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Analysis and synthesis of switched linear control systems☆

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Abstract

Switched linear systems have a long history of interest in the control community, and have attracted considerable attention recently because they are not only practically relevant, but also tangible with the rich results in the linear system theory. Rapid progress in the field has generated many new ideas and powerful tools. This paper provides a concise and timely survey on analysis and synthesis of switched linear control systems, and presents the basic concepts and main properties of switched linear systems in a systematic manner. The fundamental topics include (i) controllability and observability, (ii) system structural decomposition, (iii) feedback controller design for stabilization, and (iv) optimal control.

Keywords: Switched linear control systems; Switching signal; Stabilization; Controllability; Observability; Optimal control.

1. Introduction

Recently, switched control systems have been attracting much attention in the control community because the problems are not only academically challenging, but also of practical importance. Switched linear systems belong to a special class of hybrid control systems, which comprises a collection of subsystems described by linear dynamics (differential/difference equations), together with a switching rule that specifies the switching between the subsystems. Such systems can be used to describe a wide range of physical and engineering systems in practice. Firstly, they can be used to model systems subject to known or unknown abrupt parameter variations such as synchronously switched linear systems (Johnson, 1985), networks with periodically varying switchings (Brockett & Wood, 1974), and sudden change of system structures due to the failure of a component (Sworder, 1976; Chizeck, Willsky, & Castanon, 1986). Secondly, such systems arise naturally in the study of multi-rate sampled-data systems (Stanford, 1979; Conner & Stanford, 1984). Thirdly, the switched system can also be used to describe the overall system of a single process controlled by means of multi-controller switching. Indeed, this hybrid control scheme provides an effective mechanism to cope with highly complex systems and/or systems with large uncertainties (Kolmanovsky & McClamroch, 1996; Narendra & Balakrishnan, 1997; Leonessa, Haddad, & Chellaboina, 2001). One good example is the hybrid control scheme for nonholonomic systems which are not stabilizable by means of any individual continuous state feedback controller (Kolmanovsky & McClamroch, 1995; Hespanha & Morse, 1999; Ge, Wang, & Lee, 2003). Even for simple linear time-invariant (LTI) systems, the performance (e.g., transient response) can be improved through controllers/compensators switching (Feuer, Goodwin, & Salgado, 1997; McClamroch & Kolmanovsky, 2000; Ishii & Francis, 2002; Leith, Shorten, Leithead, Mason, & Curran, 2003).

Switched linear systems not only provides a challenging forum for academic research, but also bridges the gap between the treatment of linear systems and that of highly complex and/or uncertain systems. These systems

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are relatively easy to handle as many powerful tools from linear and multilinear analysis are applicable or extendable to cope with these systems. Moreover, the study of switched linear systems provides additional insights into some long-standing and sophisticated problems, such as intelligent control (Tanaka & Sano, 1994; Ravindranathan & Leitch, 1999; Feng & Ma, 2001) and adaptive control (Fu & Barmish, 1986; Ge, Hang, Lee, & Zhang, 2003; Kosmatopoulos & Ioannou, 1999; Hespanha, Liberzon, & Morse, 2003).

Mathematically, a switched linear control system can be described by

$$\delta x(t) = A_\sigma x(t) + B_\sigma u(t),$$
$$y(t) = C_\sigma x(t), \tag{1}$$

where $x \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}^p$ is the control input, $y \in \mathbf{R}^q$ is the output, σ is the piecewise constant switching signal taking value from the finite index set $\mathscr{I} = \{1, 2, \ldots, m\}$. For unified representation, δ denotes the derivative operator in continuous time (i.e. $\delta x(t) = (\mathrm{d}/\mathrm{d}t)x(t)$) and the shift forward operator in discrete time (i.e. $\delta x(t) = x(t+1)$).

For clarity, we denote system (1) by $\sum(C_i, A_i, B_i)_{\mathscr{I}}$. Similarly, denote by $\sum(A_i, B_i)_{\mathscr{I}}$ and $\sum(C_i, A_i)_{\mathscr{I}}$ the switched system without output and input, respectively. Matrix triplets (C_k, A_k, B_k) for $k \in \mathscr{I}$ are referred to as the subsystems of (1).

Given the structure of each individual subsystem, the overall system behavior is determined by the switching signal. In general, a switching signal σ may depend on its past value, the time and the state/output

$$\sigma(t+) = \psi(t, x(t)/y(t), \sigma(t-))$$

or it may only depend on its past value and the state/output

$$\sigma(t+) = \sigma(x(t)/y(t), \sigma(t-)), \quad \forall\, t.$$

In the latter case, the switching signal is said to be in state/output-feedback form. We assume that the switching signal is taken such that the switched system is well-posed. That is, for any initial state $x(t_0) = x_0$, the switched system always admits a solution for all forward time and there involve a finite number of switchings in any finite time interval. For a switching signal, any jump instant is said to be a switching time. Accordingly, for continuous time, at a switching time t, we have $\lim_{s\uparrow t} \sigma(s) \neq \lim_{s\downarrow t} \sigma(s)$; for discrete time, at a switching time k, we have $\sigma(k) \neq \sigma(k-1)$. The ordered sequence of the switching times is said to be the switching time sequence of the switching signal. Similarly, the ordered index sequence at the switching times is said to be the switching index sequence of the switching signal.

Switched linear systems have long been investigated in the control literature and have attracted considerable attention in the past few years. Rapid progress in the field has generated many new ideas and powerful tools. We believe that it is timely to review and integrate the recent developments in a technically sound and systematic manner. The main goal of this paper is to present in a concise way the basic concepts and the main properties of switched linear control systems. For conciseness, we restrict our attention to the open systems where the control inputs play a nontrivial role and the switching signals are governed by deterministic processes. For unforced switched systems, in which no control action is involved, or where the control inputs are given in advance, the reader is referred to the recent reviews by DeCarlo, Branicky, Pettersson, and Lennartson (2000), Liberzon and Morse (1999), and Michel (1999). For systems in which the switching signals are governed by random processes (e.g. Markov chains, where the systems are termed as jump linear systems), the reader is referred to (Kozin, 1969; Mariton, 1990) for surveys and (Fang & Loparo, 2002; do Val, Geromel, & Goncalves, 2002; Costa & Tuesta, 2003) for recent development. In this paper, we will only focus on the framework of switched linear control systems and exclude discussion of other classes of systems such as general hybrid systems and switched nonlinear systems.

For conciseness, we simply use the term 'switched systems' instead of 'switched linear control systems'.

The organization of this paper is as follows. Section 2 presents the two fundamental concepts, controllability and observability, in the design and synthesis of switched systems. In Section 3, we discuss the feedback strategies for switched systems of two distinct problems, (i) stabilization using both switching strategies and feedback control, and (ii) feedback stabilization for any switching given in an admissible set. In Section 4, optimal control of switched systems is reviewed, where appropriate switching/control strategies are designed to optimize a certain performance index. Concluding remarks are then followed in Section 5.

2. Controllability and observability

The concepts of controllability and observability play a fundamental role in the design and synthesis of linear control systems. For switched systems, controllability deals with whether or not the state can be controlled through the inputs and the switching signals, and observability deals with whether or not the initial state can be observed through the inputs, outputs and switching signals.

Let $\phi(t; t_0, x_0, u, \sigma)$ denote the state trajectory at time t of switched system (1) starting from $x(t_0) = x_0$ with input u and switching signal σ.

For completeness, the controllability and observability of switched systems are defined as follows.

Definition 1. State $x \in \mathbf{R}^n$ is controllable at time t_0, if there exist a time instant $t_f > t_0$, a switching signal $\sigma : [t_0, t_f] \to \mathscr{I}$, and input $u : [t_0, t_f] \to \mathbf{R}^p$, such that $\phi(t_f; t_0, x, u, \sigma) = 0$. The controllable set of system (1) at t_0 is the set of states which are controllable at t_0. The system is said to be (completely) controllable at time t_0, if its controllable set at t_0 is $\mathbf{R}^n$.

