THE SARKOVSKII ORDER FOR PERIODIC CONTINUA II

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ABSTRACT. In this paper it is shown that the existence of three maximal proper periodic continua for a map of a hereditarily decomposable chainable continuum onto itself implies the existence of a maximal proper periodic continuum with odd period greater than one. Hence, while the periods of such continua do follow the Sarkovskii order apart from the case in which the ambient space is the union of two maximal proper periodic continua with period two, for any nondegenerate terminal segment of the Sarkovskii order that fails to contain an odd integer greater than one, there does not exist a map of a hereditarily decomposable chainable continuum onto itself for which the set of all periods of such continua is the prescribed terminal segment. It is also shown that, for any terminal segment of the Sarkovskii order that does contain an odd integer greater than one, there is a map of $[0, 1]$ onto itself for which the set of all periods of such continua is the prescribed terminal segment.

1. INTRODUCTION

A.N. Sarkovskii's own formulation of his now famous theorem [8], [9] is roughly as follows. Consider the set of natural numbers in which the following relation is introduced: n precedes m, denoted $n \preceq m$, if and only if, for any continuous mapping of the real line into itself, the existence of a cycle of order m follows from the existence of a cycle of order n. This relation transforms the set of natural numbers into an ordered set, ordered in the following way:

$$3 \prec 5 \prec 7 \prec 9 \prec 11 \prec ... \prec 3\cdot 2 \prec 5\cdot 2 \prec ... \prec 3\cdot 2^2 \prec 5\cdot 2^2 \prec ...... \prec 2^3 \prec 2^2 \prec 2 \prec 1$$

Sarkovskii also gave examples to show that the order is sharp in the following sense. For each positive integer n, there is a map f of the real line into itself such that f has a periodic point of period m if and only if $n \preceq m$.

Piotr Minc and W.R.R. Transue [6] extended Sarkovskii's Theorem by showing that the periodic cycles for maps of hereditarily decomposable chainable continua

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follow the Sarkovskii order. The maximal proper periodic continua (defined below) of such maps take exception to the Sarkovskii order when the ambient space is the union of two maximal proper periodic continua with period two ($f(x) = -x^3$ on $[-1, 1]$ for example). The author [7] has shown that, apart from this case, the maximal proper periodic continua for such maps do indeed follow the Sarkovskii order. It is reasonable to wonder if this result, like Sarkovskii's original theorem, is sharp. More specifically, does there exist, for a given positive integer n, a map f of a hereditarily decomposable chainable continuum onto itself such that f has a maximal proper periodic continuum of period m if and only if $n \preceq m$?

For even integers n, Theorem 5.8 answers this question in the negative. If n is odd and f is an interval map, then Theorem 5.9 answers it in the affirmative. These hinge on the other main result, Theorem 5.7, according to which the existence of at least three maximal proper periodic continua is equivalent to, among other things, the existence of a maximal proper periodic continuum with odd period greater than one.

Section 2 addresses the limiting behavior of $f^n[J]$ where J is a continuum that intersects one or more maximal proper periodic continua of f. Section 3 reviews the terminology and pertinent results from Kuratowski's work on decompositions of irreducible continua. Section 4 consists of fixed-point theorems and theorems on maximal proper periodic continua that are auxiliary to the main results, which appear in Section 5.

A *compactum* is a compact metrizable space. A *continuum* is a connected compactum.

A continuum is said to be *decomposable* if and only if it is the union of two of its proper subcontinua. A *hereditarily decomposable* continuum is a continuum for which every nondegenerate subcontinuum is decomposable.

A continuum is said to be a *triod* if and only if it has a subcontinuum whose complement has at least three components. A continuum that contains no triod is *atriodic.*

A *unicoherent continuum* is a continuum M with the property that the common part of any two subcontinua of M whose union is M is a continuum. A continuum is *hereditarily unicoherent* if and only if each of its subcontinua is unicoherent.

A continuum is said to be *irreducible between* two mutually exclusive closed sets if and only if it intersects both but has no proper subcontinuum that does. If K and L are mutually exclusive subcontinua of a hereditarily unicoherent continuum

X, then X has a unique subcontinuum that is irreducible between K and L and is denoted by $\overline{KL}$. If one of K and L, say K, is equal to $\{x\}$ for some point x, then $\overline{xL}$ is used in lieu of the cumbersome $\overline{\{x\}L}$.

A *chain* is a sequence $C_1, C_2, ..., C_n$ of open sets such that $C_i \cap C_j \neq \emptyset$ if and only if $|i-j| \leq 1$. An ϵ-chain is a chain such that diameter$(C_i) \leq \epsilon$ for each i. A continuum is *chainable* if and only if it can be covered by an ϵ-chain for each positive ϵ. It is well known that chainable continua are hereditarily chainable, atriodic, hereditarily unicoherent, and irreducible between some two points.

Suppose $X_1, X_2, X_3, ...$ is a sequence of spaces and $f_1, f_2, f_3, ...$ is a sequence of maps such that f_n maps X_{n+1} into X_n for each positive integer n. The *inverse limit* $\varprojlim\{X_n, f_n\}$ is the subset of $X_1 \times X_2 \times X_3 \times ...$ to which a point $x = (x_1, x_2, x_3, ...)$ belongs if and only if $f_n(x_{n+1}) = x_n$ for each positive integer n. The maps $f_1, f_2, f_3, ...$ are called *bonding maps*, and the spaces $X_1, X_2, X_3, ...$ are called *factor spaces.*

An *inverse limit on* $[0,1]$ is an inverse limit with each factor space equal to $[0,1]$. It is well known that a continuum is chainable if and only if it is homeomorphic to an inverse limit on $[0,1]$, and that a continuum is chainable if and only if it is homeomorphic to an inverse limit on $[0,1]$ with surjective bonding maps.

Suppose f is a mapping of a continuum X onto itself. A subcontinuum K of X is said to be *periodic* under f if and only if there is a positive integer n such that $f^n[K] = K$; the *period* of K is the smallest such positive integer. A *periodic point* is a degenerate periodic continuum. A *proper periodic continuum* is a periodic continuum different from X. A *maximal proper periodic continuum* is a proper periodic continuum K such that the only periodic continuum that properly contains K is X.

Suppose $X_1, X_2, X_3, ...$ is a sequence of sets in a topological space. Then $\limsup X_n$ is the set to which a point x belongs if and only if every open set containing x intersects infinitely many terms of $X_1, X_2, X_3, ...$, and $\liminf X_n$ is the set to which a point x belongs if and only if every open set containing x intersects all but finitely many terms of $X_1, X_2, X_3, ...$ If $\limsup X_n = \liminf X_n$, then $\lim X_n$ is defined to be $\limsup X_n$.

If $\mathcal{K}$ is a collection of sets, then $\mathcal{K}^*$ denotes the union of the elements of $\mathcal{K}$.

2. Properties of continua that intersect maximal proper periodic continua

Lemma 2.1. *Suppose f is a map of a metric space X into itself, p is a positive integer, and J is a subset of X. Then* $\limsup f^n[J] = \limsup f^{np}[J] \cup \limsup f^{1+np}[J] \cup ... \cup \limsup f^{(p-1)+np}[J]$.

Lemma 2.2. *Suppose f is a map of a metric space X into itself, and suppose J and K are subsets of X. Then* $\limsup f^n[J \cup K] = \limsup f^n[J] \cup \limsup f^n[K]$.

Proof. The lemma follows from the fact that $\limsup A_n \cup \limsup B_n = \limsup(A_n \cup B_n)$ for any pair of sequences $A_1, A_2, A_3, ...$ and $B_1, B_2, B_3, ...$ of subsets of a metric space. □

Lemma 2.3. *Suppose g is a function from a compactum X into itself, f is a map from X into a metric space, and J is a subset of X. Then* $\limsup f \circ g^n[J] = f[\limsup g^n[J]]$.

