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Brief paper

Adaptive dynamic surface control of nonlinear systems with unknown dead zone in pure feedback form☆

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Abstract

In this paper, adaptive dynamic surface control (DSC) is developed for a class of pure-feedback nonlinear systems with unknown dead zone and perturbed uncertainties using neural networks. The explosion of complexity in traditional backstepping design is avoided by utilizing dynamic surface control and introducing integral-type Lyapunov function. It is proved that the proposed design method is able to guarantee semi-global uniform ultimate boundedness of all signals in the closed-loop system, with arbitrary small tracking error by appropriately choosing design constants. Simulation results demonstrate the effectiveness of the proposed approach.

Keywords: Adaptive control; Neural networks; Dynamic surface control; Pure-feedback nonlinear systems; Dead zone

1. Introduction

In the past decade, adaptive control system design for uncertain nonlinear systems has received much attention using universal function approximators such as neural networks or fuzzy logic systems to parameterize the unknown nonlinearities (Ge, Hang, Lee, & Zhang, 2001; Wang, 1994; Yesildirek & Lewis, 1995). Adaptive backstepping design was proposed for parametric strict-feedback systems with overparameterization (Kanellakipoulos, Kokotović, & Morse, 1991), which could guarantee globally stable and asymptotic tracking performance. By introducing tuning functions, the overparameterization (Kanellakipoulos et al., 1991) was eliminated (Krstić, Kanellakopoulos, & Kokotović, 1992). The adaptive backstepping design was further extended to parametric strict-feedback systems with unknown virtual control coefficients (Krsti, Kanellakopoulos, & Kokotović, 1995). Adaptive NN control was developed (Polycarpou, 1996; Polycarpou & Mears, 1998), for a class of strict-feedback nonlinear systems with virtual coefficients $g_i(\bar{x}_i) = 1, i = 1, \dots, n-1$, unknown constant control gain $g_n(\bar{x}_n) = g_n$, and $f_i(\bar{x}_i), i = 1, \dots, n$ being unknown continuous functions. Using a novel family of integral Lyapunov functions, adaptive backstepping control was presented (Ge et al., 2001; Zhang, Ge, & Hang, 2000) for a class of strict-feedback systems. The possible controller singularity problem is avoided without using projection. However, the proposed control method requires the integral operation at each recursive step in backstepping design (Zhang et al., 2000), and introduces computational complexity. As an alternative, adaptive NN control based on quadratic Lyapunov function was proposed for a class of nonlinear systems (Ge & Wang, 2002b). Adaptive NN control schemes were proposed (Ge & Wang, 2002a; Wang & Huang, 2002) for a class of uncertain nonlinear pure-feedback systems based on backstepping. Furthermore, using the input-to state stability analysis and the small gain theorem, an improved adaptive NN control for a class of completely nonaffine pure-feedback systems was presented (Wang, Hill, Ge, & Chen, 2006).

Though backstepping has become one of the most popular design methods for a large class of SISO nonlinear systems (Ge & Wang, 2002b; Kanellakipoulos et al., 1991; Krstiét al., 1995; Polycarpou, 1996; Polycarpou & Mears, 1998; Zhang et al.,

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2000), an obvious drawback in the traditional backstepping design is the problem of "explosion of complexity", which is caused by the repeated differentiations of certain nonlinear functions such as virtual controls, as pointed out (Swaroop, Hedrick, Yip, & Gerdes, 2000). To overcome the "explosion of complexity", dynamic surface control (DSC) was proposed for a class of strict-feedback nonlinear systems with known $f_i(x_1, \ldots, x_i)$ and $g_i = 1$ by introducing first-order filtering of the synthetic virtual control input at each step of traditional backstepping approach (Swaroop et al., 2000). The asymptotical tracking problem was developed for a class of strict-feedback nonlinear systems with unknown functions f_i and virtual coefficients $g_i = 1$ (Wang & Huang, 2005).

Nonsmooth, nonlinear characteristics such as dead zone, backlash, hysteresis are common in actuator and sensors such as mechanical connections, hydraulic actuators and electric servomotors. Dead zone is one of the most important nonsmooth nonlinearities in many industrial processes. Its presence severely limits system performance, and its study has been drawing much interest in the control community for a long time (Lewis, Campos, & Selmic, 2002; Selmic & Lewis, 2000; Tao & Kokotovic, 1994, 1995; Tao & Lewis, 2001; Wang, Hong, & Su, 2004). To handle systems with unknown dead zones, adaptive dead-zone inverses were proposed (Tao & Kokotovic, 1994, 1995). Under the condition that the dead-zone slopes in the positive and negative regions must be the same, robust adaptive control was developed for a class of special nonlinear systems without constructing the inverse of the dead zone (Wang et al., 2004). In the work Zhou, Wen, and Zhang (2006), adaptive output feedback control using backstepping and smooth inverse function of the dead zone was proposed for a class of SISO nonlinear systems with unknown dead zone. Adaptive neural control was developed for a class of uncertain multi-input, multi-output nonlinear state time-varying delay systems in triangular control structure with unknown nonlinear dead zones and gain signs (Zhang & Ge, 2007).

In this paper, neural network-based adaptive control is investigated for a class of perturbed pure-feedback nonlinear systems by combining dynamic surface control with backstepping. The dead-zone output is represented as a simple linear system with a static time-varying gain and bounded disturbance by introducing characteristic function. The approach removes the assumption of linear function outside the deadband without necessarily constructing a dead-zone inverse. In addition, the possible singularity of the proposed adaptive controller, which may be caused by time-varying gain function, is avoided by utilizing integral-type Lyapunov function and variable structure control strategy. Moreover, a simple control structure is presented for a class of pure-feedback nonlinear systems without calculating the integrals of Eq. (100) (Ge & Wang, 2002a), and the dimensions of variables Z_i are reduced by introducing the first-order filtering in approximating unknown functions, compared with the existing results (Ge & Wang, 2002a; Wang & Huang, 2002; Wang et al., 2006).

This paper is organized as follows. In Section 2, the problem formulation and preliminaries are presented, and a general nonlinear dead-zone model, which is used in the controller design, is introduced. In Section 3, direct adaptive dynamic surface control is developed by using MNNs for a class of SISO pure-feedback nonlinear systems with unknown dead zone and perturbed uncertainties. The closed-loop system stability is analysed. In addition, some design constants are determined. Simulation results demonstrate the effectiveness of the approach in Section 4. Section 5 contains the conclusions.

2. Problem formulation and preliminaries

Consider a class of single-input, single-output (SISO) nonlinear systems in the following form:

Plant :

$$\begin{cases} \dot{x}_i = f_i(\bar{x}_i, x_{i+1}) + d_1(\bar{x}_n, t), \quad 1 \le i \le n-2 \\ \dot{x}_{n-1} = f_{n-1}(\bar{x}_{n-1}) + g_{n-1}(\bar{x}_{n-1})x_n + d_{n-1}(\bar{x}_n, t) \\ \dot{x}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n)u + d_n(\bar{x}_n, t), \quad n \ge 3 \\ y = x_1. \end{cases} \tag{1}$$

Dead-zone:

$$u = D(v) = \begin{cases} g_r(v) & \text{if } v \ge b_r, \\ 0 & \text{if } b_l < v < b_r, \\ g_l(v) & \text{if } v \le b_l, \end{cases} \tag{2}$$

where $\bar{x}_i = [x_1, x_2, \ldots, x_i]^{\mathrm{T}} \in R^i, i = 1, \ldots, n$ are the states, $u \in R$ is the output of the dead zone, y is the output, $f_i(\bar{x}_i, x_{i+1})(i = 1, \ldots, n-2), g_{n-1}(\bar{x}_{n-1}), g_n(\bar{x}_n)$ are the unknown smooth functions; $f_{n-1}(\bar{x}_{n-1})$ and $f_n(\bar{x}_n)$ are the unknown continuous functions, $d_1(\bar{x}_n, t), d_2(\bar{x}_n, t), \ldots, d_n(\bar{x}_n, t)$ are the unknown uncertain disturbances; $v(t) \in R$ is the input to the dead zone, b_l and b_r are the unknown parameters of the dead zone.

To facilitate control system design, we need the following assumptions (Zhang & Ge, 2007):

Assumption 1. The dead-zone output, u, is not available.

Assumption 2. The dead-zone parameters, b_r and b_l, are unknown bounded constants, but their signs are known, i.e. $b_r > 0$ and $b_l < 0$.

