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Topology and its Applications 149 (2005) 177–194
Topology and its Applications
www.elsevier.com/locate/topol

Composants and the structure of periodic orbits for interval maps

David J. Ryden [1]

Department of Mathematics, Tulane University, New Orleans, LA 70118-5698, USA

Received 6 February 2004; accepted 4 August 2004

Abstract

Suppose f is a map from an interval $[a, b]$ into itself with a periodic orbit consisting of the points $p_1 < p_2 < \cdots < p_n$. This paper begins with an analysis of the structure of periodic orbits for interval maps. Blocks are defined and used to describe this structure. With these structural theorems in place, results relating blocks of $p_1, p_2, \ldots, p_n$ to irreducibility in the inverse limit of $\{[a, b], f\}$ are proved.

Assuming $p_1, p_2, \ldots, p_n$ is a Markov partition for f, necessary and sufficient conditions are given for two points of the inverse limit to belong to the same composant. This characterization of composants is used to show that the inverse limit is an $\mathbb{E}_0$-type continuum.

MSC: 54F15; 37B45

Keywords: Continuum; Indecomposable; Composant; Inverse limit; Markov map; Periodic; Block

1. Introduction

It is often useful, when considering an indecomposable continuum that is generated by an inverse limit, to have a characterization of its composants in terms of the inverse limit representation. For example, it is well known that the Brouwer–Janiszewski–Knaster continuum, sometimes called the buckethandle or the horseshoe, is homeomorphic to the

E-mail address: david_ryden@baylor.edu (D.J. Ryden).
[1] Current address: Department of Mathematics, Baylor University, Waco, TX 76798-7328, USA.

0166-8641/$ – see front matter © 2004 Published by Elsevier B.V.
doi:10.1016/j.topol.2004.08.023

inverse limit of the full tent map on $[0,1]$. Two points, x and y, of this inverse limit belong to different composants if and only if x_i and y_i straddle $\frac{1}{2}$ for infinitely many positive integers i. Bandt [1] used this characterization to show that all of the composants of this continuum are homeomorphic, with the exception of the one that contains the endpoint $(0,0,0,\ldots)$. Brucks and Diamond [2] proved a similar characterization for composants of inverse limits of certain unimodal maps and used it to describe embeddings of those inverse limits into the plane.

By far the most general result for determining whether two points of an inverse limit belong to the same composant was given by Kuykendall [4]. His result applies to inverse limits of metric continua where neither the sequence of bonding maps nor the sequence of factor spaces need be constant. The author [5] gave a procedure to construct, for an inverse limit of intervals, two points between which the continuum is irreducible. However, the procedure does not readily extend to one that produces, for an indecomposable inverse limit of intervals, a third point that belongs to a composant different from that of the first two. Furthermore, as with Kuykendall's theorem, the bonding maps and the factor spaces may vary, so neither result capitalizes upon the relationship between dynamics and continuum theory that is present in inverse limits of a single bonding map, and which was exploited by Bandt [1] and Brucks and Diamond [2].

This paper grew out of an endeavor to classify, via considerations from dynamics, the composants of $\varprojlim\{[p_1, p_n], f\}$, where f is a Markov map whose partition, $p_1, p_2, \ldots, p_n$, is a single periodic orbit of f. One of the main results of the paper, Theorem 16 in Section 4, gives such a characterization. In Section 5, this characterization is used to show that such inverse limits, when indecomposable, are $\mathbb{E}_0$ continua. Solecki [7] has shown that there are, in some sense, only two categories for the complexity of the composant equivalence relation of an indecomposable continuum; $\mathbb{E}_0$ is the simpler of the two. It is not surprising that the family of continua considered here falls into the simpler case; however, the simplicity of the family allows for the overt construction of a Borel function that relates the composant equivalence relation of $\varprojlim\{[p_1, p_n], f\}$ to $\mathbb{E}_0$. The description of these continua with inverse limits provides a natural medium for an analytic discussion of the composant equivalence relation, which Solecki considered from the viewpoint of descriptive set theory.

Sections 2 and 3 lay the groundwork for these results. Blocks are defined in Section 2, and they are used to describe behavior that is shown to occur in periodic orbits for interval maps. In Section 3, the structural theorems of Section 2 are used to prove results that relate blocks of periodic orbits to irreducibility in certain inverse limits.

Suppose f is a function from $[a,b]$ into itself. The *orbit* of a point p of $[a,b]$, denoted by $\text{orbit}(p)$, is the set $\{y\colon\ y = f^i(p) \text{ for some } i \in \mathbb{N}\}$. A point p of $[a,b]$ is said to be *periodic* provided there is a positive integer n such that $f^n(p) = p$. The *period* of a periodic point p is the smallest positive integer n such that $f^n(p) = p$.

A sequence $p_1, p_2, \ldots, p_n$ of points is said to be an *n-cycle* of f provided

(1) $a \leqslant p_1 < p_2 < \cdots < p_n \leqslant b$,
(2) p_1 is periodic with period n, and
(3) $\text{orbit}(p_1) = \{p_1, p_2, \ldots, p_n\}$.

A *map* is a continuous function. Suppose f is a map from an interval $[a,b]$ into itself. A partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a,b]$ is said to be a *Markov partition* for f provided $\{x_0, x_1, \ldots, x_n\}$ is invariant under f, and f is monotone on $[x_{i-1}, x_i]$ for each positive integer i not larger than n. A map possessing a Markov partition is called a *Markov map*.

A *continuum* is a compact connected subset of a metric space. A continuum is *irreducible* between the points p and q if and only if it has no proper subcontinuum that contains both p and q. A continuum is said to be *indecomposable* provided it is not the union of two of its proper subcontinua; otherwise, it is said to be *decomposable*. The *composant* of a point p in a continuum M is the union of all proper subcontinua of M that contain p. It is well known that the collection of distinct composants in an indecomposable continuum is uncountable and pairwise disjoint.

Suppose $X_1, X_2, X_3, \ldots$ is a sequence of metric spaces and, for each positive integer n, f_n is a continuous function from X_{n+1} into X_n. The sequence $\{X_n, f_n\}$ is called an *inverse sequence*, the spaces X_n are called *factor spaces*, and the functions f_n are called *bonding maps*. The *inverse limit* of the inverse sequence $\{X_n, f_n\}$, denoted by $\varprojlim\{X_n, f_n\}$, is the subset of the product space $\prod X_n$ to which x belongs if and only if $f_n(x_{n+1}) = x_n$ for each positive integer, n. It is well known that $\varprojlim\{X_n, f_n\}$ is a continuum if each of the factor spaces is a continuum. The *projection* of the product space $\prod X_n$ into X_n, denoted by π_n, is the function from $\prod X_n$ into X_n that satisfies $\pi_n(x) = x_n$ for each x in $\prod X_n$.

The factor spaces for all of the inverse limits in this paper are intervals. If there is a map $f : [a,b] \to [a,b]$ such that $f_i = f$ for each positive integer i, then $\varprojlim\{X_i, f_i\}$ may be denoted by $\varprojlim\{[a,b], f\}$. If $[c,d]$ is a subinterval of $[a,b]$ such that $f[c,d] = [c,d]$, then $\varprojlim\{[c,d], f\}$ denotes the subcontinuum $\varprojlim\{[c,d], f|[c,d]\}$ of $\varprojlim\{[a,b], f\}$. The *shift homeomorphism*, denoted by $\hat{f}$, is the homeomorphism from $\varprojlim\{[a,b], f\}$ onto itself that satisfies $\hat{f}(x_1, x_2, x_3, \ldots) = (f(x_1), f(x_2), f(x_3), \ldots) = (f(x_1), x_1, x_2, \ldots)$.

2. The basics of blocks

Throughout this section, it will be assumed that f is a function from $[a,b]$ into itself and that $p_1, p_2, \ldots, p_n$ is an n-cycle of f. The results of this section are of a combinatorial nature and do not even require continuity of f. Stronger assumptions regarding f will be added in Section 3 and again in Section 4.

Example 1. Consider the maps g, h, and k whose graphs appear in Fig. 1. Note that $1, 2, 3, 4, 5, 6$ is a 6-cycle of all three maps. In g and h, the cycle has the itinerary $1 \to 4 \to 6 \to 2 \to 3 \to 5 \to 1$, whereas, in k, the itinerary is $1 \to 5 \to 2 \to 6 \to 3 \to 4 \to 1$.

