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CONCERNING CONTINUA IRREDUCIBLE ABOUT FINITELY MANY POINTS

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Abstract. The purpose of this paper is to provide in a unified treatment a number of characterizations for continua and unicoherent continua that are irreducible about finitely many points, and for continua and unicoherent continua that are irreducible about n points. One of the main results, from which most of the others follow with relative ease, is that a continuum is irreducible about finitely many points if and only if every pairwise disjoint collection of nonseparating open subsets is finite. Alternate proofs for the classic results of Sorgenfrey are included in the development.

1. Introduction

The classic characterization by R. H. Sorgenfrey [6] of unicoherent continua that are irreducible about n points is given in terms of n-ods: a unicoherent continuum is irreducible about n points but no fewer than n points if and only if it is an n-od and fails to be an $n+1$-od. Equally renowned is a theorem of Sorgenfrey [7] that doesn't assume unicoherence: a continuum is irreducible about n points if and only if, for each proper decomposition of it into $n+1$ subcontinua, the union of some n of them fails to be connected. Thus irreducibility may also be characterized with decompositions. Maćkowiak [3] proved that a continuum is irreducible about finitely many points if and only if it is not the union of a monotonic collection of its proper subcontinua. One might regard this as a characterization in terms of decompositions, though not in the usual sense of the word.

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In the simple n-od, the prototypical continuum that is irreducible about n points, the the points of irreducibility are the nonseparating points. Thus it would seem natural to have a characterization of finitely irreducible continua in terms of nonseparating subcontinua. In this paper, it is shown that a continuum is irreducible about finitely many points if and only if it does not have infinitely many weakly nonseparating subcontinua, each of which has an interior point that fails to lie in the closure of the union of the others. Perhaps more natural is the following, also proved in this paper: a continuum is irreducible about finitely many points if and only if every pairwise disjoint collection of nonseparating open sets is finite.

These results provide the point of departure for the broader purpose of the paper, which is to provide, in a unified treatment that includes both new and classic results, a number of characterizations for continua and unicoherent continua irreducible about finitely many points (Section 2), and for continua and unicoherent continua irreducible about n points (Section 3). Theorems 2.1 and 3.1 summarize the results of the paper and are stated at the beginning of their respective sections to provide a sense of direction.

A *continuum* is a compact connected metrizable space. A continuum is *irreducible* about a closed set H if and only if it contains H but has no proper subcontinuum that contains H.

A continuum M is an *n-od* if and only if there is a subcontinuum K of M such that $M - K$ has at least n components. A continuum M is an *∞-od* if and only if there is a subcontinuum K of M such that $M - K$ has infinitely many components.

A continuum M is *unicoherent* if and only if it is true that if H and K are subcontinua of M such that $H \cup K = M$, then $H \cap K$ is connected.

If $\mathcal{A}$ is a collection of sets, then the notation $\mathcal{A}^*$ denotes the union of the sets belonging to $\mathcal{A}$.

If A is a subset of a continuum M, then A is a subset of M *with interior* if and only if it contains a nonempty open subset of M.

A *proper decomposition* of M is a finite collection, $M_1, M_2, ..., M_n$ of proper subcontinua of M such that $M = M_1 \cup M_2 \cup ... \cup M_n$ and no one of $M_1, M_2, ..., M_n$ is a subset of the union of the others.

A continuum is *indecomposable* if and only if it is not the union of two of its proper subcontinua; otherwise, it is *decomposable*.

The *composant* of a point p in a continuum M is the union of all proper subcontinua of M that contain p. In an indecomposable continuum, the composants partition M into uncountably many mutually disjoint subsets.

For each subset K of M, the *K-composant* of M is the union of all the composants of M that intersect K.

A *nonseparating* subset of a continuum M is a subset of M whose complement is connected. A *weakly nonseparating* subset of M is a subset of M that contains a nonempty nonseparating open subset of M.

The *limiting set* of a sequence $A_1, A_2, A_3, \ldots$ of subcontinua of a continuum M is the set to which a point p belongs if and only if every open set containing p intersects infinitely many terms of $A_1, A_2, A_3, \ldots$. The limiting set of a sequence of continua with a common point is a continuum.

