COMPOSANT STRUCTURE IN INVERSE LIMITS OF INTERVALS

DAVID J. RYDEN

Abstract. A complete classification of the composants of indecomposable continua generated by inverse limits of Markov maps is given, and it is shown that the composant equivalence relation of such continua is Borel bireducible with $\mathbb{E}_0$.

1. Introduction

It would be nice to have a complete topological classification for the family of inverse limits of Markov maps on intervals. Much recent progress on the problem has occurred at the intersection of this problem with the Ingram conjecture [6, Problem 4], which was first formulated verbally at the Spring Topology conference of 1992. The Ingram conjecture concerns the topological classification of inverse limits of tent maps. In particular, is it true that if the inverse limits of two tent maps with slope not less than one are homeomorphic, then the tent maps have the same slope? When the critical point of a tent map is preperiodic, the map is also a Markov map. In this case, Sonja Štimac [14] has confirmed the Ingram conjecture by showing that the slope of the tent map is completely determined by the topology of the inverse limit. The pursuit of more general solutions may involve movement in the direction of more general tent maps, which will no longer be Markov maps, or in the direction of more general Markov maps, which will no longer be tent maps. In the former direction, Štimac and Brian Raines have announced a positive solution to the Ingram conjecture for some tent maps with nonrecurrent critical points [15]. An open problem in the latter direction, also due to W. T. Ingram [6, Problem 8], concerns the topological classification of inverse limits of permutation maps.

Over the last decade and a half, there has been a progression of partial solutions to the Ingram conjecture ([1], [4], [7], [2], [5], [14], [15]). Some of these results are topological classification theorems for inverse limits within certain subfamilies

2000 *Mathematics Subject Classification.* Primary 54F15.

Key words and phrases. continuum, indecomposable, composant, inverse limit, Markov map, periodic, equivalence relation.

of the tent family. Others tell, for larger subfamilies where classifications have not yet been proven, that there are, in any case, uncountably many topologically distinct inverse limits. Most of the proofs for the classification theorems employ some method to describe the way composants wind through the continuum. The most recent results, those that provide topological classifications in the largest subfamilies, namely those of Kailhofer, Štimac, and Štimac and Raines, hinge on the classification given by Karen Brucks and Beverly Diamond [3] of composants in the indecomposable core of inverse limits of certain tent maps. This suggests that composant structure will play a crucial role in uncovering the deepest topological subtleties of inverse limits of Markov maps and inverse limits of tent maps.

This paper gives a complete classification of the composants of indecomposable continua arising from inverse limits of Markov maps. Each point of the continuum is identified in a natural way with an infinite sequence of symbols from a finite alphabet. Two points are then shown to belong to the same composant if and only if their corresponding sequences agree on their tails. This is curiously similar to the $\mathbb{E}_0$ equivalence relation. According to a result of Slawomir Solecki [16], $\mathbb{E}_0$ and $\mathbb{E}_1$ are the only possibilities for the composant equivalence relation of an indecomposable metric continuum up to Borel bireducibility. It is shown in this paper that the composant equivalence relation of any indecomposable inverse limit of Markov maps is indeed Borel bireducible with $\mathbb{E}_0$. Thus the classification of composants of indecomposable continua generated by inverse limits of Markov maps that is given here turns out to be an analytical description of the way in which descriptive set-theoretic ideas about the composant equivalence relation manifest themselves in an important family of continua.

Section 2 gives definitions and notation. Section 3 establishes necessary and sufficient conditions for an indecomposable inverse limit of an organic map to be indecomposable. It is also shown that Markov maps are organic. In Section 4, proper subcontinua of inverse limits of intervals with merely continuous bonding maps and their relationship to periodic continua is clarified. Necessary conditions are given for two points of an inverse limit of intervals to belong to the same composant. This paves the way for the main results of the paper, the classification of composants for indecomposable inverse limits of Markov maps and the result that such continua are $\mathbb{E}_0$ continua, which appear in Sections 5 and 6 respectively.

2. Definitions and Notation

A *continuum* is a compact connected metrizable space. A continuum is said to be *indecomposable* if and only if it is not the union of two of its proper subcontinua; otherwise it is *decomposable*. A continuum is *irreducible* between two points if and only if none of its proper subcontinua contain both points. The *composant* of a point p of a continuum X is the union of all proper subcontinua of X that contain p. The composants of an indecomposable continuum partition it into $2^{\aleph_0}$ sets.

A *map* is a continuous function. An *n-pass map* is a map f of an interval $[a, b]$ such that there are points $a = p_0 < p_1 < ... < p_n = b$ of $[a, b]$ such that $f[x_{i-1}, x_i] = [a, b]$ for each i.

A map $f : X \to X$ is said to be *semiconjugate* to the map $g : Y \to Y$ via the *semiconjugacy* $m : X \to Y$ if and only if $m \circ f = g \circ m$.

Suppose f is a mapping of a continuum X onto itself. A *periodic continuum* of f is a subcontinuum K of X for which there is a positive integer n such that $f^n[K] = K$; the *period* of K is the smallest such positive integer. A *nontrivial periodic continuum* is a periodic continuum of f is a periodic continuum different from X. A *maximal periodic continuum* is a nontrivial periodic continuum K with the property that the only periodic continuum that properly contains K is X.

A *periodic point* is a degenerate periodic continuum. A *continuum-periodic point* is a point that belongs to a nontrivial periodic continuum. A *pre continuum-periodic point* is a point that fails to belong to a nontrivial periodic continuum, but has a forward image that does.

Suppose $X_1, X_2, X_3, ...$ is a sequence of spaces and $f_1, f_2, f_3, ...$ is a sequence of maps such that f_n maps X_{n+1} into X_n for each positive integer n. The *inverse limit* of $\{X_n, f_n\}$ is the subset of $X_1 \times X_2 \times X_3 \times ...$ to which a point $x = (x_1, x_2, x_3, ...)$ belongs if and only if $f_n(x_{n+1}) = x_n$ for each positive integer n. The maps $f_1, f_2, f_3, ...$ are called *bonding maps*, and the spaces $X_1, X_2, X_3, ...$ are called *factor spaces*. If f maps a space X into itself, the *inverse limit* of f refers to the inverse limit of $\{X, f\}$.

Suppose J and K are closed (possibly degenerate) subintervals of an interval I. The notation $J < K$ means $j < k$ for every (j, k) in $J \times K$. The smallest closed subinterval of I that intersects both J and K is denoted by $\overline{JK}$. Suppose $J < K$. Then (J, K) denotes the open interval consisting of all points x such that $J < x < K$, $[J, K)$ denotes the half-open interval $J \cup (J, K)$, and $(J, K]$ and $[J, K]$ are defined similarly. If $\mathcal{K}$ is a collection of sets, then $\mathcal{K}^*$ denotes the set to which a point x belongs if and only if x belongs to some term of $\mathcal{K}$.

An *organic* map is a map f of an interval I into itself such that, for any pair x and y of points between which the inverse limit of $\{I, f\}$ is irreducible, there is a positive integer n such that $f^n[\overline{x_n y_n}] = I$.

Suppose f is a map from an interval $[a, b]$ into itself. A partition $a = p_0 < p_1 < ... < p_n = b$ of $[a, b]$ is said to be a *Markov partition* for f provided $\{p_0, p_1, ..., p_n\}$ is invariant under f, and f is monotone on $[p_{i-1}, p_i]$ for each i. A map from an interval onto itself possessing a Markov partition is called a *Markov map*. A *simple Markov map* is a Markov map for which there is a Markov partition, each periodic point of which is a maximal periodic continuum. The *critical points* of a Markov map f with Markov partition $p_0 < p_1 < ... < p_n$ are the endpoints p_0 and p_n together with any point p_j for which $f[p_{j-1}, p_{j+1}] - f(p_j)$ is connected.