Proof. Since f is continuous and X is compact, $\limsup f[K_n] = f[\limsup K_n]$ for any sequence K_1, K_2, $K_3, ...$ of sets in X. □

Lemma 2.4. *Suppose f is a map of a compactum X into itself, p is a positive integer, and J is a subset of X. Then* $f^p[\limsup f^{np}[J]] = \limsup f^{np}[J]$.

Proof. Putting both f and g in the statement of Lemma 2.3 equal to f^p gives that $f^p[\limsup f^{np}[J]] = \limsup f^p \circ f^{np}[J]$. Hence $f^p[\limsup f^{np}[J]] = \limsup f^{(n+1)p}[J] = \limsup f^{np}[J]$. □

Lemma 2.5. *Suppose f is a map of a compactum X into itself, p is a positive integer, and J is a subset of X. Then* $\limsup f^n[J] = \limsup f^{np}[J] \cup f[\limsup f^{np}[J]] \cup ... \cup f^{p-1}[\limsup f^{np}[J]]$.

Proof. For each positive integer k, putting f and g in the statement of Lemma 2.3 equal to f^k and f^p respectively gives that $f^k[\limsup f^{np}[J]] = \limsup f^{k+np}[J]$ for each such k. The conclusion of the lemma then follows from Lemma 2.1. □

Theorem 2.6. *Suppose f is a map of a continuum X onto itself. If J is a subcontinuum of X that intersects a maximal proper periodic continuum K of f and p is any multiple of the period of K, then at least one of the following holds.*

(1) $\limsup f^n[J] \cup K = X = \limsup f^{np}[J] \cup K$

(2) $\limsup f^n[J] \subset \text{Orbit}(K)^* and \limsup f^{np}[J] \subset K$

Proof. Consider the set $\limsup f^{np}[J \cup K]$. Since $f^{np}[J \cup K]$ is a continuum containing K for each positive integer n, it follows that $\limsup f^{np}[J \cup K]$ is a continuum containing K. By Lemma 2.4, $\limsup f^{np}[J \cup K]$ is a periodic continuum. Consequently, $\limsup f^{np}[J \cup K]$ is either K or X. By Lemma 2.2, $\limsup f^{np}[J \cup K] = \limsup f^{np}[J] \cup \limsup f^{np}[K]$. Substituting K for $\limsup f^{np}[K]$ gives $\limsup f^{np}[J \cup K] = \limsup f^{np}[J] \cup K$.

First consider the case in which $\limsup f^{np}[J \cup K] = X$. Then $\limsup f^{np}[J] \cup K = X$. To see that the rest of (1) holds, note that, by Lemma 2.1, $\limsup f^{np}[J] \subset \limsup f^n[J]$. Consequently, $\limsup f^n[J] \cup K = X$.

Now suppose $\limsup f^{np}[J \cup K] = K$. Then $K = \limsup f^{np}[J] \cup K$, which implies that $\limsup f^{np}[J] \subset K$. By Lemma 2.5, $\limsup f^n[J] = \limsup f^{np}[J] \cup f[\limsup f^{np}[J]] \cup ... \cup f^{p-1}[\limsup f^{np}[J]]$. Consequently, $\limsup f^n[J] \subset K \cup f[K] \cup ... \cup f^{p-1}[K] = \text{Orbit}(K)^*$. □

Theorem 2.7. *Suppose f is a map of a hereditarily unicoherent continuum X onto itself, and suppose J is a subcontinuum of X that intersects each of two disjoint maximal proper periodic continua of f. Then $\lim f^n[J] = X$.*

Proof. Denote by K and L two disjoint maximal proper periodic continua that intersect J. Let J_0 denote the unique subcontinuum of J that is irreducible between K and L. Then every subcontinuum of X that intersects both K and L contains J_0. To demonstrate the conclusion of the theorem, it suffices to show that $\lim f^n[J_0] = X$. Denote the product of the periods of K and L by p. By Theorem 2.6, either $\limsup f^{np}[J_0] \cup K = X$ or $\limsup f^{np}[J_0] \subset K$. But the latter is not possible because $f^{np}[J_0]$ intersects L for each positive integer n. Thus $\limsup f^{np}[J_0] \cup K = X$. Similarly, $\limsup f^{np}[J_0] \cup L = X$. Since K and L are disjoint, it follows that $\limsup f^{np}[J_0] = X$.

Both K and L contain a point of $f^p[J_0]$ because they both contain a point of J_0 and p is the product of their periods. Consequently $f^p[J_0]$ contains J_0. It follows that $J_0, f^p[J_0], f^{2p}[J_0], ...$ is a nondecreasing sequence. Hence $f^m[J_0]$, $f^{m+p}[J_0]$, $f^{m+2p}[J_0], ...$ is a nondecreasing sequence for each nonnegative integer m less than p. It follows that $\limsup f^{m+np}[J_0] = \liminf f^{m+np}[J_0]$, or, equivalently, that $\limsup f^{m+np}[J_0] = \lim f^{m+np}[J_0]$ for each such m. By Lemma 2.3, $\limsup f^{m+np}[J_0] = f^m[\limsup f^{np}[J_0]]$ for each m, and it has already been established that $\limsup f^{np}[J_0] = X$, therefore $\lim f^{m+np}[J_0] = X$ for each nonnegative integer m less than p. Thus $\lim f^n[J_0] = X$, and the conclusion of the theorem follows. □

3. Kuratowski Maps

The proof of Sarkovskii's Theorem depends heavily on the linearity of the real line. Chainable continua provide a context that is suitably linear for portions of Sarkovskii's proof to be extended. However, Sarkovskii's Theorem does not hold for maps of indecomposable chainable continua (see [5] and [6]). It is for hereditarily decomposable chainable continua that the folding patterns of maps with prescribed periodic orbits mimic those of maps of the real line.

K. Kuratowski's work on monotone decompositions of irreducible continua (Section 48 of [4]) – in particular, its implications for hereditarily decomposable chainable continua – is the key to describing the sufficiently linear context provided by hereditarily decomposable chainable continua for the extension of Sarkovskii's Theorem. Minc and Transue [6] make use of it, as well as the author [7] in the prequel to this article. The following Theorem summarizes the portion of Kuratowski's work that will be used here.

Theorem 3.1 (Kuratowski)**.** *Suppose X is a hereditarily decomposable irreducible continuum. Then there is a monotone map g from X onto $[0,1]$ that satisfies each of the following.*

(1) *A subcontinuum M of X is nowhere dense in X if and only if there is a $t \in [0,1]$ such that $M \subset g^{-1}(t)$.*
(2) *X is irreducible between two points if and only if they belong to $g^{-1}(0)$ and $g^{-1}(1)$ respectively.*
(3) *If $b \in (0,1)$ and $a < b < c$, then $\overline{g^{-1}(a,b)} \cap \overline{g^{-1}(b,c)}$ contains a a point of $g^{-1}(b)$, and if $b \in [0,1]$ and $a < b < c$, then $g^{-1}(b) \subset \overline{g^{-1}(a,b)} \cup \overline{g^{-1}(b,c)}$.*

A *Kuratowski map* for a hereditarily decomposable irreducible continuum is any map that satisfies the conclusion of Theorem 3.1.

A *layer* of X is a nowhere dense subcontinuum of X such that every subcontinuum of X that properly contains it has nonempty interior. Equivalently, A is a layer of X if and only if, for every Kuratowski map g of X, there is a t in $[0,1]$ such that $A = g^{-1}(t)$.

A layer A of X is an *end layer* if and only if X is irreducible between two points, one of which belongs to A. Equivalently, A is an end layer of X if and only if, for every Kuratowski map g of X, $A = g^{-1}(0)$ or $A = g^{-1}(1)$. All other layers of X are called *interior layers.*

4. Auxiliary Theorems

4.1. **A fixed-point theorem.** Theorem 4.1 is a consequence of a more general theorem of W. T. Ingram (Theorem 1 of [3]). Alternatively, it can be proved via an easy modification of O. H. Hamilton's proof that chainable continua have the fixed-point property [2].

Theorem 4.1. *Suppose f is a map of a chainable continuum X into itself. If K is a subcontinuum of X such that $K \subset f[K]$, then f has a fixed point in K.*

Notation. For subsets H and K of $\mathbb{R}$, $H < K$ means $h < k$ for every (h, k) in $H \times K$.

