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A direct method for robust adaptive nonlinear control with guaranteed transient performance

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Abstract

In this paper, the adaptive control problem is studied for a class of nonlinear systems in the presence of bounded disturbances. By utilizing a nice property of the studied systems, a novel Lyapunov-based control structure is developed, which avoids the possible control singularity problem in adaptive nonlinear control. The transient bounds of output tracking error are shown to be explicit functions of initial conditions and design parameters, and the control performance of the closed-loop system is guaranteed by suitably choosing the design parameters. Simulation study is provided to verify the theoretical results.

Keywords: Adaptive nonlinear control; Transient performance; Neural networks

1. Introduction

The research of nonlinear control systems using feedback linearization technique has motivated the development of many design approaches for exactly modelled systems [7]. To solve the control problem in the presence of linearly parameterized uncertainties, significant progress has been achieved in adaptive control for a large class of nonlinear systems [21,24,10,13]. For nonlinear systems containing unknown disturbances, several robust adaptive control schemes were developed for different kinds of nonlinear plants and system uncertainties [8,19,14]. As an alternative, an interesting adaptive strategy has been presented for nonlinear time-varying systems based on trajectory approximation technique [18,12,22], in which both unmodeled dynamics and disturbances can be dealt with. The main feature of the scheme is that controller design does not rely on a parameterized model and the exact structure information for the plants.

Recently, adaptive neural network (NN) control has been an active area and many successful design approaches have been developed [2,23,25,3,5]. Most of these neural controllers are based on feedback linearization technique, and the commonly used controller structure is $u = [-\hat{\alpha}(x) + v]/\hat{\beta}(x)$ with $\hat{\alpha}(x)$ and $\hat{\beta}(x)$ the approximators for system's nonlinearities and v the new control input. Therefore, additional precautions have to be made for avoiding the possible controller singularity problem (i.e., $\hat{\beta}(x) \neq 0$) by (i) applying a projection algorithm to project the estimated parameters in a feasible set [23], or (ii) choosing the initial NN weights sufficiently close to the ideal values [2]. In the work [25], a modified adaptive NN controller has been provided by introducing a limiting term to keep the control signal bounded regardless of the NN weight estimates.

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For a general class of non-affine nonlinear systems, stable adaptive state-feedback and output-feedback controllers were presented using Lyapunov stability and neural network parameterization techniques [4,27]. Due to the complexity of the studied systems and the approximators used, the developed schemes there require several restrictive conditions and the analytical results obtained are quite conservative. In this paper, we focus on adaptive control design for a class of unknown nonlinear systems with external disturbances using high-order neural networks (HONNs). The proposed approach ensures that regional stability of the closed-loop system is guaranteed and the controller singularity problem mentioned above is avoided. Two explicit transient bounds of system's tracking error are provided to evaluate the control performance of the proposed scheme.

The paper is organized as follows. Section 2 describes the nonlinear systems under study and the control objective. A desired feedback control (DFC) and HONNs are presented in Section 3. In Section 4, a robust adaptive NN controller is provided, and the system's stability and tracking performance are also investigated. Simulation results are given in Section 5 to show the effectiveness of the developed controller.

2. System description

Consider the SISO nonlinear system in the following form:

$$\begin{aligned} \dot{x}_i &= x_{i+1}, \quad i=1,\ldots,n-1,\\ \dot{x}_n &= \alpha(x)+\beta(x)u+d(t), \\ y &= x_1, \end{aligned} \tag{1}$$

where $x=[x_1,x_2,\ldots,x_n]^{\mathrm{T}} \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$ are state variables, system input and output, respectively; $\alpha(x)$ and $\beta(x)$ are unknown smooth functions; $d(t)$ denotes the external disturbance bounded by a known constant $d_0>0$, i.e., $|d(t)|\leqslant d_0$. In view of the fact that all physical plants operate in bounded regions, we study the adaptive control problem of system (1) whose state x belongs to a compact subset $\Omega\subset\mathbb{R}^n$. The objective is to force the output y follow a desired trajectory y_d.

Assumption 1. The sign of $\beta(x)$ is known and $|\beta(x)|>0$, $\forall x\in\Omega$.

Since the sign of $\beta(x)$ is known, we may, without losing generality, assume that $\beta(x)$ is positive in the following discussion.

Assumption 2. There exists a known smooth function $\bar{\beta}(x)$ such that $|\beta(x)|\leqslant\bar{\beta}(x)$, and $\beta(x)/\bar{\beta}(x)$ is independent of the state x_n for all $x\in\Omega$.

Remark. Let $b(x)=\beta(x)/\bar{\beta}(x)$. It follows from Assumption 2 that $0<b(x)\leqslant 1$, $\forall x\in\Omega$. Taking its time derivative, we have a nice property

$$\dot{b}(x)=\frac{\mathrm{d}[b(x)]}{\mathrm{d}t}=\frac{\partial b(x)}{\partial x}\dot{x}=\sum_{i=1}^{n-1}\frac{\partial b(x)}{\partial x_i}x_{i+1} \tag{2}$$

which only depends on the system's state x. It should be noticed that although Assumption 2 imposes an additional restriction on the class of systems, many physical systems (e.g., pendulum plants [13,26], magnetic levitation systems [11], and single link robots with flexible joints [13]) possess such a property. It will be shown later that this property plays an important role in the design of a singularity-free adaptive controller.