A subset of a separable metric space X is a *Borel set* if and only if it belongs to the smallest σ-algebra of subsets of X that contains the topology on X. A function $f : X \to Y$ is a *Borel function* provided the inverse image of every open set in Y is a Borel set in X. If Y has a countable basis $\mathcal{B}$, this is equivalent to requiring that the inverse image of each basic open set from $\mathcal{B}$ is a Borel set in X.

If E is an equivalence relation on a set X, and x and y are points of X, then the notation xEy is used to denote that x is equivalent to y under E. An equivalence relation E on a set X is said to be *Borel reducible* to an equivalence relation F on a set Y provided there is a Borel function f from X to Y such that xEy if and only if $f(x)Ff(y)$. When E is Borel reducible to F and F is Borel reducible to E, then E and F are said to be *Borel bireducible*.

The *composant equivalence relation* of an indecomposable continuum is the equivalence relation according to which two points of the continuum are equivalent if and only if they belong to the same composant.

The equivalence relation on $\{0, 1\}^{\mathbb{N}}$ according to which x and y are equivalent if and only if $x_i = y_i$ for cofinitely many positive integers i is denoted by $\mathbb{E}_0$. The equivalence relation on $\left\{\{0, 1\}^{\mathbb{N}}\right\}^{\mathbb{N}}$ according to which x and y are equivalent if and only if $x_i = y_i$ for cofinitely many positive integers i is denoted by $\mathbb{E}_1$. The composant equivalence relation of an indecomposable continuum is always Borel bireducible with one of $\mathbb{E}_0$ and $\mathbb{E}_1$.

A *partition* of an interval I is a collection $\mathcal{B}$ of mutually exclusive subsets of I whose union is I. Notice that a Markov partition is not a partition in this sense. Consequently, "Markov" will be prefixed to "partition" whenever it is a Markov partition that is intended. If each member of a partition is a Borel set, then it is said to be a *Borel partition*.

3. Indecomposability and Maximal Periodic Continua

The main result of this section is Theorem 3.4. Together with Theorem 3.7, it gives that the inverse limit of a Markov map f is indecomposable if and only if some iteration of f is a two-pass map and if and only if f has at least three maximal periodic continua. In Sections 4 and 5, the maximal periodic continua of f play an important role in identifying composants in the inverse limit of f.

3.1. **Indecomposability in inverse limits of organic maps.**

Theorem 3.1. *Suppose f is a maps of an interval I onto itself, and suppose M is the inverse limit of $\{I, f\}$. If f has at least three maximal periodic continua, then M is indecomposable.*

Proof. Let A, B, and C denote three maximal periodic continua of f, and let p denote the product of their periods. Then $f^p[A] = A$, $f^p[B] = B$, and $f^p[C] = C$; and there are fixed points a, b, and c of f^p in A, B and C respectively. Denote the inverse limit of $\{I, f^p\}$ by K, and consider the points $\alpha = (a, a, a, ...)$, $\beta = (b, b, b, ...)$, and $\gamma = (c, c, c, ...)$ of K. For each n, $\pi_n[\overline{\alpha\beta}]$ is the closure of $\cup_i f^{ip}[\overline{ab}]$ (Lemma 1 of [9]) which, by Theorem 2.7 of [13], is equal to I. Thus K is irreducible between α and β. Similarly, K is irreducible between α and γ and between β and γ. It follows that K and, hence, M are indecomposable. □

Lemma 3.2. *Suppose f is an organic map of I onto itself, and suppose M is the inverse limit of $\{I, f\}$. If M is irreducible between two points, x and y, such that x_n and y_n are both in $\text{int}(I)$ for infinitely many n, then there are points p and q in $\text{int}(I)$ such that $f^2[p, q] = I$.*

Proof. Denote I by $[a, b]$. There is a positive integer N such that $f^N[x_N, y_N] = [a, b]$, and x_N and y_N are both in (a, b). It follows that $f(a, b) \not\subset (a, b)$, $f[a, b) \not\subset [a, b)$, and $f(a, b] \not\subset (a, b]$. Since $f(a, b) \not\subset (a, b)$, there is a point $p \in (a, b)$ such that either $f(p) = a$ or $f(p) = b$. Suppose $f(p) = a$; the remainder of the proof is similar if $f(p) = b$. Since $f[a, b) \not\subset [a, b)$, there is a point q of $[a, b)$ such that $f(q) = b$. If $q \neq a$, then p and q are both in (a, b); furthermore, $f[p, q] = [a, b]$, and the conclusion of the lemma follows. If $q = a$, then $f(a) = b$. Recall that $p \in (a, b)$ and $f(p) = a$. By the Intermediate Value Theorem, there is a point $c \in (a, p)$ such that $f(c) = p$. Then $f^2(c) = a$ and $f^2(p) = b$, and the conclusion of the lemma follows. □

Theorem 3.3. *Suppose f is a map of an interval I onto itself, M is the inverse limit of $\{I, f\}$, and M is indecomposable. The following are equivalent.*

(1) *f is organic.*

(2) *f^n is a two-pass map for some positive integer n.*

(3) *There are points p and q in $\text{int}(I)$ such that $f^2[p,q] = I$.*

Proof. First consider (1) $\Rightarrow$ (3). Suppose f is organic, and denote I by $[a,b]$. If a is not a periodic point of f, then no point of $\varprojlim\{I,f\}$ has infinitely many coordinates equal to a; if a is a periodic point of f, then there are only finitely many points of $\varprojlim\{I,f\}$ with infinitely many coordinates equal to a. Similarly there are at most finitely many points of $\varprojlim\{I,f\}$ with infinitely many coordinates equal to b. Hence there are points x and y from different composants, each of which has only finitely many coordinates in $\{a,b\}$. By Lemma 3.2, (3) follows.

To see that (3) implies (2), suppose there are points p and q in $\text{int}(I)$ such that $f^2[p,q] = I$. Let x, y, and z be points from three different composants of $\varprojlim\{I,f\}$. There is a positive integer N such that $f^N[x_N, y_N]$, $f^N[x_N, z_N]$, and $f^N[y_N, z_N]$ all contain $[p,q]$. Then $f^{N+2}[x_N, y_N]$, $f^{N+2}[x_N, z_N]$, and $f^{N+2}[y_N, z_N]$ all equal I. It follows that f^{N+2} is a two-pass map.

Finally, consider (2) $\Rightarrow$ (1). Suppose n is a positive integer for which f^n is a two-pass map. Then f^{2n} is a four-pass map. Hence there are points p and q, of $\text{int}(I)$ such that $f^{2n}[p,q] = I$. To see that f is organic, suppose x and y are points from different composants of $\varprojlim\{I,f\}$. There is a positive integer N such that $f^N[x_{N+2n}, y_{N+2n}]$ contains $[p,q]$. Hence $f^{N+2n}[x_{N+2n}, y_{N+2n}]$ contains I. Thus f is organic. □

Theorem 3.4. *Suppose f is an organic map of an interval I onto itself, and suppose M is the inverse limit of $\{I,f\}$. Then the following are equivalent.*

(1) *M is indecomposable.*

(2) *f^n is a two pass map for some n.*

(3) *f has at least three maximal periodic continua.*

Proof. The proof goes (1) $\Rightarrow$ (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) $\Rightarrow$ (3). The first implication follows from Theorem 3.3, the second is well known, and the third follows from Theorem 3.1. Consider (1) $\Rightarrow$ (3), and suppose M is indecomposable. Since (1) $\Rightarrow$ (2), there is a positive integer n such that f^n is a two-pass map. Then f^{3n} is an eight-pass map. Hence there are three fixed points, $x_1 < x_2 < x_3$ of f^{3n} in the interior of I such that $f^{3n}[x_1, x_2] = f^{3n}[x_2, x_3] = I$. Denote the collection of all nontrivial periodic continua that contain x_1 by $\mathcal{K}$. Then $\overline{\mathcal{K}^*}$ is a periodic continuum by Theorem 2.9 of [12]. Since $f^{3n}[x_1, x_2] = I$ and since $f^{3n}(x_2)$, which equals x_2, is an interior point of I, there is a point y of (x_1, x_2) such that $f^{3n}[x_1, y] = I$.

