the closed-loop system described by equations (4), (5)–(7), the control (16), assumption A1, constraints C1 and C2 for trajectory $r(t)$ to satisfy, and the parameter $\lambda \neq 0, -1$, the tracking error between the end point of the beam $p(L, t)$ and the trajectory $r(t)$ is always bounded by a time-varying bound, i.e.

$$|p(L, t) - r(t)| \leq \left|\frac{y(L, t)}{1 + \lambda}\right|. \quad (38)$$

Proof. Equation (10) implies

$$|(1 + \lambda)[d(t) - r(t)]| = |w - \lambda y(L, t)| \leq |w| + |\lambda y(L, t)|. \quad (39)$$

From equation (11), we have

$$|\lambda[p(L, t) - r(t)]| = |w - [d(t) - r(t)]| \leq |w| + |d(t) - r(t)|. \quad (40)$$

Noting that $w \equiv 0$ due to Remark 1, and combining equations (39) and (40), inequality (38) follows immediately. □

Obviously, it is desirable to increase $|1 + \lambda|$ and thus reduce the error bound. However, this will increase the control effort as well, according to equation (16).

3.3. *Implementation issues.* For practical implementation of the controller in equation (16), we have the following remarks:

Remark 4. In equation (16), the term $y'''(0, t)$ represents the shear force of the beam at the base, which can be approximated using strain gauge feedback by a difference operator, as shown in Luo *et al.* (1995).

Remark 5. The controller (16) requires the acceleration measurement $\ddot{p}(L, t)$, which can be achieved by attaching an accelerometer at the tip of the beam. Usually, the accelerometer signals are quite inaccurate. However, this is not a very serious problem in our case due to the high-gain switching nature of the controller. Indeed, the parameters uncertainties or the measurement/estimation errors will not destroy the system's stability if ε is chosen large enough.

Remark 6. To cope with the discontinuity of sliding mode controllers, or the so-called chattering problem, several quasi-sliding modes methods have been discussed in Hung *et al.* (1993). For example, replacing the signum function by the saturation function sat (*) in Fig. 2 yields a continuous control. It should be noted that these quasi-sliding modes methods are not able to keep s at zero. In our simulations, the following controller is used for comparison purpose

$$f(t) = -\operatorname{sat}(s)\left[b_1|y'''(0, t)| + b_2|\lambda\ddot{p}(L, t) - (1 + \lambda)\ddot{r}(t) + k\frac{q_1}{q_2}w^{q_1/q_2 - 1}\dot{w}| + \varepsilon\right]. \quad (41)$$

which may run the risk of an unbounded control, according to Remark 2.

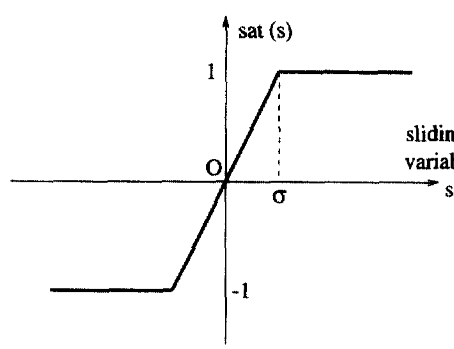

Fig. 2. Saturation function.

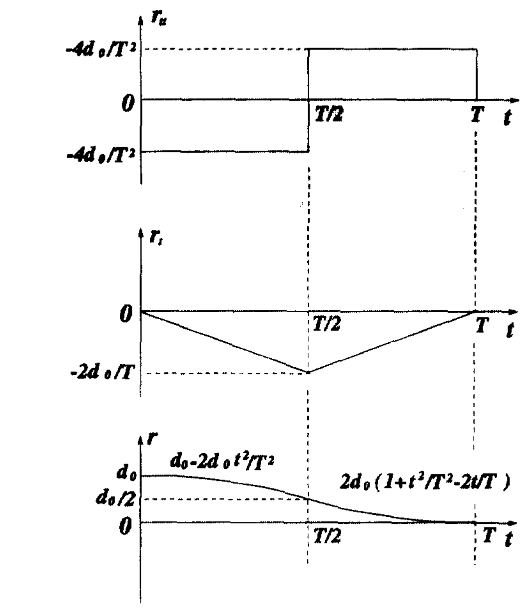

Fig. 3. The trajectory $r(t)$ with bang–bang acceleration.