Define vector x_d, e and an augmented error e_s as

$$\begin{aligned} x_d &= [y_d,\dot{y}_d,\ldots,y_d^{(n-1)}]^{\mathrm{T}},\\ e &= x-x_d=[e_1,e_2,\ldots,e_n]^{\mathrm{T}}, \quad e_s=[\Lambda^{\mathrm{T}}\quad 1]e, \end{aligned} \tag{3}$$

where $\Lambda=[\lambda_1,\lambda_2,\ldots,\lambda_{n-1}]^{\mathrm{T}}$ is chosen such that polynomial $s^{n-1}+\lambda_{n-1}s^{n-2}+\cdots+\lambda_1$ is Hurwitz. With these definitions, the tracking error may be expressed as $e_1=H(s)e_s$ with $H(s)$ being a proper stable transfer function, which implies that $e_1(t)\to 0$ as $e_s\to 0$. The time derivative of e_s can be written as

$$\dot{e}_s=\alpha(x)+v_s+b(x)\bar{u}+d(t), \tag{4}$$

where $\bar{u}=\bar{\beta}(x)u$ and $v_s=-y_d^{(n)}+[0\quad\Lambda^{\mathrm{T}}]e$.

Assumption 3. The desired trajectory vector x_d is continuous and known, and $x_d\in\Omega_d$ with Ω_d being a connected subset of Ω.

3. Desired feedback control and function approximation

In this section, supposing that nonlinear functions $\alpha(x)$ and $\beta(x)$ are known exactly, and the system has no disturbance ($d(t)=0$), we present a desired feedback control (DFC), u^*, such that the output y follows the desired trajectory $y_d(t)$ asymptotically.

Lemma 3.1. *Consider system* (1) *with Assumptions* 1–3 *satisfied and* $d(t)=0$. *If the DFC input is chosen* *as* $u^* = \bar{u}^*/\bar{\beta}(x)$ *with*

$$\bar{u}^* = -\frac{1}{b(x)}[\alpha(x) + v_s] - \left[\frac{1}{\varepsilon b(x)} + \frac{1}{\varepsilon b^2(x)} - \frac{\dot{b}(x)}{2b^2(x)}\right] e_s, \quad (5)$$

where $\varepsilon > 0$ *is a design parameter, then* $\lim_{t\to\infty} \|e(t)\| = 0$.

Proof. Substituting the DFC controller $u = u^*$ into (4) and noting $d(t) = 0$, we have

$$\dot{e}_s = -\left[\frac{1}{\varepsilon b(x)} + \frac{1}{\varepsilon b^2(x)}\right] e_s + \frac{\dot{b}(x)}{2b(x)} e_s. \quad (6)$$

Choosing a Lyapunov function candidate $V_d = e_s^2/2b(x)$ and differentiating it along (6), we obtain

$$\dot{V}_d = \frac{1}{b(x)} e_s \dot{e}_s - \frac{\dot{b}(x)}{2b^2(x)} e_s^2 = -\frac{e_s^2}{\varepsilon b(x)} - \frac{e_s^2}{\varepsilon b^2(x)}. \quad (7)$$

Since $0 < b(x) \leqslant 1$, according to the Lyapunov theorem [15], Eq. (7) implies that $\lim_{t\to\infty} |e_s(t)| = 0$. Subsequently, we have $\lim_{t\to\infty} \|e(t)\| = 0$. □

From (7), it is shown that the smaller the parameter ε is, the more negative $\dot{V}_d$ will be. Hence, the convergence rate of the tracking error can be adjusted by tuning the design parameter ε. From (2) and (5), DFC input u^* can be re-written as a function of x, e_s and v_s

$$u^* = \frac{1}{\bar{\beta}(x)} \bar{u}^*(z), \quad z = \left[x^{\mathrm{T}}, e_s, \frac{e_s}{\varepsilon}, v_s\right]^{\mathrm{T}} \in \Omega_z \subset \mathbb{R}^{n+3}, \quad (8)$$

where compact set Ω_z is defined as

$$\Omega_z = \left\{ \left(x, e_s, \frac{e_s}{\varepsilon}, v_s\right) \mid x \in \Omega;\ x_d \in \Omega_d;\ e_s = [\Lambda^{\mathrm{T}} \quad 1]e;\ v_s = -y_d^{(n)} + [0 \quad \Lambda^{\mathrm{T}}]e \right\}. \quad (9)$$

When nonlinear functions $\alpha(x)$ and $\beta(x)$ are unknown, $\bar{u}^*(z)$ is not available. In the following, neural networks shall be applied for approximating the unknown function $\bar{u}^*(z)$. It should be noted that though the elements e_s and e_s/ε in the input vector z are dependent due to constant ε, they are in different scales when a very small ε is chosen. Thus e_s/ε is also fed into NNs as an input for improving the NN approximation accuracy. Consider high-order neural networks (HONNs) [9]

$$g(W, z) = W^{\mathrm{T}} S(z), \quad W \quad \text{and} \quad S(z) \in \mathbb{R}^l,$$

$$S(z) = [s_1(z), s_2(z), \ldots, s_l(z)]^{\mathrm{T}}, \quad (10)$$

$$s_i(z) = \prod_{j \in I_i} [s(z_j)]^{d_j(i)}, \quad i = 1, 2, \ldots, l, \quad (11)$$

where $z = [z_1, z_2, \ldots, z_{n+3}]^{\mathrm{T}} \in \Omega_z \subset \mathbb{R}^{n+3}$; positive integer l denotes the NN node number, $\{I_1, I_2, \ldots, I_l\}$ is a collection of l not-ordered subsets of $\{1, 2, \ldots, n+3\}$ and $d_j(i)$ are non-negative integers, W is an adjustable synaptic weight vector, $s(z_j)$ is chosen as a hyperbolic tangent function

$$s(z_j) = \frac{e^{z_j} - e^{-z_j}}{e^{z_j} + e^{-z_j}}. \quad (12)$$

It has been shown in [17,9] that neural network $W^{\mathrm{T}} S(z)$ satisfies the conditions of the Stone–Weierstrass Theorem and can approximate any continuous function to any desired accuracy over a compact set. Because the nonlinearity $\bar{u}^*(z)$ in DFC input (8) is a continuous function on Ω_z, for an arbitrary constant $\mu_0 > 0$, there exist an integer l^* and an ideal constant weight vector W^*, such that for all $l \geqslant l^*$,

$$\bar{u}^*(z) = W^{*\mathrm{T}} S(z) + \mu_l, \quad \forall z \in \Omega_z, \quad (13)$$

where μ_l is called the NN approximation error satisfying $|\mu_l| \leqslant \mu_0$ and

$$W^* := \arg \min_{W \in \mathbb{R}^l} \left\{ \sup_{z \in \Omega_z} |W^{\mathrm{T}} S(z) - \bar{u}^*(z)| \right\}.$$

In general, the ideal NN weight W^* is unknown and needs to be estimated in controller design. In the next section, an adaptive algorithm and a robust adaptive controller shall be provided to achieve the control objective.

