CONTINUA IRREDUCIBLE ABOUT n POINTS

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Abstract. The purpose of this paper is to provide new characterizations of continua that are irreducible about n points. One of the characterizations is that a continuum is irreducible about n points if and only if every pair-wise disjoint collection of non-separating open sets has at most n elements.

1. Introduction

The following theorem is proved in [2] in response to a question of J. B. Fugate (Problem 113 of the Houston Problem Book [1]).

Theorem 1. *Suppose M is a continuum. The following are equivalent.*

(1) *The continuum M is irreducible about a finite set of points.*
(2) *Every pair-wise disjoint collection of non-separating open subsets of M is finite.*
(3) *The continuum M is not the union of a countable monotonic collection of its proper subcontinua.*
(4) *The continuum M does not have infinitely many weakly non-separating subcontinua each of which has an interior point that fails to lie in the closure of the union of the others.*

Theorem 1 gives necessary and sufficient conditions for a continuum to be irreducible about a finite set of points without reference to the exact number of points in the set. The purpose of this paper is to consider modifications of the conditions in Theorem 1 to characterize continua that are irreducible about n points. Condition (2) may be reworded as follows: Every pair-wise disjoint collection of non-separating open subsets of M has at most n elements. That this condition is both necessary and sufficient for M to be irreducible about n points is proved in Section 3. Condition (3) does not seem to have any natural modification that characterizes continua irreducible about n points. Condition (4) may be modified

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to characterize such continua, but the most natural rewording - every collection of weakly non-separating subcontinua of M each of which has an interior point that fails to lie in the union of the others has at most n elements - fails. Section 2 discusses modifications of (4) that fail to characterize continua irreducible about n points, but motivate a condition which, in Section 3, is shown to succeed in doing so.

A *continuum* is a compact connected subset of a metric space. A continuum is said to be *irreducible* about a closed set H if and only if it contains H but has no proper subcontinuum that contains H.

A continuum M is said to be an n-*od* if and only if there is a subcontinuum K of M such that $M - K$ has at least n components, and M is said to be an ∞-*od* if and only if there is a subcontinuum K of M such that $M - K$ has infinitely many components.

A *non-separating* subset of a continuum M is a nonempty subset of M whose complement is connected. A *weakly non-separating* subset of M is a subset of M that contains a nonempty non-separating open subset of M.

2. Some Examples

Consider the following condition which was suggested in the previous section.

A: Every collection of weakly non-separating subcontinua of M each of which has an interior point that fails to lie in the union of the others has at most n elements.

Consider the simple triod T in the plane that is the union of the line segment whose endpoints are $(-1, 0)$ and $(1, 0)$ with the line segment whose endpoints are $(0, 0)$ and $(0, 1)$. Denote by A_1, A_2, A_3, and A_4 the subcontinua of T that are irreducible about $(-1, 0)$ and $(0, \frac{1}{3})$, $(-1, 0)$ and $(\frac{1}{3}, 0)$, $(1, 0)$ and $(\frac{2}{3}, 0)$, and $(0, 1)$ and $(0, \frac{2}{3})$. Note that T is irreducible about three points, and that $\{A_1, A_2, A_3, A_4\}$ is a collection of weakly non-separating subcontinua of T each of which has an interior point that fails to lie in the union of the others. Consequently, (A) is not a necessary condition for a continuum that is irreducible about n points. Since (A) implies condition (3) in Theorem 2, it is sufficient.

Note that A_1 and A_2 both contain the endpoint $(-1, 0)$, but that, for each of them, the largest non-separating open subset is the interior of the subcontinuum irreducible from $(-1, 0)$ to $(0, 0)$. It is therefore reasonable to wonder if the following modification of condition (A) characterizes continua irreducible about n points.

B: Every collection of subcontinua of M each term of which fails to lie in the union of the others and has interior that is both dense in the subcontinuum and non-separating in M has at most n terms.

The simple triod of the previous example satisfies (B) with $n = 3$; however, while (A) is sufficient but not necessary for a continuum to be irreducible about n points, (B) is necessary but not sufficient. That it is necessary follows from Theorem 2; that it is insufficient, from the following example. Let M_1 be the line segment whose endpoints are $(-1, 0)$ and $(0, 0)$, M_2 be the standard Brouwer-Janiszewski-Knaster continuum constructed from the middle-thirds Cantor set in $[0, 1]$, M_3 be the line segment whose endpoints are $(1, 0)$ and $(2, 0)$, and M be $M_1 \cup M_2 \cup M_3$. Note that each of M_1 and M_3 has dense interior that is non-separating in M, but that the interior of M_2 separates M. Indeed, (B) holds for this continuum with $n = 2$, but M fails to be irreducible about 2 points.

Although the interior of M_2 separates M, it is worthwhile to note that M_2 has a dense subset that is open and non-separating in M. The set $M - A$ where A denotes the subcontinuum of M irreducible between the points $(-1, 0)$ and $(2, 0)$ is such a set. Each of M_1 and M_3 also has a dense subset that is open and non-separating in M. This suggests the following condition.

C: Every collection of subcontinua of M each term of which both fails to lie in the union of the others and has a dense subset that is open and non-separating in M has at most n terms.