The reachability relates the ability to steer a state from the origin and the notions can be defined in the same fashion.

Definition 2. State x is said to be unobservable at t_0, if for any switching signal σ, there is an input u, such that $C_\sigma\phi(t; t_0, x, u, \sigma) = C_\sigma\phi(t; t_0, 0, u, \sigma)$ for all $t \geqslant t_0$. The unobservable set of system (1) at t_0 is the set of states which are unobservable. The system is said to be (completely) observable at t_0, if its unobservable set at t_0 is null.

Note that if $t' - t = t'_0 - t_0$, $u'(t) = u(t - t'_0 + t_0)$ and $\sigma'(t) = \sigma(t - t'_0 + t_0)$ for all $t \in [t'_0, t']$, then

$$\phi(t; t_0, x_0, u, \sigma) = \phi(t'; t'_0, x_0, u', \sigma').$$

That is, the state trajectory possesses the translation invariant property. Accordingly, if x is controllable/reachable/unobservable at a time t_0, then x is controllable/reachable/unobservable at any arbitrary given instant of time.

In the above concepts of controllability and observability, both the switching signal and the control input are assumed to be design variables. Other concepts related to switching/input constraints (for instance, when the switching signal is not measurable) have also been introduced and addressed in the literature (Xu & Antsaklis, 1999; Cheng & Chen, 2003; Babaali & Egerstedt, 2003; Vidal, Chiuso, Soatto, & Sastry, 2003).

2.1. Continuous-time case

The issues of controllability and reachability were addressed by Loparo, Aslanis, and IIajek (1987) for continuous-time planar switched systems. Following that, sufficient conditions and necessary conditions for controllability and reachability were reported by Ezzine and Haddad (1988,1989a), Szigeti (1992), Sun and Zheng (2001), Xie, Zheng, and Wang (2002), and Yang (2002). Complete geometric characterizations were presented by Sun, Ge, and Lee (2002) and Xie and Wang (2003).

Recall that the controllable set of matrix pair (A, B) is the minimal A-invariant subspace that contains the image space of B. For switched system $\sum(A_i, B_i)_{\mathscr{I}}$, we have similar criteria. To this end, denote by $\mathscr{V}(A_i, B_i)_{\mathscr{I}}$ the minimum subspace of $\mathbf{R}^n$, which is invariant under all A_i, $i \in \mathscr{I}$ and which contains all image spaces of B_i, $i \in \mathscr{I}$. This subspace can be obtained recursively by

$$\mathscr{V}_1 = \sum_{i\in\mathscr{I}} \operatorname{Im} B_i,$$

$$\mathscr{V}_{j+1} = \mathscr{V}_j + \sum_{i\in\mathscr{I}} \sum_{k=0}^{n-1} A_i^k \mathscr{V}_j, \quad j = 1, 2, \ldots \tag{2}$$

and

$$\mathscr{V}(A_i, B_i)_{\mathscr{I}} = \mathscr{V}_n = \sum_{i_0,\ldots,i_{n-1}\in\mathscr{I}}^{j_1,\ldots,j_{n-1}=0,\ldots,n-1} A_{i_{n-1}}^{j_{n-1}} \cdots A_{i_1}^{j_1} \operatorname{Im} B_{i_0}, \tag{3}$$

where Im B denotes the image space of B.

The following theorem presents complete geometric criteria for the controllable set and reachable set.

Theorem 1 (Sun et al., 2002). *For switched linear system $\sum(A_i, B_i)_{\mathscr{I}}$, the reachable set and the controllable set are always identical, and they are precisely the subspace $\mathscr{V}(A_i, B_i)_{\mathscr{I}}$.*

Corollary 1. *For switched linear system $\sum(A_i, B_i)_{\mathscr{I}}$, the following statements are equivalent*:

(i) *the system is completely controllable*;
(ii) *the system is completely reachable*; *and*
(iii) $\mathscr{V}(A_i, B_i)_{\mathscr{I}} = \mathbf{R}^n$.

These criteria generalize the well-known controllability criteria for linear time-invariant systems (see, e.g., Wonham, 1979). In addition, controllability and reachability can be achieved through a single switching signal in any given time. That is, for any given $T > 0$, there exists a switching signal $\sigma : [0, T] \to \mathscr{I}$, such that for any states x_0 and x_f in $\mathscr{V}(A_i, B_i)_{\mathscr{I}}$, there exists an input $u : [0, T] \to \mathbf{R}^p$, such that $x(T; 0, x_0, u, \sigma) = x_f$. As an implication, controllability implies and hence is equivalent to small-time controllability.

It is interesting to notice that, if we restrict the number of switchings, the reachable set and controllable set are not necessarily identical. This can be seen from the following example.

Example 1 (Sun et al., 2002). Consider system $\sum(A_i, B_i)_{\mathscr{I}}$ with $n = 3$, $m = 2$, and

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{4}$$

Let $\mathscr{R}_j$ be the set of reachable states from the origin; and $\mathscr{C}_j$ be the set of controllable states which can be moved to the origin; where both cases occur within j times of switching. Simple computation gives

$$\mathscr{R}_1 = \left\{ \begin{bmatrix} a \\ at \\ 0 \end{bmatrix} : a \in \mathbf{R}, t \geqslant 0 \right\},$$

$$\mathscr{R}_2 = \left\{ \begin{bmatrix} a \\ b \\ bt \end{bmatrix} : a, b \in \mathbf{R}, t \geqslant 0 \right\},$$

$$\mathscr{R}_3 = \left\{ \begin{bmatrix} a \\ at_3 + b \\ bt_2 \end{bmatrix} : a, b \in \mathbf{R}, t_2, t_3 \geqslant 0 \right\},$$

$$\mathscr{R}_4 = \mathbf{R}^3 = \mathscr{V}(A_i, B_i)_{\mathscr{I}}.$$

Note that set $\mathscr{R}_1$ is neither a subspace nor a countable unions of subspaces. Sets $\mathscr{R}_2$ and $\mathscr{R}_3$ are strict subsets of $\mathscr{V}(A_i, B_i)_{\mathscr{I}}$, and $\mathscr{R}_3$ strictly includes $\mathscr{R}_2$ as a subset.

By analogy, the controllable counterparts are given by

$$\mathscr{C}_1 = \left\{ \begin{bmatrix} a \\ -at \\ 0 \end{bmatrix} : a \in \mathbf{R}, t \geqslant 0 \right\},$$

$$\mathscr{C}_2 = \left\{ \begin{bmatrix} a \\ b \\ -bt \end{bmatrix} : a, b \in \mathbf{R}, t \geqslant 0 \right\},$$

$$\mathscr{C}_3 = \left\{ \begin{bmatrix} a \\ -at_2 + b \\ -bt_1 \end{bmatrix} : a, b \in \mathbf{R}, t_1, t_2 \geqslant 0 \right\},$$

$$\mathscr{C}_4 = \mathbf{R}^3 = \mathscr{V}(A_i, B_i)_{\mathscr{I}}.$$

To summarize, for system (4), we have the following properties:

(i) $\mathscr{R}_j$ and $\mathscr{C}_j$ are not subspaces and $\mathscr{R}_j \neq \mathscr{C}_j$ for $j = 1, 2, 3$.
(ii) While the system has only three dimensions, it needs four switchings to transfer any arbitrary state to the origin.

Both properties indicate complex phenomena arising when switching between different subsystems. The difference between $\mathscr{R}_j$ and $\mathscr{C}_j$ is due to incomplete switching, a unique phenomenon of switched systems. As for what the minimum number of switchings required for controllability is, this is currently still an open question warranting further investigation.