Then no term of $\mathcal{K}$ contains y. It follows that $\overline{\mathcal{K}^*}$ does not contain x_2 or x_3. Thus, $\overline{\mathcal{K}^*}$ is a nontrivial and, therefore, maximal periodic continuum. Similarly, there is a maximal periodic continuum containing x_2 that fails to contain x_1 or x_3, and a maximal periodic continuum containing x_3 that fails to contain x_1 or x_2. □

3.2. **Markov maps are organic.**

Lemma 3.5. *If f is a map of an interval I onto itself, and a is an endpoint of I, then at least one of the following holds.*

(1) *There is a p in $\text{int}(I)$ such that $f^2(p) = a$.*
(2) *The endpoint a is its own unique preimage under f^2.*

Proof. Suppose I has the form $[a, b]$; the proof is similar for the form $[b, a]$. Consider the following cases: $f(a, b) = [a, b]$, $f(a, b) = [a, b)$, $f(a, b) = (a, b]$, and $f(a, b) = (a, b)$. In the first two cases, $f^2(a, b)$ contains a and (1) follows. If $f(a, b) = (a, b]$, then either (2) holds or $f^2(a, b) = [a, b]$, from which (1) follows. If $f(a, b) = (a, b)$, then $f^2(a, b) = (a, b)$, $f^2(a) = a$, and $f^2(b) = b$, from which (2) follows. □

Lemma 3.6. *Suppose f is a Markov map of an interval I onto itself, and suppose a is an endpoint of I that is its own unique preimage under f^2. Denote the inverse limit of $\{I, f\}$ by M. If M is irreducible between x and y, then $\{x_1, y_1\}$ contains a.*

Proof. Denote by P a Markov partition for f. Then $P \cup f^{-1}(P)$ is a Markov partition for f^2. Denote $P \cup f^{-1}(P)$ by $p_0, p_1, ..., p_n$. The endpoint a is equal to one of p_0 and p_n, say p_0. Then $f^2(p_0) = p_0$ and $f^2[p_1, p_n] = [p_1, p_n]$. Suppose z and w are points of M such that $\min\{z_1, w_1\} > p_0$. Choose a_1 such that $p_0 < a_1 < \min\{x_1, y_1, p_1\}$. Put a_3 equal to $\min\{x : f^2(x) = a_1\}$. Then, since $f^2[p_1, p_n] = [p_1, p_n]$ and f^2 is monotone on $[p_0, p_1]$, $p_0 < a_3 < \min\{z_3, w_3, p_1\}$ and $f^2[a_3, p_n] = [a_1, p_n]$. Proceeding inductively, one may define a sequence $[a_1, p_n], [a_3, p_n], [a_5, p_n], ...$ such that, for each k, $p_0 < a_{2k-1} < \min\{z_{2k-1}, w_{2k-1}, p_1\}$ and $f^2[a_{2k+1}, p_n] = [a_{2k-1}, p_n]$. Put A_k equal to $[a_k, p_n]$ if k is odd and $f[a_k, p_n]$ if k is even. Then the inverse limit of $\{A_k, f\}$ is a proper subcontinuum of M that contains both z and w. Since M is irreducible between x and y, it follows that $\min\{x_1, y_1\} = p_0 = a$. □

Theorem 3.7. *Every Markov map is organic.*

Proof. Suppose f is a Markov map on an interval $[a, b]$. Denote the inverse limit of $\{[a, b], f\}$ by M, and suppose x and y are points between which M is irreducible. By Lemmas 3.5 and 3.6, either $f^2(p) = a$ for some p in the interior of I or $\{x_1, y_1\}$ contains a. In the former case, since $\cup_k f^k[\overline{x_{k+3} y_{k+3}}]$ contains (a, b) by Lemma 1 of [9], there is a positive integer N such that $f^N[\overline{x_{N+3}y_{N+3}}]$ contains p. It follows that $f^{k+2}[\overline{x_{k+3}y_{k+3}}]$ contains a for $k \geq N$. The same conclusion is a triviality for the case in which $\{x_1, y_1\}$ contains a. Similarly, there is a positive integer K such that $f^{k+2}[\overline{x_{k+3}y_{k+3}}]$ contains b for $k \geq K$. Then for $L = \max\{K+3, N+3\}$, $f^{L-1}[\overline{x_L y_L}] = [a, b]$. It follows that f is organic. □

4. Proper Subcontinua in Inverse Limits of Intervals

In this section, certain maximal periodic continua of $f : I \to I$ are used to partition the interval I into mutually exclusive subintervals in such a way that any two points x and y from the same composant of the inverse limit of f have the property that, for cofinitely many i, x_i and y_i belong to the same subinterval of I. Corollary 4.9 contains the precise formulation of this result. This necessary condition for two points to belong to the same composant is shown in the next section to be sufficient for a suitably chosen partition of I when the map f is a Markov map. In short, Corollary 4.9 is half of the first main result of the paper.

4.1. **The behavior of proper subcontinua whose projections intersect maximal periodic continua.**

Theorem 4.1. *Suppose f is a map of an interval I onto itself with at least two maximal periodic continua, M is the inverse limit of $\{I, f\}$, and suppose K is a subcontinuum of M. Then $K = M$ if and only if there are two maximal periodic continua of f such that $\pi_i[K]$ intersects both of them for infinitely many i.*

Proof. If $K = M$, then $\pi_i[K]$ intersects all maximal periodic continua of f for every i. Conversely, suppose there are two maximal periodic continua, A and B, of f such that $\pi_i[K]$ intersects both of them for infinitely many i. Denote the interval that is irreducible between A and B by J. Suppose n is a positive integer. For each positive integer i, $\pi_n[K] = f^i[\pi_{n+i}[K]]$. Since $\pi_{n+i}[K]$ contains J for infinitely many i, $\pi_n[K]$ contains $f^i[J]$ for infinitely many i. By Theorem 2.7 of [13], $\lim f^i[J] = I$. Since $\pi_n[K]$ is closed, it follows that $\pi_n[K] = I$. Hence $K = M$. □

Lemma 4.2. *Suppose g is a map of a compact metric space X onto itself. If $A_1, A_2, A_3, \ldots$ is a sequence of subsets of X such that $g[A_{i+1}] = A_i$ for each positive integer i, then $g[\limsup A_i] = \limsup A_i$.*

Proof. To see that $\limsup A_i \subset g[\limsup A_i]$, suppose y is a point of $\limsup A_i$. Let $D_1, D_2, D_3, \ldots$ denote a sequence of open sets closing in on y. There is an increasing sequence $n_1, n_2, n_3, ...$ of positive integers and a sequence $y_1, y_2, y_3, ...$ of points from $A_{n_1}, A_{n_2}, A_{n_3}, ...$ respectively such that $y_i \in D_i$ for each i. Let $x_1, x_2, x_3, ...$ denote points from $A_{1+n_1}, A_{1+n_2}, A_{1+n_3}, ...$ respectively such that $g(x_i) = y_i$ for each i. Since X is compact, there is a limit point x of $x_1, x_2, x_3, ...$. Note that $x \in \limsup A_i$. Since g is continuous, $g(x) = y$. Consequently, $y \in g[\limsup A_i]$, and, hence, $\limsup A_i \subset g[\limsup A_i]$.