The proof of Theorem 4.2 that is given below is based on a suggestion by the referee.

Lemma 4.2. *Suppose X is a hereditarily decomposable inverse limit on $[0, 1]$ with π_n denoting the projection onto the n^{th} factor space. Suppose g is a Kuratowski map for X . If $K_1, K_2, ..., K_k$ is a nondegenerate sequence of closed subsets of X such that $g[K_1] < g[K_2] < ... < g[K_k]$, then there is a positive integer N such that, for each $n \geq N$, either $\pi_n[K_1] < \pi_n[K_2] < ... < \pi_n[K_k]$ or $\pi_n[K_k] < \pi_n[K_{k-1}] < ... < \pi_n[K_1]$.*

Proof. There are numbers $a_1, a_2, ..., a_{2k}$ such that $0 = a_1 < a_2 < ... < a_{2k-1} < a_{2k} = 1$ and $g[K_i] \subset [a_{2i-1}, a_{2i}]$ for $i = 1, 2, ..., k$. Set $C_j = g^{-1}[a_j, a_{j+1}]$ for $j = 1, 2, ..., 2k - 1$. Observe that $K_i \subset C_{2i-1}$ for $i = 1, 2, ..., k$. There is an integer N such that $\pi_n[C_j] \cap \pi_n[C_{j'}] \neq \emptyset$ if and only if $C_j \cap C_{j'} \neq \emptyset$ for each $n \geq N$ and each $j, j' = 1, 2, ..., 2k - 1$. Suppose that some $n \geq N$ fails to have the required property. Suppose also that $\pi_n[C_1] < \pi_n[C_3]$ (the proof in the opposite case is analogous). Let $l = 3, 5, .., 2k - 1$ be the least index such that $\pi_n[C_{l+2}] < \pi_n[C_l]$. Then $\pi_n[g^{-1}(a_{l+2})] < \pi_n[C_l]$. Hence each point of $\pi_n[g^{-1}(a_{l+2})]$ belongs to $\pi_n[C_{l+2}]$, $\pi_n[C_{l+1}]$, and $\pi_n[C_j]$ for some $j = 1, 2, ..., l - 1$, a contradiction. □

Theorem 4.3. *Suppose X is a hereditarily decomposable chainable continuum, and suppose g is a Kuratowski map for X. If f maps X into itself and there are points x and y of X such that $g \circ f(x) < g(x)$ and $g \circ f(y) > g(y)$, then f has a fixed point in $\overline{xy}$.*

Proof. Since X is chainable, it may be represented as the inverse limit of a sequence, $f_1, f_2, f_3, ...$ of maps from $[0,1]$ onto itself. The set $\{x, y, f(x), f(y)\}$ intersects between two and four layers of X. In any case, x and $f(x)$ belong to different layers, as do y and $f(y)$. By Lemma 4.2, there is a positive integer N such that, for each $n \geq N$, the order imparted to these layers by π_n is either the same as or the reverse of the order imparted by g. Consequently, for each such n, either $\pi_n \circ f(x) < \pi_n(x)$ and $\pi_n \circ f(y) > \pi_n(y)$, or $\pi_n \circ f(x) > \pi_n(x)$ and $\pi_n \circ f(y) < \pi_n(y)$.

For each positive integer n, denote $\{z \in \overline{xy} : \pi_n \circ f(z) = \pi_n(z)\}$ by K_n. Since f and π_n are continuous, each K_n is compact. For integers n not less than N, K_n is nonempty; otherwise $\overline{xy}$ would be the union of two nonempty disjoint open sets, $\{z \in \overline{xy} : \pi_n \circ f(z) < \pi_n(z)\}$ and $\{z \in \overline{xy} : \pi_n \circ f(z) > \pi_n(z)\}$, which is not possible.

Notice that if z belongs to K_{m+1} for some positive integer m, then z belongs to K_m. Consequently, $K_1, K_2, K_3, ...$ is a monotonic sequence of nonempty compact sets. The intersection of the sets is, therefore, nonempty. Each point of the intersection is a fixed point for f. □

4.2. **Sufficient conditions for maximal proper periodic continua.** This section and Section 5 make frequent use of results proved in [7]. Some are referenced without statement, but those that are used repeatedly or have complicated statements are stated here for convenience.

Theorem 4.4 ([7]; Theorem 2.9, Lemmas 3.2 & 3.3)**.** *Suppose f is a map from an atriodic and hereditarily unicoherent continuum onto itself.*

(1) *If f has at least three maximal proper periodic continua, then the collection of all maximal proper periodic continua is pairwise disjoint.*
(2) *For each periodic continuum K of f, the closure of the union of all proper periodic continua that intersect K is a periodic continuum.*
(3) *If K is a maximal proper periodic continuum of f, then, for every positive integer n, $f^n[K]$ is a maximal proper periodic continuum of f.*

Theorem 4.5 ([7]; Theorem 3.5)**.** *Suppose f is a map of a hereditarily decomposable chainable continuum X onto itself with disjoint maximal proper periodic continua.*

(1) *If f has two maximal proper periodic continua, one of which fails to lie in an end layer of X, then every proper periodic continuum of f is contained in a maximal proper periodic continuum.*

(2) *If A is an end layer of X that contains a maximal proper periodic continuum of f, then every proper periodic continuum of f lying in A is contained in a maximal proper periodic continuum.*

Theorem 4.6. *Suppose f is a map of a hereditarily decomposable chainable continuum X onto itself. If K and L are disjoint maximal proper periodic continua of f, neither of which lies in an end layer of X, then f has a maximal proper periodic continuum J that intersects $\overline{KL}$, is disjoint from both K and L, and fails to lie in an end layer of X.*

Proof. Denote by g a Kuratowski map for X, and reverse the names of K and L if necessary so that $g[K] \leq g[L]$. Put r and s equal to the maximum of $g[K]$ and the minimum of $g[L]$ respectively, and note that $0 < r \leq s < 1$. Also note that $\overline{KL} \subset g^{-1}[r,s]$.

By Theorem 2.7, $\lim f^n[\overline{KL}] = X$. Hence there is a positive integer N such that $f^N[\overline{KL}]$ intersects both $g^{-1}[0,r)$ and $g^{-1}(s,1]$. Since every open set that intersects $\overline{KL}$ intersects an interior layer of $\overline{KL}$, it follows that each of $f^{-N}\left[g^{-1}[0,r)\right]$ and $f^{-N}\left[g^{-1}(s,1]\right]$ intersects an interior layer of $\overline{KL}$. Consequently there are points a and b, each belonging to an interior layer of $\overline{KL}$, such that $f^N(a) \in g^{-1}[0,r)$ and $f^N(b) \in g^{-1}(s,1]$.

Each of $K \cap \overline{KL}$ and $L \cap \overline{KL}$ is contained in an end layer of $\overline{KL}$ since $\overline{KL}$ is irreducible between them. However, a and b belong to interior layers of $\overline{KL}$, so $\overline{ab}$ fails to intersect either K or L.

Notice that $\overline{ab} \subset \overline{KL} \subset g^{-1}[r,s] \subset \overline{f^N(a)\, f^N(b)} \subset f^N[\overline{ab}]$. By Theorem 4.1, f^N has a fixed point x in $\overline{ab}$. By (1) of Theorem 4.5, there is a maximal proper periodic continuum J containing x. Since $\overline{ab}$ fails to intersect either K or L, and x therefore lies outside of both K and L, it follows that J is distinct from both K and L. Since x lies in $\overline{KL}$, which fails to intersect either end layer of X, it follows that J intersects $\overline{KL}$ and fails to lie in an end layer of X. By (1) of Theorem 4.4, J fails to intersect either K or L. □

Lemma 4.7 ([7]; Theorem 4.2)**.** *Suppose f is a map of a chainable continuum X into itself, and suppose there are mutually exclusive subcontinua, A and B, of X such that $A \cup B \subset f[A]$ and $A \subset f[B]$. If the collection of proper periodic continua of f that intersect A and whose forward iterates all intersect $A \cup B$ is bounded away from B, then, for each positive integer p, A intersects a maximal proper periodic continuum of period p.*

Notation. For each b in $[0,1]$, denote $g^{-1}(b) \cap \overline{g^{-1}[0,b)}$ and $g^{-1}(b) \cap \overline{g^{-1}(b,1]}$ by $g^{-1}(b^-)$ and $g^{-1}(b^+)$ respectively. Notice that, for any a in $[0,b)$ and any c in $(b,1]$, $g^{-1}(b^-) = g^{-1}(b) \cap \big(g^{-1}[0,a] \cup \overline{g^{-1}(a,b)}\big) = g^{-1}(b) \cap \overline{g^{-1}(a,b)}$, and, similarly, $g^{-1}(b^+) = g^{-1}(b) \cap \overline{g^{-1}(b,c)}$.