In a sense (to be made precise below), the points of the cycle proceed in blocks of two in g and $h : \{1,2\} \to \{3,4\} \to \{5,6\} \to \{1,2\}$. For the cycle in k, this is not the case; however, blocks of three travel together: $\{1,2,3\} \to \{4,5,6\} \to \{1,2,3\}$. Note that this behavior is more easily spotted in the graphs of h and k because $1, 2, 3, 4, 5, 6$ is a Markov partition

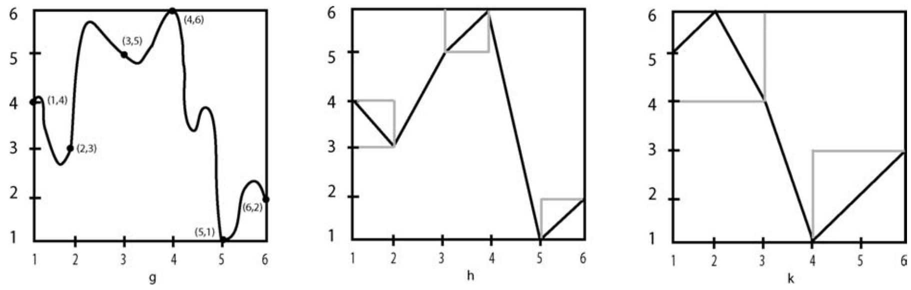

Fig. 1.

for each of them. The 6-cycle $1 \to 2 \to 5 \to 3 \to 4 \to 6 \to 1$, which does not appear in g, h, or k, has no block of length two or of length three.

Definitions. A set $\{p_{l+1}, p_{l+2}, \ldots, p_{l+k}\}$ of consecutive terms of an n-cycle $p_1, p_2, \ldots, p_n$ is called a *block of* $p_1, p_2, \ldots, p_n$ *with respect to* f if and only if, for each positive integer i, there is a set $\{p_{m+1}, p_{m+2}, \ldots, p_{m+k}\}$ of consecutive terms of $p_1, p_2, \ldots, p_n$ such that $f^i\{p_{l+1}, p_{l+2}, \ldots, p_{l+k}\} = \{p_{m+1}, p_{m+2}, \ldots, p_{m+k}\}$. Either or both of the phrases "of $p_1, p_2, \ldots, p_n$" and "with respect to f" may be dropped when context permits doing so without diminishing clarity.

Since each point of $p_1, p_2, \ldots, p_n$ has period n, $\{p_{l+1}, p_{l+2}, \ldots, p_{l+k}\}$ is a block if and only if $f^i\{p_{l+1}, p_{l+2}, \ldots, p_{l+k}\}$ is a set of consecutive terms of $p_1, p_2, \ldots, p_n$ for $i = 1, 2, \ldots, n$. Also note that if B is a block, then so is each of $f[B]$, $f^2[B]$, $f^3[B], \ldots$.

The *period* of a block $\{p_{j+1}, p_{j+2}, \ldots, p_{j+k}\}$ is the smallest positive integer i such that $f^i\{p_{j+1}, p_{j+2}, \ldots, p_{j+k}\} = \{p_{j+1}, p_{j+2}, \ldots, p_{j+k}\}$. Note that every block is periodic.

Two blocks are said to be *adjacent* if and only if they are disjoint and their union is a consecutive set of terms of $p_1, p_2, \ldots, p_n$.

Lemma 2. *Suppose f is a function from $[a, b]$ into itself with n-cycle $p_1, p_2, \ldots, p_n$. If A and B are blocks with a nonempty intersection, then one is contained in the other. Furthermore, if A and B have the same length, then they are either disjoint or identical.*

Proof. For some i, $f^i[A \cap B]$ contains p_1. Hence $p_1 \in f^i[A] \cap f^i[B]$. Since each of $f^i[A]$ and $f^i[B]$ consists of consecutive terms of $p_1, p_2, \ldots, p_n$, it follows that one of $f^i[A]$ and $f^i[B]$ is a subset of the other. Consequently, one of A and B is a subset of the other. The conclusion of the lemma follows. □

Theorem 3. *Suppose f is a function from $[a, b]$ into itself with n-cycle $p_1, p_2, \ldots, p_n$. If B is a block of length k, then k divides n, and the period of B is n/k.*

Proof. Denote the period of B by p. Then each of $B, f[B], \ldots, f^{p-1}[B]$ has period p and length k. Since all of $B, f[B], \ldots, f^{p-1}[B]$ have the same length and no two are identical, it follows from Lemma 2 that they are pairwise-disjoint. Note that $B \cup f[B] \cup$

$\cdots \cup f^{p-1}[B] = \{p_1, p_2, \ldots, p_n\}$. Consequently, $n = pk$, from which the conclusion of the theorem follows. □

Notation. Suppose $p_1, p_2, \ldots, p_n$ is an n-cycle of f, and suppose k is a positive integer that divides n. For each positive integer j not larger than $\frac{n}{k}$, $B_{k,j}$ denotes the set $\{p_{k(j-1)+1}, p_{k(j-1)+2}, \ldots, p_{k(j-1)+k} = p_{kj}\}$. In particular, $B_{k,1}$ denotes $\{p_1, p_2, \ldots, p_k\}$, and $B_{k,\frac{n}{k}}$ denotes $\{p_{n-k+1}, p_{n-k+2}, \ldots, p_n\}$.

Definition. A sequence, $B_1, B_2, \ldots, B_j$ of blocks is said to be a *block cycle* of $p_1, p_2, \ldots, p_n$ with respect to f provided

(1) for $i_1 < i_2$, every point of B_{i_1} is less than every point of B_{i_2},
(2) B_1 is periodic with period j, and
(3) $\text{orbit}(B_1) = \{B_1, B_2, \ldots, B_j\}$.

Theorem 4. *Suppose f is a function from $[a, b]$ into itself with n-cycle $p_1, p_2, \ldots, p_n$. The following are equivalent*:

(1) *There is a block of length k.*
(2) *There are $\frac{n}{k}$ blocks of length k.*
(3) $\{p_1, p_2, \ldots, p_k\}$ *is a block.*
(4) *The blocks of length k are $B_{k,1}, B_{k,2}, \ldots, B_{k,\frac{n}{k}}$.*
(5) *There is a block cycle of period $\frac{n}{k}$.*
(6) $B_{k,1}, B_{k,2}, \ldots, B_{k,\frac{n}{k}}$ *is a block cycle.*

Proof. The proof goes (1) → (3) → (6) → (4) → (2) → (1), and (6) → (5) → (1). Assume (1), and suppose B is a block of length k. Recall that forward images of blocks are blocks. Then for suitably chosen i, $f^i[B] = \{p_1, p_2, \ldots, p_k\}$; (3) follows.

It was shown in the proof of Theorem 3 that if B is any block of length k, then $B, f[B], \ldots, f^{p-1}[B]$, where p denotes the period of B, are disjoint blocks of length k whose union is $\{p_1, p_2, \ldots, p_n\}$. Consequently, the terms of $B, f[B], \ldots, f^{p-1}[B]$ are, up to rearrangement, the terms of $B_{k,1}, B_{k,2}, \ldots, B_{k,\frac{n}{k}}$. It follows that (3) implies (6).

Suppose $B_{k,1}, B_{k,2}, \ldots, B_{k,\frac{n}{k}}$ is a block cycle. Since $B_{k,1} \cup B_{k,2} \cup \cdots \cup B_{k,\frac{n}{k}} = \{p_1, p_2, \ldots, p_n\}$, it follows from Lemma 2, that there are no other blocks of length k. Thus (6) implies (4).

It is trivial that (4) implies (2), (2) implies (1), and (6) implies (5). To complete the proof of the theorem, it remains only to show that (5) implies (1). Let B denote the block in the cycle that contains p_1. Then $B = \{p_1, p_2, \ldots, p_j\}$ for some positive integer j. By Theorem 3, B has period $\frac{n}{j}$, and, by hypothesis, B has period $\frac{n}{k}$. It follows that $j = k$. □

3. Blocks and irreducibility

Henceforth, it will be assumed that f is continuous, which is to say that f is a map of $[a, b]$ into itself for which $p_1, p_2, \ldots, p_n$ is an n-cycle. Theorem 6 may be regarded as the main result of the section, as it is fundamental to the proofs of many results in Sections 4 and 5. Theorem 5, which is used to prove Theorem 6, also plays a pivotal role in [6] where it is used to relate block structure of periodic orbits to decomposability in inverse limits.

Definition. A *maximal block* is a block for which the one and only block that properly contains it is $\{p_1, p_2, \dots, p_n\}$.