For the purpose of showing the reverse containment, suppose y is a point of $g[\limsup A_i]$, and suppose D is an open set containing y. Then $y = g(x)$ for some point x in $\limsup A_i$. Since $g^{-1}[D]$ is an open set containing x, there are infinitely many terms of $A_1, A_2, A_3, ...$ that intersect $g^{-1}[D]$. For each i such that $g^{-1}[D]$ intersects A_{i+1}, D intersects A_i. Hence there are infinitely many terms of $A_1, A_2, A_3, ...$ that intersect D. It follows that $y \in \limsup A_i$, and, therefore, that $g[\limsup A_i] \subset \limsup A_i$. □

Lemma 4.3. *Suppose f is a map of an interval I onto itself, and suppose K is a proper subcontinuum of the inverse limit of $\{I, f\}$. If J is a maximal periodic continuum of f with period p, and $\pi_i[K]$ intersects J for infinitely many i, then there are positive integers n and N such that, for $i \geq N$, $\pi_i[K]$ intersects J if and only if $i = n + pj$ for some j.*

Proof. By the Pigeonhole Principal, there is a nonnegative integer n less than p such that $\pi_{n+pj}[K]$ intersects J for infinitely many j. Then $\pi_{n+pj}[K]$ intersects J for every j. Suppose, for the purpose of establishing contradiction, that there is an integer m in $\{0, 1, ..., p-1\} - \{n\}$ such that $\pi_{m+pj}[K]$ intersects J for infinitely many j. Then, depending upon whether $n < m$ or $m < n$, either $\pi_{n+pj}[K]$ intersects both J and $f^{m-n}[J]$ for infinitely many j, or $\pi_{m+pj}[K]$ intersects both J and $f^{n-m}[J]$ for infinitely many j. In either case, Theorem 4.1 implies that K is not a proper subcontinuum of the inverse limit, contrary to hypothesis. Hence, for each integer m in $\{1, 2, ..., p-1\} - \{n\}$, $\pi_{m+pj}[K]$ intersects J for only finitely many j. The conclusion of the Lemma follows. □

Theorem 4.4. *Suppose f is a map of an interval I onto itself with at least three maximal periodic continua, and suppose J is a maximal periodic continuum of f. If K is a proper subcontinuum of the inverse limit of $\{I, f\}$ and D is an open set containing J, then there is a positive integer N such that $\pi_i[K] \subset D$ whenever i is a positive integer not less than N for which $\pi_i[K]$ intersects J.*

Proof. The result is trivial if there are only finitely many positive integers for which $\pi_i[K]$ intersects J. Suppose $\pi_i[K]$ intersects J for infinitely many i. Denote the period of J by p. By Lemma 4.3, there are positive integers n and N_1 such that, for $i \geq N_1$, $\pi_i[K]$ intersects J if and only if $i = n + pj$ for some j. Notice that $\limsup(J \cup \pi_{n+pj}[K])$ is a continuum. Since $f^p[J \cup \pi_{n+p(j+1)}[K]] = J \cup \pi_{n+pj}[K]$, $\limsup(J \cup \pi_{n+pj}[K])$ is a periodic continuum by Lemma 4.2. Note that $\limsup(J \cup \pi_{n+pj}[K]) = J \cup \limsup \pi_{n+pj}[K]$. Since J is maximal, $J \cup \limsup \pi_{n+pj}[K]$ is either a subset of J or equal to I. By Theorem 5.4 of [13], there are infinitely many maximal periodic continua of f. Hence, there is a maximal periodic continuum J' different from J that contains an interior point of I. It follows that if $J \cup \limsup \pi_{n+pj}[K] = I$, then $\pi_{n+pj}[K]$ intersects J' for infinitely many positive integers j. Since $\pi_{n+pj}[K]$ intersects J for cofinitely many j, it follows from Theorem 4.1 that K fails to be a proper subcontinuum of the inverse limit, contrary to hypothesis. Consequently, $J \cup \limsup \pi_{n+pj}[K]$ is a subset of J. It follows that there are only finitely many values of j for which $\pi_{n+pj}[K]$ intersects the complement of D, or, equivalently, that there is a positive integer N_2 such that $\pi_{n+pj}[K] \subset D$ for $n + pj$ not less than N_2. The conclusion of the theorem follows with N equal to the maximum of N_1 and N_2. □

4.2. **The location of projections of proper subcontinua relative to maximal periodic continua and pre continuum-periodic points.**

Theorem 4.5. *Suppose f is a map of an interval $[a, b]$ onto itself with at least three maximal periodic continua, M is the inverse limit of $\{[a, b], f\}$, and x is a pre continuum-periodic point of f. If K is a proper subcontinuum of M, then there is a positive integer N such that, for each i not less than N, either $\pi_i[K] \subset [a, x)$ or $\pi_i[K] \subset (x, b]$.*

Proof. It suffices to show that if K is a subcontinuum of M such that $x \in \pi_i[K]$ for infinitely many i, then $K = M$. Let m denote a positive integer such that there is a maximal periodic continuum J that contains $f^m(x)$. Denote the period of J by p. Since $x \in \pi_i[K]$ for infinitely many i, it follows that $f^m(x) \in \pi_i[K]$ for infinitely many i. Consequently, $\pi_i[K]$ intersects $\text{orbit}(J)^*$ for every i. Then there is a term $\tilde{J}$ of $\text{orbit}(J)$ such that $\pi_i[K]$ both contains x and intersects $\tilde{J}$ for infinitely many i. Since x is pre continuum-periodic, x is not in $\tilde{J}$. Let D denote an open set containing $\tilde{J}$ that fails to contain x. Since there are infinitely many positive integers i such that $\pi_i[K]$ intersects the maximal periodic continuum $\tilde{J}$ but fails to lie in D, it follows from Theorem 4.4 that $K = M$. □

Definition. Suppose f is a map of an interval I onto itself, and suppose J is a nontrivial periodic continuum with period p.

- J is a *type-L* periodic continuum provided there is an open set D containing J such that J lies to the left of $f^p[D] - J$.
- J is a *type-R* periodic continuum provided there is an open set D containing J such that J lies to the right of $f^p[D] - J$.
- J is a *type-C* periodic continuum provided it is neither Type L nor Type R.

Theorem 4.6. *Suppose f is a map of an interval $[a, b]$ onto itself with at least three maximal periodic continua, and suppose J is a type-L maximal periodic continuum of f. If K is a proper subcontinuum of the inverse limit of $\{[a, b], f\}$, then there is a positive integer N such that, for each i not less than N, either $\pi_i[K] \subset [a, J)$ or $\pi_i[K] \subset [J, b]$.*

Proof. First note that the conclusion of the theorem is trivial if $\pi_i[K]$ intersects J for only finitely many i. Suppose $\pi_i[K]$ intersects J for infinitely many i. Denote the period of J by p, and let D denote an open set containing J such that J lies to the left of $f^p[D]-J$. By Theorem 4.4, there is a positive integer N_1 such that $\pi_i[K] \subset D$ whenever i is a positive integer not less than N_1 for which $\pi_i[K]$ intersects J. There are positive integers n and N_2 such that, for $i \geq N_2$, $\pi_i[K]$ intersects J if and only if $i = n + pj$ for some j by Lemma 4.3. Denote the maximum of N_1 and N_2 by N.