Like (B), condition (C) is necessary but not sufficient for a continuum to be irreducible about n points. The necessity of (C) follows from Theorem 2. To see that it is insufficient consider the following example. Let C denote the standard Cantor set in $[0, 1]$, and let M be $(C \times [0, 1]) \cup ([0, 1] \times \{0\})$. Then M satisfies (C) with $n = 2$, but M fails to be irreducible about two points. However, when (C) is coupled with the condition that M is not an ∞-od, the result characterizes continua irreducible about n points.

3. Continua Irreducible About n Points

Sorgenfrey [3] proved in 1944 that a unicoherent continuum is irreducible about n points if and only if it fails to be an $n+1$-od. Simple examples exist to show that this result does not remain true if "unicoherent continuum" is replaced by "continuum." Condition (2) of the following theorem seems to be a very natural generalization of

Sorgenfrey's criterion for unicoherent continua to the class of all continua. In 1946, Sorgenfrey [4] published another result that is included in Theorem 2 for the sake of completeness.

Theorem 2. *Suppose M is a continuum, and n is a positive integer not less than two. The following are equivalent.*

(1) *The continuum M is irreducible about n points.*
(2) *Every pair-wise disjoint collection of non-separating open subsets of M has at most n members.*
(3) *The continuum M is not an ∞-od, and every collection of subcontinua of M each term of which both fails to lie in the union of the others and has a dense subset that is open and non-separating in M has at most n terms.*
(4) *(Sorgenfrey [4]) For each proper decomposition of M into $n+1$ continua, the union of some n of them fails to be connected.*

Sorgenfrey proved the equivalence of (1) and (4). The remainder of the proof goes (1) $\Rightarrow$ (2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) $\Rightarrow$(2). Suppose (1) holds. Denote by P a set of n points about which M is irreducible. If $\mathcal{A}$ is a pair-wise disjoint collection of non-separating open subsets of M, then each term of $\mathcal{A}$ contains an element of P. It follows that every pair-wise disjoint collection of non-separating open subsets of M has at most n terms. Hence (1) implies (2).

Conversely, suppose (2) holds. Then by (1) and (2) of Theorem 1, M is irreducible about finitely many points. Let k denote the smallest number of points about which M is irreducible, and $P = \{p_1, p_2, ..., p_k\}$ denote a set of points about which M is irreducible. To show that (1) holds, it suffices to show that $k \leq n$. Since $n \geq 2$, it is trivial that $k \leq n$ if $k = 2$. Suppose $k \geq 3$. For each positive integer i not greater than k, let M_i be a subcontinuum of M that is irreducible about $P - \{p_i\}$. Note that $M_i \cup M_j = M$ for $i \neq j$ since M_i and M_j both contain $P - \{p_i, p_j\}$, and $M_i \cup M_j$ contains P. It follows that $(M - M_i) \cap (M - M_j) = \emptyset$ for $i \neq j$. Therefore, the collection $\{M - M_1, M - M_2, ..., M - M_k\}$ is a pair-wise disjoint collection of non-separating open sets with k members. By (2), $k \leq n$.

To see that (2) implies (3), suppose (3) does not hold. Then either M is an ∞-od, or there is a collection $\mathcal{A} = \{M_1, M_2, ..., M_{n+1}\}$ of subcontinua of M each term of which both fails to lie in the union of the others and has a dense subset that is open and non-separating in M. In the former case, it is clear that (1) does not hold; hence (2) also does not hold. Suppose the latter. For each i, let D_i

denote a dense subset of M_i that is open and non-separating in M. Since D_1 is a dense subset of M_1, and M_1 does not lie in $M_2 \cup M_3 \cup ... \cup M_{n+1}$, it follows that D_1 does not lie in $M_2 \cup M_3 \cup ... \cup M_{n+1}$. Then $(M - D_1) \cup M_2 \cup ... \cup M_{n+1}$ is a proper subset of M. Since each of $M_2, M_3, ..., M_{n+1}$ fails to lie in D_1, $M - D_1$ intersects each of $M_2, M_3, ..., M_{n+1}$; furthermore, $M - D_1$ is a continuum. Hence, $(M - D_1) \cup M_2 \cup ... \cup M_{n+1}$ is a proper subcontinuum of M. Denote its complement by D_1'. Note that D_1' is a subset of M_1 which is open and non-separating in M, and that D_1' fails to intersect M_i for $i \neq 1$. Similarly, there exist non-separating open sets $D_1', D_2', ..., D_{n+1}'$ such that $D_i' \subset M_i$, and D_i' fails to intersect M_j for $i \neq j$. It follows that $\{D_1', D_2', ..., D_{n+1}'\}$ is a pair-wise disjoint collection of non-separating open subsets of M, so (2) does not hold. Consequently, (2) implies (3).

Finally, suppose (3) holds. Suppose D is a non-separating subset of M. Since $M - D$ is a continuum, and no subcontinuum of M separates M into infinitely many components, D has finitely many components. Denote by D^* a component of D, and note that D^* is open and that the union of $M - D$ with all of the components of D different from D^* is connected. Therefore, D^* is a connected non-separating open set. Then $\overline{D^*}$ is a subcontinuum of M with a dense subset that is both open and non-separating in M. Hence, to each non-separating open subset of M, there corresponds a subcontinuum of M with a dense subset that is both open and non-separating in M. Consequently, (3) implies (2). □

References

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