Notation. If M is a continuum that is either an interval or an inverse limit of intervals, and x and y are points of M, then $\overline{xy}$ denotes the smallest subcontinuum of M that contains both x and y.

Theorem 5. *Suppose f is a map from $[a, b]$ into itself with n-cycle $p_1, p_2, \dots, p_n$. There is a positive integer N such that, for each pair, x and y, of points belonging to different maximal blocks, $[p_1, p_n] \subset f^N[\overline{xy}]$.*

Proof. First note that $[p_1, p_n] \subset f^k[p_1, p_n]$ for every positive integer k. Consequently, if $[p_1, p_n] \subset f^N[\overline{xy}]$, then, for every integer k greater than N, $[p_1, p_n] \subset f^{k-N}[p_1, p_n] \subset f^k[\overline{xy}]$. Thus, since there are only finitely many pairs of points from maximal blocks, it suffices to show that for each pair, x and y, belonging to different maximal blocks, there is a positive integer N such that $[p_1, p_n] \subset f^N[\overline{xy}]$.

Since f is one-to-one on $\{p_1, p_2, \dots, p_n\}$, it follows that, for each positive integer i, $f^{i+1}[\overline{xy}] \cap \{p_1, p_2, \dots, p_n\}$ has at least as many points as $f^i[\overline{xy}] \cap \{p_1, p_2, \dots, p_n\}$. Consequently, there is a positive integer K such that, for $i \geqslant K$, $f^i[\overline{xy}] \cap \{p_1, p_2, \dots, p_n\}$ and $f^K[\overline{xy}] \cap \{p_1, p_2, \dots, p_n\}$ have the same cardinality. Denote Kn by N, and consider the set $f^N[\overline{xy}] \cap \{p_1, p_2, \dots, p_n\}$.

Notice that $f^N[\overline{xy}] \cap \{p_1, p_2, \dots, p_n\}$ contains both x and y. Applying f^i to $f^N[\overline{xy}] \cap \{p_1, p_2, \dots, p_n\}$ gives $f^{N+i}[\overline{xy}] \cap \{p_1, p_2, \dots, p_n\}$. Since $f^{N+i}[\overline{xy}]$ is connected for each nonnegative integer i, $f^{N+i}[\overline{xy}] \cap \{p_1, p_2, \dots, p_n\}$ is a set of consecutive terms of $p_1, p_2, \dots, p_n$ for each such i. Consequently $f^N[\overline{xy}] \cap \{p_1, p_2, \dots, p_n\}$ is a block. But, by hypothesis, no maximal block contains both x and y, so $f^N[\overline{xy}] \cap \{p_1, p_2, \dots, p_n\}$ is the trivial block, $\{p_1, p_2, \dots, p_n\}$. It follows that $[p_1, p_n] \subset f^N[\overline{xy}]$. □

Theorem 6. *Suppose f is a map from $[p_1, p_n]$ into itself with n-cycle $p_1, p_2, \dots, p_n$, and suppose K is a subcontinuum of $\varprojlim\{[p_1, p_n], f\}$. The following are equivalent.*

(1) $K = \varprojlim\{[p_1, p_n], f\}$.
(2) *For infinitely many positive integers i, $\pi_i[K]$ intersects at least two maximal blocks.*
(3) *There are points x and y of K such that x_i and y_i belong to different maximal blocks for infinitely many positive integers i.*

Proof. The proof goes (3) → (2) → (1) → (3). Note that (3) → (2) is trivial. Suppose (2) holds. Then there are points, x and y, that belong to different maximal blocks such that $\overline{xy} \subset \pi_i[K]$ for infinitely many positive integers i. By Theorem 5, there is a positive integer N such that $f^N[\overline{xy}] = [p_1, p_n]$. Suppose k is a positive integer, and let j denote a positive integer greater than k such that $\overline{xy} \subset \pi_{j+N}[K]$. Applying f^N gives $[p_1, p_n] \subset \pi_j[K]$, and further applying f^{j-k} gives $[p_1, p_n] \subset \pi_k[K]$. Hence $\pi_k[K] = [p_1, p_n]$ for each positive integer k, from which (1) follows.

Finally, suppose (1) holds. Let B denote a maximal block of $p_1, p_2, \dots, p_n$, and denote its period by j. Notice that $f[B]$ and B are disjoint. Consider points x and y of K that satisfy $x_{ij} \in B$ for each i and $y = \hat{f}(x)$. Then $y_{ij} \in f[B]$ for each i, and (3) follows. □

The next theorem follows from Theorem 4 of [6], but the proof given there involves much that is unnecessary in the context of this paper. Hence a shorter proof is given.

Theorem 7. *Suppose f is a map from $[p_1, p_n]$ onto itself with n-cycle $p_1, p_2, \dots, p_n$. If f has at least three maximal blocks, then $\varprojlim\{[p_1, p_n], f\}$ is indecomposable.*

Proof. Let x be a point of $\varprojlim\{[p_1, p_n], f\}$ such that x_i belongs to a maximal block for each positive integer i, and let $y = \hat{f}(x)$ and $z = \hat{f}^2(x)$. Then, for each i, x_i, y_i, and z_i belong to three different maximal blocks. It follows from (3) of Theorem 6 that $\varprojlim\{[p_1, p_n], f\}$ is irreducible between any two of x, y, and z. □

Example 8. Theorem 7 is a generalization of the following theorem by Ingram [3, Theorem 11]: If $n > 3$, k is an integer such that $f^k(p_1) = p_2$, and n and k are relatively prime, then $\varprojlim\{[p_1, p_n], f\}$ is indecomposable. Under the hypotheses of Ingram's theorem, the maximal blocks of f have length one; hence, there are at least three maximal blocks, and the Hypothesis of Theorem 7 is satisfied.

Consider the maps g and h of Fig. 1. Both generate indecomposable inverse limits by Theorem 7. Ingram's theorem does not yield the indecomposability of these inverse limits. However, the purpose of Ingram's criteria is not only to detect indecomposability. Under the additional hypothesis that $p_1, p_2, \dots, p_n$ forms a Markov partition for f, Ingram's criteria also imply that every nondegenerate subcontinuum of $\varprojlim\{[p_1, p_n], f\}$ is an arc [3, Theorem 13]. Since the map h generates an inverse limit that contains a $\sin 1/x$-curve, it is to be expected that Ingram's theorem does not apply to h.

The map k of Fig. 1 produces a decomposable inverse limit by Theorem 9 in the next section.

4. Markov maps and composant structure

For the duration of this article, it will be assumed that $p_1, p_2, \dots, p_n$ is a Markov partition for f. Consequently, f is a map of $[p_1, p_n]$ onto itself. The main result of this section is Theorem 16, which gives necessary and sufficient conditions for two points of $\varprojlim\{[p_1, p_n], f\}$ to belong to the same composant. A more general version of the sufficiency of those conditions is proved in Theorem 10. Both of these theorems assume in their hypotheses that the Markov partition, $p_1, p_2, \dots, p_n$ of f is a periodic orbit with at least three maximal blocks. According to Theorem 9, this hypothesis is equivalent to the assumption that $\varprojlim\{[p_1, p_n], f\}$ is indecomposable.

The following theorem is a restatement of [6, Theorem 7]. It is included here for completeness with a sketch of the proof that appears in [6].

Theorem 9. *Suppose f is a Markov map with an n-cycle, $p_1, p_2, \ldots, p_n$, that forms a Markov partition for f. Then $\varprojlim\{[p_1, p_n], f\}$ is indecomposable if and only if f has at least three maximal blocks.*

Proof. If f has at least three maximal blocks, then $\varprojlim\{[p_1, p_n], f\}$ is indecomposable by Theorem 7. Suppose there are exactly two maximal blocks, B_1 and B_2. Then $f[B_1] = B_2$ and $f[B_2] = B_1$. Hence there is a fixed point q of f between B_1 and B_2 such that $f[p_1, q] = [q, p_n]$ and $f[q, p_n] = [p_1, q]$. It follows that $\varprojlim\{[p_1, p_n], f\}$ is decomposable. □

Notation. For each block $B = \{p_{l+1}, p_{l+2}, \ldots, p_{l+k}\}$, B^* refers to $[p_{l+1}, p_{l+k}]$, the smallest interval containing B. Note that if f is monotone between each two consecutive points of B, then $f[B^*] = (f[B])^*$. For convenience, $(f[B])^*$ will be denoted by $f[B]^*$.

Definition. A *partition* of an interval $[a, b]$ is a finite sequence, $I_1, I_2, \ldots, I_m$, of nondegenerate connected sets such that

(1) for $l_1 < l_2$, every point of I_{l_1} is less than every point of I_{l_2}, and
(2) $I_1 \cup I_2 \cup \cdots \cup I_m = [a, b]$.