Suppose i is a positive integer not less than N. If $\pi_i[K]$ fails to intersect J, it follows that either $\pi_i[K] \subset [a, J)$ or $\pi_i[K] \subset [J, b]$. If $\pi_i[K]$ does intersect J, then there are positive integers n and j such that $i = n + pj$. Note that $n + p(j + 1) > N_2$. Consequently, $\pi_{n+p(j+1)}[K] \subset D$. It follows that J lies to the left of $f^p[\pi_{n+p(j+1)}[K]] - J$. However, $f^p[\pi_{n+p(j+1)}[K]] = \pi_{n+pj}[K] = \pi_i[K]$, so J lies to the left of $\pi_i[K] - J$. Thus $\pi_i[K] \subset [J, b]$. □

Theorem 4.7. *Suppose f is a map of an interval $[a, b]$ onto itself with at least three maximal periodic continua, and suppose J is a type-R maximal periodic continuum of f. If K is a proper subcontinuum of the inverse limit of $\{[a, b], f\}$, then there is a positive integer N such that, for each i not less than N, either $\pi_i[K] \subset [a, J]$ or $\pi_i[K] \subset (J, b]$.*

4.3. **Necessary conditions for points to lie in the same composant.**

Definition. Suppose f is a map of an interval $[a, b]$ onto itself, and suppose that, for some positive integer $n \geq 2$ and each nonnegative integer $i \leq n$, x_i is one of the following: an endpoint of $[a, b]$, a pre continuum-periodic point of f, or a type-L or type-R maximal periodic continuum. Suppose further that x_0 contains a, x_n contains b, and $x_0 < x_1 < ... < x_n$. The *partition of* $[a, b]$ *generated by* $x_0, x_1, ..., x_n$, denoted by $\mathcal{B}(x_0, x_1, ..., x_n)$, is the collection $B_1, B_2, ..., B_n$ of subsets of I that satisfies the following.

$$B_1 = \begin{cases} [x_0, x_1) & \text{if} \quad x_1 \text{ is type L} \\ [x_0, x_1] & \text{if} \quad x_1 \text{ is type R or pre continuum-periodic} \end{cases}$$

$$B_n = \begin{cases} (x_{n-1}, x_n] & \text{if} \quad x_{n-1} \text{ is type R or pre continuum-periodic} \\ [x_{n-1}, x_n] & \text{if} \quad x_{n-1} \text{ is type L} \end{cases}$$

and, for each i in $\{1, 2, ..., n - 1\}$,

$$B_i = \begin{cases} [x_{i-1}, x_i) & \text{if} \quad x_{i-1} \text{ is type L, and} \\ & \qquad x_i \text{ is type L} \\ [x_{i-1}, x_i] & \text{if} \quad x_{i-1} \text{ is type L, and} \\ & \qquad x_i \text{ is type R or pre continuum-periodic} \\ (x_{i-1}, x_i] & \text{if} \quad x_{i-1} \text{ is type R or pre continuum-periodic, and} \\ & \qquad x_i \text{ is type R or pre continuum-periodic} \\ (x_{i-1}, x_i) & \text{if} \quad x_{i-1} \text{ is type R or pre continuum-periodic, and} \\ & \qquad x_i \text{ is type L} \end{cases}$$

Remark. Every collection of sets of the form $\mathcal{B}(x_0, x_1, ..., x_n)$ as defined above is a Borel partition of I.

Theorem 4.8. *Suppose f is a map of an interval $[a, b]$ onto itself with at least three maximal periodic continua, suppose each of $x_0 < x_1 < ... < x_n$ is either a pre continuum-periodic point, one of a and b, or a type-L or type-R periodic continuum, and suppose $a \in x_0$ and $b \in x_n$. If K is a proper subcontinuum of the inverse limit of $\{[a, b], f\}$, then $\pi_i[K]$ is a subset of some term of $\mathcal{B}(x_0, x_1, ..., x_n)$ for each of cofinitely many i.*

Proof. It follows from Theorems 4.1, 4.5, 4.6, and 4.7 that there is a positive integer N such that each of the following holds for $i \geq N$.

(1) $\pi_i[K]$ intersects at most one maximal periodic continuum in $x_0, x_1, ..., x_n$.
(2) $\pi_i[K]$ fails to intersect each pre continuum-periodic point in $x_0, x_1, ..., x_n$.

(3) $\pi_i[K] \subset [x, b]$ if x is a type-L maximal periodic continuum in $\{x_0, x_1, ..., x_n\}$ that intersects $\pi_i[K]$.
(4) $\pi_i[K] \subset [a, x]$ if x is a type-R maximal periodic continuum in $\{x_0, x_1, ..., x_n\}$ that intersects $\pi_i[K]$.

Suppose i a positive integer i not less than N. If $\pi_i[K]$ fails to intersect each term of $x_1, x_2, ..., x_{n-1}$, then $\pi_i[K]$ is a subset of some term of $\mathcal{B}(x_0, x_1, ..., x_n)$. Suppose $\pi_i[K]$ intersects some term x_j of $x_1, x_2, ..., x_{n-1}$. By (1) and (2), $\pi_i[K]$ fails to intersect each term of $x_1, x_2, ..., x_{n-1}$ different from x_j, and, by (2), x_j is either a type L or type R maximal periodic continuum. Suppose x_j is type L. Then $\pi_i[K] \subset [x_j, b]$ by (3). According to the definition of $\mathcal{B}(x_0, x_1, ..., x_n)$, B_{j+1} contains x_j because x_j is type L. Consequently B_{j+1} is either $[x_j, x_{j+1}]$ or $[x_j, x_{j+1})$. Since $pi_i[K]$ fails to intersect each term of $x_1, x_2, ..., x_{n-1}$, different from x_j, either $x_{j+1} = b$ or $\pi_i[K] \subset [x_j, x_{j+1})$. In the former case, $\pi_i[K] \subset [x_j, x_{j+1}] = B_{j+1}$, and in the latter case, $\pi_i[K] \subset [x_j, x_{j+1}) \subset B_{j+1}$. Similarly, if x_j is type R, then $\pi_i[K] \subset B_j$. Consequently, for each $i \geq N$, $\pi_i[K]$ is a subset of some term of $\mathcal{B}(x_0, x_1, ..., x_n)$. □

Corollary 4.9. *Suppose f is a map of an interval $[a, b]$ onto itself with at least three maximal periodic continua, suppose each of $x_0 < x_1 < ... < x_n$ is either a pre continuum-periodic point, one of a and b, or a type-L or type-R periodic continuum, and suppose $a \in x_0$ and $b \in x_n$. If y and z are two points from the same composant of the inverse limit of $\{[a, b], f\}$, then y_i and z_i belong to the same term of $\mathcal{B}(x_0, x_1, ..., x_n)$ for each of cofinitely many i.*

Proof. By the previous theorem, for each of cofinitely many i, $\pi_i[\overline{yz}]$ is a subset of some term of $\mathcal{B}(x_0, x_1, ..., x_n)$. It follows that y_i and z_i are in the same term of $\mathcal{B}(x_0, x_1, ..., x_n)$ for each such i. □

5. The Classification of Composants for Indecomposable Inverse Limits of Markov Maps

Theorem 5.2 gives the classification of composants for inverse limits of simple Markov maps, which is relatively easy to formulate on the basis of the notation from the previous section. The classification for Markov maps in general requires simplifying the map to a simple Markov map via a monotone semiconjugacy, identifying $\mathcal{B}(x_0, x_1, ..., x_n)$ for the simple map, and lifting $\mathcal{B}(x_0, x_1, ..., x_n)$ back through the semiconjugacy. The result is Corollary 5.5. In Section 5.3, this process is carried out for a nontrivial Markov map.