Theorem 4.8. *Suppose f is a map of a hereditarily decomposable chainable continuum X onto itself, g is a Kuratowski map for X, and $b \in [0,1]$. If $g^{-1}(b)$ contains two disjoint maximal proper periodic continua K and L, and one of $g^{-1}(b^+)$ and $g^{-1}(b^-)$ intersects both K and L, then $\overline{KL}$ intersects a maximal proper periodic continuum of f with period not a power of two that is distinct from both K and L.*

Proof. For convenience, assume that K and L both intersect $g^{-1}(b^+)$. It follows from Theorem 2.7 that there is a multiple N of period(K)·period(L) such that $f^N[\overline{KL}]$ intersects $g^{-1}(b,1]$. Then there are points x, y, and z from distinct interior layers of $\overline{KL}$ such that $f^N(x)$, $f^N(y)$, and $f^N(z)$ belong to $g^{-1}(b,1]$ and such that the layer containing y separates the layer containing x from the layer containing z. Since x and z belong to distinct interior layers of $\overline{KL}$, one of $\overline{Kx} \cap \overline{Lz}$ and $\overline{Kz} \cap \overline{Lx}$ is empty, say $\overline{Kx} \cap \overline{Lz}$.

Since $f^N[\overline{Kx}]$ intersects both $g^{-1}(b,1]$ and $g^{-1}(b)$ it follows that $g^{-1}(b^+) \subset f^N[\overline{Kx}]$ and, hence, that $\overline{KL} \subset f^N[\overline{Kx}]$. Similarly, $\overline{KL} \subset f^N[\overline{zL}]$. It follows that $\overline{Kx} \cup \overline{zL} \subset f^N[\overline{Kx}]$, and $\overline{Kx} \subset f^N[\overline{zL}]$.

For the purpose of applying Lemma 4.7, consider the collection $\mathcal{K}$ of proper periodic continua that intersect $\overline{Kx}$ whose forward iterates under f^N all intersect $\overline{Kx} \cup \overline{zL}$. Suppose, for the purpose of establishing contradiction, that there is a term Q of $\mathcal{K}$ that contains y. Then $f^N[Q]$ contains $f^N(y)$ and intersects $\overline{Kx} \cup \overline{zL}$. Consequently $f^N[Q]$ is a proper periodic continuum that contains $g^{-1}(b^+)$ and, therefore, intersects both K and L. Since K is maximal, $K \cup f^N[Q] = X$. Hence $f^N[Q]$ is a proper periodic continuum that properly contains L, contrary to the maximality of L. Consequently each term of $\mathcal{K}$ fails to contain y, and the terms of $\mathcal{K}$ are bounded away from $\overline{zL}$.

By Lemma 4.7, f^N has a maximal proper periodic continuum J that intersects $\overline{Kx}$ with period three. Then the period p of J under f divides $3N$. If p were not a multiple of three, then p would divide N giving $f^N[J] = J$, which is not true. Hence p is a multiple of three and, therefore, not a power of two. Since x belongs to $\overline{KL}$, $\overline{Kx}$ is a subset of $\overline{KL}$. Consequently, J intersects $\overline{KL}$. Since $f^N[K] = K$, $f^N[L] = L$, and $f^N[J] \neq J$, J is distinct from both K and L. □

Lemma 4.9. *Suppose f is a map of a hereditarily decomposable chainable continuum X onto itself, K and L are disjoint maximal proper periodic continua of f, g is a Kuratowski map for X, and $b \in [0,1]$. If $b \in (0,1)$, or if $f[g^{-1}(b)] \cap g^{-1}(b) \neq \emptyset$, then there is an odd positive integer N such that $g^{-1}(b) \subset f^N[\overline{KL}]$.*

Proof. If $b \in (0,1)$, then, by Theorem 2.7, $f^n[\overline{KL}]$ intersects both $g^{-1}[0,b)$ and $g^{-1}(b,1]$ for cofinitely many n. The conclusion of the lemma follows. Suppose $b \in \{0,1\}$ and $f[g^{-1}(b)] \cap g^{-1}(b) \neq \emptyset$. By Theorem 2.7, there is a multiple n of the period of K such that $f^k[\overline{KL}]$ intersects $g^{-1}(0,1)$ for all $k \geq n$. Then $f^n[\overline{KL}]$ and $f^{n+1}[\overline{KL}]$ both contain contain $g^{-1}(b)$ and the conclusion of the lemma follows. □

Theorem 4.10. *Suppose f is a map of a hereditarily decomposable chainable continuum X onto itself, g is a Kuratowski map for X, and $b \in [0,1]$. Suppose further that $g^{-1}(b)$ contains two disjoint maximal proper periodic continua K and L, and one of $g^{-1}(b^+)$ and $g^{-1}(b^-)$ intersects both K and L. If $b \in (0,1)$, or if $f[g^{-1}(b)] \cap g^{-1}(b) \neq \emptyset$, then $\overline{KL}$ intersects a maximal proper periodic continuum of f with odd period greater than one.*

Proof. For convenience, assume that K and L both intersect $g^{-1}(b^+)$. By Theorem 4.8, there is a maximal proper periodic continuum Q that intersects $\overline{KL}$ but is distinct from both K and L. By (1) of Theorem 4.4, K, Q, and L are pairwise disjoint. Each end layer of $\overline{KL}$ intersects exactly one of K and L. Since Q does not intersect either K or L, it fails to intersect at least one of the two end layers of $\overline{KL}$, say the one that intersects L.

It follows from Theorem 2.7 that there is a multiple N_1 of period(K)·period(L) such that $f^{N_1}[\overline{KQ}]$ and $f^{N_1}[\overline{QL}]$ both intersect $g^{-1}(b,1]$. Then there is a point x from an interior layer of $\overline{KQ}$ and a point y from a layer of $\overline{KL}$ that separates Q from L such that $f^{N_1}(x)$ and $f^{N_1}(y)$ both belong to $g^{-1}(b,1]$. Then $f^{N_1}[\overline{Kx}]$ intersects both $g^{-1}(b)$ and $g^{-1}(b,1]$, so $g^{-1}(b^+) \subset f^{N_1}[\overline{Kx}]$. Similarly, $g^{-1}(b^+) \subset f^{N_1}[\overline{yL}]$. Notice that x and y both belong to $g^{-1}(b^+)$. Consequently, $\overline{Kx} \cup \overline{yL} \subset f^{N_1}[\overline{Kx}]$ and $\overline{Kx} \cup \overline{yL} \subset f^{N_1}[\overline{yL}]$.

By Lemma 4.9, there is an odd positive integer N_2 such that $g^{-1}(b) \subset f^{N_2}[\overline{KL}]$. Hence $g^{-1}(b) \subset f^{N_2}[g^{-1}(b^+)] \subset f^{N_1+N_2}[\overline{Kx}]$ and, similarly, $g^{-1}(b) \subset f^{N_1+N_2}[\overline{yL}]$. Consequently, $\overline{Kx} \cup \overline{yL} \subset f^{N_1+N_2}[\overline{Kx}]$ and $\overline{Kx} \cup \overline{yL} \subset f^{N_1+N_2}[\overline{yL}]$. Since N_1 is greater than one and at least one of N_1 and $N_1 + N_2$ is odd, it follows that there is an odd integer N greater than one such that $\overline{Kx} \cup \overline{yL} \subset f^N[\overline{Kx}]$ and $\overline{Kx} \cup \overline{yL} \subset f^N[\overline{yL}]$.