Assumption 3. The functions, $g_l(v)$ and $g_r(v)$, are smooth, and there exist unknown positive constants, k_{l0}, k_{l1}, k_{r0}, and k_{r1} such that

$$0 < k_{l0} \le g_l'(v) \le k_{l1}, \quad \forall v \in (-\infty, b_l] \tag{3}$$

$$0 < k_{r0} \le g_r'(v) \le k_{r1}, \quad \forall v \in [b_r, +\infty) \tag{4}$$

and $\beta_0 \le \min\{k_{l0}, k_{r0}\}$ is a known positive constant, where $g_l'(v) = \frac{\mathrm{d}g_l(z)}{\mathrm{d}z}|_{z=v}$ and $g_r'(v) = \frac{\mathrm{d}g_r(z)}{\mathrm{d}z}|_{z=v}$.

For convenience, $g_l(v)$ and $g_r(v)$ in (3) and (4) are assumed to be true for $v \in (-\infty, b_r]$, and for $v \in [b_l, +\infty)$, respectively.

Based on Assumption 3, the dead zone (2) can be rewritten as follows (Zhang & Ge, 2007):

$$u = D(v) = K^{\mathrm{T}}(t)\Phi(t)v + d(v), \tag{5}$$

where

$$\Phi(t) = [\varphi_r(t), \varphi_l(t)]^{\mathrm{T}}, \tag{6}$$

$$\varphi_r(t) = \begin{cases} 1 & \text{if } v(t) > b_l, \\ 0 & \text{if } v(t) \le b_l, \end{cases} \tag{7}$$

$$\varphi_l(t) = \begin{cases} 1 & \text{if } v(t) < b_r, \\ 0 & \text{if } v(t) \ge b_r, \end{cases} \tag{8}$$

$$K(t) = [K_r(v(t)), K_l(v(t))]^{\mathrm{T}}, \tag{9}$$

$$K_r(v(t)) = \begin{cases} 0 & \text{if } v(t) \le b_l, \\ g'_{ir}(\xi_r(v(t))) & \text{if } b_l < v(t) < +\infty, \end{cases} \tag{10}$$

$$K_l(v(t)) = \begin{cases} g'_l(\xi_l(v(t))) & \text{if } -\infty < v(t) < b_r, \\ 0 & \text{if } v(t) \ge b_r, \end{cases} \tag{11}$$

$$d(v) = \begin{cases} -g'_r(\xi_r(v))b_r & \text{if } v \ge b_r, \\ -[g'_l(\xi_l(v)) + g'_r(\xi_r(v))]v & \text{if } b_l < v < b_r, \\ -g'_l(\xi_l(v))b_l & \text{if } v \le b_l, \end{cases} \tag{12}$$

$\xi_l(v) \in (v, b_l)$, if $v < b_l$; $\xi_l(v) \in (b_l, v)$, if $b_l \le v < b_r$; $\xi_r(v) \in (b_r, v)$, if $b_r < v$; $\xi_r(v) \in (v, b_r)$, if $b_l < v \le b_r$, and $|d(v)| \le p^*$, p^* is an unknown positive constant with $p^* = (k_{r1} + k_{l1})\max\{b_r, -b_l\}$.

The control objective is to design adaptive control $u(t)$ for system (1) such that the output y follows the specified desired trajectory y_d.

For the control of pure feedback system (1), define

$$g_i(\bar{x}_i, x_{i+1}) = \frac{\partial f_i(\bar{x}_i, x_{i+1})}{\partial x_{i+1}}, i = 1, \ldots, n-2. \tag{13}$$

Assumption 4. The signs of $g_i(\bar{x}_i, x_{i+1}), i = 1, \ldots, n-2$, $g_{n-1}(\bar{x}_{n-1})$ and $g_n(\bar{x}_n)$ are known, and there exist constants g_{i0} and g_{i1} such that (i) $|g_i(\cdot)| \ge g_{i0} > 0, \forall \bar{x}_n \in R^n$, and (ii) $|g_i(\cdot)| \le g_{i1}, \forall \bar{x}_n \in \Omega_{\bar{x}_n} \subset R^n$, where $\Omega_{\bar{x}_n}$ is a compact region, $i = 1, \ldots, n$. Without loss of generality, we shall assume that $g_i(\cdot) \ge g_{i0} > 0, \forall \bar{x}_n \in R^n, i = 1, \ldots, n$.

Assumption 5. There exist unknown constants, $g_{id} > 0$, such that $|\dot{g}_i(\cdot)| \le g_{id}, \forall \bar{x}_n \in \Omega_{\bar{x}_n} \subset R^n$ where $\Omega_{\bar{x}_n}$ is a compact region, $i = 1, \ldots, n$.

Assumption 6. The desired trajectory vectors are continuous and available, and $[y_d, \dot{y}_d, \ddot{y}_d]^{\mathrm{T}} \in \Omega_d$ with known compact set $\Omega_d = \{[y_d, \dot{y}_d, \ddot{y}_d]^{\mathrm{T}} : \ y_d^2 + \dot{y}_d^2 + \ddot{y}_d^2 \le B_0\} \subset R^3$, whose size B_0 is a known positive constant.

Assumption 7. There exist unknown positive constants, p_i^*, such that $\forall(\bar{x}_n, t) \in R^n \times R_+$, $|d_i(\bar{x}_n, t)| \le p_i^* \rho_i(\bar{x}_i)$ with $\rho_i(\bar{x}_i)$ being known positive smooth functions, $i = 1, \ldots, n$.

In this paper, $\|\cdot\|$ denotes the 2-norm, $\|\cdot\|_F$ denotes the Frobenius norm, $\|A\|_1 = \sum_{k=1}^{l} |a_k|$ with $A = [a_1, \ldots, a_l] \in R^l$, $\lambda_{\min}(B)$ and $\lambda_{\max}(B)$ denote the smallest and largest eigenvalues of a square matrix B, respectively. Throughout this paper, let $\tilde{(\cdot)} = \hat{(\cdot)} - (\cdot)$.

3. Adaptive neural network control

In this section, we will combine the dynamic surface control (Swaroop et al., 2000) with backstepping and adaptive control for the nth-order systems described by (1). Similar to traditional backstepping, the design of adaptive control laws is based on the following change of coordinates: $z_1 = x_1 - y_d, z_i = x_i - \omega_i, i = 2, \ldots, n$, where ω_i is the output of a first-order filter with α_{i-1} as the input, and α_{i-1} is an intermediate control which shall be developed for the corresponding $(i-1)$th subsystem. Finally, an overall control law u is constructed at step n. The major difference of our approach from the previous design method (Ge & Wang, 2002a,b; Wang & Huang, 2002; Wang et al., 2006; Zhang et al., 2000) is to replace, at each step of recursion, the quantity $\dot{\alpha}_{i-1}$ by $\dot{\omega}_i$ in determining the virtual control α_i. As a result, the operation of differentiation can be replaced by simpler algebraic operation.

Before proceeding with the adaptive control, some notations are presented below. Let $\Omega_{Z_i} \subset R^{i+2}$ be a given compact set, and $\hat{W}_{hi}^{\mathrm{T}} S_i(\hat{V}_{hi}^{\mathrm{T}} \bar{Z}_i)$ be the approximation of the three-layer neural networks on the compact Ω_{Z_i} to $h_i(Z_i)$ as discussed (Funahashi, 1989; Zhang et al., 2000), where unknown continuous function $h_i(Z_i)$ will be given later. Then, we have

$$h_i(Z_i) = \hat{W}_{hi}^{\mathrm{T}} S_i(\hat{V}_{hi}^{\mathrm{T}} \bar{Z}_i) - \tilde{W}_{hi}^{\mathrm{T}}(\hat{S}_{hi} - \hat{S}'_{hi} \hat{V}_{hi}^{\mathrm{T}} \bar{Z}_i) - \hat{W}_{hi}^{\mathrm{T}} \hat{S}'_{hi} \tilde{V}_{hi}^{\mathrm{T}} \bar{Z}_i + \psi_{hi}, \tag{14}$$

where

$$Z_1 = [x_1, z_1, \dot{y}_d]^{\mathrm{T}}, \quad Z_i = [\bar{x}_i, z_i, \dot{\omega}_i]^{\mathrm{T}}, \quad \bar{Z}_i = [Z_i^{\mathrm{T}}, 1]^{\mathrm{T}} \tag{15}$$