2. Continua Irreducible about Finitely Many Points

Theorem 2.1. *Suppose M is a continuum. The following are equivalent.*

(1) *The continuum M is irreducible about a finite set of points.*
(2) *Every pairwise disjoint collection of nonseparating open subsets of M is finite.*
(3) *The continuum M does not have infinitely many weakly nonseparating subcontinua, each of which has an interior point that fails to lie in the closure of the union of the others.*
(4) *(Maćkowiak) The continuum M is not the union of a countable monotonic collection of its proper subcontinua.*

Furthermore, if M is unicoherent, then the previous conditions are also equivalent to each of the following.

(5) *The continuum M does not have infinitely many subcontinua with a common point, each of which has an interior point that fails to lie in the closure of the union of the others.*
(6) *The continuum M is not the union of infinitely many subcontinua with a common point, each of which has an interior point that fails to lie in the closure of the union of the others.*
(7) *(Sorgenfrey) For some positive integer n, M fails to be an n-od.*
(8) *The continuum M fails to be an ∞-od.*

It is the goal of Sections 2.1 and 2.2 to prove Theorem 2.1. The equivalence of the first three conditions is considered in Section 2.1; that of (5) through (8), in Section 2.2. Maćkowiak [3] proved the equivalence of (1) and (4). One can prove (4) $\Rightarrow$ (3) to obtain an alternate proof of Maćkowiak's Theorem, but that proof is omitted here.

2.1. **Finite Irreducibility.** The proof of the equivalence of the first three conditions of Theorem 2.1 goes (1) $\Rightarrow$ (3) $\Rightarrow$ (1) and (1) $\Rightarrow$ (2) $\Rightarrow$ (1). The proofs of (1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (2) are short and straightforward. The other two implications follow from Theorems 2.9 and 2.10 respectively. It should be noted that condition (3) implies that the complement of every subcontinuum of M has finitely many components. This fact will be used without further reference.

Lemma 2.2. *Suppose M is a continuum such that the complement of each subcontinuum of M has finitely many components. If A is a proper subcontinuum of M that fails to lie in the interior of any proper subcontinuum of M, and $\overline{M-A}$ is not the union of finitely many indecomposable continua, then there are a set B and a connected set D such that*

(1) *B and D are nonempty open subsets of $M-A$,*
(2) *B is has finitely many components,*
(3) *$D = M - (A \cup \overline{B})$, and*
(4) *$\overline{D}$ is not the union of finitely many indecomposable continua.*

Proof. First note that if H is a subcontinuum of M with interior, then H intersects A; otherwise, the closure of the component of $M-H$ containing A would be a proper subcontinuum of M containing A in its interior. If $M-A$ has more than one component, then the closure of some one of them fails to be the union of finitely many indecomposable continua. Then the conclusion of the lemma holds for D equal to this component, and B equal to the union of the remaining components of $M-A$.

Suppose $M-A$ is connected. Then $\overline{M-A}$ is a decomposable continuum; hence, there is a proper subcontinuum K of $\overline{M-A}$ that contains a nonempty open subset of $\overline{M-A}$. It follows that K contains a nonempty open subset of M. Hence, by the initial remark, $A \cup K$ is a continuum. Consequently, the complement of $A \cup K$ has finitely many components. Denote them by $D_1, D_2, ..., D_k$. Note that each of $\overline{D}_1, \overline{D}_2, ..., \overline{D}_k$ intersects A. Hence $M - (A \cup \overline{D}_1 \cup \overline{D}_2 \cup ... \cup \overline{D}_k)$ has finitely many components. Denote them by $D_{k+1}, D_{k+2}, ..., D_n$. It follows that $\overline{D}_1 \cup \overline{D}_2 \cup ... \cup \overline{D}_n = \overline{M-A}$. Note that $n \geq 2$. By hypothesis, $\overline{M-A}$ is not the union of finitely many indecomposable continua, so there is a term $\overline{D}_m$ of $\overline{D}_1, \overline{D}_2, ..., \overline{D}_n$ that fails to be the union of finitely many indecomposable continua.