5.1. **The classification for simple Markov maps.**

Lemma 5.1. *Suppose f is a simple Markov map of an interval I onto itself with Markov partition $p_0, p_1, ..., p_n$, and suppose j is a point of $\{1, 2, ..., p_{n-1}\}$ for which p_j is periodic. Then p_j is a type-L or type-R maximal periodic continuum if and only if there is a critical point in the orbit of p_j.*

Proof. Suppose there is a critical point in the orbit of p_j, and denote the period of p_j by p. For each nonnegative integer i not larger than p, let D_i denote an open interval that contains $f^i(p_j)$ but fails to contain any other term of $p_0, p_1, ..., p_n$. Then there is an open interval D containing p_j such that $f^i[D] \subset D_i$ for each i. Since the orbit of p_j contains a critical point, there is a nonnegative integer m less than p such that $f^m(p_j)$ is a critical point. Then $f^{m+1}(p_j)$ is an endpoint of $f[D_m]$. Since $f^{m+1}[D] \subset f[D_m]$, it follows that $f^{m+1}(p_j)$ is an endpoint of $f^{m+1}[D]$. Furthermore $f^{m+1}(p_j)$ is the only point of $p_0, p_1, ..., p_n$ that lies in $f^{m+1}[D]$, so f is monotone on $f^{m+1}[D]$. If $m + 1 = p$, then it follows that p_j is a type L or type R periodic point. If $m + 1 < p$, then note that $f^{m+2}(p_j)$ is an endpoint of $f^{m+2}[D]$ and $f^{m+2}[D]$ fails to intersect any other term of $p_0, p_1, ..., p_n$. Proceeding inductively, it follows that $f^p(p_j)$ is an endpoint of $f^p[D]$. Consequently, p_j is either a type-L or type-R periodic continuum. That it is maximal follows from the definition of simple Markov map.

Now suppose the orbit of p_j does not contain a critical point. Then the orbit of p_j fails to intersect $\{p_0, p_n\}$. Denote the period of p_j by p. Since no point of the orbit of p_j is a critical point, it follows that each of f^p and f^{2p} is monotone on some interval containing p_j. Then f^{2p} is monotonically nondecreasing.

Suppose, for the purpose of establishing contradiction, that there is a point $y \in (p_{j-1}, p_j) \cup (p_j, p_{j+1})$ such that $f^p(y) = f^p(p_j)$, or, equivalently, such that $f^p(y) = p_j$. Then there is a point $x \in (p_{j-1}, p_j) \cup (p_j, p_{j+1})$ such that $f^p(x) = y$. Hence $f^{2p}(x) = p_j$. For convenience, it will be assumed that $x \in (p_j, p_{j+1})$. A similar argument leads to a contradiction if $x \in (p_{j-1}, p_j)$. Then $f^{2p}(x) < x$. Let s denote $\sup\{z \geq p_j : f^{2p}$ is monotone on $[p_j, z]\}$. Then $f^{2p}(s)$ is one of $p_0, p_1, ..., p_n$. Since f^{2p} is nondecreasing on $[p_j, s]$, it follows that $f^{2p}(s) \geq p_{j+1}$. If $s \leq p_{j+1}$, then $[p_j, s] \subset f^{2p}[p_j, s] = f^{2p}[x, s]$. Consequently, there is a fixed point c of f^{2p} on $[x, s]$. If $s > p_{j+1}$, then f^{2p} is monotone on $[p_j, p_{j+1}]$. Hence $f^{2p}(p_{j+1}) > p_{j+1}$, from which it follows that $[p_j, p_{j+1}] \subset f^{2p}[p_j, p_{j+1}] = f^{2p}[x, p_{j+1}]$. Consequently, there is a fixed point c of f^{2p} on $[x, p_j + 1]$. In either case f^{2p} is monotone on $[p_j, c]$, and p_j and c are both fixed by f^{2p}. Therefore, $[p_j, c]$ is a periodic continuum, which contradicts the hypothesis that f is a simple Markov map. Hence, for each point $y \in (p_{j-1}, p_j) \cup (p_j, p_{j+1})$, $f^p(y) \neq p_j$.

To see that p_j is neither type L nor type R, suppose D is an open set containing p_j. Then $D \cap (p_{j-1}, p_{j+1})$ contains an open interval (a, b) containing p_j such that $f^p(a, b) \subset (p_{j-1}, p_{j+1})$. Consequently, f^p is monotone on $[a, b]$. By the conclusion of the previous paragraph, $f^p(a) \neq p_j$ and $f^p(b) \neq p_j$. Hence p_j is between $f^p(a)$ and $f^p(b)$, from which it follows that p_j separates $f^p(a, b)$ and, thus, $f^p[D]$. Consequently, p_j is neither type L nor type R. □

Definition. An *essential Markov partition* for a Markov map f is a Markov partition for f, each point of which is in the orbit of a critical point of f.

Suppose f is a map of an interval I into itself, and suppose the inverse limit M of $\{I, f\}$ is indecomposable. Then a partition $\mathcal{B}$ of the interval I is said to *determine the composants of* M if and only if, for each pair x and y of points of M, x and y belong to the same composant of M if and only if x_i and y_i belong to the same term of $\mathcal{B}$ for each of cofinitely many i.

Remark. The essential Markov partition for a Markov map f is the smallest of all Markov partitions for f and is unique. It is the union of all orbits of critical points of f and can be obtained from any Markov partition of f by removing all points that are not in the orbit of a critical point of f.

If $p_0, p_1, ..., p_n$ is the essential Markov partition for a simple Markov map, then each periodic point in $\{p_0, p_1, ..., p_n\}$ is a type L or type R maximal periodic continuum by Lemma 5.1, each preperiodic point in $\{p_0, p_1, ..., p_n\}$ is pre continuum-periodic, and $\mathcal{B}(p_0, p_1, ..., p_n)$ is well defined.

Theorem 5.2. *Suppose g is a simple Markov map with essential Markov partition $p_0, p_1, ..., p_n$, M is the inverse limit of $\{[p_0, p_n], g\}$, and M is indecomposable. Then $\mathcal{B}(p_0, p_1, ..., p_n)$ determines the composants of M.*

Proof. Since M is indecomposable, $n > 1$. Consequently, $\mathcal{B}(p_0, p_1, ..., p_n)$ has at least two terms. If x and y are points from the same composant of M, then x_i and y_i belong to the same term of $\mathcal{B}(p_0, p_1, ..., p_n)$ for each of cofinitely many i by Corollary 4.9. Conversely, since g is monotone on each term of $\mathcal{B}(p_0, p_1, ..., p_n)$, it follows that if x and y are two points of M such that x_i and y_i belong to the same term of $\mathcal{B}(p_0, p_1, ..., p_n)$ for each of cofinitely many i, then there is an arc in M, which is to say a proper subcontinuum of M, that contains both x and y. □

5.2. **The classification for Markov maps in general.**

Lemma 5.3. *Every Markov map is semiconjugate to a simple Markov map via a monotone semiconjugacy.*

Proof. Suppose f is a Markov map with Markov partition $p_0, p_1, ..., p_n$. Denote by $K_0, K_1, ..., K_k$ the collection of all maximal periodic continua that contain a point of $p_0, p_1, ..., p_n$ and all points of $p_0, p_1, ..., p_n$ that do not belong to a maximal periodic continuum, enumerated so that $K_0 < K_1 < ... < K_k$. There is a monotone quotient map m that identifies two points of $[p_0, p_n]$ if and only if they belong to the same term of $K_0, K_1, ..., K_k$ and a map g, defined on $m[p_0, p_n]$, such that $m \circ f = g \circ m$ (Lemma 4.5 of [12]).