4. Controller design and performance analysis

Let $\hat{W}$ be an estimate of the ideal NN weight W^*. We propose the direct adaptive controller

$$u = \frac{1}{\bar{\beta}(x)} \hat{W}^{\mathrm{T}} S(z) \quad (14)$$

and the robust updating algorithm for NN weights as

$$\dot{\hat{W}} = -\Gamma[S(z)e_s + \sigma \hat{W}], \quad (15)$$

where $\Gamma = \Gamma^{\mathrm{T}} > 0$ is an adaptation gain matrix and $\sigma > 0$ is a constant. In adaptive law (15), σ-modification [6] is used to improve the robustness of the adaptive controller in the presence of the NN approximation error and the external disturbances.

Substituting controller (14) into (4), error equation (4) can be re-written as

$$\dot{e}_s = \alpha(x) + v_s + b(x)\hat{W}^{\mathrm{T}} S(z) + d(t). \tag{16}$$

Adding and subtracting $b(x)\bar{u}^*(z)$ on the right-hand side of (16) and noting (13), we have

$$\dot{e}_s = \alpha(x) + v_s + b(x)[\hat{W}^{\mathrm{T}} S(z) - W^{*\mathrm{T}} S(z) - \mu_l] + b(x)\bar{u}^*(z) + d(t), \quad \forall z \in \Omega_z. \tag{17}$$

Substituting (5) into (17) leads to

$$\dot{e}_s = b(x)\left[\frac{\dot{b}(x)}{2b^2(x)}e_s - \frac{e_s}{\varepsilon b(x)} - \frac{e_s}{\varepsilon b^2(x)} + \tilde{W}^{\mathrm{T}} S(z) - \mu_l\right] + d(t), \quad \forall z \in \Omega_z, \tag{18}$$

where $\tilde{W} = \hat{W} - W^*$. Since NN approximation (13), and Assumptions 1 and 2 are only valid on the compact set Ω, it is necessary to guarantee the system's states remaining in Ω for all time. In the following theorem, we show that for appropriate initial condition $x(0)$ and suitably choosing design parameters, adaptive controller (14)–(15) guarantees $x \in \Omega$, $\forall t \geqslant 0$.

Theorem 4.1 (Stability). *For the closed-loop adaptive system consisting of plant* (1), *controller* (14) *and NN weight adaptive law* (15), *there exist a compact set* $\Omega_0 \subset \Omega$, *and positive constants* l^*, ε^* *and* λ^* *such that if*

(i) *the initial condition* $x(0) \in \Omega_0$, *and*
(ii) *the design parameters are chosen such that* $l \geqslant l^*$, $\varepsilon \leqslant \varepsilon^*$, *and* $\bar{\lambda} < \lambda^*$ *with* $\bar{\lambda}$ *the largest eigenvalue of* Γ^{-1},

then, $\hat{W}(t) \in L_\infty$ *and* $x(t) \in \Omega$, $\forall t \geqslant 0$.

Proof. The proof includes two steps. We shall firstly assume that $x \in \Omega$ holds for all time, and find the upper bounds of system states. Later, for the appropriate initial condition $x(0)$ and controller parameters, we prove that state x indeed remains in the compact set Ω for all $t \geqslant 0$.

Step 1: Suppose that $x \in \Omega$, $\forall t \geqslant 0$, then NN approximation (13), Assumptions 1 and 2 are valid. Consider the Lyapunov function candidate

$$V_s = \frac{1}{2}\left[\frac{e_s^2}{b(x)} + \tilde{W}^{\mathrm{T}}\Gamma^{-1}\tilde{W}\right]. \tag{19}$$

Differentiating (19) along (15) and (18) yields

$$\begin{aligned}\dot{V}_s &= \frac{e_s\dot{e}_s}{b(x)} - \frac{\dot{b}(x)}{2b^2(x)}e_s^2 + \tilde{W}^{\mathrm{T}}\Gamma^{-1}\dot{\hat{W}} \\ &= -\frac{e_s^2}{\varepsilon b(x)} - \frac{e_s^2}{\varepsilon b^2(x)} + \frac{d(t)}{b(x)}e_s - \mu_l e_s - \sigma\tilde{W}^{\mathrm{T}}\hat{W}.\end{aligned} \tag{20}$$

Using the facts that

$$\begin{aligned}2\tilde{W}^{\mathrm{T}}\hat{W} &= \|\tilde{W}\|^2 + \|\hat{W}\|^2 - \|W^*\|^2 \\ &\geqslant \|\tilde{W}\|^2 - \|W^*\|^2,\end{aligned} \tag{21}$$

$$\frac{d(t)}{b(x)}e_s \leqslant \frac{e_s^2}{2\varepsilon b^2(x)} + \frac{\varepsilon}{4}d(t)^2,$$
$$|\mu_l e_s| \leqslant \frac{e_s^2}{\varepsilon b(x)} + \frac{\varepsilon}{2}\mu_l^2 b(x)$$

and noting that $|\mu_l| \leqslant \mu_0$, $|d(t)| \leqslant d_0$ and $0 < b(x) \leqslant 1$, the following inequality holds

$$\dot{V}_s \leqslant -\frac{e_s^2}{2\varepsilon b(x)} - \frac{\sigma}{2}\|\tilde{W}\|^2 + \frac{\varepsilon}{2}\mu_0^2 + \frac{\varepsilon}{4}d_0^2 + \frac{\sigma}{2}\|W^*\|^2. \tag{22}$$