The sets $A, D_1, D_2, ..., D_n$ are pairwise disjoint, so $\cup_{i \neq m} \overline{D}_i \cup A$ fails to intersect D_m. It follows that $D_m \subset M - (\cup_{i \neq m} \overline{D}_i \cup A)$. Since $\overline{D}_1 \cup \overline{D}_2 \cup ... \cup \overline{D}_n \cup A = M$, it follows further that $D_m \subset M - (\cup_{i \neq m} \overline{D}_i \cup A) \subset \overline{D}_m$. The set D_m is connected, so $M-(\cup_{i\neq m}\overline{D}_i \cup A)$ is also connected. Then the conclusion of the lemma follows for $D = M-(\cup_{i\neq m}\overline{D}_i \cup A)$ and $B = \cup_{i\neq m} D_i$. □

Theorem 2.3. *Suppose M is a continuum that does not have infinitely many subcontinua with a common point each of which has an interior point that fails to lie in the closure of the union of the others. If A is a proper subcontinuum of M that fails to lie in the interior of any proper subcontinuum of M, then $\overline{M-A}$ is the union of finitely many weakly nonseparating indecomposable continua.*

Proof. First note that if H is a subcontinuum of M with interior, then H intersects A; otherwise, the closure of the component of $M-H$ containing A would be a proper subcontinuum of M containing A in its interior.

For the purpose of establishing a contradiction, suppose $\overline{M-A}$ is not the union of finitely many indecomposable continua. Then, by Lemma 2.2, there are sets B_1 and D_1 such that

(1) B_1 and D_1 are nonempty open subsets of $M-A$,
(2) B_1 has finitely many components,
(3) $D_1 = M-(A\cup\overline{B_1})$, and
(4) $\overline{D_1}$ is not the union of finitely many indecomposable continua.

It follows from (1), (2), and the initial remark that the closure of each component of B_1 intersects A. Consequently $A\cup\overline{B}_1$ is a continuum. By (3) and (4), $\overline{M-(A\cup\overline{B}_1)}$ is not the union of finitely many indecomposable continua.

Applying Lemma 2.2 to the subcontinuum $A\cup\overline{B}_1$ yields a nonempty open subset B_2 of $M-(A\cup\overline{B}_1)$ such that B_2 has finitely many components and $\overline{M-(A\cup\overline{B}_1\cup\overline{B}_2)}$ is not the union of finitely many indecomposable continua. Furthermore, $A\cup\overline{B}_2$ is a continuum.

Proceeding inductively yields a sequence $B_1, B_2, B_3, ...$ of mutually exclusive open subsets of $M-A$ such that $A\cup B_i$ is a continuum for each positive integer i. Then $A\cup\overline{B}_1$, $A\cup\overline{B}_2$, $A\cup\overline{B}_3, ...$ is a sequence of subcontinua with a common point, each of which contains an interior point not in the closure of the union of the others. This contradicts the hypothesis of the theorem, so the assumption that $\overline{M-A}$ is not the union of finitely many indecomposable continua is false.

Thus there is a finite collection of indecomposable continua whose union is $\overline{M-A}$. Choosing $K_1, K_2, ..., K_n$ to be a minimal such collection gives that $\cup_{i\neq j}K_i$ is a proper closed subset of $\overline{M-A}$ for each $j=1,2,...,n$. It follows that $M-(\cup_{i\neq j}K_i\cup A)$ is a nonempty open subset of K_j for each such j. Recall that every subcontinuum of M with interior intersects A. Consequently, $\cup_{i\neq j}K_i\cup A$ is a continuum and K_j is weakly nonseparating for each $j=1,2,...,n$. □

In his paper *On Subsets of Indecomposable Continua* [1], H. Cook proved that if K is a closed point set in an indecomposable continuum I, then the K-composant of I is the sum of countably many closed point sets. Much of what occurs in the proof of the following theorem is a rewording of Cook's proof; the rest accounts for differences in context.

Theorem 2.4. *If K is a closed subset of an indecomposable continuum I that fails to intersect every composant of I, then the K-composant of I is the union of countably many closed nowhere-dense sets.*

PROOF. Denote by $C(K)$ the K-composant of I , and let p be a point of $I - C(K)$. Let $D_1, D_2, D_3, ...$ be a decreasing sequence of open subsets of $I - K$ whose only common point is p. For each positive inter i, let C_i denote the union of all components of $I - D_i$ that intersect K. Note that, for each i, C_i is a subset of $C(K)$. Every point of $C(K)$ is connected to K by a subcontinuum of I that fails to contain p and is, therefore, a subset of some term of $C_1, C_2, C_3, ...$ Consequently, $C(K) = C_1 \cup C_2 \cup C_3 \cup ....$ To see that each term of $C_1, C_2, C_3, ...$ is closed, suppose i is given, and suppose q is a limit point of C_i. Then there is a sequence $q_1, q_2, q_3, ...$ of points of C_i that converge to q and a sequence $H_1, H_2, H_3, ...$ of components of $I - D_i$ such that H_j contains both q_j and a point of K for each positive integer j. Then the limiting set of $H_1, H_2, H_3, ...$ is a continuum that contains both q and a point of K, and lies in $I - D_i$. Thus q is in C_i. It remains only to show that each term of $C_1, C_2, C_3, ...$ is nowhere dense. Note that $C(K)$ lies in the closure of its complement. For each i, C_i is a closed subset of $C(K)$ and, therefore, nowhere dense. $\square$