For observability, let $\mathscr{O}(C_i, A_i)_{\mathscr{I}}$ be the minimal subspace which is invariant under A_i^{T}, $i \in \mathscr{I}$ and which contains image spaces of C_i^{T}, $i \in \mathscr{I}$. Let

$$\begin{aligned} \mathscr{U}(C_i, A_i)_{\mathscr{I}} &= (\mathscr{O}(C_i, A_i)_{\mathscr{I}})^{\perp} \\ &= \{y \in \mathbf{R}^n : \langle x, y \rangle \\ &= 0, \quad \forall\, x \in \mathscr{O}(C_i, A_i)_{\mathscr{I}}\}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbf{R}^n$.

Theorem 2 (Sun et al., 2002). *For switched linear system $\sum(C_i, A_i, B_i)_{\mathscr{I}}$, the unobservable set is subspace $\mathscr{U}(C_i, A_i)_{\mathscr{I}}$.*

Corollary 2. *For switched linear system $\sum(C_i, A_i, B_i)_{\mathscr{I}}$, the following statements are equivalent*:

(i) *the system is completely observable*;
(ii) *system $\sum(A_i^{\mathrm{T}}, C_i^{\mathrm{T}})_{\mathscr{I}}$ is completely controllable and/or reachable*; *and*
(iii) $\mathscr{O}(C_i, A_i)_{\mathscr{I}} = \mathbf{R}^n$.

The corollary establishes the principle of duality, namely, if we term the system $\sum(A_i^{\mathrm{T}}, C_i^{\mathrm{T}})_{\mathscr{I}}$ as the dual system of $\sum(C_i, A_i)_{\mathscr{I}}$, then the complete observability of a switched system is equivalent to the complete reachability (controllability) of its dual system.

2.2. Discrete-time case

Reachability and observability of discrete-time systems have been extensively studied by Stanford and his co-workers (Stanford & Conner, 1980; Conner & Stanford, 1984; Conner & Stanford, 1987; Barker, Conner, & Stanford, 1988). As in the time-invariant case, the reachable set and controllable set do not coincide in general. Moreover, these sets are not necessarily to be subspaces, as in the continuous-time case.

Simple analysis shows that the reachable set is a union of countable subspaces of $\mathbf{R}^n$. Each such subspace is said to be a component of the reachable set. A component is said to be maximal, if it is not a strict subset of any other component. The reachable set is then the union of its maximal components.

The following example shows that the reachable set is the union of a possibly infinite number of maximal components. As a consequence, reachability cannot be achieved via any single switching signal in finite time.

Example 2 (*Conner & Stanford, 1984*). Let $n=3$ and $m=2$ and

$$A_1 = \begin{bmatrix} \cos 1 & -\sin 1 & 0 \\ \sin 1 & \cos 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix};$$

$$A_2 = 0, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Routine calculation shows that the reachable set is

$$\bigcup_{i=1}^{\infty} \operatorname{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\sin i \\ \cos i \\ 0 \end{bmatrix} \right\},$$

where each subspace is a maximal component.

Although the structure of controllable and reachable sets may be quite complicated in general, it is simple for an important special class of systems. System $\sum(A_i, B_i)_{\mathscr{I}}$ is said to be reversible, if all matrices A_i, $i = 1, \ldots, m$ are nonsingular. As proved in Fliess (1992), any causal discrete-time (input–output) system can be realized by means of a reversible state variable representation. In addition, sampled-data systems are always reversible.

Theorem 3 (Ge, Sun, & Lee, 2001). *Suppose that a discrete-time switched system $\sum(A_i, B_i)_{\mathscr{I}}$ is reversible, then the reachable set always coincides with the controllable set, and is precisely the subspace $\mathscr{V}(A_i, B_i)_{\mathscr{I}}$.*

Corollary 3. *For a reversible switched system $\sum(A_i, B_i)_{\mathscr{I}}$, the following statements are equivalent*:

(i) *the system is completely controllable*;
(ii) *the system is completely reachable*; *and*
(iii) $\mathscr{V}(A_i, B_i)_{\mathscr{I}} = \mathbf{R}^n$.

Note that the criteria are in the same form as those in the continuous-time case.

The observability criterion can be obtained by the principle of duality.

Theorem 4. *For a reversible switched system $\sum(C_i, A_i)_{\mathscr{I}}$, the unobservable set is $\mathscr{U}(C_i, A_i)_{\mathscr{I}}$.*

2.3. Sampling and regular switching

For a continuous-time switched system

$$\dot{x}(t) = A_\sigma x(t) + B_\sigma u(t), \tag{5}$$

each linear time-invariant subsystem (A_k, B_k) can be sampled as

$$x_{k+1} = C_k^\tau x_k + D_k^\tau u_k,$$

where τ is the sampling period, and

$$x_k = x(k\tau), \quad u_k = u(k\tau), \quad C_k^\tau = \mathrm{e}^{A_k \tau},$$
$$D^\tau = \int_0^\tau \exp(tA_k)\,\mathrm{d}t\, B_k.$$

If we sample all the subsystems (A_k, B_k), $k \in \mathscr{I}$ with a (unified) period τ, we have m sampled-data systems (C_k^τ, D_k^τ), $k \in \mathscr{I}$. This motivates the following definition.

Definition 3. For a positive real number τ, the discrete-time switched linear system

$$x_{k+1} = C_\sigma^\tau x_k + D_\sigma^\tau u_k \tag{6}$$

is said to be the sampled-data system of the continuous-time system (5) (with sampling period τ).

It can be seen that the sampled-data system corresponds to the original system (5) with piecewise constant control inputs. That is, if we set the control input to be

$$u(t) = u_j, \quad \text{for } t \in [j\tau, (j+1)\tau), \quad j = 0, 1, \ldots,$$

then, the system states coincide at the sampling times:

$$x(j\tau) = x_j, \quad \text{for } j = 0, 1, \ldots.$$

The following theorem provides a sufficient condition for the problem of sampling without loss of controllability.

Theorem 5 (Sun, 2004b). *Let $\mathscr{C}_c$ and $\mathscr{C}_s$ denote the controllable subspaces of systems* (5) *and* (6), *respectively. Suppose that $s_{k,l} = \delta_{k,l} + \omega_{k,l}\sqrt{-1}$, $l = 1, \ldots, n$ are the eigenvalues of matrix A_k, then under the condition*

$$\exp(s_{k,l}\tau) \neq \exp(s_{k,i}\tau), \ \forall\, l, \ i = 1, \ldots, n,$$
$$k \in \mathscr{I}, \ s_{k,l} \neq s_{k,i},$$

or equivalently,

$$\tau \neq \frac{\imath\pi}{\omega_{k,l} - \omega_{k,i}}, \ \forall\, s_{k,l} \neq s_{k,i} \ \textit{with}\ \delta_{k,l} = \delta_{k,i}, \ \textit{and}$$
$$\forall\, \imath = \pm 1, \pm 2, \ldots,$$

we have

$$\mathscr{C}_c = \mathscr{C}_s.$$

This theorem exhibits that controllability is preserved under almost any sampling period.

Another topic is to achieve controllability by means of regular switching. By regular switching, we mean either of the following types.

Switching signal σ is said to be

- Cyclic, if there is a subset $\{j_1, \ldots, j_s\}$ of $\mathscr{I}$, such that the switching index sequence is

  $\{j_1, \ldots, j_s, j_1, \ldots, j_s, \ldots\}$.

- Synchronous, if there exist a base rate ω, and a sequence of natural numbers $\{\mu_1, \mu_2, \ldots\}$, such that the switching time sequence is

  $\{0, \mu_1\omega, \mu_2\omega, \ldots\}$.

- Of single-rate, if there is a base rate ω, such that the switching time sequence is

  $\{0, \omega, 2\omega, 3\omega, \ldots\}$.

It is clear that regular switching is interesting from the viewpoint of implementation. The following theorem points out how regular we can expect for controllability.