Considering (19) and $\tilde{W}^{\mathrm{T}}\Gamma^{-1}\tilde{W} \leqslant \bar{\lambda}\|\tilde{W}\|^2$ (since $\bar{\lambda}$ is the largest eigenvalue of Γ^{-1}), we obtain

$$\dot{V}_s \leqslant -\frac{1}{\alpha_0}V_s + \frac{\varepsilon}{2}\mu_0^2 + \frac{\varepsilon}{4}d_0^2 + \frac{\sigma}{2}\|W^*\|^2,$$

where $\alpha_0 = \max\{\varepsilon, \bar{\lambda}/\sigma\}$. Solving the above inequality using Lemma B.5 in [10], we have

$$\begin{aligned}V_s(t) &\leqslant \mathrm{e}^{-t/\alpha_0}V_s(0) + \left(\frac{\varepsilon}{2}\mu_0^2 + \frac{\varepsilon}{4}d_0^2 + \frac{\sigma}{2}\|W^*\|^2\right) \times \int_0^t \mathrm{e}^{-(t-\tau)/\alpha_0}\,\mathrm{d}\tau \\ &\leqslant \mathrm{e}^{-t/\alpha_0}V_s(0) + \alpha_0\left(\frac{\varepsilon}{2}\mu_0^2 + \frac{\varepsilon}{4}d_0^2 + \frac{\sigma}{2}\|W^*\|^2\right), \\ &\quad \forall t \geqslant 0.\end{aligned} \tag{23}$$

Since $V_s(0)$ is bounded, inequality (23) shows that e_s and $\hat{W}(t)$ are bounded. By (19) and $0 < b(x) \leqslant 1$, it follows that $|e_s| \leqslant \sqrt{2V_s b(x)} \leqslant \sqrt{2V_s}$. Combining with (23), we obtain

$$|e_s(t)| \leqslant \mathrm{e}^{-t/2\alpha_0}\sqrt{2V_s(0)} + \sqrt{\alpha_0}\left(\varepsilon\mu_0^2 + \frac{\varepsilon}{2}d_0^2 + \sigma\|W^*\|^2\right)^{1/2}, \quad \forall t \geqslant 0. \tag{24}$$

Next, we determine an upper bound of the error vector e. Define $\zeta = [e_1, e_2, \ldots, e_{n-1}]^{\mathrm{T}}$. One state-space representation of mapping $e_s = [\Lambda^{\mathrm{T}} \ \ 1]e$ can be expressed as

$$\dot{\zeta} = A_s\zeta + b_s e_s,$$

where A_s is a stable matrix (since $s^{n-1} + \lambda_{n-1}s^{n-2} + \cdots + \lambda_1$ is Hurwitz) and $b_s = [0, 0, \ldots, 0, 1]^{\mathrm{T}}$. In addition, two constants $k_0 > 0$ and $\lambda_0 > 0$ can be found such that $\|\mathrm{e}^{A_s t}\| \leqslant k_0 \mathrm{e}^{-\lambda_0 t}$ [16]. The solution for ζ is

$$\zeta(t) = \mathrm{e}^{A_s t}\zeta(0) + \int_0^t \mathrm{e}^{A_s(t-\tau)} b_s e_s(\tau)\,\mathrm{d}\tau.$$

It follows that

$$\|\zeta(t)\| \leqslant k_0\|\zeta(0)\|\mathrm{e}^{-\lambda_0 t} + k_0 \int_0^t \mathrm{e}^{-\lambda_0(t-\tau)}|e_s(\tau)|\,\mathrm{d}\tau. \tag{25}$$

Considering (24) and $\mathrm{e}^{-t/2\alpha_0}\sqrt{2V_s(0)} \leqslant \sqrt{2V_s(0)}$, we have

$$\begin{aligned}\|\zeta(t)\| &\leqslant k_0\|\zeta(0)\|\mathrm{e}^{-\lambda_0 t} + k_0 \Bigg[\sqrt{2V_s(0)} \\ &\quad + \sqrt{\alpha_0}\left(\varepsilon\mu_0^2 + \frac{\varepsilon}{2}d_0^2 + \sigma\|W^*\|^2\right)^{1/2}\Bigg] \\ &\quad \times \int_0^t \mathrm{e}^{-\lambda_0(t-\tau)}\,\mathrm{d}\tau \\ &\leqslant k_0\|\zeta(0)\| + \frac{k_0}{\lambda_0}\Bigg[\sqrt{2V_s(0)} + \sqrt{\alpha_0}\Big(\varepsilon\mu_0^2 \\ &\quad + \frac{\varepsilon}{2}d_0^2 + \sigma\|W^*\|^2\Big)^{1/2}\Bigg], \quad \forall t \geqslant 0. \end{aligned} \tag{26}$$

Noting $e_s = [\Lambda^{\mathrm{T}} \ \ 1]e$ and $e = [\zeta^{\mathrm{T}} \ \ e_n]^{\mathrm{T}}$, we obtain

$$\|e\| \leqslant \|\zeta\| + |e_n| \leqslant (1 + \|\Lambda\|)\|\zeta\| + |e_s|$$