Remark 7. For the end-point tracking control, the use of sat(*) will lead to a larger error bound, as shown below. When sat(*) is used, s and subsequently w do not stay at zero, but are bounded by a small number. We assume $|w| \leq \delta$. Inequalities (39) and (40) become

$$|d(t) - r(t)| \leq \left|\frac{\lambda}{1+\lambda}\right| |y(L,t)| + \frac{\delta}{|1+\lambda|}$$

and

$$|p(L,t) - r(t)| \leq \frac{|d(t) - r(t)|}{|\lambda|} + \frac{\delta}{|\lambda|}$$

The error bound is then given by

$$|p(L,t) - r(t)| \leq \left|\frac{y(L,t)}{1+\lambda}\right| + \frac{\delta}{|\lambda|} + \frac{\delta}{|\lambda||1+\lambda|},$$

which is larger than the bound given in equation (38).

Remark 8. In proofs of Theorems 1 and 2, we have not invoked any model truncation procedure. The controller in equation (16) is thus PDEs-based. The problem of spillovers, from which the conventional truncated-model-based methods suffer, is essentially avoided.

4. *Computer simulations*

The system parameters are chosen as: $L = 1.0$ m, EI $= 2.0$ Nm^2, $\rho = 0.1$ kg/m and $M_b = 0.2$ kg.

4.1. *Regulation. Case* 1. In this case, finite-time regulation is considered. According to constraints C1–C3, we construct the trajectory $r(t)$ with bang–bang-type acceleration, as shown in Fig. 3. Let $d_0 = d(0) = 1.0$ m and $T = 3.0$ s. Controller parameters are $q_1 = 3, q_2 = 5, \varepsilon = 0.5, k = 2.0$ and $\lambda = 1.01$. Controllers with both sgn(*) and sat(*) are simulated. As pointed out in Remark 6, the use of sat(*) may yield an unbounded control. The results corresponding to sat(*), though are bounded in our simulations, are presented here just for illustration and comparison purpose. For the sat(*), σ is chosen to be 0.01. System performance is plotted in Fig. 4. The control efforts are given in Fig. 5. Clearly, when sat(*) is used, the chattering of $f(t)$ is eliminated.

Case 2. In this case, a more tough time-constraint for regulation is considered by reducing T to 1.0 s for the $r(t)$ given in Fig. 3. We set $\varepsilon = 0.8$ and $\sigma = 0.03$. Other parameters are kept unchanged. System performance is given in Fig. 6, showing that finite-time regulation is still achieved for sgn(*), though the difference between $r(t)$ and $p(L,t)$ has become visible due to the reduction in time-constraint T. Figure 7 shows the control signals, which are larger compared with that in Case 1. This is expected since the regulation is required to be completed in a shorter time interval.

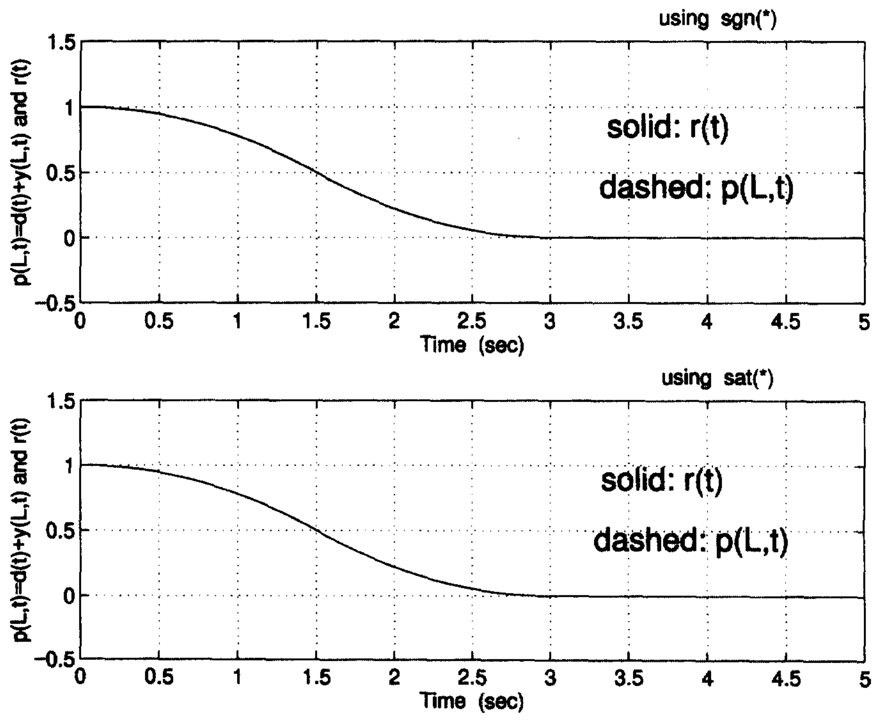

Fig. 4. Tip trajectory $p(L,t)$ in Case 1.