Consider the collection $\mathcal{K}$ of all proper periodic continua of f^N that intersect $\overline{Kx}$. Since Q is a maximal proper periodic continuum, there does not exist a proper periodic continuum that intersects both Q and its complement; for if there did, such a continuum would contain the complement of Q and, therefore, properly contain the maximal proper periodic continuum K. Since $\overline{Kx}$ and Q are disjoint, $\mathcal{K}$ has no member that intersects Q. It follows that $\mathcal{K}$ is bounded away from $\overline{yL}$.

By Lemma 4.7, f^N has a maximal proper periodic continuum J that intersects $\overline{Kx}$ with period three. Then the period p of J under f divides $3N$, which is odd. Hence p is odd. Since $f^N[J] \neq J$, $p > 1$. Since x belongs to $\overline{KL}$, $\overline{Kx}$ is a subset of $\overline{KL}$. Consequently, J intersects $\overline{KL}$. □

5. Maximal proper periodic Continua and the Sarkovskii Order

5.1. **Three maximal proper periodic continua implies infinitely many.**

Lemma 5.1. *Suppose B is a subcontinuum of a hereditarily decomposable chainable continuum X that fails to intersect an end layer of X, and suppose g is a Kuratowski map for X. Denote the maximum and minimum values of $g[B]$ by b_1 and b_2 respectively. Then at least one of $\overline{g^{-1}(0, b_1)}$ and $\overline{g^{-1}(b_2, 1)}$ intersects B, and, if $b_1 \neq b_2$, then both do.*

Proof. If $b_1 = b_2$, then one of $\overline{g^{-1}(0, b_1)}$ and $\overline{g^{-1}(b_2, 1)}$ intersects B by (3) of Theorem 3.1. If $b_1 < b_2$, then $B \cap \overline{g^{-1}(b_2, 1)}$ contains $\overline{g^{-1}(b_1, b_2)} \cap \overline{g^{-1}(b_2, 1)}$, which is nonempty by the same result. Similarly $\overline{g^{-1}(0, b_1)}$ intersects B. □

Lemma 5.2. *Suppose X is a hereditarily decomposable continuum. If $\mathcal{K}$ is a finite nondegenerate pairwise-disjoint collection of subcontinua of X, then there are $A, B \in \mathcal{K}$ such that $\overline{AB}$ fails to intersect any term of $\mathcal{K} - \{A, B\}$.*

Proof. Denote by $P(n)$ the proposition that the lemma is true if $\mathcal{K}$ contains exactly n elements. Then $P(2)$ is trivial. Suppose $n \geq 2$ is an integer for which $P(n)$ is true. To see that $P(n+1)$ is true, suppose $K_1, K_2, ..., K_{n+1}$ are pairwise disjoint subcontinua of X. Then there are distinct positive integers i and j not greater than n such that $\overline{K_iK_j}$ fails to intersect any term of $\{K_1, K_2, ..., K_n\} - \{K_i, K_j\}$. If $\overline{K_iK_j}$ fails to intersect K_{n+1}, then it follows that $\overline{K_iK_j}$ fails to intersect any term of $\{K_1, K_2, ..., K_{n+1}\} - \{K_i, K_j\}$. If $\overline{K_iK_j}$ does intersect K_{n+1}, then denote by M a subcontinuum of $\overline{K_iK_j}$ that is irreducible with respect to the property of intersecting each of K_i, K_j, and K_{n+1}, and note that M fails to intersect any term of $\{K_1, K_2, ..., K_{n+1}\} - \{K_i, K_j, K_{n+1}\}$. Since X is hereditarily decomposable, $M = M_1 \cup M_2$ where M_1 and M_2 are proper subcontinua of M. Then neither M_1 nor M_2 intersects all three of K_i, K_j, and K_{n+1}. Consequently, neither of M_1 and M_2 intersects more than two terms of $K_1, K_2, ..., K_{n+1}$. Hence $P(n+1)$ is true, and, by induction, $P(n)$ is true for each integer $n \geq 2$. □

Lemma 5.3. *Suppose f is a map of a hereditarily decomposable chainable continuum X onto itself, and suppose g is a Kuratowski map for X. If A, B, and C are maximal proper periodic continua of f such that $g[A] = 0$, $g[B] = [b_1, b_2]$, $g[C] = 1$, and $0 < b_1 \leq b_2 < 1$, then either f has a maximal proper periodic continuum with odd period greater than one or f has a maximal proper periodic continuum distinct from A, B, and C that intersects an interior layer of X.*

Proof. One of $\overline{g^{-1}(0, b_1)}$ and $\overline{g^{-1}(b_2, 1)}$ intersects B by Lemma 5.1. The two cases are symmetric, so only one will be considered.

Step 1: From this point forward, p will denote the product of the periods of A, B, and C, and it will be assumed that $\overline{g^{-1}(0, b_1)}$ intersects B. Then either there is a maximal proper periodic continuum distinct from A, B, and C that intersects an interior layer of X, or $g \circ f^p(x) \geq g(x)$ for every x in $g^{-1}(0, b_1)$.

For the purpose of establishing contradiction, suppose $f^p[g^{-1}(0, b_1)] \subset \overline{g^{-1}(0, b_1)}$. Then $f^p[\overline{g^{-1}(0, b_1)} \cup B] \subset \overline{g^{-1}(0, b_1)} \cup B$. It follows that $\limsup f^{np}\left[\overline{g^{-1}(0, b_1)} \cup B\right]$ is a proper subset of X. But $\overline{g^{-1}(0, b_1)} \cup B$ intersects both A and B, so, by Theorem 2.7, $\lim f^{np}\left[\overline{g^{-1}(0, b_1)} \cup B\right] = X$. This contradiction establishes the existence of a point x' of $g^{-1}(0, b_1)$ such that $g \circ f^p(x') \geq b_1$. Hence $g \circ f^p(x') > g(x')$.

Either $g \circ f^p(x) \geq g(x)$ for every x in $g^{-1}(0, b_1)$, or there is a point y' of $g^{-1}(0, b_1)$ such that $g \circ f^p(y') < g(y')$. In the latter case, it follows from Theorem 4.3 that there is a fixed point in $\overline{x'y'}$, which, by (1) of Theorem 4.5, belongs to a maximal proper periodic continuum. Since $\overline{x'y'}$ is a subset of $g^{-1}(0, b_1)$, it follows that there is a maximal proper periodic continuum distinct from A, B, and C that intersects an interior layer of X.

Step 2: From this point forward it will be assumed that $g \circ f^p(x) \geq g(x)$ for every x in $g^{-1}(0, b_1)$. Then either there is a maximal proper periodic continuum distinct from A, B, and C that intersects an interior layer of X, or $g \circ f^p(x) \leq g(x)$ for every $x \in g^{-1}(b_2, 1)$.

If $\overline{g^{-1}(b_2,1)}$ intersects B, then, by an argument similar to that of Step 1, either there is a maximal proper periodic continuum distinct from A, B, and C that intersects an interior layer of X, or $g \circ f^p(x) \leq g(x)$ for every x in $g^{-1}(b_2,1)$.