$\hat{W}_{hi} = [\hat{w}_{hi1}, \ldots, \hat{w}_{hil_i}]^{\mathrm{T}} \in R^{l_i}$ and $\hat{V}_{hi} = [\hat{v}_{hi1}, \ldots, \hat{v}_{hil_i}] \in R^{(i+3)\times l_i}$ denote the estimates of W_{hi}^* and V_{hi}^*, respectively,

$$\hat{S}_{hi} = S_i(\hat{V}_{hi}^{\mathrm{T}} \bar{Z}_i) = [s(\hat{v}_{hi1}^{\mathrm{T}} \bar{Z}_1), \ldots, s(v_{hi,l_i-1}^{\mathrm{T}} \bar{Z}_i), 1]^{\mathrm{T}} \tag{16}$$

with $s(z_\alpha) = 1/(1 + \mathrm{e}^{-\gamma_{i0} z_\alpha})$, and constant $\gamma_{i0} > 0$, $\hat{S}'_{hi} = \mathrm{diag}\{\hat{s}'_{hi1}, \ldots, \hat{s}'_{hi,l_i-1}, 0\}$ with $\hat{s}'_{hik} = s'(\hat{v}_{hik}^{\mathrm{T}} \bar{Z}_i) = d[s(z_\alpha)]/dz_\alpha|_{z_\alpha = \hat{v}_{hik}^{\mathrm{T}} \bar{Z}_i}$, $k = 1, \ldots, l_i - 1$; and the NN node number $l_i > 1$; W_{hi}^* and V_{hi}^* are ideal constant weights, which are defined as follows:

$$(W_{hi}^*, V_{hi}^*) = \arg \min_{\substack{W_{hi} \in \Omega_{whi} \\ V_{hi} \in R^{(i+3)\times l_i}}} [\sup_{Z_i \in \Omega_{Z_i}} |W_{hi}^{\mathrm{T}} S_i(V_{hi}^{\mathrm{T}} \bar{Z}_i) - h_i(Z_i)|] \tag{17}$$

with

$$\Omega_{whi} = \{\hat{W}_{hi} \mid \|\hat{W}_{hi}\| \le M_{hi}\}, \tag{18}$$

$M_{hi} > 0$, $\psi_{hi} = -d_{uhi} + \varepsilon_{hi}(Z_i)$, and the residual term d_{uhi} is bounded by

$$|d_{uhi}| \le \|V_{hi}^*\|_F \|\bar{Z}_i \hat{W}_{hi}^{\mathrm{T}} \hat{S}'_{hi}\|_F + \|W_{hi}^*\| \|\hat{S}'_{hi} \hat{V}_{hi}^{\mathrm{T}} \bar{Z}_i\| + \|W_{hi}^*\|_1 \tag{19}$$

and the approximation error $\varepsilon_{hi}(Z_i)$ satisfies $|\varepsilon_{hi}(Z_i)| \le \varepsilon_{hi}^*, \forall Z_i \in \Omega_{Z_i}$, with constant $\varepsilon_{hi}^* > 0$.

For clarity, the following notations are defined:

$$\bar{z}_i = [z_1, \ldots, z_i]^{\mathrm{T}}, \quad \bar{y}_j = [y_2, \ldots, y_j]^{\mathrm{T}}, \tag{20}$$

$$\bar{\hat{W}}_{hi} = [\hat{W}_{h1}^{\mathrm{T}}, \ldots, \hat{W}_{hi}^{\mathrm{T}}]^{\mathrm{T}}, \quad \mathrm{col}(A) = [a_1^{\mathrm{T}}, \ldots, a_q^{\mathrm{T}}]^{\mathrm{T}}, \tag{21}$$

$$\bar{\hat{V}}_{hi} = [\mathrm{col}(\hat{V}_{h1})^{\mathrm{T}}, \ldots, \mathrm{col}(\hat{V}_{hi})^{\mathrm{T}}]^{\mathrm{T}}, \tag{22}$$

where $i = 1 \ldots, n$, $y_j = \omega_j - \alpha_{j-1}$, $j = 2, \ldots, n$, $A = [a_1, \ldots, a_q] \in R^{m \times q}$.

Step 1: Let $\omega_1 = y_d$. Then, we have $z_1 = x_1 - \omega_1$. The time derivative of z_1 is

$$\dot{z}_1 = f_1(x_1, x_2) + d_1(\bar{x}_n, t) - \dot{\omega}_1. \tag{23}$$

From Assumption 4, we know that $\partial f_1(x_1, x_2)/\partial x_2 \geq g_{10} > 0$ for all $(x_1, x_2) \in R^2$. Define

$$\nu_1 = -\dot{\omega}_1. \tag{24}$$

Considering the fact that $\partial \nu_1/\partial x_2 = 0$, the following inequality holds:

$$\frac{\partial[f_1(x_1, x_2) + \nu_1]}{\partial x_2} \geq g_{10} > 0. \tag{25}$$

According to Lemma 1 (Ge & Wang, 2002a), for every value of x_1 and ν, there exists a smooth ideal control input $x_2 = \alpha_1^*(x_1, \nu_1)$ such that

$$f_1(x_1, \alpha_1^*) + \nu_1 = 0. \tag{26}$$

Using mean value theorem, there exists $\lambda_1 (0 < \lambda_1 < 1)$ such that

$$f_1(x_1, x_2) = f_1(x_1, \alpha_1^*) + g_{1\lambda_1}(x_2 - \alpha_1^*), \tag{27}$$

where $g_{1\lambda_1} = g_1(x_1, x_{2\lambda_1})$ with $x_{2\lambda_1} = \lambda_1 x_2 + (1 - \lambda_1)\alpha_1^*$. Combining Eqs. (23)–(27) yields

$$\dot{z}_1 = g_{1\lambda_1}(x_2 - \alpha_1^*) + d_1(\bar{x}_n, t). \tag{28}$$

In order to design direct adaptive control, define a smooth scalar function as follows:

$$V_{z1} = \frac{1}{g_{1\lambda_1}} z_1^2. \tag{29}$$

Differentiating V_{z1} with respect to time t, we obtain

$$\dot{V}_{z1} = \frac{2 z_1 \dot{z}_1}{g_{1\lambda_1}} - \frac{\dot{g}_{1\lambda_1} z_1^2}{2 g_{1\lambda_1}^2}. \tag{30}$$

According to Assumption 7, and using Young's Inequality, we obtain

$$\begin{aligned} \frac{|z_1 d_1(\bar{x}_n, t)|}{g_{1\lambda_1}} &\leq \frac{|z_1| p_1^* \rho_1(x_1)}{g_{1\lambda_1}} \leq \frac{z_1^2 \rho_1^2(x_1)}{2 g_{1\lambda_1}^2} + \frac{p_1^{*2}}{2} \\ &\leq \frac{z_1^2 \rho_1^2(x_1)}{2 g_{10}^2} + \frac{p_1^{*2}}{2}. \end{aligned} \tag{31}$$

Substituting Eqs. (28) and (31) into Eq. (30), we have

$$\begin{aligned} \dot{V}_{z1} &= z_1[x_2 - \alpha_1^*] + \frac{z_1 d_1(x, t)}{g_{1\lambda_1}} - \frac{\dot{g}_{1\lambda_1} z_1^2}{2 g_{1\lambda_1}^2} \\ &\leq z_1[x_2 + h_1(Z_1)] + \frac{p_1^{*2}}{2} - \frac{\dot{g}_{1\lambda_1} z_1^2}{2 g_{1\lambda_1}^2}, \end{aligned} \tag{32}$$

where $h_1(Z_1) = -\alpha_1^* + \frac{z_1 \rho_1^2(x_1)}{2 g_{10}^2}$.

Choose a virtual control α_1 as follows:

$$\alpha_1 = -k_1 z_1 - \hat{W}_{h1}^{\mathrm{T}} S_1(\hat{V}_{h1}^{\mathrm{T}} \bar{Z}_1), \tag{33}$$

where k_1 is a design constant that we will choose later.