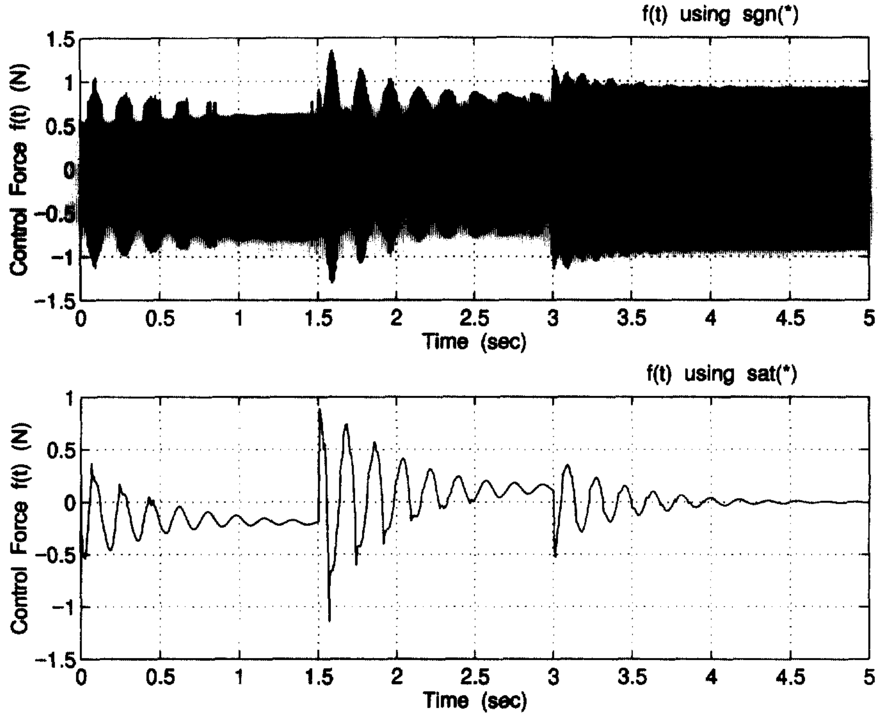

Fig. 5. Control force $f(t)$ in Case 1.

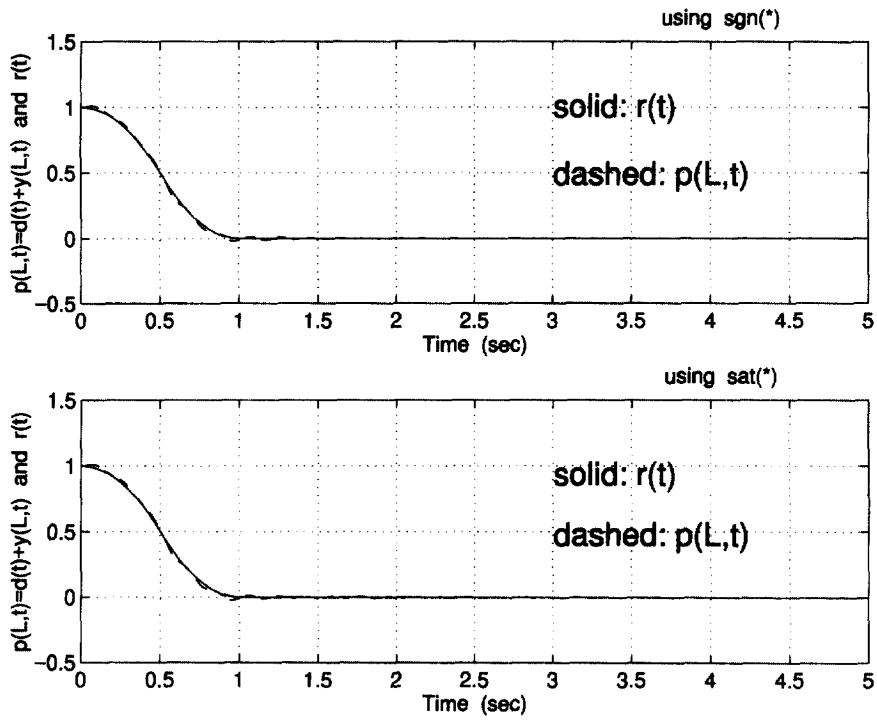

Fig. 6. Tip trajectory $p(L, t)$ in Case 2.