It should be noted that the word “partition” has been used in connection with Markov maps to denote a finite set of points in an interval, whereas it refers here to a finite pairwise-disjoint cover for an interval. From this point forward, when the former meaning is intended, the phrase “Markov partition” will always be used.

Theorem 10. *Suppose f is a Markov map with an n-cycle, $p_1, p_2, \ldots, p_n$, that forms a Markov partition for f, and suppose $p_1, p_2, \ldots, p_n$ has j maximal blocks for some $j \geqslant 3$. Suppose further that $I_1, I_2, \ldots, I_{j-1}$ is a partition of $[p_1, p_n]$, the two endpoints of each term of which belong respectively to adjacent maximal blocks. If x and y are points of $\varprojlim\{[p_1, p_n], f\}$ such that x_i and y_i belong to the same term of $I_1, I_2, \ldots, I_{j-1}$ for all but finitely many i, then x and y belong to the same composant.*

Proof. Each of $B_2, B_3, \ldots, B_{j-1}$ contains the common endpoint of each of some two of $I_1, I_2, \ldots, I_{j-1}$. Thus, for every i such that $\overline{x_i y_i}$ contains one of $B_2, B_3, \ldots, B_{j-1}$ in its interior, it follows that x_i and y_i belong to different terms of $I_1, I_2, \ldots, I_{j-1}$. Consequently, the hypothesis of the theorem requires that there be at most finitely many i such that $\overline{x_i y_i}$ contains one of $B_2, B_3, \ldots, B_{j-1}$ in its interior.

Next, it will be shown that there are at most finitely many i such that $\overline{x_i y_i}$ intersects two distinct maximal blocks. Suppose to the contrary that, for each of infinitely many i, $\overline{x_i y_i}$

intersects two distinct maximal blocks, A and B. Then either $x_i \in A^*$ and $y_i \in B^*$ for infinitely many i, or $y_i \in A^*$ and $x_i \in B^*$ for infinitely many i, the former of which only will be considered here. It follows from the considerations of the preceding paragraph that A and B are consecutive blocks. Thus $A \cup B$ is a set of consecutive points of $p_1, p_2, \ldots, p_n$ that properly contains A; furthermore, $A \cup B$ is a proper subset of $\{p_1, p_2, \ldots, p_n\}$ since $p_1, p_2, \ldots, p_n$ has at least three maximal blocks. Consequently, $A \cup B$ is not a block, which is to say that there is a positive integer i_0 such that $f^{i_0}[A]$ and $f^{i_0}[B]$ fail to be consecutive. Since $f^{i_0}[A^*] = f^{i_0}[A]^*$ and $f^{i_0}[B^*] = f^{i_0}[B]^*$, it follows that $x_{i-i_0} \in f^{i_0}[A]^*$ and $y_{i-i_0} \in f^{i_0}[B]^*$ both hold for each of infinitely many i. Hence, $\overline{x_i y_i}$ intersects each of two nonconsecutive maximal blocks for infinitely many positive integers i, the impossibility of which was established in the preceding paragraph. Thus, there are at most finitely many i such that $\overline{x_i y_i}$ intersects two distinct maximal blocks.

Let K denote a positive integer such that $\overline{x_i y_i}$ fails to contain a maximal block in its interior for each positive integer i not less than K and such that $\overline{x_i y_i}$ intersects the same number of maximal blocks for every $i > K$. Then either $\overline{x_i y_i}$ intersects exactly one maximal block for $i > K$ or it intersects zero maximal blocks for $i > K$. Only the former case will be considered here. An argument for the latter case may easily be constructed by simplifying what follows.

Suppose $\overline{x_i y_i}$ intersects exactly one block for $i \geqslant K$, and, for each such i, denote by A_i the unique maximal block that intersects $\overline{x_i y_i}$. It will be shown that $f[A_{i+1}^* \cup \overline{x_{i+1} y_{i+1}}] = A_i^* \cup \overline{x_i y_i}$ for $i \geqslant K$. Suppose a positive integer i not less than K is given. Since $\overline{x_{i+1} y_{i+1}}$ intersects A_{i+1}^*, but does not contain A_{i+1}^* in its interior, either $x_{i+1} \in A_{i+1}^*$ or $y_{i+1} \in A_{i+1}^*$. Suppose $x_{i+1} \in A_{i+1}^*$; the other case is similar. Let a denote the point of A_{i+1}^* that is closest to y_{i+1}. Then $A_{i+1}^* \cup \overline{x_{i+1} y_{i+1}} = A_{i+1}^* \cup \overline{a y_{i+1}}$. Note that f is monotone on $\overline{a y_{i+1}}$ and that x_i and $f(a)$ are both in A_i^*. Then $f[A_{i+1}^* \cup \overline{x_{i+1} y_{i+1}}] = f[A_{i+1}^*] \cup f[\overline{a y_{i+1}}] = A_i^* \cup \overline{f(a) y_i} = A_i^* \cup \overline{x_i y_i}$.

Since $f[A_{i+1}^* \cup \overline{x_{i+1} y_{i+1}}] = A_i^* \cup \overline{x_i y_i}$ for $i \geqslant K$, there is a subcontinuum M of $\varprojlim\{[p_1, p_n], f\}$ such that $\pi_i[M] = A_i^* \cup \overline{x_i y_i}$ for $i \geqslant K$ and $\pi_i[M] = f^{K-i}[A_K^* \cup \overline{x_K y_K}]$ for $i < K$. Furthermore M is a proper subcontinuum of $\varprojlim\{[p_1, p_n], f\}$ because $\pi_K[M]$ intersects only one of at least three maximal blocks and is, therefore, a proper subset of $[p_1, p_n]$. Since both x and y belong to M, it follows that x and y belong to the same composant. □

Lemma 11. *Suppose f is a Markov map with an n-cycle, $p_1, p_2, \ldots, p_n$, that forms a Markov partition for f, and suppose B is a block and I is an interval that contains B, but does not intersect any other block. If B^* fails to separate I, then $f[B]^*$ fails to separate $f[I]$.*

Proof. Denote the points of B by $p_{l+1}, p_{l+2}, \ldots, p_{l+k}$. Either I is a subset of $[p_{l+1}, p_{l+k+1})$ or I is a subset of $(p_l, p_{l+k}]$. The proofs for the two cases are similar so only the latter case is proved here. Let q denote the point of $(p_l, p_{l+1}]$ for which $[q, p_{l+k}] = I$. Since f is a Markov map, $f[B^*] = f[B]^*$. Consequently, $f[I] = f[q, p_{l+1}] \cup f[B^*] = f[q, p_{l+1}] \cup f[B]^*$. Since f is monotone on $[p_l, p_{l+1}]$ and, hence, on $[q, p_{l+1}]$, the endpoints of the interval $f[q, p_{l+1}]$ are $f(q)$ and $f(p_{l+1})$. Note that the latter of these belongs to $f[B]^*$. It follows that $f[B]^*$ does not separate $f[I]$. □

Theorem 12. *Suppose f is a Markov map with an n-cycle, $p_1, p_2, \ldots, p_n$, that forms a Markov partition for f. If K is a proper subcontinuum of $\varprojlim\{[p_1, p_n], f\}$, and B is a maximal block, then there is a positive integer N such that, for $i \geqslant N$, B^* fails to separate $\pi_i[K]$.*

Proof. Note that if m_1 and m_2 are positive integers and $m_1 < m_2$, then $\pi_{m_1}[K]$ contains at least as many points of $\{p_1, p_2, \ldots, p_n\}$ as $\pi_{m_2}[K]$, and $\pi_{m_1}[K]$ intersects at least as many blocks as $\pi_{m_2}[K]$. Consequently, there is a positive integer N such that $\pi_m[K]$ contains the same number of points of $\{p_1, p_2, \ldots, p_n\}$ for every $m \geqslant N$ and such that $\pi_m[K]$ intersects the same number of blocks for every $m \geqslant N$. Since K is a proper subcontinuum of $\varprojlim\{[p_1, p_n], f\}$, it follows from (2) → (1) of Theorem 6 that, for $m \geqslant N$, $\pi_m[K]$ does not intersect more than one maximal block. Hence there are two possibilities: either $\pi_m[K]$ intersects zero blocks for $m \geqslant N$, or $\pi_m[K]$ intersects one block for $m \geqslant N$. In the former case, the conclusion of the theorem follows.