Theorem 6 (Sun, 2004b). *Suppose that $\sum(A_i, B_i)_{\mathscr{I}}$ is a continuous-time system or a reversible discrete-time system, t_1 and t_2 are any given time instants with $t_2 > t_1$, and x_1 and x_2 are two any given states in the controllable subspace $\mathscr{V}(A_i, B_i)_{\mathscr{I}}$ of the system. Then, there is a cyclic and synchronous switching signal σ, and a piecewise constant control input u, such that*

$$\phi(t_2; t_1, x_1, u, \sigma) = x_2.$$

The above theorem demonstrates that it suffices to exploit cyclic and synchronous switching for the purpose of controllability. In general, it is impossible to use more regular switching, for example, cyclic and single-rate switching, to achieve controllability. Counterexamples in this regard can be found in Sun (2004b).

2.4. Canonical decomposition

As in the standard linear system theory, the controllability/observability criteria enable us to decompose a switched system into several modes each with clear system structure. By exploiting the similarities between the criteria for continuous-time systems and the counterparts for reversible discrete-time systems, we are able to formulate a unifying framework applicable to the treatment of the two classes of systems.

Suppose that T is a nonsingular $n \times n$ real matrix. By letting $\bar{x} = Tx$, system (1) is changed into

$$\delta\bar{x}(t) = TA_\sigma T^{-1}\bar{x}(t) + TB_\sigma u(t),$$
$$y(t) = C_\sigma T^{-1}\bar{x}(t). \tag{7}$$

This equation describes the same system dynamics represented in different bases of the state space. Thus, the two systems are equivalent under the coordinate transformation $\bar{x} = Tx$. It can also be shown that complete controllability/observability are invariant under a change-of-basis transformation.

The following theorem shows that any switched linear system permits a structural decomposition in terms of controllability and observability.

Theorem 7 (Sun, 2004a). *Switched system* $\sum(C_i, A_i, B_i)_{\mathscr{I}}$ *is equivalent to a system in the form*

$$\sum\left([0\ \bar{C}_{i2}\ 0\ \bar{C}_{i4}], \begin{bmatrix} \bar{A}_{i11} & \bar{A}_{i12} & \bar{A}_{i13} & \bar{A}_{i14} \\ 0 & \bar{A}_{i22} & 0 & \bar{A}_{i24} \\ 0 & 0 & \bar{A}_{i33} & \bar{A}_{i34} \\ 0 & 0 & 0 & \bar{A}_{i44} \end{bmatrix}, \begin{bmatrix} \bar{B}_{i1} \\ \bar{B}_{i2} \\ 0 \\ 0 \end{bmatrix}\right)_{\mathscr{I}}. \tag{8}$$

In addition, the switched system

$$\sum\left(\begin{bmatrix} \bar{A}_{i11} & \bar{A}_{i12} \\ 0 & \bar{A}_{i22} \end{bmatrix}, \begin{bmatrix} \bar{B}_{i1} \\ \bar{B}_{i2} \end{bmatrix}\right)_{\mathscr{I}}$$

is completely controllable, and the switched system

$$\sum\left([\bar{C}_{i2}\ \bar{C}_{i4}], \begin{bmatrix} \bar{A}_{i22} & \bar{A}_{i24} \\ 0 & \bar{A}_{i44} \end{bmatrix}\right)_{\mathscr{I}}$$

is completely observable.

The form in (8) is said to be the canonical form of the switched system. It divides the state variables into four parts. The first is controllable but unobservable, the second is both controllable and observable, the third is both uncontrollable and unobservable, and the fourth is observable but uncontrollable. The partition is the same as in the linear time-invariant case (see, e.g., Wonham, 1979). Similar system decomposition was obtained for a class of discrete-time switched linear systems (Stanford & Conner, 1980; Conner & Stanford, 1984).

By Theorem 7, each switched system admits an equivalent system in the canonical form. This paves the way for further investigation of synthesis problems such as feedback stabilization.

3. Feedback stabilization

Feedback stabilization strategies for switched systems may be broadly classified into two categories. If the switching signal is a design variable, then the problem of stabilization is to design both switching strategies and feedback control laws to stabilize the switched systems. Otherwise, the problem is to design appropriate feedback control laws to make the closed-loop systems stable for any switching signal given in an admissible set.

3.1. Stabilization via switching signal and feedback control input

For switched system $\sum(C_i, A_i, B_i)_{\mathscr{I}}$, the problem of (linear state) feedback stabilization is to find a piecewise linear feedback law

$$u(t) = F_\sigma x(t)$$

and switching signal σ, such that the closed-loop system

$$\delta x(t) = (A_\sigma + B_\sigma F_\sigma)x(t)$$

is uniformly asymptotically stable. The problem of (dynamical) output feedback stabilization can be formulated in the same fashion.

These problems have been investigated for discrete-time systems (Stanford, 1979; Conner & Stanford, 1984; Barker et al., 1988) and for continuous-time systems (Ezzine & Haddad, 1989a; Sun & Zheng, 2001; Savkin & Evans, 2002).

While the problem of feedback stabilization is still open in general, elegant solutions of the dynamic output feedback stabilization problem have been obtained for an important special class of switched systems that are formed by single nonswitched processes, but with multiple switchable control devices and sensors (Savkin & Evans, 2002). A mathematical description is given by

$$\dot{x}(t) = Ax(t) + B_\sigma u(t),$$
$$y(t) = C_\sigma x(t),$$

or $\sum(C_i, A, B_i)_{\mathscr{I}}$ for conciseness.

Suppose that the controllable mode is observable, i.e. the system permits the canonical form

$$\dot{\bar{x}}_1 = \bar{A}_{11}\bar{x}_1 + \bar{A}_{12}\bar{x}_2 + \bar{B}_\sigma u,$$
$$\dot{\bar{x}}_2 = \bar{A}_{22}\bar{x}_2,$$
$$y = \bar{C}_\sigma\bar{x}_1 + \hat{C}_\sigma\bar{x}_2, \tag{9}$$

where switched system $\sum(\bar{A}_{11}, \bar{B}_i)_{\mathscr{I}}$ is controllable and switched system $\sum(\bar{C}_i, \bar{A}_{11})_{\mathscr{I}}$ is observable.

We seek a dynamic output feedback control law

$$u(t) = F_\sigma \hat{x}_1(t),$$
$$\dot{\hat{x}}_1(t) = \bar{A}_{11}\hat{x}_1(t) + L_\sigma[y(t) - \bar{C}_\sigma \hat{x}_1] + \bar{B}_\sigma u(t), \tag{10}$$

where the gain matrices F_i and L_i are to be determined.

Note that the dynamic output device is an observer which is also a switched system. Let $\omega = [\bar{x}_1, \hat{x}_1, \bar{x}_2]^{\mathrm{T}}$, and, for $i = 1, \ldots, m$,

$$\Omega_i = \begin{bmatrix} \bar{A}_{11} & \bar{B}_i F_i & \bar{A}_{12} \\ L_i \bar{C}_i & \bar{A}_{11} + \bar{B}_i F_i - L_i \bar{C}_i & L_i \hat{C}_i \\ 0 & 0 & \bar{A}_{22} \end{bmatrix}. \tag{11}$$

The closed-loop system is given by

$$\dot{\omega}(t) = \Omega_\sigma \omega(t). \tag{12}$$

Fix a set of weighted factors $w_i > 0$, $i \in \mathscr{I}$ with $\sum_{i \in \mathscr{I}} w_i = 1$. Let

$$\bar{B} = [w_1 \bar{B}_1, \ldots, w_m \bar{B}_m], \quad \bar{C} = \begin{bmatrix} w_1 \bar{C}_1 \\ \vdots \\ w_m \bar{C}_m \end{bmatrix}$$

and

$$F = \begin{bmatrix} F_1 \\ \vdots \\ F_m \end{bmatrix}, \quad L = [L_1, \ldots, L_m].$$

On one hand, the controllability and observability of switched system $\sum(\bar{C}_i, \bar{A}_{11}, \bar{B}_i)_{\mathscr{I}}$ implies that the LTI system $(\bar{C}, \bar{A}_{11}, \bar{B})$ is completely controllable and observable. On the other hand, the average matrix of Ω_i, $i \in \mathscr{I}$ under weighted factors w_i, $i \in \mathscr{I}$ can be computed as

$$\Omega = \sum_{i \in \mathscr{I}} w_i \Omega_i = \begin{bmatrix} \bar{A}_{11} & \bar{B}F & \bar{A}_{12} \\ L\bar{C} & \bar{A}_{11} + \bar{B}F - L\bar{C} & L\hat{C} \\ 0 & 0 & \bar{A}_{22} \end{bmatrix}. \tag{13}$$