Since f is invariant on $K_0 \cup K_1 \cup .... \cup K_k$, g is invariant on $m[K_0] \cup m[K_1] \cup .... \cup m[K_k]$. Since m is monotone, $m[p_0, p_n]$ is an arc and can be ordered so that $m[K_0] < m[K_1] < ... < m[K_k]$. Since f is monotone and m is injective on $\overline{K_{i-1}K_i}$ for each i, g is monotone on $\overline{m[K_{i-1}]m[K_i]}$ for each i. Each of $m[K_0], m[K_1], ..., m[K_k]$ is degenerate by definition of m. Thus g is a Markov map.

It remains only to show that g is simple. Suppose L is a nontrivial periodic continuum of g that contains one of $m[K_0], m[K_1], ..., m[K_k]$, say $m[K_i]$. Denote the period of L by p. For each positive integer n, $m \circ f^{np}[m^{-1}[L]] = g^{np} \circ m[m^{-1}[L]] = L$. Putting n equal to one gives $f^p[m^{-1}[L]] \subset m^{-1}[L]$. Then $f^p[m^{-1}[L]], f^{2p}[m^{-1}[L]], f^{3p}[m^{-1}[L]], ...$ is a decreasing sequence of continua. Denote the set of common points to the terms of this sequence by K. Then K is a continuum, $m[K] = L$, and $f^p[K] = K$. It follows that the nontrivial periodic continuum L is just the point $m[K_i]$; otherwise K would be a nontrivial periodic continuum that properly contains K_i, contrary to the definition of K_i. □

Theorem 5.4. *Suppose X and Y are continua, $f : X \to X$ and $g : Y \to Y$ are continuous surjections, m is a monotone map such that $m \circ f = g \circ m$. Denote the inverse limits of $\{X, f\}$ and $\{Y, g\}$ by M and N respectively. If N is indecomposable and $B_1, B_2, ..., B_n$ is a partition of Y that determines its composants, then M is indecomposable and $m^{-1}(B_1), m^{-1}(B_2), ..., m^{-1}(B_n)$ determines its composants.*

Proof. The monotone map m is surjective since g is surjective. Hence m induces a monotone map m^* from M onto N. It follows that M is indecomposable and that two points x and y of M belong to the same composant if and only if $m^*(x)$ and $m^*(y)$ belong to the same composant of N. Furthermore, for each i, x_i and y_i belong to the same term of $m^{-1}(B_1), m^{-1}(B_2), ..., m^{-1}(B_n)$ if and only if $m(x_i)$ and $m(y_i)$, which is to say $\pi_i(m^*(x))$ and $\pi_i(m^*(y))$, belong to the same term of $B_1, B_2, ..., B_n$. The conclusion of the theorem follows. □

Definition. Suppose f is a Markov map of an interval onto itself, $m \circ f = g \circ m$ where m is monotone and g is the simplification of f, and $p_0, p_1, ..., p_n$ is the essential Markov partition of g. The *partition of I generated by f*, denoted by $\mathcal{B}(f)$, is given by $\mathcal{B}(f) = \{m^{-1}(B) : B \in \mathcal{B}(p_0, p_1, ..., p_n)\}$.

Corollary 5.5. *The composants of a Markov map f are determined by $\mathcal{B}(f)$.*

5.3. **An example.** Consider the maps f and g whose graphs appear in Figure One. There are two cycles of maximal periodic continua for f that intersect its Markov partition: $[1, 2]$, $[6, 7]$, $[13, 14]$ and $[4, 5]$, $[11, 12]$. Shrinking each of these intervals down to points yields the map g, the simplification of f guaranteed by Lemma 5.3. The essential Markov partition for g consists of the Markov partition it inherits from f, namely $\{a, b, c, d, e, f, g, h, i\}$, less those points that fail to lie in the orbit of a critical point, namely the preperiodic point e and the type-C periodic continua c and h. Hence the essential Markov partition for g is $\{a, b, d, f, g, i\}$. Among these points, a is the only type-L periodic continuum, d and i are type-R periodic continua, and b, f, and g are pre continuum-periodic. Hence $\mathcal{B}(g)$ consists of the intervals $[a, b]$, $(b, d]$, $(d, f]$, $(f, g]$, and $(g, i]$; and $\mathcal{B}(f)$ consists of the intervals $[1, 3]$, $(3, 7]$, $(7, 9]$, $(9, 10]$, and $(10, 14]$.

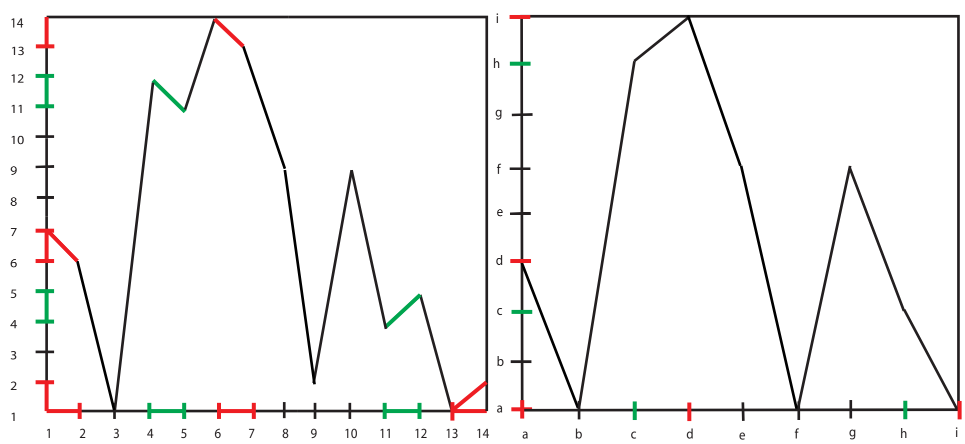

Figure One

6. The Composant Equivalence Relation of Indecomposable Inverse Limits of Markov Maps

The composant equivalence relation of an indecomposable continuum was first studied by James T. Rogers Jr. [10] to tackle the classic question as to whether or not the composants of an indecomposable continuum admit a Borel transversal, answering it in the negative for Knaster continua and solenoids. Alexander S. Kechris and Alain Louveau [8] showed that every hypersmooth equivalence relation on a Polish space is either smooth or Borel bireducible with one of two canonical forms, $\mathbb{E}_0$ and $\mathbb{E}_1$. Only in the smooth case is there a Borel transversal to the equivalence classes. Rogers had shown that the composant equivalence relation of an indecomposable continuum is the union of countably many compact equivalence relations. A tweaking of his argument gives that the composant equivalence relation is hypersmooth. Slawomir Solecki [16] made this observation and demonstrated the impossibility of a composant equivalence relation being smooth by embedding $\mathbb{E}_0$ into it. In so doing, he not only demonstrated the nonexistence of Borel transversals for all indecomposable continua, but also established that, up to Borel bireducibility, $\mathbb{E}_0$ and $\mathbb{E}_1$ are the only possibilities for the composant equivalence relation. Hereditarily indecomposable continua have the more complicated $\mathbb{E}_1$ type composant structure, while Knaster continua have the simpler $\mathbb{E}_0$ type.

Using the classification of composants of indecomposable inverse limits of Markov maps from the previous section, it is not difficult to show that the composant equivalence relation of such continua is Borel reducible to $\mathbb{E}_0$. This follows immediately from Theorem 6.3, which is stated and proved in a more general setting.