4.2. *End-point tracking. Case* 3. In this case, we test the bounded end-point tracking control of the system. The trajectory to follow is selected to be $r(t) = 0.5 - 0.5 * \cos(\omega t)$ with $\omega = 2\pi/2.5$, which satisfies constraint C1 and C2. Similar remarks can be made for uses of sgn(*) and sat(*), and only the results corresponding to sgn(*) are presented here for simplicity. All system and control parameters, except λ are the same as those in Case 1. Firstly, λ is set to be 0.5, then $\lambda = 1.1$ is used to obtain a smaller error bound. The tracking performances and tracking errors are given in Fig. 8. It can be observed that the larger λ yields a smaller tracking error, which verifies the theoretical analysis. Figure 9 shows that

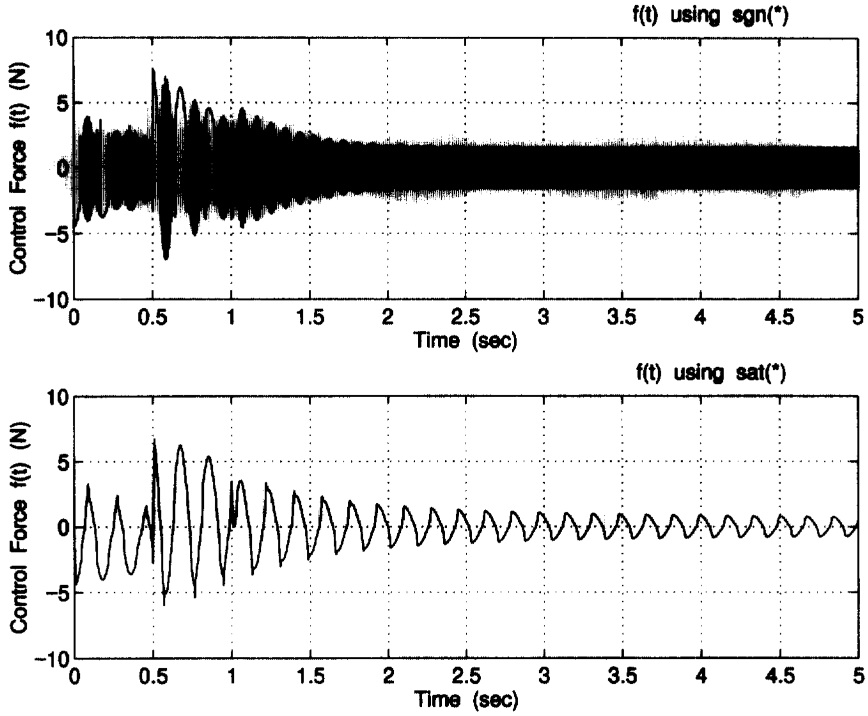

Fig. 7. Control force $f(t)$ in Case 2.

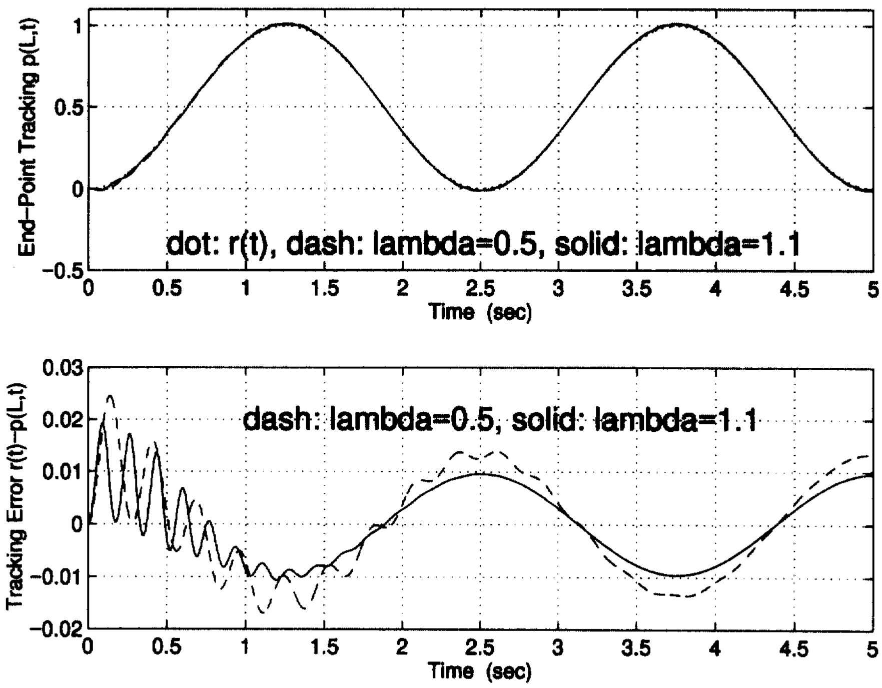

Fig. 8. End-point tracking performance in Case 3.

a larger control effort is needed to reduce the tracking error.