Through a similarity transformation, a block triangular matrix can be obtained:

$$\begin{bmatrix} I_{n_1} & 0 & 0 \\ I_{n_1} & -I_{n_1} & 0 \\ 0 & 0 & I_{n-n_1} \end{bmatrix} \Omega \begin{bmatrix} I_{n_1} & 0 & 0 \\ I_{n_1} & -I_{n_1} & 0 \\ 0 & 0 & I_{n-n_1} \end{bmatrix}^{-1} = \begin{bmatrix} \bar{A}_{11} + \bar{B}F & -\bar{B}F & \bar{A}_{12} \\ 0 & \bar{A}_{11} - L\bar{C} & \bar{A}_{12} - L\hat{C} \\ 0 & 0 & \bar{A}_{22} \end{bmatrix},$$

where n_1 is the dimension of the controllable mode. Therefore, the characteristic polynomial of the average matrix is the product of those of the state feedback, state estimator and uncontrollable modes. As a consequence, the design of feedback gain matrices and observer gain matrices can be implemented separately, and the average system is asymptotically stable if and only if $\bar{A}_{22}$ is Hurwitz. By the well-known Campbell–Baker–Hausdorff formula, the stability of the average system implies stabilizability of the switched system via a periodic switching law (Ezzine & Haddad, 1989b; Tokarzewski, 1987) or a state-feedback switching law (Wicks, Peleties, & DeCarlo, 1998). Thus, we have the following theorem:

Theorem 8 (*Sun & Ge, 2003*). *System* (9) *is dynamic output feedback stabilizable if and only if matrix* $\bar{A}_{22}$ *is Hurwitz.*

When the system is completely controllable, we can select the feedback and observer gain matrices, such that the average system possesses the pre-assigned poles. This means that, for the resultant switched system with sufficiently high switching frequency, the rate of convergence of the system state can be arbitrarily assigned.

The above approach clearly exhibits how structural decomposition leads to a powerful yet elegant way of treating switched systems.

Recall that complete controllability implies feedback stabilizability for a linear time-invariant system. Is this also true for a switched system? While the question has not been fully resolved for general systems, some confirmative results have been obtained for special classes of systems, as highlighted in the following theorems.

Theorem 9 (*Sun & Zheng, 2001*). *If the summation of the controllable set of all the individual subsystems is the total state space, then the switched system is linear state feedback stabilizable.*

Theorem 10 (*Conner & Stanford, 1984*). *If a third-order discrete-time switched system is reversible and controllable, then it is linear state feedback stabilizable.*

Some controllable higher-order systems have also been proven to be feedback stabilizable (Conner & Stanford, 1984).

All these results were obtained from structural decomposition, and the stabilizing laws were designed in constructive ways.

For a general switched system in controllability canonical form

$$\dot{x}_1 = A_{1\sigma} x_1 + A_{2\sigma} x_2 + B_\sigma u,$$
$$\dot{x}_2 = A_{3\sigma} x_2,$$

a necessary condition for stabilizability is that the uncontrollable mode system $\sum(A_{3i})_{\mathscr{I}}$ is stabilizable via suitable switching signal. Is this condition also sufficient? Or equivalently, does there exist a feedback control law and a switching strategy which simultaneously stabilize the controllable and uncontrollable modes? This is a challenging question that has, thus far, not been addressed in the literature.

3.2. Stabilization via feedback control

In many practical situations the switching signals are not design variables but are arbitrary in an admissible set. In this case, the problem of stabilization is to design the control input that stabilizes the system under any possible switching signal.

An interesting issue is to investigate the relationship between controllability and stabilizability. Suppose that in the switched system, each individual subsystem is controllable. How can we design a feedback control law to ensure closed-loop stability under as many switching signals as possible? The following result presents a partial solution for this problem.

Theorem 11 (*Li, Wen, & Soh, 2003*). *For continuous-time switched system $\sum(C_i, A_i, B_i)_{\mathscr{I}}$, suppose that each subsystem is controllable and observable. Then, for any given scalar $\tau > 0$, there is a dynamic output feedback law such that the closed-loop system is stable under every switching signal with dwell time τ.*

Recall that a switching signal with dwell time τ means that the interval between any two consecutive switching times has a length greater than or equal to τ. The above theorem is based on the observation that, for a controllable pair (A, B) and positive time τ, it is always possible to appropriately choose a high gain matrix F, such that

$$\|e^{(A+BF)t}\| < \varphi, \quad \forall\, t \geqslant \tau,$$

where $\varphi < 1$ is any given positive real number, and $\|\cdot\|$ denotes the standard 2-norm. In other words, we can decrease the norm as much as we wish in the given time τ using high gain feedback control (Zhao & Zheng, 1999). Similarly, for an observable pair (C, A) and $\tau > 0$, we can construct a standard observer with a gain matrix L satisfying the squashing inequality (Hespanha, 2001)

$$\|e^{(A-LC)t}\| < \varphi, \quad \forall\, t \geqslant \tau.$$

By incorporating the observer into the feedback loop, the closed-loop system possesses the contraction property which guarantees that the switched system is stable under every switching signal with dwell time τ.

From the above result, a question arises very naturally: Can the controllability of the individual subsystems guarantee that, with a suitable feedback control law, the switched system is stable under any switching signal? In other words, does the above theorem still hold if we let $\tau = 0$? As we know, controllability means that all the eigenvalues and some of the eigenvectors can be freely assigned for each individual subsystem, while stability under arbitrary switching requires that the eigenstructures of the subsystems are in a 'compatible' mode (see, e.g., Shorten & Cairbre, 2002). Unfortunately, flexibilities in individual subsystems are generally insufficient to produce the required compatible mode, as shown in the following example.

Example 3 (*Fang and Loparo, 2002*). Consider the switched system $\sum(A_i, B_i)_{\{1,2\}}$ given by

$$A_1 = \begin{bmatrix} 0.5 & 10 \\ 0 & 0.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix};$$
$$A_2 = \begin{bmatrix} 0.5 & 0 \\ 10 & 0.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

This system is not mean square stabilizable if the switching is governed by some Markov process. This clearly implies that the switched system is not feedback stabilizable under an arbitrary switching signal.

3.3. Lyapunov approach for stabilization and robustness

Due to the intrinsic discontinuous nature of a switched system, a smooth Lyapunov function may not permit a smooth derivative along the system trajectories at the switching instants. This leads to a difficulty in differentiating the Lyapunov function. To overcome this, a possible way is to first 'smooth' the system in some sense and then establish some connection between the switched system and smoothed system. Fortunately, this can be achieved by means of the classical Filippov–Wazewski Theorem (Aubin & Cellina, 1984; Fryszkowski & Rzezuchowski, 1991) and its recent extension (Ingalls, Sontag, & Wang, 2003). To see this, let

$$\mathscr{W} = \left\{ w \in \mathbf{R}^m : w_i \geqslant 0, i = 1, \ldots, m, \sum_{i=1}^m w_i \leqslant 1 \right\},$$
$$A(w) = \sum_{i=1}^m w_i A_i, \;\; B(w) = \sum_{i=1}^m w_i B_i, \;\; w \in \mathscr{W}$$

and

$$\mathscr{A}(x) = \{A(w)x + B(w)u : w \in \mathscr{W}, u \in \mathbf{R}^p\}, \quad x \in \mathbf{R}^n.$$

Let us consider the differential inclusion system

$$\dot{x}(t) \in \mathscr{A}(x(t)) \tag{14}$$

and the polytopic linear uncertain system

$$\dot{x} = A(w(t))x(t) + B(w(t))u(t), \tag{15}$$

where $w(\cdot) \in \mathscr{W}$ is a piecewise continuous (uncertain) index function. Note that such systems can be connected to switched systems in a one-to-one manner. Furthermore, any solution of the switched system is a solution of the polytopic linear uncertain system, and any solution of the polytopic linear uncertain system is a solution of the differential inclusion system.