Suppose $\overline{g^{-1}(b_2,1)}$ fails to intersect B. Then, by Lemma 5.1, $b_1 = b_2$. Suppose, for the purpose of establishing contradiction, that $g \circ f^p(x) \geq g(x)$ for each x in $g^{-1}(b_2,1)$. Then $g \circ f^p(x) \geq g(x)$ for each x in $\overline{g^{-1}(b_2,1)}$. Recall from the assumption of Step 2 that $g \circ f^p(x) \geq g(x)$ for every x in $g^{-1}(0,b_1)$. Hence $g \circ f^p(x) \geq g(x)$ for every x in $\overline{g^{-1}(0,b_1)}$. Every point of $g^{-1}(b_2)$ belongs to one of $\overline{g^{-1}(b_2,1)}$ and $\overline{g^{-1}(0,b_2)}$ by (3) of Theorem 3.1. Since $b_1 = b_2$, it follows that $g \circ f^p(x) \geq g(x) \geq b_2$ for each x in $g^{-1}[b_2,1]$. That is to say, $f^p\left[g^{-1}[b_2,1]\right] \subset g^{-1}[b_2,1]$, from which it follows that $\limsup f^{np}\left[g^{-1}[b_2,1]\right] \subset g^{-1}[b_2,1]$. But $g^{-1}[b_2,1]$ contains both B and C, so $\lim f^{np}\left[g^{-1}[b_2,1]\right] = X$ by Theorem 2.7, a contradiction. Consequently, there is a point x'' of $g^{-1}(b_2,1)$ such that $g \circ f^p(x'') < g(x'')$.

As in the second paragraph of the proof of Step 1, it follows from Theorems 4.3 and 4.5 that either there is a maximal proper periodic continuum distinct from A, B, and C that intersects an interior layer of X, or $g \circ f^p(x) \leq g(x)$ for x in $g^{-1}(b_2,1)$.

Step 3: From this point forward it will be assumed that $g \circ f^p(x) \leq g(x)$ for every $x \in g^{-1}(b_2,1)$. Then $f^p\left[g^{-1}[0,b_2]\right] = X$ and $f^p\left[g^{-1}[b_1,1]\right] = X$.

Denote $\max\{g \circ f^p(x) : x \in g^{-1}[0,b_2]\}$ by m. Then $f^p\left[g^{-1}[0,b_2]\right] \subset g^{-1}[0,m]$, and, since $g \circ f^p[B]$ contains b_2, $m \geq b_2$. It follows from the assumption of Step 3 that $f^p\left[g^{-1}(b_2,m]\right] \subset g^{-1}[0,m]$. Consequently $f^p\left[g^{-1}[0,m]\right] = f^p\left[g^{-1}[0,b_2]\right] \cup f^p\left[g^{-1}(b_2,m]\right] \subset g^{-1}[0,m]$. Both A and B intersect $g^{-1}[0,b_2]$, so $\lim f^{np}\left[g^{-1}[0,b_2]\right] = X$ by Theorem 2.7. Then $\lim f^{np}\left[g^{-1}[0,m]\right] = X$ and $f^p\left[g^{-1}[0,m]\right] \subset g^{-1}[0,m]$, which is possible only if $g^{-1}[0,m] = X$. Hence $m = 1$, and $f^p\left[g^{-1}[0,b_2]\right] = X$. Similarly, $f^p\left[g^{-1}[b_1,1]\right] = X$.

Step 4: $f^p\left[g^{-1}[0,b_1]\right] = X$.

If $b_1 = b_2$, this follows immediately from Step 3. Suppose $b_1 \neq b_2$, and suppose $x \in g^{-1}(b_1,b_2]$. It follows from (3) of Theorem 3.1 that every point of $g^{-1}(b_2)$ belongs to the closure of one of $g^{-1}(b_1,b_2)$ and $g^{-1}(b_2,1)$. Hence x belongs to the closure of one of $g^{-1}(b_1,b_2)$ and $g^{-1}(b_2,1)$. In the former case, $x \in B$, so $g \circ f^p(x) \leq b_2 < 1$. In the latter case, it follows from the assumption of Step 3 that $g \circ f^p(x) \leq g(x) \leq b_2 < 1$. Since $f^p\left[g^{-1}[0,b_2]\right] = X$, and $g \circ f^p(x) < 1$ for $x \in g^{-1}(b_1,b_2]$, there is a point z of $g^{-1}[0,b_1]$ such that $g \circ f^p(z) = 1$. Recall that

A is a subset of $g^{-1}[0,b_1]$, and $g \circ f^p[A] = g[A] = 0$. Then X is irreducible between $f^p[A]$ and $f^p(z)$, and $f^p\left[g^{-1}[0,b_1]\right] = X$.

Step 5: There are a multiple q of p and a number $\beta < b_1$ such that $f^q\left[g^{-1}[0,\beta]\right] = X$.

Since $\overline{g^{-1}[0,b_1)}$ intersects both A and B, it follows from Theorem 2.7 that there is a positive integer n such that $f^{np}\left[\overline{g^{-1}[0,b_1)}\right]$ intersects $g^{-1}(b_1,1]$. Hence there is a point $y \in g^{-1}[0,b_1)$ such that $g \circ f^{np}(y) \in (b_1,1]$. Since $g \circ f^{np}[A] = 0$ and $g \circ f^{np}(y) > b_1$, it follows, for $\beta = g(y)$, that $g^{-1}[0,b_1] \subset f^{np}\left[g^{-1}[0,\beta]\right]$. Recall that $f^p\left[g^{-1}[0,b_1]\right] = X$ by Step 4. Consequently $X = f^p\left[g^{-1}[0,b_1]\right] \subset f^{(n+1)p}\left[g^{-1}[0,\beta]\right]$.

Step 6: Either f has a maximal proper periodic continuum with odd period greater than one or f has a maximal proper periodic continuum distinct from A, B, and C that intersects an interior layer of X.

By Step 5, there are a multiple q of p and a number $\beta < b_1$ such that $f^q\left[g^{-1}[0,\beta]\right] = X$. It follows from Step 3 that $f^q\left[g^{-1}[b_1,1]\right] = X$. There is an odd integer $r \geq q$ such that $f^r\left[g^{-1}[0,\beta]\right] = X$ and $f^r\left[g^{-1}[b_1,1]\right] = X$. Consider the collection $\mathcal{K}$ of proper periodic continua for f^r that intersect $g^{-1}[0,\beta]$. If $\mathcal{K}$ is not bounded away from $g^{-1}[b_1,1]$, then there is a proper periodic continuum that contains $g^{-1}\left(\frac{\beta+b_1}{2}\right)$, which, by (1) of Theorem 4.5, is contained in a maximal proper periodic continuum. Thus there is a maximal proper periodic continuum of f distinct from A, B, and C that intersects an interior layer of X. If $\mathcal{K}$ is bounded away from $g^{-1}[b_1,1]$, then, by Theorem 4.7, f^r has a maximal proper periodic continuum K of period three. Since $3r$ is odd, it follows that the period of K under f is odd. Since $f^r[K] \neq K$, it follows that the period of K is greater than one. $\square$

Lemma 5.4. *Suppose f is a map of a hereditarily decomposable chainable continuum X onto itself. If f has at least three maximal proper periodic continua, then it has at least four.*

Proof. Denote by A, B, and C three maximal proper periodic continua of f. By (1) of Theorem 4.4, the maximal proper periodic continua of f are pairwise disjoint. The proof will be divided into three cases.

First suppose that some two of A, B, and C intersect an interior layer of X. If the third fails to intersect an interior layer, then it fails to intersect the continuum irreducible between the first two. If all three intersect an interior layer of X, then, by Lemma 5.2, the continuum irreducible between some two of them fails to intersect the third. In either case, it follows from Theorem 4.6 that there is a fourth maximal proper periodic continuum.

Next suppose that some two of A, B, and C, say A and B, lie in a single end layer of X. By Lemma 5.2, the continuum irreducible between some two of them fails to intersect the third. Whichever two these may be, they both lie in the same end layer with A and B. It follows from Theorem 4.8 that there is a fourth maximal proper periodic continuum.