The update NN weights are employed as follows:

$$\dot{\hat{W}}_{hi} = \begin{cases} \gamma_{whi}[(\hat{S}_{hi} - \hat{S}'_{hi} \hat{V}_{hi}^{\mathrm{T}} \bar{Z}_i) z_i - \sigma_{whi} \hat{W}_{hi}] \\ \quad \text{if } \|\hat{W}_{hi}\| < M_{hi} \text{ or } \|\hat{W}_{hi}\| = M_{hi} \\ \quad \text{and } \hat{W}_{hi}^{\mathrm{T}}(\hat{S}_{hi} - \hat{S}'_{hi} \hat{V}_{hi}^{\mathrm{T}} \bar{Z}_i) z_i \leq 0; \\ \gamma_{whi} \Big[(\hat{S}_{hi} - \hat{S}'_{hi} \hat{V}_{hi}^{\mathrm{T}} \bar{Z}_i) z_i \\ \quad - \frac{\hat{W}_{hi} \hat{W}_{hi}^{\mathrm{T}}}{\|\hat{W}_{hi}\|^2} (\hat{S}_{hi} - \hat{S}'_{hi} \hat{V}_{hi}^{\mathrm{T}} \bar{Z}_i) z_i - \sigma_{whi} \hat{W}_{hi} \Big] \\ \quad \text{if } \|\hat{W}_{hi}\| = M_{hi} \text{ and } \hat{W}_{hi}^{\mathrm{T}}(\hat{S}_{hi} \\ \quad - \hat{S}'_{hi} \hat{V}_{hi}^{\mathrm{T}} \bar{Z}_i) z_i > 0, \end{cases} \tag{34}$$

$$\dot{\hat{V}}_{hi} = \gamma_{vhi}[\bar{Z}_i \hat{W}_{hi}^{\mathrm{T}} \hat{S}'_{hi} z_i - \sigma_{vhi} \hat{V}_{hi}], \tag{35}$$

where subscript $i = 1$, and constants $\gamma_{whi}, \gamma_{vhi}, \sigma_{whi}, \sigma_{vhi}$ are strictly positive.

Define ω_2 in such a way that

$$\tau_2 \dot{\omega}_2 + \omega_2 = \alpha_1, \quad \omega_2(0) = \alpha_1(0), \tag{36}$$

where τ_2 is a design constant that we will choose later. From Eq. (36), we have $\dot{\omega}_2 = -\frac{y_2}{\tau_2}$. Since $x_2 = z_2 + y_2 + \alpha_1 = z_2 + y_2 - k_1 z_1 - \hat{W}_{h1}^{\mathrm{T}} S_1(\hat{V}_{hi}^{\mathrm{T}} \bar{Z}_1)$, using Eq. (32) and Young's Inequality, we obtain

$$\begin{aligned} \dot{V}_{z1} \leq{}& (-k_1 + 2) z_1^2 + \frac{1}{4} z_2^2 + \frac{1}{4} y_2^2 - z_1 \tilde{W}_{h1}^{\mathrm{T}}(\hat{S}_{h1} \\ & - \hat{S}'_{h1} \hat{V}_{h1}^{\mathrm{T}} \bar{Z}_1) - z_1 \hat{W}_{h1}^{\mathrm{T}} \hat{S}'_{h1} \tilde{V}_{h1}^{\mathrm{T}} \bar{Z}_1 + \frac{p_1^{*2}}{2} \\ & + |z_1| \eta_1(z_1, \hat{W}_{h1}, \mathrm{col}(\hat{V}_{h1}), y_d, \dot{y}_d), \end{aligned} \tag{37}$$

where continuous function $\eta_1(z_1, \hat{W}_{h1}, \mathrm{col}(\hat{V}_{h1}), y_d, \dot{y}_d)$ satisfies

$$\left| \psi_{h1} - \frac{\dot{g}_{1\lambda_1} z_1}{2 g_{1\lambda_1}^2} \right| \leq \eta_1(z_1, \hat{W}_{h1}, \mathrm{col}(\hat{V}_{h1}), y_d, \dot{y}_d). \tag{38}$$

From Young's Inequality, we have $|z_1| \eta_1 \leq z_1^2 + \frac{1}{4} \eta_1^2$. Therefore, we obtain

$$\begin{aligned} \dot{V}_{z1} \leq{}& (-k_1 + 3) z_1^2 + \frac{1}{4} z_2^2 + \frac{1}{4} y_2^2 \\ & - z_1 \tilde{W}_{h1}^{\mathrm{T}}(\hat{S}_{h1} - \hat{S}'_{h1} \hat{V}_{h1}^{\mathrm{T}} \bar{Z}_1) - z_1 \mathrm{tr}\{\tilde{V}_{h1}^{\mathrm{T}} \bar{Z}_1 \hat{W}_{h1}^{\mathrm{T}} \hat{S}'_{h1}\} \\ & + \frac{p_1^{*2}}{2} + \frac{1}{4} \eta_1^2. \end{aligned} \tag{39}$$

Noting Assumption 7, we have

$$\dot{y}_2 = -\frac{y_2}{\tau_2} + \left(\dot{\hat{W}}_{h1}^{\mathrm{T}}\hat{S}_{h1} + \hat{W}_{h1}^{\mathrm{T}}\dot{\hat{S}}_{h1} + k_1\dot{z}_1\right), \tag{40}$$

$$\left|\dot{y}_2 + \frac{y_2}{\tau_2}\right| \le \xi_2(\bar{z}_2, y_2, \hat{W}_{h1}, \mathrm{col}(\hat{V}_{h1}), y_d, \dot{y}_d, \ddot{y}_d), \tag{41}$$

where $\xi_2(\bar{z}_2, y_2, \hat{W}_{h1}, \mathrm{col}(\hat{V}_{h1}), y_d, \dot{y}_d, \ddot{y}_d)$ is a continuous function.

From Eqs. (40) and (41), we obtain

$$y_2\dot{y}_2 \le -\frac{y_2^2}{\tau_2} + |y_2|\xi_2 \le -\frac{y_2^2}{\tau_2} + y_2^2 + \frac{1}{4}\xi_2^2. \tag{42}$$

Step i ($2 \le i \le n-2$): The time derivative of z_i is

$$\dot{z}_i = f_i(\bar{x}_i, x_{i+1}) + d_i(\bar{x}_n, t) - \dot{\omega}_i. \tag{43}$$

From Assumption 4, we know that $\partial f_i(\bar{x}_i, x_{i+1})/\partial x_{i+1} \ge g_{i0} > 0$ for all $\bar{x}_{i+1} \in R^{i+1}$. Define $\nu_i = -\dot{\omega}_i$. Considering the fact that $\partial \nu_i/\partial x_{i+1} = 0$, the following inequality holds:

$$\frac{\partial[f_i(\bar{x}_i, x_{i+1}) + \nu_i]}{\partial x_{i+1}} \ge g_{i0} > 0. \tag{44}$$

According to Lemma 1 (Ge & Wang, 2002a), for every value of $\bar{x}_i$ and ν_i, there exists a smooth ideal control input $x_{i+1} = \alpha_i^*(\bar{x}_i, \nu_i)$ such that

$$f_i(\bar{x}_i, \alpha_i^*) + \nu_i = 0. \tag{45}$$

Using mean value theorem, there exists $\lambda_i (0 < \lambda_i < 1)$ such that

$$f_i(\bar{x}_i, x_{i+1}) = f_i(\bar{x}_i, \alpha_i^*) + g_{i\lambda_i}(x_{i+1} - \alpha_i^*), \tag{46}$$

where $g_{i\lambda_i} = g_i(\bar{x}_i, x_{(i+1)\lambda_i})$ with $x_{(i+1)\lambda_i} = \lambda_i x_{i+1} + (1-\lambda_i)\alpha_i^*$.

Combining Eqs. (43)–(46) yields

$$\dot{z}_i = g_{i\lambda_i}(x_{i+1} - \alpha_i^*) + d_i(\bar{x}_n, t). \tag{47}$$

Define a smooth scalar function as follows:

$$V_{zi} = \frac{1}{2g_{i\lambda_i}} z_i^2. \tag{48}$$

Differentiating V_{zi} with respect to time t, we obtain

$$\dot{V}_{zi} = \frac{z_i \dot{z}_i}{g_{i\lambda_i}} - \frac{\dot{g}_{i\lambda_i} z_i^2}{2g_{i\lambda_i}^2}. \tag{49}$$

According to Assumption 7, and using Young's Inequality, we obtain

$$\frac{|z_i d_i(\bar{x}_n, t)|}{g_{i\lambda_i}} \le \frac{z_i^2 \rho_i^2(\bar{x}_i)}{2g_{i0}^2} + \frac{p_i^{*2}}{2}. \tag{50}$$

Substituting Eqs. (47) and (50) into Eq. (49), we have

$$\dot{V}_{zi} \le z_i[x_{i+1} + h_i(Z_i)] + \frac{p_i^{*2}}{2} - \frac{\dot{g}_{i\lambda_i} z_i^2}{2g_{i\lambda_i}^2}, \tag{51}$$

where $h_i(Z_i) = -\alpha_i^* + \frac{z_i \rho_i^2(\bar{x}_i)}{2g_{i0}^2}$, $Z_i = [\bar{x}_i, z_i, \dot{\omega}_i] \in R^{i+2}$.