Suppose $\pi_m[K]$ intersects exactly one block for $m \geqslant N$, and suppose i is an integer not less than N. If $\pi_i[K]$ does not contain B, then B^* fails to separate $\pi_i[K]$. Suppose $\pi_i[K]$ contains B. Since f^n is the identity on $\{p_1, p_2, \ldots, p_n\}$, it follows that $\pi_{i+n}[K]$ also contains B. Let k denote the length of B. By (4) → (6) of Theorem 4, there is a nonnegative integer l less than n such that $f^l[B] = \{p_1, p_2, \ldots, p_k\}$. It follows that $f^{n-l}\{p_1, p_2, \ldots, p_k\} = B$ and that $\pi_{i+n-l}[K]$ contains $[p_1, p_k]$.

For each positive integer not larger than $n - l$, let $P(m)$ denote the proposition that $[f^m\{p_1, p_2, \ldots, p_k\}]^*$ does not separate $\pi_{i+(n-l)-m}[K]$, and note that from $P(n - l)$ it follows that B^* fails to separate $\pi_i[K]$. It will be shown by way of induction that $P(n - l)$ is true. First consider $P(0)$. Since $[f^0\{p_1, p_2, \ldots, p_k\}]^* = [p_1, p_k]$, which fails to separate any subinterval of $[p_1, p_n]$, it follows that $P(0)$ is true.

Suppose m is a nonnegative integer such that $P(0), P(1), \ldots, P(m)$ are all true. If $m \geqslant n - l$, then $P(n - l)$ follows. Suppose $m < n - l$, and consider $P(m + 1)$. Since $i + (n - l) - m > i \geqslant N$, $\pi_{i+(n-l)-m}[K]$ intersects exactly one block. It has been noted that $\pi_{i+n-l}[K]$ contains $[p_1, p_k]$. Consequently, $\pi_{i+(n-l)-m}[K]$ contains the block $f^m\{p_1, p_2, \ldots, p_k\}$ but fails to intersect any other block. By $P(m)$, $[f^m\{p_1, p_2, \ldots, p_k\}]^*$ fails to separate $\pi_{i+(n-l)-m}[K]$. Therefore, by Lemma 11, $f[f^m\{p_1, p_2, \ldots, p_k\}]^*$ fails to separate $f[\pi_{i+(n-l)-m}[K]]$, which is to say that $[f^{m+1}\{p_1, p_2, \ldots, p_k\}]^*$ fails to separate $\pi_{i+(n-l)-(m+1)}[K]$. Thus $P(m + 1)$ is true. Proceeding inductively yields that $P(n - l)$ is true and, hence, that B^* fails to separate $\pi_i[K]$. □

Lemma 13. *Suppose f is a Markov map with an n-cycle, $p_1, p_2, \ldots, p_n$, that forms a Markov partition for f. If $p \in \{p_1, p_2, \ldots, p_n\}$ and K is a proper subcontinuum of $\varprojlim\{[p_1, p_n], f\}$ such that $p \in \pi_i[K]$ for infinitely many positive integers i, then there are a positive integer l and a nonnegative integer m such that*

(1) *$\pi_{m+i}[K]$ contains p if and only if i is a multiple of l, and*
(2) *the period of the maximal-block cycle divides l.*

Proof. Since f is one-to-one on $\{p_1, p_2, \ldots, p_n\}$, $\pi_i[K]$ contains at least as many points of $\{p_1, p_2, \ldots, p_n\}$ as $\pi_{i+1}[K]$ does for each positive integer i. Hence there are positive integers N and k such that $\pi_i[K]$ contains exactly k points of $\{p_1, p_2, \ldots, p_n\}$ for $i \geqslant N$. By hypothesis, $k \neq 0$. For each positive integer i, let K_i denote $\pi_i[K] \cap \{p_1, p_2, \ldots, p_n\}$. Since $f[\pi_{i+1}[K]] = \pi_i[K]$, and $f\{p_1, p_2, \ldots, p_n\} = \{p_1, p_2, \ldots, p_n\}$, it follows that $f[K_{i+1}] \subset K_i$ for each positive integer i. But f is one-to-one on K_{i+1}, and K_i and K_{i+1} contain the same number of points for $i \geqslant N$; consequently, $f[K_{i+1}] = K_i$ for $i \geqslant N$.

The points of K_i are consecutive points of $p_1, p_2, \ldots, p_n$ for each i because $\pi_i[K]$ is connected for each i. Since each of $p_1, p_2, \ldots, p_n$ has period n, a set of consecutive points of $p_1, p_2, \ldots, p_n$ is a block if and only if its image under each of $f, f^2, \ldots, f^{n-1}$ is also a set of consecutive points of $p_1, p_2, \ldots, p_n$. It follows that K_i is a block for $i \geqslant N + n - 1$. Since forward images of blocks are also blocks, it follows further that K_i is a block for $i \geqslant N$.

Let m denote an integer not less than N for which $p \in K_m$. Recall that, for each non-negative integer i, the length of K_{m+i} is k. By Theorem 3, the period of any block of length k is $\frac{n}{k}$. Denote $\frac{n}{k}$ by l. Then, for any block B of length k, $f^i[B] = B$ if and only if i is a multiple of l. Consequently, $f^i[K_{m+i}] = K_{m+i}$ if and only if i is a multiple of l. However, $f^i[K_{m+i}] = K_m$ for every i, so $K_m = K_{m+i}$ if and only if i is a multiple of l. Hence, K_{m+i} contains p if and only if i is a multiple of l, which completes the proof of (1).

Denote the period of the maximal-block cycle by j and the length of a maximal block by k'. Then $\{p_1, p_2, \ldots, p_k\} \subset \{p_1, p_2, \ldots, p_{k'}\}$, and, by Theorem 3, $n = jk'$. By (1) $\to$ (3) of Theorem 4, $\{p_1, p_2, \ldots, p_{k'}\}$ is a maximal block. Then $f^j\{p_1, p_2, \ldots, p_{k'}\} = \{p_1, p_2, \ldots, p_{k'}\}$. Each point of $\{p_1, p_2, \ldots, p_{k'}\}$ has period k' under f^j because each has period n under f and $jk' = n$. Thus $p_1, p_2, \ldots, p_{k'}$ is a k'-cycle of f^j. Since $\{p_1, p_2, \ldots, p_k\}$ is a block with respect to f it is also a block with respect to f^j. Hence, $\{p_1, p_2, \ldots, p_k\}$ is a block of $p_1, p_2, \ldots, p_{k'}$ with respect to f^j. Then by Theorem 3, $k' = ck$ for some positive integer c. Recall that $n = lk$ and $n = jk'$. It follows that $lk = jkc$, or, equivalently, that $l = jc$. Thus (2) holds. □

Lemma 14. *Suppose f is a Markov map with an n-cycle, $p_1, p_2, \ldots, p_n$, that forms a Markov partition for f, and suppose B is a maximal block with period j. There are a point $q \in B$ and a proper subcontinuum H of $\varprojlim\{[p_1, p_n], f\}$ such that*

(1) *$\pi_{ij}[H]$ properly contains B^* for each positive integer i,*
(2) *$\pi_i[H]$ intersects B if and only if i is a multiple of j, and*
(3) *either q is the left endpoint of $\pi_{ij}[H]$ for each i or q is the right endpoint of $\pi_{ij}[H]$ for each i.*

Proof. Let H' denote the subcontinuum of $\varprojlim\{[p_1, p_n], f\}$ that satisfies $\pi_{ij}[H'] = B^*$ for each positive integer i, and let H'' denote a proper subcontinuum of $\varprojlim\{[p_1, p_n], f\}$ that properly contains H'. Then there is a positive integer N_1 such that $\pi_{ij}[H'']$ properly contains B^* for i not less than N_1. By (2) $\to$ (1) of Theorem 6, there is a positive integer N_2 such that $\pi_i[H'']$ intersects at most one maximal block for i not less than N_2. By Theorem 12, there is a positive integer N_3 such that B^* fails to separate $\pi_i[H'']$ for i not less than N_3. Consider the continuum H defined by $\hat{f}^{-(N_1+N_2+N_3)j}[H'']$.

Note that $\pi_i[H] = \pi_{i+(N_1+N_2+N_3)j}[H'']$ for each positive integer i. Since $ij + (N_1 + N_2 + N_3)j = (i + N_1 + N_2 + N_3)j$ and $i + N_1 + N_2 + N_3 \geqslant N_1$ for each i, it follows that $\pi_{ij}[H]$ properly contains B^* for each i. Thus H satisfies (1).