Finally suppose that some two of A, B, and C, say A and C, lie in different end layers of X, while the third, B, intersects an interior layer of X. Since B does not intersect either A or C, it follows that B does not intersect either end layer of X. Denote by g a Kuratowski map for X, and switch the names of A and C if necessary so that $g[A] = 0$ and $g[C] = 1$. Let $b_1 = \min g[B]$ and $b_2 = \max g[B]$. Then $0 < b_1 \leq b_2 < 1$. By Lemma 5.3, either there is a fourth maximal proper periodic continuum, or there is a maximal proper periodic continuum with odd period greater than one. In the latter case, since the maximal proper periodic continua of f follow the Sarkovskii order [7, Theorem 4.9], it follows that there are infinitely many maximal proper periodic continua. □

Theorem 5.5. *Suppose f is a map of a hereditarily decomposable chainable continuum X onto itself. If f has at least three maximal proper periodic continua, then it has infinitely many.*

Proof. It suffices to show that f has at least n maximal proper periodic continua for each positive integer n. By Lemma 5.4, f has at least four. Suppose n is an integer not less than four such that f has at least n maximal proper periodic continua, $K_1, K_2, ..., K_n$. Then either there is an end layer of X that contains at least two terms of $K_1, K_2, ..., K_n$, or there are at least two maximal proper periodic continua that fail to lie in either end layer of X. Then by Lemma 5.2, there is a pair, A and B, of terms of $K_1, K_2, ..., K_n$ such that $\overline{AB}$ fails to intersect any term of $K_1, K_2, ..., K_n$ different from A and B; furthermore, A and B may be chosen so that either both lie in the same end layer of X or both fail to lie in either end layer of X. Theorem 4.8 guarantees the existence of a maximal proper periodic continuum different from $K_1, K_2, ..., K_n$ in the former case; Theorem 4.6, in the latter. By induction, f has at least n maximal proper periodic continua for each positive integer n. □

5.2. **Three maximal proper periodic continua implies the existence of one with odd period greater than one.**

Theorem 5.6. *Suppose f is a continuous map from a hereditarily decomposable chainable continuum X onto itself. If f has at least three maximal proper periodic continua, then it has a maximal proper periodic continuum with odd period greater than one.*

Proof. By Theorem 5.5, f has infinitely many maximal proper periodic continua, and, by (1) of Theorem 4.4, the collection of maximal proper periodic continua is pairwise disjoint. Let g denote a Kuratowski map for X. Consider the following three cases: (1) there is a maximal proper periodic continuum K such that K and $f[K]$ both belong to a single end layer of X, (2) there is a maximal proper periodic continuum K such that K and $f[K]$ belong to opposite end layers of X, and (3) there are infinitely many maximal proper periodic continua that intersect an interior layer of X. In Case (1), it follows from Theorem 4.10 that there is a maximal proper periodic continuum with odd period greater than one.

Case (2) leads either to the conclusion of the theorem or to Case (3). To that end, suppose there is a maximal proper periodic continuum K such that K and $f[K]$ belong to opposite end layers of X. Since the maximal proper periodic continua for a map of a hereditarily decomposable continuum onto itself with disjoint such continua follow the Sarkovskii order [7, Theorem 4.9], it follows that there is a maximal proper periodic continuum L with period one. It is not possible that L properly contains either end layer, since it does not properly contain either K or $f[K]$. Hence L either lies in an end layer or fails to intersect either end layer. If L lies in an end layer, then it follows from Theorem 4.10 that there is a maximal proper periodic continuum with odd period greater than one. If L fails to intersect either end layer, then, by Lemma 5.3, either there is maximal proper periodic continuum with odd period greater than one, or there is a maximal proper periodic continuum distinct from L that intersects an interior layer of X. In the former case, the conclusion of the theorem holds. In the latter, Theorem 4.6 gives that there is a third maximal proper periodic continuum that intersects an interior layer of X. An inductive application of Lemma 5.2 and Theorem 4.6 gives that there are infinitely many maximal proper periodic continua that intersect an interior layer of X, which is to say (3) holds.

Thus, to complete the proof of the theorem, it suffices to show that if there are infinitely many maximal proper periodic continua that intersect an interior layer of X, then there is a maximal proper periodic continuum with odd period greater than one. The remainder of the proof falls into three cases: either all but finitely many of the maximal proper periodic continua of f intersect more than one layer of X, or X has infinitely many layers that contain a maximal proper periodic continuum, or there is an interior layer of X that contains infinitely many maximal proper periodic continua.

First suppose all but finitely many of the maximal proper periodic continua of f intersect more than one layer of X, denote the collection of all such continua by $\mathcal{K}$. Each term of $\mathcal{K}$ that intersects an end layer of X contains that end layer of X, and the terms of $\mathcal{K}$ are pairwise disjoint. Hence there are there are infinitely many terms of $\mathcal{K}$ that fail to intersect an end layer of X.

Suppose, for the purpose of establishing contradiction, that there are two terms, K and L, of $\mathcal{K}$ such that $g[K] \cap g[L]$ is nonempty. If $g[K] \cap g[L]$ is nondegenerate and t is in its interior, then K and L both contain $g^{-1}(t)$, which is not possible because the terms of $\mathcal{K}$ are pairwise disjoint. If $g[K] \cap g[L] = \{t\}$ for some $t \in [0,1]$, then, since each of $g[K]$ and $g[L]$ are nondegenerate, one of K and L contains $g^{-1}(t^-)$, and the other, $g^{-1}(t^+)$. Consequently, K and L have a common point, contrary to the pairwise-disjointness of $\mathcal{K}$. Thus, for each pair, K and L, of terms of $\mathcal{K}$, $g[K]$ and $g[L]$ are disjoint.

Denote by A, B, C, D, and E five terms of $\mathcal{K}$ such that $0 < g[A] < g[B] < g[C] < g[D] < g[E] < 1$. By Theorem 2.7, there is a positive integer N such that each of $f^N[\overline{AB}]$ and $f^N[\overline{DE}]$ contains $\overline{AB} \cup \overline{DE}$; furthermore, N may be chosen to be odd. The collection of proper periodic continua that intersect $\overline{AB}$ contains no term that intersects C, so it is bounded away from $\overline{DE}$. Then, by Lemma 4.7, f^N has a maximal proper periodic continuum J with period three. It follows that J is a maximal proper periodic continuum of f whose period is a divisor of $3N$ and, therefore, odd. Since $f^N[J] \neq J$, it follows that the period of J is not one.

Suppose there are infinitely many layers that contain a maximal proper periodic continuum of f. Then there are five maximal proper periodic continua, A, B, C, D, and E such that $0 < g[A] < g[B] < g[C] < g[D] < g[E] < 1$, and, by the argument in the previous paragraph, there is a maximal proper periodic continuum with odd period greater than one.

Finally suppose there is an interior layer, $g^{-1}(b)$, of X that contains infinitely many maximal proper periodic continua, then one of $g^{-1}(b^+)$ and $g^{-1}(b^-)$ intersects two maximal proper periodic continua, and the conclusion of the theorem follows from Theorem 4.10. □

5.3. **Equivalences to the existence of three maximal proper periodic continua.**

Theorem 5.7. *Suppose f is a continuous map from a hereditarily decomposable chainable continuum X onto itself. The following are equivalent.*

(1) *There is a maximal proper periodic continuum with odd period greater than one.*
(2) *There is a maximal proper periodic continuum with period greater than two.*
(3) *There is a maximal proper periodic continuum K such that $f[K] \cap K = \emptyset$.*
(4) *There are at least three maximal proper periodic continua.*
(5) *There are infinitely many maximal proper periodic continua.*

Proof. The proof goes (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1). The first of these implications is trivial, and the last two follow from Theorems 5.5 and 5.6 respectively. To demonstrate (2) $\Rightarrow$ (3), suppose K is a maximal proper periodic continuum whose period is greater than two. By (3) of Theorem 4.4, K, $f[K]$, and $f^2[K]$ are all maximal proper periodic continua. They are distinct because the period of K is greater than two. Hence they are disjoint by (1) of Theorem 4.4, from which (3) follows.

Consider (3) $\Rightarrow$ (4), and suppose K is a maximal proper periodic continuum such that $f[K] \cap K = \emptyset$. If the period of K is greater than two, then (4) follows from (3) of Theorem 4.4. Suppose the period of K is equal to two. Then $\overline{Kf[K]} \subset f[\overline{Kf[K]}]$. Hence there is a fixed-point x of f in $\overline{Kf[K]}$. Notice that x belongs to neither K nor $f[K]$.