Choose a virtual control α_i as follows:

$$\alpha_i = -k_i z_i - \hat{W}_{hi}^{\mathrm{T}} S_i(\hat{V}_{hi}^{\mathrm{T}} \bar{Z}_i), \tag{52}$$

where k_i is a design constant that we will choose later.

The NN weights $\hat{W}_{hi}$ and $\hat{V}_{hi}$ are determined by Eqs. (34) and (35) with subscript i.

Define ω_{i+1} in such a way that

$$\tau_{i+1}\dot{\omega}_{i+1} + \omega_{i+1} = \alpha_i, \quad \omega_{i+1}(0) = \alpha_i(0), \tag{53}$$

where τ_{i+1} is a design constant that we will choose later.

From Eq. (53), we have $\dot{\omega}_{i+1} = -\frac{y_{i+1}}{\tau_{i+1}}$. Since $x_{i+1} = z_{i+1} + y_{i+1} + \alpha_i = z_{i+1} + y_{i+1} - k_i z_i - \hat{W}_{hi}^{\mathrm{T}} S_i(\hat{V}_{hi}^{\mathrm{T}} \bar{Z}_i)$, using Eq. (51) and Young's Inequality, we obtain

$$\begin{aligned}\dot{V}_{zi} \le{} & (-k_i + 2) z_i^2 + \frac{1}{4} z_{i+1}^2 + \frac{1}{4} y_{i+1}^2 \\ & - z_i \tilde{W}_{hi}^{\mathrm{T}}(\hat{S}_{hi} - \hat{S}_{hi}' \hat{V}_{hi}^{\mathrm{T}} \bar{Z}_i) - z_i \hat{W}_{hi}^{\mathrm{T}} \hat{S}_{hi}' \tilde{V}_{hi}^{\mathrm{T}} \bar{Z}_i + \frac{p_i^{*2}}{2} \\ & + |z_i| \eta_i(\bar{z}_i, \bar{y}_i, \bar{\hat{W}}_{hi}, \bar{\hat{V}}_{hi}, y_d, \dot{y}_d),\end{aligned} \tag{54}$$

where continuous function $\eta_i(\bar{z}_i, \bar{y}_i, \bar{\hat{W}}_{hi}, \bar{\hat{V}}_{hi}, y_d, \dot{y}_d)$ satisfies

$$|\psi_{hi} - \frac{\dot{g}_{i\lambda_i} z_i^2}{2g_{i\lambda_i}^2}| \le \eta_i(\bar{z}_i, \bar{y}_i, \bar{\hat{W}}_{hi}, \bar{\hat{V}}_{hi}, y_d, \dot{y}_d). \tag{55}$$

From Young's Inequality, we have $|z_i|\eta_i \le z_i^2 + \frac{1}{4}\eta_i^2$. Therefore, we obtain

$$\begin{aligned}\dot{V}_{zi} \le{} & (-k_i + 3) z_i^2 + \frac{1}{4} z_{i+1}^2 + \frac{1}{4} y_{i+1}^2 \\ & - z_i \tilde{W}_{hi}^{\mathrm{T}}(\hat{S}_{hi} - \hat{S}_{hi}' \hat{V}_{hi}^{\mathrm{T}} \bar{Z}_i) - z_i \mathrm{tr}\{\tilde{V}_{hi}^{\mathrm{T}} \bar{Z}_i \hat{W}_{hi}^{\mathrm{T}} \hat{S}_{hi}'\} \\ & + \frac{p_i^{*2}}{2} + \frac{1}{4}\eta_i^2.\end{aligned} \tag{56}$$

Noting Assumption 7, we have

$$\dot{y}_{i+1} = -\frac{y_{i+1}}{\tau_{i+1}} + \left(\dot{\hat{W}}_{hi}^{\mathrm{T}} \hat{S}_{hi} + \hat{W}_{hi}^{\mathrm{T}} \dot{\hat{S}}_{hi} + k_i \dot{z}_i\right), \tag{57}$$

$$\begin{aligned}&\left|\dot{y}_{i+1} + \frac{y_{i+1}}{\tau_{i+1}}\right| \\ &\quad \le \xi_{i+1}(\bar{z}_{i+1}, \bar{y}_{i+1}, \bar{\hat{W}}_{hi}, \bar{\hat{V}}_{hi}, y_d, \dot{y}_d, \ddot{y}_d),\end{aligned} \tag{58}$$

where $\xi_{i+1}(\bar{z}_{i+1}, \bar{y}_{i+1}, \bar{\hat{W}}_{hi}, \bar{\hat{V}}_{hi}, y_d, \dot{y}_d, \ddot{y}_d)$ is a continuous function.

From Eqs. (57) and (58), we obtain

$$\begin{aligned}y_{i+1}\dot{y}_{i+1} &\le -\frac{y_{i+1}^2}{\tau_{i+1}} + |y_{i+1}|\xi_{i+1} \\ &\le -\frac{y_{i+1}^2}{\tau_{i+1}} + y_{i+1}^2 + \frac{1}{4}\xi_{i+1}^2.\end{aligned} \tag{59}$$

Step $n-1$: The time derivative of z_{n-1} is

$$\begin{aligned}\dot{z}_{n-1} ={} & f_{n-1}(\bar{x}_{n-1}) + g_{n-1}(\bar{x}_{n-1}) x_n \\ & + d_{n-1}(\bar{x}_n, t) - \dot{\omega}_{n-1}.\end{aligned} \tag{60}$$

Define a smooth scalar function as follows:

$$V_{z(n-1)} = \int_0^{z_{n-1}} \sigma g_{n-1}^{-1}(\bar{x}_{n-2}, \sigma + \omega_{n-1}) \mathrm{d}\sigma. \quad (61)$$

By Second Mean Value Theorem for Integrals, $V_{z(n-1)}$ can be rewritten as $V_{z(n-1)} = z_{n-1}^2/2g_{n-1}(\bar{x}_{n-2}, \lambda_{z(n-1)}z_{n-1} + \omega_{n-1})$ with $\lambda_{z(n-1)} \in (0, 1)$. Because $0 < g_{(n-1)0} \leq g_{n-1}(\bar{x}_{n-1})$, it is shown that $V_{z(n-1)}$ is positive definitive with respect to z_{n-1}.

Differentiating $V_{z(n-1)}$ with respect to time t, applying Assumption 7 and Eq. (60), we obtain

$$\begin{aligned}\dot{V}_{z(n-1)} = {} & \frac{z_{n-1}}{g_{n-1}(\bar{x}_{n-1})}\dot{z}_{n-1} + \sum_{j=1}^{n-2} d_j(x,t) z_{n-1}^2 \\ & \times \int_0^1 \theta \frac{\partial g_{n-1}^{-1}(\bar{x}_{n-2}, z_{n-1}\theta + \omega_{n-1})}{\partial x_j} \mathrm{d}\theta \\ & + z_{n-1}^2 \int_0^1 \theta \left\{ \sum_{j=1}^{n-2} \frac{\partial g_{n-1}^{-1}(\bar{x}_{n-2}, z_{n-1}\theta + \omega_{n-1})}{\partial x_j} \right. \\ & \left. \times\ f_j(\bar{x}_j, x_{j+1}) \right\} \mathrm{d}\theta + \frac{\dot{\omega}_{n-1} z_{n-1}}{g_{n-1}(\bar{x}_{n-1})} \\ & - \int_0^1 \frac{\dot{\omega}_{n-1} z_{n-1}}{g_{n-1}(\bar{x}_{n-2}, z_{n-1}\theta + \omega_{n-1})} \mathrm{d}\theta. \end{aligned} \quad (62)$$

In view of Assumption 7, and using Young's Inequality, we obtain

$$\begin{aligned} & \left| d_j(x,t) z_{n-1}^2 \int_0^1 \theta \frac{\partial g_{n-1}^{-1}(\bar{x}_{n-2}, z_{n-1}\theta + \omega_{n-1})}{\partial x_j} \mathrm{d}\theta \right| \\ & \leq \frac{z_{n-1}^4 \rho_j^2(\bar{x}_j)}{2} \left[\int_0^1 \theta \frac{\partial g_{n-1}^{-1}(\bar{x}_{n-2}, z_{n-1}\theta + \omega_{n-1})}{\partial x_j} \mathrm{d}\theta \right]^2 \\ & \quad + \frac{p_j^{*2}}{2}. \end{aligned} \quad (63)$$