Substituting (24) and (26) into the above inequality leads to

$$\begin{aligned}\|e\| &\leqslant k_\lambda\|\zeta(0)\| + \left(1 + \frac{k_\lambda}{\lambda_0}\right) \\ &\quad \times \left[\frac{e_s^2(0)}{b(x(0))} + \bar{\lambda}\|\tilde{W}(0)\|^2\right]^{1/2} + \sqrt{\alpha_0}\left(1 + \frac{k_\lambda}{\lambda_0}\right) \\ &\quad \times \left(\varepsilon\mu_0^2 + \frac{\varepsilon}{2}d_0^2 + \sigma\|W^*\|^2\right)^{1/2}, \end{aligned} \tag{27}$$

where $k_\lambda = k_0(1 + \|\Lambda\|)$. Since k_λ, α_0, λ_0, and σ are positive constants, and $\zeta(0)$ and $e_s(0)$ depend on $x(0) - x_d(0)$, we conclude that there exists a positive constant $R(\varepsilon, \bar{\lambda}, x(0), \tilde{W}(0))$ depending on ε, $\bar{\lambda}$, $x(0)$ and $\tilde{W}(0)$ such that

$$\|e\| \leqslant R(\varepsilon, \bar{\lambda}, x(0), \tilde{W}(0)), \quad \forall t \geqslant 0 \tag{28}$$

Step 2: In the following, we shall find the conditions such that $x \in \Omega$, $\forall t \geqslant 0$. Firstly, define a compact set

$$\Omega_0 := \{x(0) \,|\, \{x \,|\, \|x - x_d\| < R(0, 0, x(0), 0)\} \subset \Omega, \quad x_d \in \Omega_d\}. \tag{29}$$

It is easy to see that for all $x(0) \in \Omega_0$ and $x_d \in \Omega_d$, we have $x \in \Omega$, $\forall t \geqslant 0$. Then, for the system with $x(0) \in \Omega_0$, bounded $\hat{W}(0)$ and $x_d \in \Omega_d$, the following constants λ^* and ε^* can be determined:

$$\lambda^* := \sup_{\bar{\lambda} \in \mathbb{R}^+} \{\bar{\lambda} | \{x \,|\, \|x - x_d\| < R(0, \bar{\lambda}, x(0), \tilde{W}(0))\} \subset \Omega, \ x_d \in \Omega_d\} \tag{30}$$

and

$$\varepsilon^* := \sup_{\varepsilon \in \mathbb{R}^+} \{\varepsilon \,|\, \{x \,|\, \|x - x_d\| \leqslant R(\varepsilon, \bar{\lambda}, x(0), \tilde{W}(0)), \ \bar{\lambda} < \lambda^*\} \subset \Omega, \ x_d \in \Omega_d\}. \tag{31}$$

Therefore, for the initial condition $x(0) \in \Omega_0$, bounded initial NN weight $\hat{W}(0)$, and the desired signal $x_d \in \Omega_d$, if the controller parameters are chosen such that $l \geqslant l^*$, $\bar{\lambda} < \lambda^*$ and $\varepsilon \leqslant \varepsilon^*$, then system state x indeed stays in Ω for all time. This completes the proof. □

Remarks. 1. The result obtained in Theorem 4.1 is regionally stable because the initial states are required to be within Ω_0. This is reasonable because the neural network approximation is only valid on a compact set and Assumptions 1 and 2 hold on Ω. The stability analysis of Theorem 4.1 also provides a method to estimate the allowed initial state region for a given controller. For controller (14) with parameters ε, l, and Γ satisfying condition (ii) in Theorem 4.1, the following compact set can be found:

$$\bar{\Omega}_0 = \{x(0) \,|\, \|x - x_d\| \leqslant R(\varepsilon, \bar{\lambda}, x(0), \tilde{W}(0))\} \subset \Omega, \quad x_d \in \Omega_d\}.$$

Then, we say that $\bar{\Omega}_0$ is the largest region of initial states for achieving a stable closed-loop system under the given controller.

2. Theorem 4.1 shows that the initial NN weights of the adaptive controller can be set to any bounded values, which relaxes the strong condition required in [2], where the initial values of the NN weights are required to be chosen sufficiently close to the ideal values.

The above discussion only presents the boundedness of the signals in the adaptive system, no transient performance of the system's output is provided. In practical applications, the transient response is often more important.

Theorem 4.2 (Transient performance). *For the closed-loop adaptive system* (1), (14) *and* (15), *if* $x(0) \in \Omega_0$ *and the design parameters* $l \geqslant l^*$, $\bar{\lambda} < \lambda^*$ *and* $\varepsilon \leqslant \varepsilon^*$, *then*

(i) *the transient bound of tracking error*

$$|e_1(t)| \leqslant k_0 \left[\|\zeta(0)\| e^{-\lambda_0 t} + t e^{-\lambda_s t} \sqrt{2V_s(0)} + \frac{\sqrt{\alpha_0}}{\lambda_0} \left(\varepsilon \mu_0^2 + \frac{\varepsilon}{2} d_0^2 + \sigma \|W^*\|^2 \right)^{1/2} \right], \tag{32}$$

where $\lambda_s = \min\{\lambda_0, 1/2\alpha_0\}$, $\alpha_0 = \max\{\varepsilon, \bar{\lambda}/\sigma\}$, k_0 *and* λ_0 *are positive constants defined in Theorem* 4.1, *and positive constant* $V_s(0)$ *depends on* $x(0)$ *and* $\hat{W}(0)$,

(ii) *the* L_∞ *tracking error bound is*

$$\sup_{t \geqslant 0} |e_1(t)| \leqslant k_0 \left[\|\zeta(0)\| + \frac{\sqrt{2V_s(0)}}{\lambda_0} + \frac{\sqrt{\alpha_0}}{\lambda_0} \times \left(\varepsilon \mu_0^2 + \frac{\varepsilon}{2} d_0^2 + \sigma \|W^*\|^2 \right)^{1/2} \right]. \tag{33}$$