Consider (2). Since $i + (N_1 + N_2 + N_3)j \geqslant N_2$, $\pi_i[H]$ intersects at most one block for each i. In light of (1), it suffices to show that if i is a positive integer that is not a multiple of j, then $\pi_i[H]$ fails to intersect B. Suppose i is such a positive integer. Then there is a positive integer m less than j such that $i+m$ is a multiple of j. Since H satisfies (1), $\pi_{i+m}[H]$ intersects B. Then $\pi_i[H]$ intersects the block $f^m[B]$. But $f^m[B]$ is not B because B has period j and m is a positive integer less than j. Thus the only block that $\pi_i[H]$ intersects is not B. Consequently, H satisfies (2).

Finally consider (3). Since $i + (N_1 + N_2 + N_3)j \geqslant N_3$, B^* fails to separate $\pi_i[H]$ for each i. It follows that B^* fails to separate $\pi_{ij}[H]$ for each i. It has already been shown that, for each i, $\pi_{ij}[H]$ properly contains B^*. Consequently, for each i, B^* and $\pi_{ij}[H]$ share exactly one endpoint; denote it by q_i.

It will now be shown that $q_i = q_1$ for each i. Consider the continuum $\hat{f}^j[H]$. Note that $\pi_{ij}[\hat{f}^j[H]] = \pi_{(i-1)j}[H]$ for i not less than 2. It follows that both $\pi_{ij}[H]$ and $\pi_{ij}[\hat{f}^j[H]]$ contain B^* for each i. Since B^* contains at least one point of the n-cycle $p_1, p_2, \dots, p_n$, there are infinitely many positive integers i such that both $\pi_i[H]$ and $\pi_i[\hat{f}^j[H]]$ contain p_1. Consequently, one of $\pi_i[H]$ and $\pi_i[\hat{f}^j[H]]$ is a subset of the other for infinitely many, and hence for every, positive integer i. In particular, for each i, one of $\pi_{ij}[H]$ and $\pi_{ij}[\hat{f}^j[H]] = \pi_{(i-1)j}[H]$ is a subset of the other. Now suppose that a positive integer i not less than 2 is given. Since each of $\pi_{ij}[H]$ and $\pi_{(i-1)j}[H]$ has exactly one endpoint in common with B^*, namely q_i and q_{i-1} respectively, and both properly contain B^*, it follows that $q_{i-1} = q_i$. Consequently, $q_i = q_1$ for each i.

Denote q_1 by q. Then q is either the left endpoint of B^* or the right endpoint of B^*. In either case q is a point of B. In the former case q is the left endpoint of $\pi_{ij}[H]$ for each i, and in the latter case q is the right endpoint of $\pi_{ij}[H]$ for each i. Hence q and H satisfy (3). □

Theorem 15. *Suppose f is a Markov map with an n-cycle, $p_1, p_2, \dots, p_n$, that forms a Markov partition for f. Each maximal block contains a unique point q such that one of the following holds for every proper subcontinuum K of $\varprojlim\{[p_1, p_n], f\}$:*

(1) *For all but finitely many i, either q is the left endpoint of $\pi_i[K]$ or $q \notin \pi_i[K]$.*
(2) *For all but finitely many i, either q is the right endpoint of $\pi_i[K]$ or $q \notin \pi_i[K]$.*

Proof. Suppose B is a maximal block, and denote the period of B by j. Let H and q denote a proper subcontinuum of $\varprojlim\{[p_1, p_n], f\}$ and a point of B respectively that satisfy the conclusion of Lemma 14. Then by (3) of Lemma 14, either q is the left endpoint of

$\pi_{ij}[H]$ for each i or q is the right endpoint of $\pi_{ij}[H]$ for each i. The two cases are similar, so only the former will be considered in detail.

Suppose K is a proper subcontinuum of $\varprojlim\{[p_1, p_n], f\}$. By Lemma 13, there is a multiple l of j and a nonnegative integer m such that $\pi_{m+i}[K]$ contains q if and only if i is a multiple of l. By Theorem 12, there is a positive integer N such that B^* fails to separate $\pi_i[K]$ for $i \geqslant N$. Consider the continuum $K' = \hat{f}^{-(m+lN)}[K]$, and note that $\pi_i[K'] = \pi_{i+m+lN}[K]$ for each i. Consequently, $\pi_i[K']$ contains q if and only if i is a multiple of l, and B^* fails to separate $\pi_i[K']$ for each i.

Since l is a multiple of j, it follows that both $\pi_i[H]$ and $\pi_i[K']$ contain q if i is a multiple of l, and, further, that q is the left endpoint of $\pi_i[H]$ if i is a multiple of l. Note that $p_1 \in \mathrm{orbit}(q)$. Then $\pi_i[H]$ and $\pi_i[K']$ both contain p_1 for infinitely many i. For such i, either $\pi_i[H] \subset \pi_i[K']$ or $\pi_i[K'] \subset \pi_i[H]$. If the former holds for infinitely many i, then $\pi_i[H] \subset \pi_i[K']$ for all i; otherwise, $\pi_i[K'] \subset \pi_i[H]$ for all i. These two cases will, for the moment, be treated separately.

First suppose $\pi_i[K'] \subset \pi_i[H]$ for all i. Since q is in both $\pi_i[K']$ and $\pi_i[H]$ if i is a multiple of l, and q is the left endpoint of $\pi_i[H]$ for all such i, it follows that q is the left endpoint of $\pi_i[K']$ if i is a multiple of l.

The same conclusion holds if $\pi_i[H] \subset \pi_i[K']$ for all i; consider the following. Recall that H satisfies the conclusion of Lemma 14 and that l is a multiple of j. It follows that $\pi_i[H]$ properly contains B^* if i is a multiple of l. Since q is the left endpoint of $\pi_i[H]$ for each such i and, hence, the left endpoint of B^*, the right endpoint of B^* is an interior point of $\pi_i[H]$ for multiples i of l. Consequently, the right endpoint of B^* is an interior point of $\pi_i[K']$ if i is a multiple of l. But B^* fails to separate $\pi_i[K']$ for each i, so, if i is a multiple of l, the left endpoint of B^*, namely q, is also the left endpoint of $\pi_i[K']$.

Thus q is the left endpoint of $\pi_i[K']$ for each multiple i of l. It has been noted that $\pi_i[K']$ contains q if and only if i is a multiple of l. Hence, for every i, either q is the left endpoint of $\pi_i[K']$ or $q \notin \pi_i[K']$. Since $\pi_i[K'] = \pi_{i+m+lN}[K]$ for each i, it follows that, for all but finitely many i, either q is the left endpoint of $\pi_i[K]$ or $q \notin \pi_i[K]$. Had it been assumed in the first paragraph that q is the right endpoint of $\pi_{ij}[H]$ for each i, then a similar argument would show that if K is a proper subcontinuum of $\varprojlim\{[p_1, p_n], f\}$, then, for all but finitely many i, either q is the right endpoint of $\pi_i[K]$ or $q \notin \pi_i[K]$.

To see that q is unique, suppose q' is another point of B. Recall that q and H satisfy the conclusion of Lemma 14. Then, $\pi_{ij}[H]$ properly contains B^* for each positive integer i, and q is an endpoint of $\pi_{ij}[H]$ for each i. By the former, $\pi_{ij}[H]$ contains both q and q', and at most one of them is an endpoint of $\pi_{ij}[H]$ for each i. Thus, by the latter, q' fails to be an endpoint of $\pi_{ij}[H]$ for each i. There are infinitely many positive integers i such that q' is a point of $\pi_i[H]$, but not an endpoint, so q' does not satisfy the conclusion of the theorem. Consequently, q is unique. □

Theorem 16. *Suppose f is a Markov map with an n-cycle, $p_1, p_2, \ldots, p_n$, that forms a Markov partition for f, and suppose $p_1, p_2, \ldots, p_n$ has j maximal blocks for some $j \geqslant 3$. Then there is a partition, $I_1, I_2, \ldots, I_{j-1}$, of $[p_1, p_n]$, the two endpoints of each term of which belong respectively to adjacent maximal blocks, such that two points, x and y of*

$\varprojlim\{[p_1, p_n], f\}$ belong to the same composant if and only if x_i and y_i belong to the same term of $I_1, I_2, \dots, I_{j-1}$ for all but finitely many positive integers i.

Proof. Denote the maximal-block cycle of $p_1, p_2, \dots, p_n$ by $B_1, B_2, \dots, B_j$. By Theorem 15, for each positive integer l not larger than j, there is a unique point q_l of the maximal block B_l such that one of the following holds for every proper subcontinuum K of $\varprojlim\{[p_1, p_n], f\}$.