The following result is a special case of Theorem 1 in (Ingalls et al., 2003).

Lemma 1. *Fix $\xi \in \mathbf{R}^n$ and let $z : [0, \infty) \to \mathbf{R}^n$ be a solution of*

$$\dot{z}(t) \in \mathscr{A}(z(t)), \quad z(0) = \xi.$$

Let $r : [0, \infty) \to \mathbf{R}$ be a continuous function satisfying $r(t) > 0$ for all $t \geqslant 0$. Then, there exist an η with $\|\eta - \xi\| \leqslant r(0)$, and a solution $x : [0, \infty) \to \mathbf{R}$ of

$$\dot{x}(t) \in \{A_1 x(t), \ldots, A_m x(t)\}, \quad x(0) = \eta,$$

such that

$$\|z(t) - x(t)\| \leqslant r(t), \quad \forall\, t \in [0, \infty).$$

This lemma implies that the asymptotic stabilizabilities of the switched linear system $\sum(A_i, B_i)_{\mathscr{I}}$, of the differential system (14), and of the polytopic linear uncertain system (15), are in fact equivalent. This deduction can also be established through the converse Lyapunov theorems (Lin, Sontag, & Wang, 1996; Dayawansa & Martin, 1999; Mancilla-Aguilar & Garcia, 2000).

From the above deduction, the rich results on the robustness of the uncertain systems (e.g., Molchanov & Pyatniskiy, 1989; Blanchini, 1994, 1995, 2000; Gurvits, 1995; Pyatniskiy & Rapoport, 1996) are directly applicable to the stability of switched systems. In particular, Blanchini (2000) established the following result.

Theorem 12 (*Blanchini, 2000*). *For the polytopic linear uncertain system* (15), *the following statements are equivalent*:

(i) *there is a locally Lipschitz stabilizing controller of form $u = \Phi(x, w)$ for the system*;
(ii) *there is a globally Lipschitz stabilizing law of form $u = \Phi(x)$*;
(iii) *there is a gain scheduling controller of the form*

$$u(t) = D(w(t))x(t) + E(w(t))z(t),$$
$$\dot{z}(t) = G(w(t))x(t) + F(w(t))z(t); \tag{16}$$

(iv) *there is a control Lyapunov function of polyhedral form*; *and*
(v) *there exist an $r \geqslant n$, positive real numbers τ_i, $i \in \mathscr{I}$, a matrix $X \in \mathbf{R}^{n\times r}$ of full row rank, a matrix $U \in \mathbf{R}^{p\times r}$, and a set of matrices $\{P_i\}_{i\in\mathscr{I}}$ in $\mathbf{R}^{r\times r}$ with $\|P_i\|_1 < 1$, such that*

$$A_i X + B_i U = \tau_i^{-1} X(P_i - I_n), \quad \forall\, i \in \mathscr{I}.$$

The theorem tells us some important facts. Firstly, if the polytopic system is stabilizable via $\Phi(x, w)$, then it is stabilizable by means of $\Phi(x)$. This means that the switched system is stabilizable and the knowledge (availability) of the switching signal provides no advantage for the stabilizability of the polytopic linear uncertain system and hence that of the switched system. Secondly, in the case where the switching signal is available online, then it is sufficient to seek the gain-scheduling controllers of form (16). Thirdly, the stabilizability is reduced to searching for polyhedral control Lyapunov functions or solving coupled matrix equalities. This reduction is very useful since there are already many works in the literature involving the finding of polyhedral control Lyapunov functions. For example, numerical algorithms to compute the polyhedral Lyapunov functions were developed by Brayton and Tong (1979, 1980) and Polanski (2000). High-order homogeneous positive definite polynomial functions were also proposed as Lyapunov candidates (Blanchini & Miani, 1999). Once such Lyapunov functions are found, the corresponding stabilizing feedback control laws can be explicitly constructed.

However, as the computation of a polyhedral/polynomial Lyapunov function is usually not tractable in practice, an alternative way is to find some less universal but more tractable classes of Lyapunov functions including the standard quadratic functions. It is clear that, if there exist a scalar $\gamma > 0$ and a positive definite matrix P, such that

$$A_i P + P A_i^{\mathrm{T}} - \gamma P^{\mathrm{T}} B_i B_i^{\mathrm{T}} P < 0, \quad \forall\, i \in \mathscr{I},$$

then the problem of stabilization admits a solution $u(t) = F_\sigma x(t)$, where $F_i = (\gamma/2) B_i^{\mathrm{T}} P$, $i \in \mathscr{I}$. In addition, the closed-loop system is quadratically stable under an arbitrary switching signal. This approach has been exploited by Kar (2002) and Cheng (2004). A more general scheme is the parameter-dependent quadratic Lyapunov function approach developed by Daafouz and Bernussou (2001) and Daafouz, Riedinger, and Iung (2002). The main point is to associate a quadratic Lyapunov function to each subsystem and reduce the existence of such Lyapunov functions to the compatibility of a set of linear matrix inequalities (LMIs). The following theorem presents a sufficient condition for feedback stabilizability along this line.

Theorem 13 (*Daafouz et al., 2002*). *Discrete-time switched system $\sum(A_i, B_i)_{\mathscr{I}}$ is feedback stabilizable if there exist symmetric positive definite matrices S_i, and matrices G_i, R_i, such that the LMIs*

$$\begin{bmatrix} G_i + G_i^{\mathrm{T}} - S_i & (A_i G_i + B_i R_i)^{\mathrm{T}} \\ A_i G_i + B_i R_i & S_j \end{bmatrix} > 0$$

admit a solution for all $i, j \in \mathscr{I}$. The stabilizing state feedback gain matrices are $F_i = R_i G_i^{-1}$.

Besides its utility in stability analysis and design, the Lyapunov approach is also a powerful tool to address robustness issues for switched systems with uncertainties. The polyhedral Lyapunov functions are utilized to study the ultimate boundedness of system (15) with additional uncertainties (Blanchini, 1994, 1995). As we have mentioned, these results are applicable to switched linear systems with perturbations. In particular, for the perturbed switched system

$$\dot{x} = A_\sigma x + B_\sigma u + E v(t),$$

where v stands for the system perturbation, a convex and compact set X is globally attractive (all state trajectories go into the set in finite time) if, and only if the system admits a polyhedral control Lyapunov function outside set X (Blanchini, 1994, 1995). Other robustness criteria were presented for output-feedback control (Savkin & Evans, 1998; Li, Wang, Niemann, & Tanaka, 1999; Daafouz & Bernussou, 2002) and for disturbance rejection (Shamma & Xiong, 1999).

4. Optimal control

For the switched linear control system, the aim of optimal control is to seek appropriate switching/control strategies to optimize a certain performance index. If the switching signal is given and fixed, the switched system is in fact a time-varying control system. Hence, the problem is reduced to the conventional optimal control problem and can be addressed using either the classic maximum principle or the dynamic programming approach. However, if the switching signal is a design variable or generated by an event-driven (state-feedback) switching device, complication arises. While recently developed general versions of maximum principle apply to switched systems (Sussmann, 2001; Piccoli & Sussmann, 2000), they do not provide further insights on how to find the optimal switching strategy.

For the case when the switching signal obeys a state-space-partition-based law, there have been some noteworthy contributions in the literature, as highlighted here: Witsenhausen (1966) discussed the conditions for a solution to be well-behaved so that the variational methods can be applied; Seidman (1987) and Branicky, Borkar, and Mitter (1998) presented some results on the existence of optimal trajectories; Egerstedt, Ogren, Shakernia, and Lygeros (2000) proposed tight lower bounds for the minimum energy control problem; and Riedinger, Kratz, Iung, and Zanne (1999) proved that the optimal trajectories are asymptotically stable for a class of switched linear systems under quadratic cost functions.