Proof. If $x(0) \in \Omega_0$, $l \geqslant l^*$, $\bar{\lambda} < \lambda^*$, and $\varepsilon \leqslant \varepsilon^*$, Theorem 4.1 ensures that $x \in \Omega, \forall t \geqslant 0$, subsequently NN approximation (13), Assumptions 1 and 2 are valid. Therefore, inequalities (20)–(25) hold. Since $\zeta = [e_1, e_2, \ldots, e_{n-1}]^{\mathrm{T}}$, it follows from (24) and (25) that

$$|e_1(t)| \leqslant \|\zeta(t)\| \leqslant k_0 \|\zeta(0)\| e^{-\lambda_0 t} + k_0 \sqrt{2V_s(0)} \times \int_0^t e^{-\lambda_0 (t-\tau)} e^{-\tau/2\alpha_0} \, d\tau + \frac{k_0 \sqrt{\alpha_0}}{\lambda_0} \times \left(\varepsilon \mu_0^2 + \frac{\varepsilon}{2} d_0^2 + \sigma \|W^*\|^2 \right)^{1/2}. \tag{34}$$

For the integral in the above inequality, we have

$$\int_0^t e^{-\lambda_0 (t-\tau)} e^{-\tau/2\alpha_0} \, d\tau = \begin{cases} (\lambda_0 - \frac{1}{2\alpha_0})^{-1} (e^{-\frac{t}{2\alpha_0}} - e^{-\lambda_0 t}) & \text{if } \lambda_0 \neq \dfrac{1}{2\alpha_0}, \\ t e^{-\lambda_0 t} & \text{if } \lambda_0 = \dfrac{1}{2\alpha_0}. \end{cases} \tag{35}$$

By applying Mean Value Theory [1], there exists a constant λ_s' between λ_0 and $1/2\alpha_0$ such that $e^{-\lambda_0 t} = e^{-t/2\alpha_0} - (\lambda_0 - 1/2\alpha_0) t e^{-\lambda_s' t}$. As $\lambda_s = \min\{\lambda_0, 1/2\alpha_0\}$, we have $\lambda_s \leqslant \min\{\lambda_0, \lambda_s'\}$. Combining with (35), it can be shown that

$$\int_0^t e^{-\lambda_0 (t-\tau)} e^{-\tau/2\alpha_0} \, d\tau \leqslant t e^{-\lambda_s t}.$$

Substituting the above inequality into (34), we arrive at (32).

Because $|e_1(t)| \leqslant \|\zeta(t)\|$, the L_∞ tracking error bound (33) can be obtained from (26) directly. □

Remarks. 1. It is shown from (32) and (33) that large initial errors $e(0)$ and $\tilde{W}(0)$ may lead to a large tracking error during the initial period of adaptation. In view of (32) and (33), output tracking error may be reduced by choosing small design parameters ε and $\bar{\lambda}$ (i.e., by increasing the controller gain and adaptation gain Γ).

2. Taking the limits of both sides of (32) and noting the facts that $\lim_{t \to \infty} e^{-\lambda_0 t} = 0$ and $\lim_{t \to \infty} t e^{-\lambda_s t} = 0$, we have

$$\lim_{t \to \infty} |e_1(t)| \leqslant k_0 \frac{\sqrt{\alpha_0}}{\lambda_0} \left(\varepsilon \mu_0^2 + \frac{\varepsilon}{2} d_0^2 + \sigma \|W^*\|^2 \right)^{1/2}. \tag{36}$$

Because $\alpha_0 = \max\{\varepsilon, \bar{\lambda}/\sigma\}$ and $\bar{\lambda}$ is defined as the largest eigenvalue of Γ^{-1}. The above inequality indicates that the tracking error converges to a small residual set which depends on the NN approximation error μ_l, external disturbance $d(t)$ and controller parameters ε, σ and Γ. The decreases in ε and σ or increases in the adaptive gain Γ and NN node number l result in a better tracking performance.

3. It should be pointed out that HONNs used in this paper may be replaced by any other linear approximator such as radial basis function networks [5,20], spline functions [16] or fuzzy systems [23], while the stability and performance properties of the adaptive system are still valid.

5. Simulation study

To verify the effectiveness of the proposed approach, the developed adaptive controller is applied to a pendulum plant with a variable length $l(\theta)$ [26] as shown in Fig. 1. The plant dynamics can be expressed

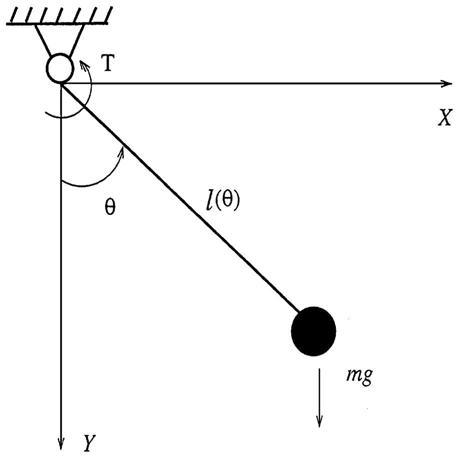


Fig. 1. Variable length pendulum plant with $l(\theta) = l_0 + l_1\cos(\theta)$, $l_1/l_0 = 0.5$, $g/l_0 = 10$ and $ml_0^2 = 1$.

in the form of system (1) with

$$\alpha(x)= \frac{0.5\sin x_1(1+0.5\cos x_1)x_2^2 - 10\sin x_1(1+\cos x_1)}{0.25(2+\cos x_1)^2},$$

$$\beta(x) = \frac{1}{0.25(2+\cos x_1)^2}, \qquad d(t) = d_1(t)\cos x_1,$$

where $x=[x_1,x_2]^T=[\theta,\dot{\theta}]^T$, $u=T$ and $d_1(t)=\cos(3t)$. The initial states $[x_1(0),x_2(0)]^T = [0,0]^T$, and the reference signal $y_d=(\pi/6)\sin(t)$. The operation range of the system is chosen as

$$\Omega = \left\{(x_1,x_2)\Big| |x_1| \leqslant \frac{\pi}{2},\ |x_2| \leqslant 4\pi\right\}.$$

It can be checked that Assumptions 1 and 2 are satisfied and $\frac{4}{9} \leqslant \beta(x) \leqslant 1$, $\forall x \in \Omega$.