(1) For all but finitely many i, either q_l is the left endpoint of $\pi_i[K]$ or $q_l \notin \pi_i[K]$.
(2) For all but finitely many i, either q_l is the right endpoint of $\pi_i[K]$ or $q_l \notin \pi_i[K]$.

For each l, if q_l satisfies (1) for every proper subcontinuum K of $\varprojlim\{[p_1, p_n], f\}$, then q_l will be said to be of type L; otherwise, q_l will be said to be of type R. Note that $q_1 = p_1$ and $q_j = p_n$.

For each positive integer l less than j, define I_l to be

- $[q_l, q_{l+1}]$ if q_l is type L and q_{l+1} is type R,
- $[q_l, q_{l+1})$ if q_l is type L and q_{l+1} is type L,
- $(q_l, q_{l+1}]$ if q_l is type R and q_{l+1} is type R,
- (q_l, q_{l+1}) if q_l is type R and q_{l+1} is type L.

Since $q_1 = p_1$, $q_j = p_n$, and $q_1 < q_2 < \cdots < q_j$, it follows that $I_1, I_2, \dots, I_{j-1}$ is a partition of $[p_1, p_n]$.

Suppose x and y are points of $\varprojlim\{[p_1, p_n], f\}$ such that x_i and y_i belong to the same term of $I_1, I_2, \dots, I_{j-1}$ for all but finitely many i. Then by Theorem 10, x and y belong to the same composant of $\varprojlim\{[p_1, p_n], f\}$.

To prove the converse, suppose x and y are points of the same composant of $\varprojlim\{[p_1, p_n], f\}$, and consider the continuum $\overline{xy}$. Since no two of $q_1, q_2, \dots, q_j$ belong to the same maximal block, it follows from (2) $\to$ (1) of Theorem 6 that there is an integer N_0 such that $\pi_i[\overline{xy}]$ contains at most one of $q_1, q_2, \dots, q_j$. Corresponding to each integer l for which q_l is type L, there is a positive integer N_l such that, for each $i \geqslant N_l$, either q_l is the left endpoint of $\pi_i[\overline{xy}]$ or $q_l \notin \pi_i[\overline{xy}]$; and corresponding to each integer l for which q_l is type R, there is a positive integer N_l such that, for each $i \geqslant N_l$, either q_l is the right endpoint of $\pi_i[\overline{xy}]$ or $q_l \notin \pi_i[\overline{xy}]$. Let N denote the largest of $N_0, N_1, N_2, \dots, N_j$.

It will now be shown that, for each $i \geqslant N$, $\pi_i[\overline{xy}]$ is a subset of one of $I_1, I_2, \dots, I_{j-1}$. Suppose a value for i not less than N is given. If $\pi_i[\overline{xy}]$ fails to intersect $\{q_1, q_2, \dots, q_j\}$, then the containment of $\pi_i[\overline{xy}]$ by of one of $I_1, I_2, \dots, I_{j-1}$ follows immediately. Suppose $\pi_i[\overline{xy}]$ does intersect $\{q_1, q_2, \dots, q_j\}$. Since $i \geqslant N \geqslant N_0$, $\pi_i[\overline{xy}]$ intersects only one element of $\{q_1, q_2, \dots, q_j\}$; denote it by q_{l^*}. If q_{l^*} is type L, then $q_l \neq q_j$, $[q_{l^*}, q_{l^*+1}) \subset I_{l^*}$, and $\pi_i[\overline{xy}] \subset [q_{l^*}, q_{l^*+1})$, from which it follows that $\pi_i[\overline{xy}] \subset I_{l^*}$. Similarly, if q_{l^*} is type R, then $\pi_i[\overline{xy}] \subset I_{l^*-1}$. Thus, for a given integer i not less than N, $\pi_i[\overline{xy}]$ is a subset of one of $I_1, I_2, \dots, I_{j-1}$. Consequently, for each $i \geqslant N$, x_i and y_i belong to the same term of $I_1, I_2, \dots, I_{j-1}$. □

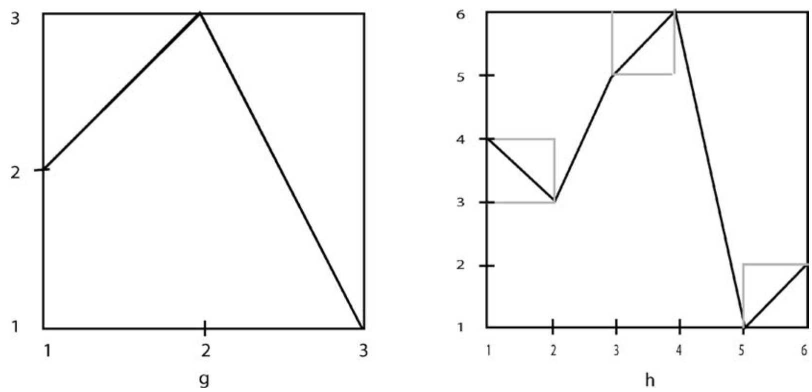

Fig. 2.

Example 17. The inverse limit of the map g, whose graph appears in Fig. 2, is a familiar three-endpoint indecomposable continuum. The map h from Fig. 2 is similar to g, but in place of the period-three orbit, h has a period-three maximal-block cycle. Theorem 16 does not provide enough information to determine completely the sets $I_1, I_2, \ldots, I_k$ of the partition whose existence it guarantees. However, for the mappings

$$g:[1,3]\to[1,3] \quad \text{and} \quad h:[1,6]\to[1,6],$$

one may verify as follows that the partitions are $I_1=[1,2)$, $I_2=[2,3]$ and $I_1=[1,3)$, $I_2=[3,6]$, respectively.

Since the maximal blocks of the orbit $1\to2\to3\to1$ each consist of a single point, the partition of $[1,3]$ is either $I_1=[1,2)$, $I_2=[2,3]$ or $I_1=[1,2]$, $I_2=(2,3]$. There is a subcontinuum K of $\varprojlim\{[1,3],g\}$ such that, for each i, $\pi_{3i}[K]$ is a closed interval whose left endpoint is 2. Then K contains both $(1,3,2,1,3,2,\ldots)$ and a point $(x_1,x_2,x_3,\ldots)$ such that $x_{3i}>2$ for each i. It follows that the partition is not $I_1=[1,2]$, $I_2=(2,3]$.

Similarly, there is a subcontinuum K of $\varprojlim\{[1,6],h\}$ such that $\pi_{3i}[K]$ is a closed interval whose left endpoint is 3 and whose right endpoint is greater than 4. Then K contains both $(2,6,4,1,5,3,2,\ldots)$ and a point x such that $x_{6i}>4$ for each i. Hence the partition is $I_1=[1,3)$, $I_2=[3,6]$.

5. The space of composants of $\varprojlim\{[p_1,p_n],f\}$

Recently, Solecki [7] has shown that the composant equivalence relation for an indecomposable continuum is Borel bireducible with one of two canonical forms, $\mathbb{E}_0$ and $\mathbb{E}_1$ (see definitions below). He goes on to note that the composant equivalence relations of Knaster continua correspond to $\mathbb{E}_0$, the simpler of the two forms, while those of hereditarily indecomposable continua correspond to $\mathbb{E}_1$. It would be interesting to know whether the class of $\mathbb{E}_0$-type continua includes all indecomposable inverse limits with a single Markov bonding map. It will be shown in this section that this is true in the special case where the

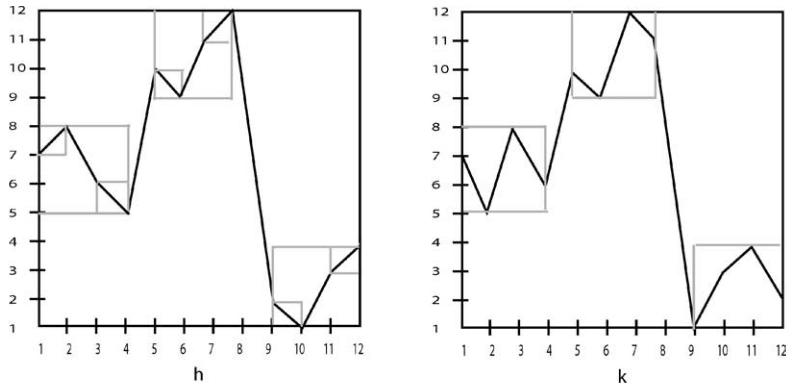

Fig. 3.