Substituting Eqs. (60) and (63) into Eq. (62) yields

$$\dot{V}_{z(n-1)} \leq z_{n-1}[x_n + h_{n-1}(Z_{n-1})] + \sum_{j=1}^{n-1} \frac{p_j^{*2}}{2}, \quad (64)$$

where

$$\begin{aligned} h_k(Z_k) = {} & \frac{f_k(\bar{x}_k)}{g_k(\bar{x}_k)} + z_k^3 \sum_{j=1}^{k-1} \frac{\rho_j^2(\bar{x}_j)}{2} \\ & \times \left[\int_0^1 \theta \frac{\partial g_k^{-1}(\bar{x}_{k-1}, z_k\theta + \omega_k)}{\partial x_j} \mathrm{d}\theta \right]^2 \\ & + z_i \int_0^1 \theta \left\{ \sum_{j=1}^{k-1} \frac{\partial g_k^{-1}(\bar{x}_{k-1}, z_k\theta + \omega_k)}{\partial x_j} f_j(\bar{x}_j, x_{j+1}) \right\} \mathrm{d}\theta \\ & + \frac{z_k \rho_k^2(\bar{x}_k)}{2g_k^2(\bar{x}_k)} - \int_0^1 \frac{\dot{\omega}_k}{g_k(\bar{x}_{k-1}, z_k\theta + \omega_k)} \mathrm{d}\sigma, \end{aligned} \quad (65)$$

$$Z_k = [\bar{x}_k^{\mathrm{T}}, z_k, \dot{\omega}_k]^{\mathrm{T}} \in R^{k+2} \quad (66)$$

with $k = n - 1$.

Choose a virtual control α_{n-1} as follows:

$$\alpha_{n-1} = -k_{n-1}z_{n-1} - \hat{W}_{h(n-1)}^{\mathrm{T}} S_{n-1}(\hat{V}_{h(n-1)}^{\mathrm{T}} \bar{Z}_{n-1}), \quad (67)$$

where $k_{n-1} > 0$ is a design constant to be determined later, $\hat{W}_{hn}$ and $\hat{V}_{hn}$ are the estimates of W_{hn}^* and $\hat{V}_{hn}^*$, respectively, which are determined by Eqs. (34) and (35) with $i = n - 1$.

Define ω_n in such a way that

$$\tau_n \dot{\omega}_n + \omega_n = \alpha_{n-1}, \quad \omega_n(0) = \alpha_{n-1}(0), \quad (68)$$

where τ_n is a design constant to be chosen later.

Noting that $y_n = \omega_n - \alpha_{n-1}$,we know that $\dot{\omega}_n = -\frac{y_n}{\tau_n}$. Since $z_n = x_n - \omega_n$, it follows that

$$x_n = z_n + y_n - k_{n-1}z_{n-1} - \hat{W}_{h(n-1)}^{\mathrm{T}} S_{n-1}(\hat{V}_{h(n-1)}^{\mathrm{T}} \bar{Z}_{n-1}). \quad (69)$$

Substituting Eqs. (14) and (69) into (64), using Young's Inequality, and by induction for some continuous function $\eta_{n-1}(\bar{z}_{n-1}, \bar{y}_{n-1}, \bar{\hat{W}}_{h(n-1)}, \bar{\hat{V}}_{h(n-1)}, y_d, \dot{y}_d)$, we obtain

$$\begin{aligned} \dot{V}_{z(n-1)} \leq {} & (-k_{n-1} + 2)z_{n-1}^2 + \frac{1}{4}(z_n^2 + y_n^2) \\ & - z_{n-1} \tilde{W}_{h(n-1)}^{\mathrm{T}} (\hat{S}_{h(n-1)} - \hat{S}'_{h(n-1)} \hat{V}_{h(n-1)}^{\mathrm{T}} \bar{Z}_{n-1}) \\ & - z_{n-1} \hat{W}_{h(n-1)}^{\mathrm{T}} \hat{S}'_{h(n-1)} \tilde{V}_{h(n-1)}^{\mathrm{T}} \bar{Z}_{n-1} + \sum_{j=1}^{n-1} \frac{p_j^{*2}}{2} \\ & + |z_{n-1}| \eta_{n-1}(\bar{z}_{n-1}, \bar{y}_{n-1}, \bar{\hat{W}}_{h1}, \bar{\hat{V}}_{h(n-1)}, y_d, \dot{y}_d), \end{aligned} \quad (70)$$

where continuous function η_{n-1} satisfies $|\psi_{h(n-1)}| \leq \eta_{n-1}$. From Young's Inequality, we have $|z_{n-1}|\eta_{n-1} \leq z_{n-1}^2 + \frac{1}{4}\eta_{n-1}^2$. Therefore,

$$\begin{aligned} \dot{V}_{z(n-1)} \leq {} & (-k_{n-1} + 3)z_{n-1}^2 + \frac{1}{4}(z_n^2 + y_n^2) - z_{n-1} \\ & \times \tilde{W}_{h(n-1)}^{\mathrm{T}} (\hat{S}_{h(n-1)} - \hat{S}'_{h(n-1)} \hat{V}_{h(n-1)}^{\mathrm{T}} \bar{Z}_{n-1}) - z_{n-1} \mathrm{tr} \\ & \times \{ \tilde{V}_{h(n-1)}^{\mathrm{T}} \bar{Z}_{n-1} \hat{W}_{h(n-1)}^{\mathrm{T}} \hat{S}'_{h(n-1)} \} + \sum_{j=1}^{n-1} \frac{p_j^{*2}}{2} + \frac{1}{4}\eta_{n-1}^2. \end{aligned} \quad (71)$$

Noting Assumption 7, we obtain

$$\begin{aligned} \dot{y}_n = {} & -\frac{y_n}{\tau_n} + \left(\dot{\hat{W}}_{h(n-1)}^{\mathrm{T}} \hat{S}_{h(n-1)} \right. \\ & \left. +\ \hat{W}_{h(n-1)}^{\mathrm{T}} \dot{\hat{S}}_{h(n-1)} + k_{n-1}\dot{z}_{n-1} \right). \end{aligned} \quad (72)$$

In view of Eq. (72) and by induction for some continuous function ξ_n, we have

$$\left| \dot{y}_n + \frac{y_n}{\tau_n} \right| \leq \xi_n(\bar{z}_n, \bar{y}_n, \bar{\hat{W}}_{h(n-1)}, \bar{\hat{V}}_{h(n-1)}, y_d, \dot{y}_d, \ddot{y}_d). \quad (73)$$

From Eqs. (72) and (73), we obtain

$$y_n \dot{y}_n \leq -\frac{y_n^2}{\tau_n} + |y_n|\xi_n \leq -\frac{y_n^2}{\tau_n} + y_n^2 + \frac{1}{4}\xi_n^2. \quad (74)$$

Step n: The final control law will be constructed in this step. Since $z_n = x_n - \omega_n$, the time derivative of z_n is

$$\dot{z}_n = f_n(\bar{x}_n) + g_n(\bar{x}_n)u + d_n(x,t) - \dot{\omega}_n. \tag{75}$$

Similarly, define a smooth scalar function as follows:

$$V_{zn} = \int_0^{z_n} \sigma g_n^{-1}(\bar{x}_{n-1}, \sigma + \omega_n)\mathrm{d}\sigma. \tag{76}$$

By Second Mean Value Theorem for Integrals, V_{zn} can be rewritten as $V_{zn} = z_n^2/2g_n(\bar{x}_{n-1}, \lambda_{zn}z_n + \omega_n)$ with $\lambda_{zn} \in (0,1)$. Because $0 < g_{n0} \le g_n(\bar{x}_n)$, it is shown that V_{zn} is positive definitive with respect to z_n.

Similar to the discussion at the $(n-1)$th step, we have

$$\dot{V}_{zn} \le z_n[K^{\mathrm{T}}(t)\Phi(t)v + d(v) + h_n(Z_n)] + \sum_{j=1}^{n} \frac{p_j^{*2}}{2}, \tag{77}$$

where $h_n(Z_n)$ and Z_n are determined by Eqs. (65) and (66) with $k = n$.