The adaptive controller (14)–(15) used in this example is described as: $\bar{\beta}(x) = 1$; $\Lambda = 10.0$; the input vector is $z = [x_1,x_2,e_s,e_s/\varepsilon,v_s]^T$; a two-order neural network is selected with the elements $s_i(z)$ in (10) choosing as $s(z_1)$, $s(z_2)$, $s(z_3)$, $s(z_4)$, $s(z_5)$, and the possible combinations of them (i.e., $s(z_i)s(z_j)$ with $i,j = 1,2,3,4,5$). The total number of the NN nodes is 20. The parameter σ in adaptive law (15) is taken as $\sigma = 0.005$ and the initial NN weight $\hat{W}(0)=0.0$. In order to find the transient bounds of x_1 and x_2, we suppose that the designed HONNs satisfies $\|W^*\| \leqslant 6.0$ and $\mu_0 \leqslant 0.5$ on compact set Ω_z.

Next, constants λ^* and ε^* (defined in (30) and (31), respectively) shall be specified such that if $\bar{\lambda} < \lambda^*$ and $\varepsilon \leqslant \varepsilon^*$, the requirement $(x_1,x_2) \in \Omega$ holds for all time. Because the plant is of second order, we have $\zeta = x_1 - y_d$. It follows from $\Lambda = 10.0$ that k_0 and λ_0 in (26) are $k_0 = 1.0$ and $\lambda_0 = 10.0$. Then, inequality (26) shows that

$$|x_1| \leqslant |\zeta| + |y_d| \leqslant |\zeta(0)| + \frac{\sqrt{2V_s(0)}}{\lambda_0} + \frac{\sqrt{\alpha_0}}{\lambda_0}\left(\varepsilon\mu_0^2 + \frac{\varepsilon}{2}d_0^2 + \sigma\|W^*\|^2\right)^{1/2} + |y_d|. \tag{37}$$

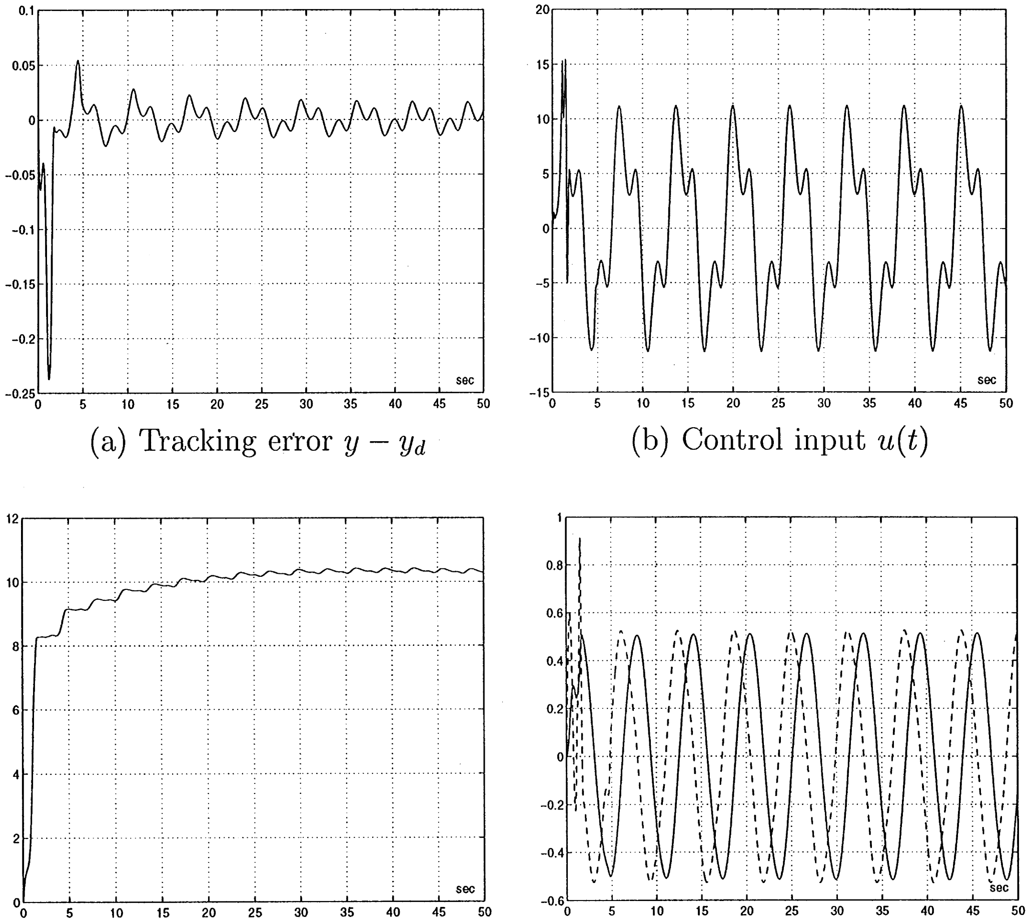
Noticing $[x_1(0),x_2(0)]^T=[0,0]^T$ and $x_d(0)=[0,\pi/6]^T$, we have $\zeta(0)=0$, $e_s(0)=\pi/6$ and $|y_d| \leqslant \pi/6$. Through a suitable calculation using (37), we may find $\lambda^* = 0.505$ and $\varepsilon^* = 0.2571$ such that for $\varepsilon < 0.2571$ and $\bar{\lambda} < 0.505$ (i.e., all of the eigenvalues of the adaptive gain matrix Γ^{-1} are chosen smaller than 0.505), the state $|x_1| \leqslant \pi/2$, $\forall t \geqslant 0$. Considering $x_2 = e_s - \zeta + \dot{y}_d$ and inequality (24), we obtain

$$|x_2| \leqslant |e_s| + |\zeta| + |\dot{y}_d| \leqslant \left(1 + \frac{1}{\lambda_0}\right)\left[\sqrt{2V_s(0)} + \sqrt{\alpha_0}\left(\varepsilon\mu_0^2 + \frac{\varepsilon}{2}d_0^2 + \sigma\|W^*\|^2\right)^{1/2}\right]. \tag{38}$$

For the choice of $\bar{\lambda} < 0.505$ and $\varepsilon \leqslant 0.2571$, the above inequality shows $|x_2| < 11.5192 < 4\pi$, $\forall t \geqslant 0$. Therefore, the state (x_1,x_2) do remain in the required set Ω for all time.