Generally, optimization of switched control systems can be divided into two schemes according to the switching mechanisms. The first is when the switching signal is generated by a state-feedback switching law which is typically decided by the partition of the state space, the systems are known as piecewise affine (PWA) systems. The other scheme includes the switched systems where the switching signals are free design parameters and hence are time-driven in nature. The optimal control problems for the two schemes are quite different and we will briefly review the recent developments one by one.

4.1. Optimal control of piecewise affine systems

For discrete-time piecewise affine (PWA) systems, the optimal control problems have been extensively investigated in the literature. For notational convenience, here we restrict our attention to a special class of PWA systems described by

$$x(t+1) = A_i x(t) + B_i u(t), \text{ if } \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathscr{XU}_i, \quad (17)$$

$$E_1 x(t) + E_2 u(t) \leqslant E_3, \quad (18)$$

where $\{\mathscr{XU}_i\}_{i=1}^m$ is a partition of the state space into a number of compact polyhedral regions with no common interiors and each polyhedron contains the origin, matrices A_i, B_i and E_1, E_2 are in compatible dimensions, and E_3 is a column vector. Note that the system is subject to the input and state constraint (18).

Given a time horizon N, let us define the following quadratic cost function:

$$J(u(0),\ldots,u(N-1),x(0)) = x^{\mathrm{T}}(N)Px(N) + \sum_{k=0}^{N-1}(x^{\mathrm{T}}(k)Qx(k) + u^{\mathrm{T}}(k)Ru(k)),$$

where $P, Q \geqslant 0$ and $R>0$. The problem of optimal control for the PWA system is to seek for control sequence $u(0),\ldots,u(N-1)$ to minimize the cost function:

$$J^*(x(0)) \overset{\text{def}}{=} \min_{u(0),\ldots,u(N-1)} J(u(0),\ldots,u(N-1),x(0)) \quad \text{subject to (17), (18) and } x(N) \in \mathscr{X}_f,$$

where $\mathscr{X}_f$ is the terminal region. Any optimal control sequence

$$U_N^*(x_0) = \{u^*(x_0),\ldots,u_{N-1}^*(x_0)\}$$

is said to be a minimizer of the cost function.

Suppose that the optimal control problem admits at least one minimizer for each feasible x_0. The following result characterizes the structural properties of the optimal control law.

Theorem 14 (*Borrelli, Baotic, Bemporad, & Morari, 2003*). *The solution of the above optimal control problem is a feedback control law of the form*

$$u_k^*(x(k)) = F_k^i x(k) + G_k^i \quad \text{if } x(k) \in \mathscr{X}_k^i,$$

where $\{\mathscr{X}_k^i\}_{i=1}^{N_k}$ *is a partition of the set* $\mathscr{X}_k$ *of feasible state* $x(k)$ *and the closure* $\bar{\mathscr{X}}_k^i$ *of* $\mathscr{X}_k^i$ *is in the form*

$$\bar{\mathscr{X}}_k^i = \{x : x^{\mathrm{T}} L_k^i(j)x + M_k^i(j)x \leqslant K_k^i(j), \; j=1,\ldots,n_k^i\}, \; i=1,\ldots,N_k.$$

In addition, if the minimizer $U_N^*(x_0)$ *is unique for all* x_0, *then,* $\{\mathscr{X}_k^i\}_{i=1}^{N_k}$ *is a polyhedral partition of the set* $\mathscr{X}_k$ *of feasible state* $x(k)$.

The theorem illustrates that the optimal control law possesses a PWA form and the partition is defined by quadratic surfaces. In the case that the minimizer is unique, each region of the partition turns out to be a polyhedron, which in general is much simpler than a region defined by quadratic surfaces.

To solve the optimal control problem, a quadratic-parametric-programming based approach was developed to solve the Hamilton–Jacobi–Bellman (HJB) equation associated with the problem (Borrelli, 2003), and an approach based on the multiparametric mixed integer quadratic program (mp-MIQP) was established (Bemporad & Morari, 1999). For detailed technical developments in more general framework as well as several nice case studies, the reader is referred to (Borrelli et al., 2003; Morari, Baotic, & Borrelli, 2003).

For a class of continuous-time PWA systems with state+input constraints

$$\begin{matrix} \dot{x} = A_i x + B_i u \\ G_i x + H_i u \geqslant 0 \end{matrix} \quad \text{if } x \in \mathscr{X}_i, \tag{19}$$

where $\{\mathscr{X}_i\}_{i\in\mathscr{I}}$ is a partition of the state space into a number of closed polyhedral regions each of which contains the origin, piecewise linear quadratic optimal control was proposed in Rantzer and Johansson (2000). For the polyhedrons $\mathscr{X}_i$'s, we can construct matrices E_i and F_i such that

$$E_i x \geqslant 0, \quad x \in \mathscr{X}_i \quad \text{and} \quad F_i x = F_j x, \ x \in \mathscr{X}_i \cap \mathscr{X}_j.$$

The optimal control problem is to bring the system to $x(\infty) = 0$ from an arbitrary initial state $x(0)$, while limiting the piecewise quadratic cost

$$J(x_0, u) = \int_0^\infty (x^{\mathrm{T}} Q_i x + u^{\mathrm{T}} R_i u)\, \mathrm{d}t,$$

where $Q_i \geqslant 0$, $R_i > 0$, and $i(t)$ is defined so that $x(t) \in \mathscr{X}_{i(t)}$.

Theorem 15 (*Rantzer & Johansson, 2000*). *Assume that the existence of symmetric matrices T and U_i, such that U_i has nonnegative entries, while $P_i = F_i^{\mathrm{T}} T F_i$, $i \in \mathscr{I}$, satisfy*

$$\begin{bmatrix} A_i^{\mathrm{T}} P_i + P_i A_i + Q_i & P_i B_i \\ B_i^{\mathrm{T}} P_i & R_i \end{bmatrix} - \begin{bmatrix} E_i & 0 \\ G_i & H_i \end{bmatrix}^{\mathrm{T}} \times U_i \begin{bmatrix} E_i & 0 \\ G_i & H_i \end{bmatrix} > 0, \ i \in \mathscr{I}.$$

Then, every continuous and piecewise continuously differentiable trajectory $x(t) \in \cup \mathscr{X}_i$ of system (19) *with $x(\infty) = 0$, $x(0) = x_0 \in \mathscr{X}_{i_0}$ satisfies*

$$J(x_0, u) \geqslant \sup_{T, U_i} x_0^{\mathrm{T}} P_{i_0} x_0.$$

The theorem gives a lower bound on the optimal cost. The computation of an upper bound was also discussed in Rantzer and Johansson (2000).

4.2. Optimal control of time-driven switched systems

Suppose that the switching signal is an independent design parameter. The optimal control problem is to find both the optimal control input and optimal switching signal to optimize the value/cost function. As the switching signal is a discontinuous function of time and maybe highly nonlinear, the optimization is usually extremely involved and nonconvex in nature.

For a discrete-time switched system

$$x(t+1) = A_\sigma x(t) + B_\sigma u(t), \ \ x(0) = x_0,$$

the task is to find a switching signal σ and a feedback control law $u = F_\sigma x$ to minimize the cost function

$$J_N(x_0, u(0), \ldots, u(N-1), \sigma) = x(N)^{\mathrm{T}} Q_{\sigma_N} x(N) + \sum_{i=0}^{N-1} \begin{bmatrix} x(i) \\ u(i) \end{bmatrix}^{\mathrm{T}} Q_{\sigma(i)} \begin{bmatrix} x(i) \\ u(i) \end{bmatrix},$$

where N is the terminal time, $Q_N \geqslant 0$, and $Q_i > 0$, $i \in \mathscr{I}$.