Choose the control law v as follows:

$$v = -\frac{1}{\beta_0}[k_n|z_n| + |\hat{W}_{hn}^{\mathrm{T}} S_n(\hat{V}_{hn}^{\mathrm{T}} \bar{Z}_n)|]\mathrm{sgn}(z_n), \tag{78}$$

where $k_n > 0$ is the design constant, $\hat{W}_{hn}$ and $\hat{V}_{hn}$ are the estimates of W_{hn}^* and $\hat{V}_{hn}^*$, respectively, which are determined by Eqs. (34) and (35) with $i = n$.
Since $K^{\mathrm{T}}(t)\Phi(t) \ge \beta_0$, we obtain $z_n K^{\mathrm{T}}(t)\Phi(t)v \le -k_n z_n^2 + |\hat{W}_{hn}^{\mathrm{T}} S_n(\hat{V}_{hn}^{\mathrm{T}} \bar{Z}_n)z_n|$. As shown in Eq. (70) at the $(n-1)$th step, we have

$$\dot{V}_{zn} \le (-k_n + 1)z_n^2 - z_n \tilde{W}_{hn}^{\mathrm{T}}(\hat{S}_{hn} - \hat{S}'_{hn}\hat{V}_{hn}^{\mathrm{T}}\bar{Z}_n) - z_n \mathrm{tr}\{\tilde{V}_{hn}^{\mathrm{T}} \bar{Z}_n \hat{W}_{hn}^{\mathrm{T}} \hat{S}'_{hn}\} + \sum_{j=1}^{n}\frac{p_j^{*2}}{2} + \frac{1}{4}\eta_n^2, \tag{79}$$

where continuous function $\eta_n(\bar{z}_n, \bar{y}_n, \bar{\hat{W}}_{hn}, \bar{\hat{V}}_{hn}, y_d, \dot{y}_d)$ satisfies $|d(v) + \psi_{hn}| \le \eta_n$.

Define a compact set as follows:

$$\Omega_i = \{[\bar{z}_i^{\mathrm{T}}, \bar{y}_i^{\mathrm{T}}, \bar{\hat{W}}_{hi}^{\mathrm{T}}, \bar{\hat{V}}_{hi}^{\mathrm{T}}]^{\mathrm{T}} : V_i \le p\} \subset R^{p_i}, \tag{80}$$

where p is a positive constant specified by the designer, $p_i = 2i - 1 + \sum_{j=1}^{i}(j+4)l_i$, $i = 1, \ldots, n$, and

$$V_1 = V_{z1} + \frac{1}{2\gamma_{wh1}}\tilde{W}_{h1}^{\mathrm{T}}\tilde{W}_{h1} + \frac{1}{2\gamma_{vh1}}\mathrm{tr}(\tilde{V}_{h1}^{\mathrm{T}}\tilde{V}_{h1}), \tag{81}$$

$$V_i = \sum_{j=1}^{i}\left[V_{zj} + \frac{1}{2\gamma_{whj}}\tilde{W}_{hj}^{\mathrm{T}}\tilde{W}_{hj} + \frac{1}{2\gamma_{vhj}}\mathrm{tr}(\tilde{V}_{hj}^{\mathrm{T}}\tilde{V}_{hj})\right] + \frac{1}{2}\sum_{j=2}^{i} y_j^2, \quad i = 2, \ldots, n. \tag{82}$$

It is easy to see that $\Omega_1 \times R^{p_n - p_1} \supset \Omega_2 \times R^{p_n - p_2} \supset \cdots \supset \Omega_{n-1} \times R^{p_n - p_{n-1}} \supset \Omega_n$. Since $\Omega_d \times \Omega_i$ and $\Omega_d \times \Omega_{i+1}$ are two compact sets in R^{p_i} and $R^{p_{i+1}}$, respectively, η_i has a maximum H_i and on $\Omega_d \times \Omega_i$, and ξ_{i+1} has a maximum M_{i+1} and on $\Omega_d \times \Omega_{i+1}$.

Theorem 1. *Consider the closed-loop system consisting of the plant (1) under Assumptions 1–7, the controller (78), and adaptation laws (34) and (35). For bounded initial conditions, there exist constants $p > 0$, $k_i > 0$, $\tau_i > 0$, $\gamma_{whi} > 0$, $\gamma_{vhi} > 0$, $\sigma_{whi} > 0$, and $\sigma_{vhi} > 0$, satisfying $V_n \le p$, such that the overall closed-loop control system is semi-globally stable in the sense that all of the signals in the closed-loop system are bounded, and the tracking error is smaller than a prescribed error bound, and k_i and τ_i satisfy*

$$\begin{cases} k_i \ge 3\frac{1}{4} + \frac{\alpha_0}{2g_{i0}}, \quad i = 1, \ldots, n \\ \frac{1}{\tau_i} \ge 1\frac{1}{4} + \alpha_0, \quad i = 2, \ldots, n \\ \alpha_0 \ge \min\{\gamma_{wh1}\sigma_{wh1}, \ldots, \gamma_{whn}\sigma_{whn}, \gamma_{vh1}\sigma_{vh1}, \\ \quad \ldots, \gamma_{vhn}\sigma_{vhn}\}. \end{cases} \tag{83}$$

Proof. Consider the Lyapunov function candidate $V = V_n$. Differentiating V with respect to time t leads to

$$\dot{V} = \sum_{i=1}^{n}\left[\dot{V}_{zi} + \frac{1}{\gamma_{whi}}\tilde{W}_{hi}^{\mathrm{T}}\dot{\hat{W}}_{hi} + \frac{1}{\gamma_{vhi}}\mathrm{tr}\{\tilde{V}_{hi}^{\mathrm{T}}\dot{\hat{V}}_{hi}\}\right] + \sum_{i=1}^{n-1}\left[y_{i+1}\dot{y}_{i+1}\right]. \tag{84}$$

Substituting Eqs. (39), (42), (56), (59), (71), (74) and (79) into Eq. (84), and applying Eqs. (34) and (35), it follows that:

$$\begin{aligned}\dot{V} \le\ & (-k_1+3)z_1^2 + \sum_{i=2}^{n-1}\left[\left(-k_i + 3\frac{1}{4}\right)z_i^2\right] + (-k_n+1)z_n^2 \\ & + \sum_{i=1}^{n}\left[-\sigma_{whi}\tilde{W}_{hi}^{\mathrm{T}}\hat{W}_{hi} - \sigma_{vhi}\mathrm{tr}\{\tilde{V}_{hi}^{\mathrm{T}}\hat{V}_{hi}\}\right] \\ & - \sum_{i=1}^{n}\left[I_{whi}\frac{\tilde{W}_{hi}^{\mathrm{T}}\hat{W}_{hi}\hat{W}_{hi}^{\mathrm{T}}}{\|\hat{W}_{hi}\|^2}(\hat{S}_{hi} - \hat{S}'_{hi}\hat{V}_{hi}^{\mathrm{T}}\bar{Z}_i)z_i\right] \\ & + \sum_{i=1}^{n-1}\left[-\frac{y_{i+1}^2}{\tau_{i+1}} + 1\frac{1}{4}y_{i+1}^2 + \frac{1}{4}\xi_{i+1}^2\right] \\ & + \sum_{i=1}^{n}\left[\frac{p_i^{*2}}{2} + \frac{1}{4}\eta_i^2\right] + \sum_{j=1}^{n-2}\frac{p_j^{*2}}{2} + \sum_{j=1}^{n-1}\frac{p_j^{*2}}{2},\end{aligned} \tag{85}$$

where $I_{whi} = 0$ (or 1), if the first (second) condition is true in Eq. (34). If $I_{whi} = 1$, then $\|\hat{W}_{hi}\| = M_{hi}$ and $\hat{W}_{hi}^{\mathrm{T}}(\hat{S}_{hi} - \hat{S}'_{hi}\hat{V}_{hi}^{\mathrm{T}}\bar{Z}_i)z_i > 0$. Therefore, $\tilde{W}_{hi}^{\mathrm{T}}\hat{W}_{hi} = \frac{1}{2}[\|\tilde{W}_{hi}\|^2 + \|\hat{W}_{hi}\|^2 - \|W_{whi}^*\|] \ge 0$. Furthermore, we have

$$I_{whi}\frac{\tilde{W}_{hi}^{\mathrm{T}}\hat{W}_{hi}\hat{W}_{hi}^{\mathrm{T}}}{\|\hat{W}_{hi}\|^2}(\hat{S}_{hi} - \hat{S}'_{hi}\hat{V}_{hi}^{\mathrm{T}}\bar{Z}_i)z_i \ge 0. \tag{86}$$