Fig. 2 presents the simulation results for the designed NN controller with $\Gamma = \text{diag}\{2.0\}$ and $\varepsilon = 0.25$. Although no exact model of the pendulum plant is available and the initial NN weights are set to zero, through the NN learning phase, it is shown that the tracking error given in Fig. 2(a) converges to a small region after 15 s. The boundedness of the NN weight estimates, control input and system states are shown in Figs. 2(b)–(d), respectively.

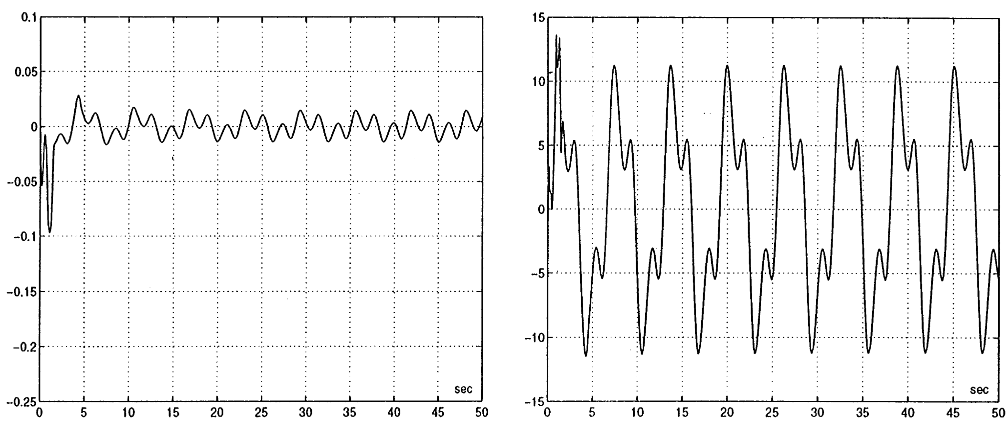
To study the control performance for different design parameters, the following two cases have been investigated. Firstly, we increase the adaptation gain from $\Gamma = \text{diag}\{2.0\}$ to diag{6.0} with all other parameters fixed. The simulation results are given in Fig. 3. Comparing Fig. 3(a) with Fig. 2(a), it can be seen that smaller tracking error in Fig. 3(a) is obtained during the initial period (from 0 to 10 s). This confirms that fast adaptation may improve the transient performance of the adaptive system. Secondly, we reduce the control parameter ε from 0.25 to 0.05 and keep other simulation parameters be the same as in the first



(a) Tracking error $y - y_d$ (b) Control input $u(t)$

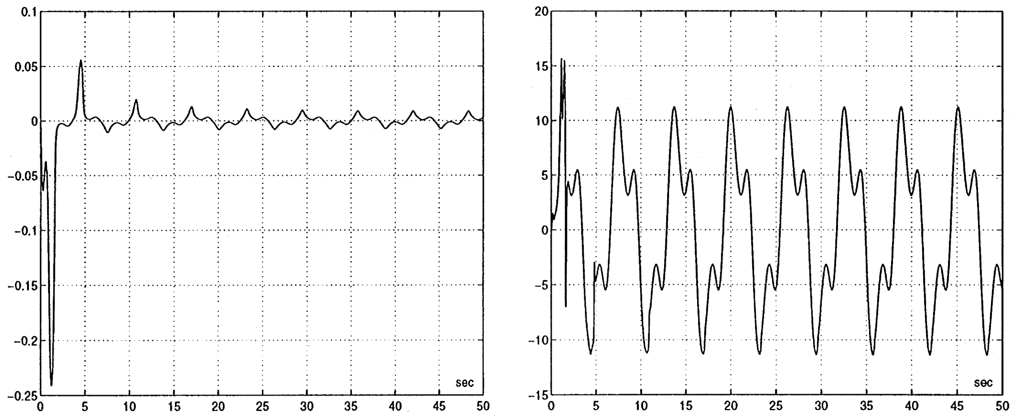
(c) HONN weight $\|\hat{W}\|$ (d) States x_1("—") and x_2("- -")

Fig. 2. Responses of the robust adaptive NN controller ($\varepsilon = 0.25$ and $\Gamma = \text{diag}\{2.0\}$).



(a) Tracking error $y - y_d$ (b) Control input $u(t)$

Fig. 3. Responses of the robust adaptive NN controller ($\varepsilon = 0.25$ and $\Gamma = \text{diag}\{6.0\}$).



(a) Tracking error $y - y_d$ (b) Control input $u(t)$

Fig. 4. Responses of the robust adaptive NN controller ($\varepsilon = 0.05$ and $\Gamma = \text{diag}\{2.0\}$).

simulation. Fig. 4 provides the simulation results. Comparing the tracking errors in Figs. 4(a), 2(a) and 3(a), we find that smaller final tracking error is achieved for smaller ε.

6. Conclusion

In this paper, for a class of nonlinear systems containing bounded disturbances, a robust adaptive control scheme has been presented using neural networks. It has been shown that for appropriately chosen controller parameters, stability of the closed-loop adaptive system can be guaranteed. In addition, explicit transient bounds for output tracking error have been provided, and the effectiveness of the proposed scheme is illustrated through the application to the pendulum plant control.

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