Lemma 6.1. *There is a continuous function, γ, from $\Pi_{i\in\mathbb{N}}\{1,2,...,n_i\}$ into $\{0,1\}^{\mathbb{N}}$ such that, for any two points x and y of $\Pi_{i\in\mathbb{N}}\{1,2,...,n_i\}$, $x_i = y_i$ for cofinitely many i if and only if $\pi_i(\gamma(x)) = \pi_i(\gamma(y))$ for cofinitely many i.*

Proof. For each positive integer i, denote by g_i the function that assigns to each element of $\{1,2,...,n_i\}$ its unique binary representation with $\lceil \log_2(n_i)\rceil$ characters, where $\lceil \log_2(n_i)\rceil$ denotes the smallest integer that is strictly larger than $\log_2(n_i)$. For each $x = x_1x_2x_3...$ in $\Pi_{i\in\mathbb{N}}\{1,2,...,n_i\}$, define $\gamma(x)$ to be the concatenation $g_1(x_1)g_2(x_2)g_3(x_3)....$ □

Lemma 6.2. *If there is a Borel function χ from an indecomposable continuum M into $\prod_{i\in\mathbb{N}}\{1,2,...,n_i\}$ such that two points x and y belong to the same composant of M if and only if $\pi_i \circ \chi(x) = \pi_i \circ \chi(y)$ for cofinitely many i, then M is an $\mathbb{E}_0$ continuum.*

Proof. The composant equivalence relation of M is reducible to $\mathbb{E}_0$ via the Borel function $\gamma \circ \chi$, where γ is the map guaranteed by Lemma 6.1. □

Theorem 6.3. *Suppose* $\{X_i, f_i\}$ *is an inverse sequence,* M *is the inverse limit of* $\{X_i, f_i\}$*, and* M *is indecomposable. If, for each positive integer* i*, there is a Borel partition* $B_1^i, B_2^i, ..., B_{n_i}^i$ *of* X_i *such that two points,* x *and* y*, of* M *belong to the same composant if and only if* x_i *and* y_i *belong to the same term of* $B_1^i, B_2^i, ..., B_{n_i}^i$ *for cofinitely many positive integers* i*, then* M *is an* $\mathbb{E}_0$ *continuum.*

Proof. For each positive integer i and each x in M, let $\chi_i(x)$ denote the integer l for which x_i belongs to B_l^i, which is to say, $x_i \in B_l^i$ if and only if $l = \chi_i(x)$. For each x in M, let $\chi(x)$ denote the sequence $\chi_1(x), \chi_2(x), \chi_3(x), ...$ Then χ is a function from M into $\Pi_{i\in\mathbb{N}}\{1, 2, ..., n_i\}$, and x and y belong to the same composant of M if and only if $\chi_i(x) = \chi_i(y)$ for cofinitely many positive integers i.

By Lemma 6.2, it remains only to show that χ is a Borel function. To that end, denote by $\mathcal{B}$ the collection to which a subset of $\Pi_{i\in\mathbb{N}}\{1, 2, ..., n_i\}$ belongs if and only if its inverse image under χ is a Borel subset of M. Note that $\mathcal{B}$ is a σ-algebra (Theorem 1.12, [11]). Since $\Pi_{i\in\mathbb{N}}\{1, 2, ..., n_i\}$ is second countable, $\mathcal{B}$ contains all open subsets of $\Pi_{i\in\mathbb{N}}\{1, 2, ..., n_i\}$ if and only if it contains all subbasic open subsets of $\Pi_{i\in\mathbb{N}}\{1, 2, ..., n_i\}$. Thus, to show that χ is a Borel function, it suffices to show that $\mathcal{B}$ contains all subbasic open subsets of $\Pi_{i\in\mathbb{N}}\{1, 2, ..., n_i\}$.

Suppose D is a subbasic open set in $\Pi_{i\in\mathbb{N}}\{1, 2, ..., n_i\}$. Then there are a positive integer N and a subset, A, of $\{1, 2, ..., n_N\}$ such that $D = \{z \in \Pi_{i\in\mathbb{N}}\{1, 2, ..., n_i\} : z_N \in A\}$. Denote $\{B_l^N : l \in A\}^*$ by B_A^N, and note that it is a Borel set. It has already been noted that $x_N \in B_l^N$ if and only if $\chi_N(x) = l$. It follows that $x_N \in B_A^N$ if and only if $\chi_N(x) \in A$. Hence,

$$\begin{aligned} \chi^{-1}[D] &= \{x \in M : \chi(x) \in D\} \\ &= \{x \in M : \chi_N(x) \in A\} \\ &= \{x \in M : x_N \in B_A^N\} \\ &= \pi_1^{-1}[B_A^N]. \end{aligned}$$

Since B_A^N is a Borel set, $\chi^{-1}[D]$ is a Borel set. Hence $\mathcal{B}$ contains all subbasic open sets in $\Pi_{i\in\mathbb{N}}\{1, 2, ..., n_i\}$, from which it follows that χ is a Borel function. □

Corollary 6.4. *Every inverse limit of Markov maps is an* $\mathbb{E}_0$ *continuum.*

References

[1] Marcy Barge and Beverly Diamond, *Homeomorphisms of inverse limit spaces of one-dimensional maps*, Fund. Math. 146 (1995), pp. 171 - 187.

[2] Louis Block, Slagjana Jakimovik, Lois Kailhofer, James Keesling, *On the classification of inverse limits of tent maps*, Fund Math. 187 (2005), pp. 171 - 192.

[3] Karen M. Brucks and Beverly Diamond, *A symbolic representation of inverse limit spaces for a class of unimodal maps*, Continua with the Houston Problem Book, pp. 207 - 225.

[4] Henk Bruin, *Inverse limit spaces of post-critically finite tent maps*, Fund. Math. 165 (2000), pp. 125 - 138.

[5] Chris Good and Brian Raines, *Continuum many tent map inverse limits with homeomorphic postcritical ω-limit sets*, Fund. Math. 191 (2006), no. 1, pp. 1-21.

[6] W. T. Ingram, *Inverse limits and dynamical systems*, Problem 4, to appear in a forthcoming book on open problems in topology.

[7] Lois Kailhofer, *A classification of inverse limit spaces of tent maps with periodic critical points*, Fund. Math. 177 (2003), no. 2, pp. 95 - 120.

[8] Alexander S. Kechris and Alain Louveau, *The classification of hypersmooth Borel equivalence relations*, Journal of the AMS 10, (1997), no. 1, pp. 215 - 242.

[9] D. P. Kuykendall, *Irreducibility and indecomposability in inverse limits*, Fundamenta Mathematicae 80 (1973), pp. 265 - 270.

[10] James T. Rogers, Jr., *Borel transversals and ergodic measures on indecomposable continua*, Topology and its Applications 24 (1986), pp. 217 - 227.

[11] Walter Rudin, *Real and Complex Analysis*, third edition, McGraw-Hill, Inc.

[12] David J. Ryden, *The Sarkovskii order for periodic continua*, preprint.

[13] David J. Ryden, *The Sarkovskii order for periodic continua II*, preprint.

[14] Sonja Štimac, *Structure of inverse limit spaces of tent maps with finite critical orbit*, Fund. Math. 191 (2006), no. 2, pp. 125 - 150.

[15] Sonja Štimac, *On the inverse limits of tent maps with critical points approaching periodic points*, talk given at the 40th Spring Topology and Dynamical Systems Conference, University of North Carolina at Greensboro, March 23 -25, 2006.

[16] Slawomir Solecki, *The space of composants of an indecomposable continuum*, Advances in Mathematics 166 (2002), pp. 149 - 192.

Department of Mathematics, Baylor University, Waco, Texas 76798-7328

E-mail address: David_Ryden@baylor.edu