Either K and $f[K]$ lie in opposite end layers of X, or they do not. If not, then by either (1) or (2) of Theorem 4.5, there is a maximal proper periodic continuum J that contains x. Since neither K nor $f[K]$ contains x, J is a third maximal proper periodic continuum of f.

Suppose K and $f[K]$ lie in opposite end layers of X. Denote by A the end layer of X that contains K. The remainder of the proof falls into two cases: either A is thrown onto X by some iterate of f, or it is not. If not, then by Theorem 3.6 of

[7] there is a maximal proper periodic continuum that contains x and is therefore distinct from K and $f[K]$.

Finally, suppose some iterate of f throws A onto X. Then there is a positive integer n such that $f^n[A] = X$; furthermore, since (3) implies that the period of K is not equal to one, n may be chosen so that it is not a multiple of the period of K. Then f^n has a fixed point y in A. Notice that y belongs neither to K nor to $f[K]$. By (2) of Theorem 4.5, y belongs to a maximal proper periodic continuum M. Consequently, M is a third maximal proper periodic continuum of f. □

5.4. **The existence and nonexistence of maps having maximal proper periodic continua with a prescribed set of periods.**

Sections 2.1 and 2.2 of [1] contain a proof of Sarkovskii's Theorem. Included there are a couple of proofs that the Sarkovskii order is sharp, one of which culminates in the following: If f is a Markov map whose Markov partition is a Štefan cycle with period n, then f has a periodic point of period m if and only if $n \preceq m$. The proof of Theorem 5.9 consists in large measure in showing that the set of periods of maximal proper periodic continua of f is the same as the set of periods of periodic points of f. Most of the following definitions may be found in Sections 1.1 through 2.1 of [1].

Definitions. For any finite subset F of $[0,1]$, the enumeration $x_1, x_2, ..., x_n$ of the terms of F such that $x_k < x_{k+1}$ for $k = 1, 2, ..., n-1$ is the *spacial labelling* of F.

Suppose f is a map of $[0,1]$ into itself. A *periodic cycle* P of f of period n is a periodic point p of period n together with its forward iterates, that is $P = \{f^{k-1}(p) : k = 1, 2, ..., n\}$. For each q in P, the enumeration $q_1, q_2, ..., q_n$ of the terms of P where $q_k = f^{k-1}(q)$ for $k = 1, 2, ..., n$ is a *temporal labelling* of P.

A *Štefan cycle* is a periodic cycle of odd period greater than one for which there is a temporal labelling $p_1, p_2, ..., p_n$ such that either

$$p_n < p_{n-2} < ... < p_5 < p_3 < p_1 < p_2 < p_4 < ... < p_{n-3} < p_{n-1}$$

or

$$p_{n-1} < p_{n-3} < ... < p_4 < p_2 < p_1 < p_3 < p_5 < ... < p_{n-2} < p_n.$$

A finite subset P of $[0,1]$ containing both 0 and 1 is said to be a *Markov partition* for f provided $f[P] \subset P$, and f is monotone between members of P that are consecutive according to the spacial labelling of P. A map possessing a Markov partition is called a *Markov map*. Notice that, for each odd integer n greater than one, there is a Markov map whose partition is a Štefan cycle with period n. Such a map has a periodic point of period m if and only if $n \preceq m$ [1, Proposition 2.2.3].

Theorem 5.8. *Suppose X is a hereditarily decomposable chainable continuum, and n, a positive integer. If there is a map f of X onto itself such that f has a maximal proper periodic continuum of period m if and only if $n \preceq m$, then n is odd.*

Proof. If $n = 1$, then, of course, n is odd. Suppose n is greater than one. Since $n \preceq 1$, it follows that f has a maximal proper periodic continuum of period one. Hence f has at least $n+1$ maximal proper periodic continua and, by Theorem 5.7, has a maximal proper periodic continuum of odd period greater than one. It follows that n is odd. □

Theorem 5.9. *Suppose n is a positive integer. The following are equivalent.*

(1) *The integer n is odd.*
(2) *There is a map f of $[0,1]$ onto itself such that f has a maximal proper periodic continuum of period m if and only if $n \preceq m$.*

Proof. Theorem 5.8 gives that (2) implies (1). To see the converse, suppose n is odd. If $n = 1$, then (2) holds by dint of the map of $[0,1]$ onto itself given by $f(x) = x^2$, which has exactly two maximal proper periodic continua, $\{0\}$ and $\{1\}$, both with period one. Suppose n is an odd integer greater than one, and denote by f a Markov map of $[0,1]$ onto itself whose Markov partition is a Štefan cycle with period n. Then there is a temporal labelling $p_1, p_2, ..., p_n$ of the partition such that either $0 = p_n < p_{n-2} < ... < p_5 < p_3 < p_1 < p_2 < p_4 < ... < p_{n-3} < p_{n-1} = 1$ or $0 = p_{n-1} < p_{n-3} < ... < p_4 < p_2 < p_1 < p_3 < p_5 < ... < p_{n-2} < p_n = 1$.

To see that p_n belongs to a maximal proper periodic continuum of f with period n, consider the collection $\mathcal{K}$ of all proper periodic continua of f that contain p_n. Then $\mathcal{K}^*$ is a periodic continuum by (2) of Theorem 4.4. To see that $\mathcal{K}^*$ fails to contain p_{n-2}, suppose K is a periodic continuum of f that contains both p_n and p_{n-2}. Then $f^2[K]$ contains both p_2 and p_n. Consequently, $f^2[K]$ contains both p_1 and p_2, from which it follows that $f^n[K]$ contains both p_{n-1} and p_n. Hence $f^n[K] = [0,1]$ and $K \notin \mathcal{K}$. It follows that $\mathcal{K}^*$ is a maximal proper periodic continuum of f that contains p_n, but fails to contain any other point of the orbit of p_n. The forward images of $\mathcal{K}^*$ are all maximal proper periodic continua by (3) of Theorem 4.4, and each forward image contains exactly one of $p_1, p_2, ..., p_n$. Since $n \geq 3$, the maximal proper periodic continua of f are disjoint by (1) of Theorem 4.4.

Now consider (2). If $n \preceq m$, then f has a maximal proper periodic continuum of period m since the maximal proper periodic continua of f follow the Sarkovskii order [7, Theorem 4.9]. Conversely, if f has a maximal proper periodic continuum of period m, then, since the maximal proper periodic continua of f are disjoint, f has a periodic point of period m. By Proposition 2.2.3 of [1], f has a periodic point of period m if and only if $n \preceq m$. □

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References

[1] Lluís Alsedà, Jaume Llibre, and MichałMisiurewicz, *Combinatorial dynamics and entropy in dimension one*, Second ed., World Scientific (2000).

[2] O.H. Hamilton, *A fixed point theorem for pseudo-arcs and certain other metric continua*, Proc. Amer. Math. Soc. 2 (1951), pp. 173-174.

[3] W.T. Ingram, *Concerning periodic points in mappings of continua*, Proc. Amer. Math. Soc. 104 no.2 (1988), pp. 643 - 649.

[4] K. Kuratowski, *Topology*, Vol. II, Academic Press, New York, Polish Scientific Publishers, Warsaw, 1968.

[5] Wayne Lewis, *Periodic homeomorphisms of chainable continua*, Fund. Math. 117 (1983), pp. 81-84.

[6] Piotr Minc and W.R.R. Transue, *Sarkovskii's theorem for heredtarily decomposable chainable continua*, Trans. Amer. Math. Soc. 315 (1989), pp. 173 - 188.

[7] David J. Ryden, *The Sarkovskii order for periodic continua*, Top. Appl. 154 no. 11 (2007), pp. 2253 - 2264.

[8] A.N. Sharkovskii, *Coexistence of cycles of a continuous map of the line into itself*, Ukrain. Mat. Z. 16 (1964), pp. 61 - 71. (Russian, English summary)

[9] A.N. Sharkovskii, *Coexistence of cycles of a contiuous map of the line into itself*, International Journal of Bifurcation and Chaos, Vol. 5 No. 5 (1995) pp. 1263 - 1273. (English translation)

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