By completion of squares, the following inequalities hold:

$$-\sigma_{whi}\tilde{W}_{hi}^{\mathrm{T}}\hat{W}_{hi} \le -\frac{\sigma_{whi}\|\tilde{W}_{hi}\|^2}{2} + \frac{\sigma_{whi}\|W_{hi}^*\|^2}{2}, \tag{87}$$

$$-\sigma_{vhi}\mathrm{tr}\{\tilde{V}_{hi}^{\mathrm{T}}\hat{V}_{hi}\} \le -\frac{\sigma_{vhi}\|\tilde{V}_{hi}\|_F^2}{2} + \frac{\sigma_{vhi}\|V_{hi}^*\|_F^2}{2}. \tag{88}$$

Let

$$\mu = \frac{3}{2}\sum_{i=1}^{n-2} p_i^{*2} + p_{n-1}^{*2} + \frac{1}{2}p_n^{*2} + \frac{1}{4}\sum_{i=1}^{n} H_i^2 + \frac{1}{4}\sum_{i=1}^{n-1} M_{i+1}^2 + \sum_{i=1}^{n}\left[\frac{\sigma_{whi}\|W_{hi}^*\|^2}{2} + \frac{\sigma_{vhi}\|V_{hi}^*\|_F^2}{2}\right]. \quad (89)$$

Substituting Eqs. (83) and (86)–(89) into (85), we obtain

$$\dot{V} \le -\alpha_0 V + \mu. \quad (90)$$

If $V = V_n = p$, then $\eta_i^2 \le H_i$ and $\xi_{i+1}^2 \le M_{i+1}^2$. If $V = p$ and $\alpha_0 > \frac{\mu}{p}$, then $\dot{V} \le 0$. It implies that $V(t) \le p, \forall t \ge 0$ for $V(0) \le p$. Multiplying Eq. (90) by $e^{\alpha_0 t}$ yields

$$\frac{d}{dt}(V(t)e^{\alpha_0 t}) \le e^{\alpha_0 t}\mu. \quad (91)$$

Integrating Eq. (91) over $[0, t]$, we have

$$0 \le V(t) \le \frac{\mu}{\alpha_0} + \left[V(0) - \frac{\mu}{\alpha_0}\right]e^{-\alpha_0 t}. \quad (92)$$

Therefore, all signals of the closed-loop system, i.e., z_i, y_i and $\|\hat{V}_{hi}\|_F$ are uniformly ultimately bounded. Furthermore, x_i, α_i and ω_{i+1} are also uniformly ultimately bounded. From Eqs. (83) and (89), we know that for any given constants $B_0, p, M_{hi}, \sigma_{whi}, \sigma_{vhi}$ and l_i, $\frac{\mu}{\alpha_0}$ can be made arbitrarily small by choosing γ_{whi} and γ_{vhi}. This leads to z_1 arbitrarily small. □

Remark 1. The design constants k_i, τ_i and α_0 determined by Eq. (83) are only a sufficient condition. They provide a guideline for the designers. From Eqs. (83) and (89), some suggestions are given for the choice of some key design parameters for any given constants B_0, p, and compact sets Ω_{Z_i} $(i = 1\ldots, n)$, which are used to define the ideal weights W_{hi}^* and V_{hi}^*, and M_{hi}:

(i) increasing $\gamma_{wh1}, \ldots, \gamma_{whn}$ helps to increase α_0, subsequently reduces μ/α_0;
(ii) decreasing $\sigma_{wh1}, \ldots, \sigma_{whn}, \sigma_{vh1}, \ldots, \sigma_{vhn}$ helps to reduce μ, and reduces μ/α_0.

4. Simulation results

To demonstrate the effectiveness of the proposed approach, we consider the following nonlinear system:

$$\begin{cases} \dot{x}_1 = x_1 + x_2 + \dfrac{x_2^3}{5} + d_1, \\ \dot{x}_2 = x_3 + \dfrac{x_3^3}{2} + d_2, \\ \dot{x}_3 = x_1x_2x_3 + (1 + 0.1\sin(0.5x_1x_2x_3))u + \dfrac{u^3}{7} + d_3, \\ y = x_1, \end{cases} \quad (93)$$

where $d_1 = 0.1\sin(x_1x_2x_3t)$, $d_2 = 0.1\cos(x_2x_3t)$, $d_3 = 0.2\cos(0.5x_2t)$. The desired tracking trajectory is taken as $y_d(t) = 0.5[\sin(t) + \sin(0.5t)]$.

The design parameters of the controller are taken as $l_1 = 10, l_2 = 15, l_3 = 20, \gamma_{10} = \gamma_{20} = \gamma_{30} = 3.5, k_1 = 5, k_2 = 15, k_3 = 25, \tau_2 = \tau_3 = 0.01, M_{hi} = 10, i = 1, 2, 3, \sigma_{wh1} = \sigma_{wh2} = \sigma_{wh3} = \sigma_{vh1} = \sigma_{vh2} = \sigma_{vh3} = 0.01, \gamma_{wh1} = \gamma_{wh2} = \gamma_{wh3} = 12.5, \gamma_{vh1} = \gamma_{vh2} = \gamma_{vh3} = 20$. The initial conditions: $x_i(0) = 0$, $\hat{W}_{hi}(0) = 0$, $\hat{V}_{hi}(0) = 0$, $i = 1, 2, 3$. We choose $D(v)$ as follows:

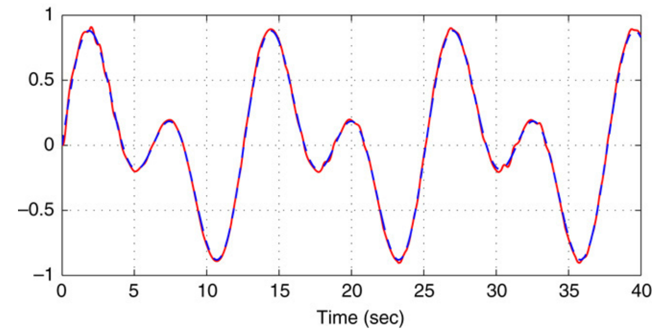

Fig. 1. Output y (solid line) follows desired trajectory y_d (dashed line).

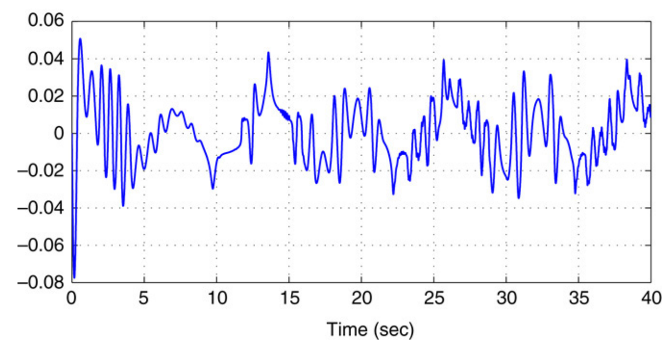

Fig. 2. Tracking error $y - y_d$.

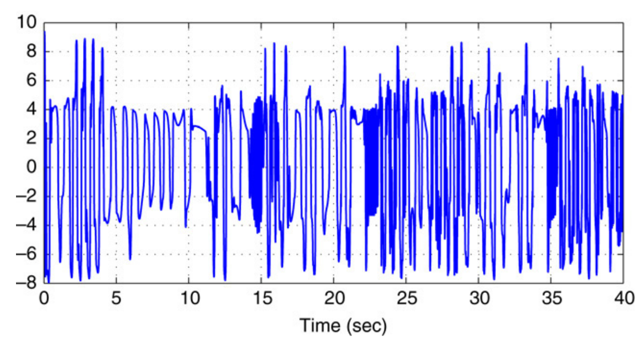

Fig. 3. Control signal v.

$$u = D(v) = \begin{cases} (1 - 0.3\sin(v))(v - 2.5) & \text{if } v \ge 2.5, \\ 0 \quad \text{if } -1.5 < v < 2.5, \\ (0.8 - 0.2\cos(v))(v + 1.5) & \text{if } v \le -1.5. \end{cases} \quad (94)$$

Simulation results are shown in Figs. 1–3. From Figs. 1 and 2, it can be see that fairly good tracking performance is obtained.

5. Conclusions

Novel dynamic surface direct adaptive control has been presented by using multilayer neural networks for a class of pure-feedback nonlinear systems with unknown dead-zone input and perturbed uncertainties. The controller has been made to be free from singularity problem by introducing integral-type Lyapunov function. The developed adaptive control can guarantee that all signals involved are semi-globally uniformly ultimately bounded. Simulation results have been provided to show the effectiveness of the proposed approach.

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