If N is small, we can find an optimal solution by doing backward recursion of the cost using dynamic programming, evaluating all possible choice of switching signals. However, the searching will grow exponentially for larger N. To avoid combinatoric explosion, Lincoln and Bernhardsson (2002) proposed a method for efficient pruning of the search tree. Though the algorithm greatly reduces the searching time, it still runs in an exponential time.

For continuous-time systems, a common approach is based on some discretization of time space and/or state space into grids and then use optimal/suboptimal search methods for the resultant discrete-time systems (Hedlund & Rantzer, 2002). However, this indirect approach may lead to computational combinatoric explosion and inaccuracy of the solution. In view of this, a direct approach was recently developed to analyze and compute the optimal strategy for switched linear systems (Xu & Antsaklis, 2004). In the parameterization-based scheme, (Xu & Antsaklis, 2004) proposed the following two-stage optimization methodology.

Given a continuous-time switched control system

$$\dot{x}(t) = A_\sigma x(t) + B_\sigma u(t), \quad x(t_0) = x_0$$

and a fixed end-time t_f, the optimal control problem is to find a piecewise continuous input u, and a switching signal σ, such that the quadratic cost function

$$J(\sigma, u) = \frac{1}{2} x(t_f)^{\mathrm{T}} Q_f x(t_f) + \int_{t_0}^{t_f} \left(\frac{1}{2} x^{\mathrm{T}} Q x + \frac{1}{2} u^{\mathrm{T}} R u \right) \mathrm{d}t \tag{20}$$

is minimized, where $Q_f \geqslant 0$, $Q \geqslant 0$ and $R > 0$.

In the problem, we need to find an optimal control solution (σ^*, u^*) such that

$$J(\sigma^*, u^*) = \min_{\sigma, u} J(\sigma, u).$$

Note that if we fix the switching signal, then the problem reduces to a conventional optimal control problem for linear time-varying systems. This idea leads to the two-stage strategy for solving the problem.

Stage 1: Fixing a switching signal, solve the optimal control problem for the corresponding time-varying system.

Stage 2: Regarding the optimal control for each switching signal as a function

$$J_1(\sigma) = \min_{u \in \mathscr{U}} J(\sigma, u),$$

minimize J_1 with respect to the switching signal σ.

The two-stage optimization method actually provides a basic framework for approaching the optimization problems of switched and hybrid systems (see also Gokbayrak & Cassandras, 2000). In the follows, we briefly illustrate the two-stage strategy through a simple example.

For the switched system

$$\dot{x}(t) = A_1 x(t) + B_1 u, \quad t_0 < t < t_1,$$
$$\dot{x}(t) = A_2 x(t) + B_2 u, \quad t_1 \leqslant t \leqslant t_f,$$

where t_0 and t_f are given, we are to find optimal switching time t_1 and optimal input u to minimize the quadratic cost function

$$J = x^{\mathrm{T}}(t_f) Q_f x(t_f) + \int_{t_0}^{t_f} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^{\mathrm{T}} Q \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t.$$

By introducing a state variable x_{n+1} corresponding to the switching instant t_1, and a new scaled time variable τ with

$$t = t_0 + (x_{n+1} - t_0)\tau, \text{ for } 0 \leqslant \tau \leqslant 1,$$
$$t = x_{n+1} + (t_f - x_{n+1})(\tau - 1), \text{ for } 1 \leqslant \tau \leqslant 2,$$

the problem is converted into finding optimal x_{n+1} and optimal control u for system

$$\left.\begin{aligned} \frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau} &= (x_{n+1} - t_0) A_1 x + B_1 u, \\ \frac{\mathrm{d}x_{n+1}}{\mathrm{d}\tau} &= 0, \end{aligned}\right\} 0 \leqslant \tau < 1,$$
$$\left.\begin{aligned} \frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau} &= (t_f - x_{n+1}) A_2 x + B_2 u, \\ \frac{\mathrm{d}x_{n+1}}{\mathrm{d}\tau} &= 0, \end{aligned}\right\} 1 \leqslant \tau \leqslant 2,$$

with the quadratic cost function

$$J = x(2)^{\mathrm{T}} Q_f x(2) + \int_0^1 (x_{n+1} - t_0) \begin{bmatrix} x(\tau) \\ u(\tau) \end{bmatrix}^{\mathrm{T}} Q \begin{bmatrix} x(\tau) \\ u(\tau) \end{bmatrix} \mathrm{d}\tau + \int_1^2 (t_f - x_{n+1}) \begin{bmatrix} x(\tau) \\ u(\tau) \end{bmatrix}^{\mathrm{T}} Q \begin{bmatrix} x(\tau) \\ u(\tau) \end{bmatrix} \mathrm{d}\tau.$$

The latter can be addressed by solving parameterized general Riccati equations which can be derived using the dynamic programming. To determine the optimal switching time, we need to know $\partial J / \partial x_{n+1}$, that is, the derivative of J with respect to the switching instant. This derivative can be obtained based on the solution of the ordinary differential equations obtained by differentiating the Riccati equations with respect to the switching instants. Together with the corresponding HJB equations, the optimal solution can be obtained by solving a set of differential–algebraic equations. The reader is referred to Xu and Antsaklis (2002, 2004) for details in more general schemes.

Note that the above method also applies to the optimal control problem with several subsystems and more than one switchings.

5. Concluding remarks

State-of-the-art development on the analysis and synthesis of switched linear control systems has been briefly introduced in this paper. The topics include controllability/observability issues, feedback stabilization, and optimal control.

The theory of controllability and observability is quite complete. It has been shown that the inherent structures of continuous-time systems and reversible discrete-time systems are similar to that of linear time-invariant systems. By means of the controllability/observability criteria, we can transform a switched system into the canonical form, which greatly simplifies the feedback synthesis problems.

The problem of feedback stabilization has been investigated by a few researchers and is still in the early stage of development. Stabilizing design for switched control systems is challenging since both the control input and the switching mechanism have to be taken into account, and thus the interaction between them must be fully understood. When the switching signal is a design variable, several constructive design procedures have been proposed for certain classes of systems based on the system structural decomposition. Although the theory is far from complete, there is no doubt that further progress will take place along this line, and it is possible to develop a parallel theory to the standard linear system theory. In this regard, a key question is whether or not controllability implies feedback stabilizability. If yes, the next step is to integrate the controllable and uncontrollable modes using a single switching signal. While a thorough treatment seems quite involved, the existing results indicate that the approach is promising.

In the case where the switching signal is not a design variable but belongs to a prescribed admissible set, the problem of feedback stabilization is then to find a feedback control law to ensure system stability under any switching signal. This problem is closely related to the robust control problem of the polytopic uncertain linear systems which can be addressed by utilizing the Lyapunov approach. The stabilizability is inferred from the existence of certain polyhedral control Lyapunov functions. If we restrict ourselves to quadratic or piecewise quadratic stabilizability, then the design of stabilizing controllers can be converted into the solving of certain linear matrix inequalities which permit efficient numerical solutions. Another approach is to first design local controllers based on the structural properties of the individual subsystems, and then analyze the stability of the closed-loop unforced switched system. The two approaches are complementary in some sense and may be combined in the way that, we design the local controllers by means of standard linear system technique, and analyze the closed-loop system using the Lyapunov approach.

The optimal control problem of switched systems is in general difficult to solve due to the involvement of the switching signal. For piecewise affine systems with state/input constraints, some efficient computational schemes have been established. For time-driven switched systems, the two-stage optimization method applies to the finite-time horizon optimal control problem with finite number of switchings. Despite these achievements, there still lacks a general theory on the existence and uniqueness of the optimal control strategy. In addition, much work needs to be done towards building an understanding of what an optimal switching signal should look like. At the same time, efficient numerical algorithms are to be developed for practical applications.

To summarize, there have been many good results reported in the literature and more are emerging. However, some fundamental issues are still in need of further explorations. As many powerful tools from linear and multilinear analysis are applicable or extendable to cope with these systems, it seems reasonable to expect a comprehensive switched system theory which extends the standard linear theory on one hand and applies to more real world problems on the other.

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