5. *Conclusion*

A quasi-tracking approach has been presented for finite-time control of a distributed-parameter system: a translational mass–beam system. With the terminal sliding mode technique, it has been shown that finite-time regulation can be achieved. Moreover, the same approach also guarantees bounded end-point tracking performance of the system. Furthermore, the approach is PDEs-based, and does not suffer from the problem of spillovers existing in truncated-model-based controllers. A set of satisfactory simulation results are obtained and verify the theoretical analysis.

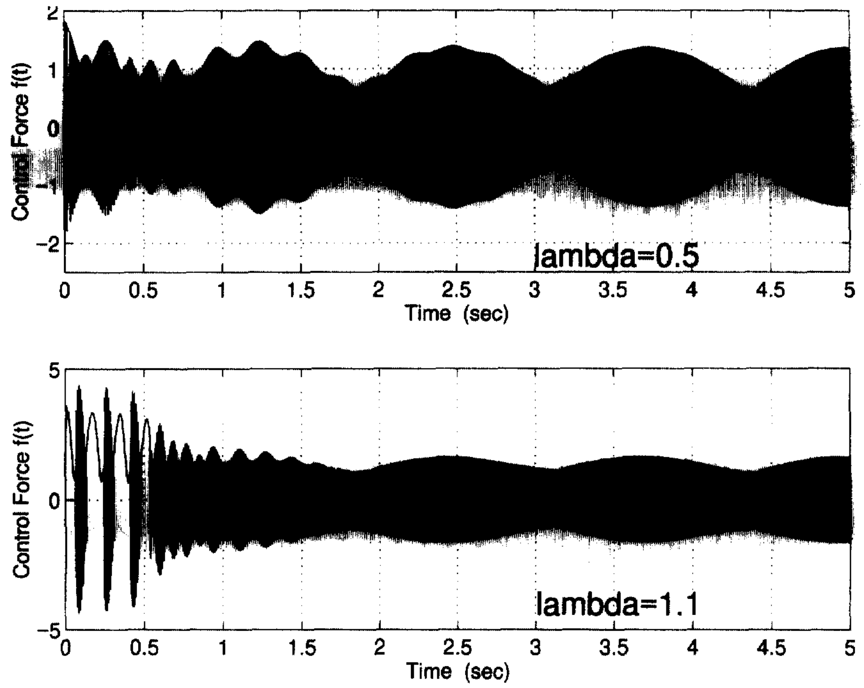

Fig. 9. Control force $f(t)$ in Case 3.

References

Ge, S. S., T. H. Lee and G. Zhu (1996). Energy-based robust controller design for multi-link flexible robots. *Mechatronics*, **6**(7), 779–798.

Hung, J. Y., W. B. Gao and J. C. Hung (1993). Variable structure control: a survey. *IEEE Trans. Ind. Electron.*, **40**(1), 2–22.

Kreyszig, E. (1993). *Advanced Engineering Mathematics*, 7th edn, Chap. 11. Wiley, New York.

Luo, Z. H. (1993). Direct strain feedback control of flexible robot arms: new theoretical and experimental results. *IEEE Trans. Automat. Control*, **38**(11), 1610–1622.

Luo, Z. H., Kitamura and B. Z. Guo (1995). Shear force feedback control of flexible robot arms. *IEEE Trans. Robotics Automat.*, **11**(5), 760–765.

Nathan, P. J. and S. N. Singh (1989). Variable structure control of a robotic arm with flexible links. *Proc. IEEE Int. Conf. Robotics and Automation*, Vol. 2, pp. 882–887.

Slotine, J. E. and W. P. Li (1991). *Applied Nonlinear Control*, Chap. 7. Prentice-Hall, Englewood Cliffs, NJ.

Venkataraman, S. T. and S. Gulati (1992). Control of Nonlinear systems using terminal sliding modes. *Proc. American Control Conf.*, pp. 891–893.

Venkataraman, S. T. and S. Gulati (1993). *ASME J. Dynamic Systems Measurement Control*, **115**, 554–560.

Yeung, K. S. and Y. P. Chen (1989). Regulation of a one-link flexible robot arm using sliding-mode technique. *Int. J. Control*, **49**(6), 1965–1978.

Young, K. D. (ed.) (1993). *Variable Structure Control for Robotics and Aerospace Applications*, Chap. 12, Elsevier, Amsterdam.

Zhihong, M., A. P. Paplinski and H. R. Wu (1994). A robust MIMO terminal sliding mode control scheme for rigid robotic manipulators. *IEEE Trans. Automat. Control*, **39**(12), 2464–2469.