Markov partition for the map is an n-cycle. As in the previous section, $p_1, p_2, \dots, p_n$ is both an n-cycle of f and a Markov partition for f.

Definition. Recall that a function $f: X \to Y$ is a *Borel function* provided the inverse image of every open set in Y is a Borel set in X. If Y has a countable basis $\mathcal{B}$, this is equivalent to requiring that the inverse image of each basic open set from $\mathcal{B}$ is a Borel set in X.

If E is an equivalence relation on a set X, and x and y are points of X, then the notation xEy is used to denote that x is equivalent to y under E. An equivalence relation E on a set X is said to be *Borel reducible* to an equivalence relation F on a set Y provided there is a Borel function f from X to Y such that xEy if and only if $f(x)Ff(y)$. When E is Borel reducible to F and F is Borel reducible to E, then E and F are said to be *Borel bireducible*.

The *composant equivalence relation*, denoted by E_C, on an indecomposable continuum is the equivalence relation according to which two points, x and y, of the continuum are equivalent if and only if they belong to the same composant.

The equivalence relation on $\{0,1\}^{\mathbb{N}}$ according to which x and y are equivalent if and only if $x_i = y_i$ for all but finitely many positive integers i is denoted by $\mathbb{E}_0$.

The definition of $\mathbb{E}_1$ is included for completeness, but it is unnecessary for what follows. The equivalence relation on $\{\{0,1\}^{\mathbb{N}}\}^{\mathbb{N}}$ according to which x and y are equivalent if and only if $x_i = y_i$ for all but finitely many positive integers i is denoted by $\mathbb{E}_1$.

Lemma 18. *There is a continuous function, γ, from $\{1,2,\dots,l\}^{\mathbb{N}}$ into $\{0,1\}^{\mathbb{N}}$ such that, for any two points x and y of $\{1,2,\dots,l\}^{\mathbb{N}}$, $x_i = y_i$ for all but finitely many i if and only if $\pi_i(\gamma(x)) = \pi_i(\gamma(y))$ for all but finitely many i.*

Proof. A suitable function γ will be constructed here, but the proof that it satisfies the requirements of the lemma is left to the reader. Denote by g the function that assigns to each element of $\{1,2,\dots,l\}$ its unique binary representation with $\lceil \log_2(l) \rceil$ characters, where $\lceil \log_2(l) \rceil$ denotes the smallest integer that is strictly larger than $\log_2(l)$. For each $x = x_1x_2x_3\dots$ in $\{1,2,\dots,l\}^{\mathbb{N}}$, define $\gamma(x)$ to be the concatenation $g(x_1)g(x_2)g(x_3)\dots$. $\square$

Theorem 19. *Suppose f is a Markov map with an n-cycle, $p_1, p_2, \dots, p_n$, that forms a Markov partition for f. If $\varprojlim\{[p_1, p_n], f\}$ is indecomposable, then the composant equivalence relation on $\varprojlim\{[p_1, p_n], f\}$ is Borel bireducible with $\mathbb{E}_0$.*

Proof. By Corollary 3.3 of [7], $\mathbb{E}_0$ is Borel reducible to the composant equivalence relation of any indecomposable continuum. Thus, it is necessary only to show that the composant equivalence relation on $\varprojlim\{[p_1, p_n], f\}$, hereafter denoted by E_C, is Borel reducible to $\mathbb{E}_0$.

Denote by j the number of maximal blocks of $p_1, p_2, \dots, p_n$. It follows from Theorem 7 of [6] and the assumption that $\varprojlim\{[p_1, p_n], f\}$ is indecomposable that j is not less than three. By Theorem 16, there is a partition, $I_1, I_2, \dots, I_{j-1}$, of $[p_1, p_n]$ such that two points, x and y, of $\varprojlim\{[p_1, p_n], f\}$ belong to the same composant if and only if x_i and y_i belong to the same term of $I_1, I_2, \dots, I_{j-1}$ for each of all but finitely many i.

For each positive integer i and each x in $\varprojlim\{[p_1, p_n], f\}$, let $\chi_i(x)$ denote the integer l for which x_i belongs to I_l, which is to say, $x_i \in I_l$ if and only if $l = \chi_i(x)$. For each x in $\varprojlim\{[p_1, p_n], f\}$, let $\chi(x)$ denote the sequence $\chi_1(x), \chi_2(x), \chi_3(x), \dots$. Then χ is a function from $\varprojlim\{[p_1, p_n], f\}$ into $\{1, 2, \dots, j-1\}^{\mathbb{N}}$, and $x E_C y$ if and only if $\chi_i(x) = \chi_i(y)$ for all but finitely many positive integers i.

Before showing that χ is a Borel function, it is worthwhile to recall that there is a countable basis for the topology of the compact metric space $\{1, 2, \dots, j-1\}^{\mathbb{N}}$. In particular, the collection of all open sets of the form $\pi_1^{-1}[A_1] \cap \pi_2^{-1}[A_2] \cap \dots \cap \pi_n^{-1}[A_n]$ for some positive integer n and some finite sequence $A_1, A_2, \dots, A_n$ of subsets of $\{1, 2, \dots, j-1\}$ is such a basis. Consequently, to show that the inverse images of open sets under χ are Borel sets, it suffices to show that inverse images of such basic open sets under χ are Borel sets.

Suppose a basic open set D is given. Then there are a positive integer N and subsets $A_1, A_2, \dots, A_N$ of $\{1, 2, \dots, j-1\}$ such that $D = \{z \in \{1, 2, \dots, j-1\}^{\mathbb{N}}\colon z_i \in A_i \text{ for } i = 1, 2, \dots, N\}$. For any subset A of $\{1, 2, \dots, j-1\}$, let I_A denote $\bigcup\{I_i\colon i \in A\}$. It has been noted that $x_i \in I_l$ if and only if $\chi_i(x) = l$. It follows, for each subset A of $\{1, 2, \dots, j-1\}$, that $x_i \in I_A$ if and only if $\chi_i(x) \in A$. Hence,

$$\begin{aligned}
\chi^{-1}[D] &= \bigl\{x \in \varprojlim\bigl\{[p_1, p_n], f\bigr\}\colon\ \chi(x) \in D\bigr\} \\
&= \bigl\{x \in \varprojlim\bigl\{[p_1, p_n], f\bigr\}\colon\ \chi_i(x) \in A_i \text{ for } i = 1, 2, \dots, N\bigr\} \\
&= \bigl\{x \in \varprojlim\bigl\{[p_1, p_n], f\bigr\}\colon\ x_i \in I_{A_i} \text{ for } i = 1, 2, \dots, N\bigr\} \\
&= \pi_1^{-1}[I_{A_1}] \cap \pi_2^{-1}[I_{A_2}] \cap \dots \cap \pi_N^{-1}[I_{A_N}].
\end{aligned}$$

Each of $I_1, I_2, \dots, I_{j-1}$ is either an open interval, a half-open interval, or a closed interval; hence each is a Borel set. Then each of $I_{A_1}, I_{A_2}, \dots, I_{A_N}$, being the union of finitely many terms of $I_1, I_2, \dots, I_{j-1}$, is a Borel set. Consequently, each of $\pi_1^{-1}[I_{A_1}], \pi_2^{-1}[I_{A_2}], \dots, \pi_N^{-1}[I_{A_N}]$ is a Borel set, and, hence, $\chi^{-1}[D]$ is a Borel set. Thus inverse images of basic open sets under χ are Borel. It follows that inverse images of all open sets under χ are Borel, and, finally, that χ is a Borel function.

By Lemma 18, there is a continuous function, γ, from $\{1, 2, \dots, j-1\}^{\mathbb{N}}$ into $\{0, 1\}^{\mathbb{N}}$ such that, for any two points x and y of $\{1, 2, \dots, j-1\}^{\mathbb{N}}$, $x_i = y_i$ for all but finitely many

i if and only if $\pi_i(\gamma(x)) = \pi_i(\gamma(y))$ for all but finitely many i. Consider the function $\gamma \circ \chi$. Note that $\gamma \circ \chi$ is a Borel function from $\varprojlim\{[p_1, p_n], f\}$ into $\{0,1\}^{\mathbb{N}}$ and that $\pi_i(\gamma \circ \chi(x)) = \pi_i(\gamma \circ \chi(y))$ for all but finitely many i if and only if $\pi_i(\chi(x)) = \pi_i(\chi(y))$ for all but finitely many i. It has already been established that the latter is true if and only if $x E_C y$. Consequently, E_C is Borel reducible to $\mathbb{E}_0$. $\square$

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