Theorem 2.5. *Suppose M is a continuum and I is an indecomposable subcontinuum of M with nonempty interior. If the boundary of I intersects every composant of I, then some subcontinuum of M separates M into infinitely many mutually separated sets.*

PROOF. If $M - I$ has infinitely many components, then the conclusion of the theorem holds. Suppose $M - I$ has finitely many components. Denote the closures of the components of $M - I$ by $C_1, C_2, ..., C_N$. For $I = 1, 2, ..., N$, let $\mathcal{C}_i$ denote the union of all composants of I that intersect C_i. Since the boundary of I intersects every composant of I, it follows that $\mathcal{C}_1 \cup \mathcal{C}_2 \cup ... \cup \mathcal{C}_N = I$. It follows from the Baire Category Theorem that one of $\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_N$ fails to be the union of countably many closed nowhere-dense (relative to I) sets. Renaming if necessary, suppose it is $\mathcal{C}_1$. Then, by Theorem 2.4, C_1 intersects every composant of I.

Let x denote a point of $C_2 \cap I$. Since C_1 intersects every composant of I, there is a proper subcontinuum I_2 of I that contains both x and a point of C_1. It follows that I_2 is a nowhere-dense subcontinuum of I that intersects both C_1 and C_2. Similarly there are nowhere-dense subcontinua $I_3, I_4, ..., I_N$ of I such that each I_n intersects both C_1 and C_n. Denote by A the continuum $C_1 \cup (I_2 \cup C_2) \cup (I_3 \cup C_3) \cup ... \cup (I_N \cup C_N)$. Then A is a proper subcontinuum of M, every point of $M - A$ is a point of I, and A intersects every composant of I.

Let p denote a point of I that is not in A, and let $D_1, D_2, D_3, ...$ denote a decreasing sequence of open subsets of $M - A$ whose only common point is p. For each positive integer n, denote by A_n the component of $M - D_n$ that contains A. Note that $A_1, A_2, A_3, ...$ is an increasing sequence of subcontinua of M. Suppose q is a point of I from a different composant than that of p. Then there is a proper subcontinuum of I that contains both q and a point of A but misses p. It follows that for some n, A_n contains q. The composant of I that contains p is the union of a countable monotonic collection of subcontinua of I each member of which intersects A. Denote such a collection by $B_1, B_2, B_3, ...$ Then $M = (A_1 \cup B_1) \cup (A_2 \cup B_2) \cup ...$ where for each positive integer n, $A_n \cup B_n$ is a proper subcontinuum of M that contains A and is a subset of $A_{n+1} \cup B_{n+1}$.

By the Baire Category Theorem, there is an integer K such that $A_K \cup B_K$ contains a nonempty open subset of I. Note that $M - (A_K \cup B_K) = I - I \cap (A_K \cup B_K)$. Suppose $I - I \cap (A_K \cup B_K)$ has finitely many components. Then each is open in I, and the closure of each is a proper subcontinuum of I with interior, which contradicts the assumption that I is indecomposable. It follows that $A_K \cup B_K$ separates M into infinitely many mutually separated sets. $\square$

Theorem 2.6. *A continuum has infinitely many subcontinua with a common point each of which has an interior point that fails to lie in the closure of the union of the others if and only if it is the union of countably many of its subcontinua with a common point each of which has an interior point that fails to lie in the closure of the union of the others.*

Proof. It is trivial that the former follows from the latter. To prove the converse, suppose M is a continuum and $A_1, A_2, A_3, ...$ is a sequence of subcontinua of M with a common point each of which has an interior point that fails to lie in the closure of the union of the other terms of the sequence. Denote the limiting set of $A_1, A_2, A_3, ...$ by A_0, and denote $A_0 \cup A_1 \cup A_2 \cup ...$ by A. Note that A_0, $A_0 \cup A_1$, and A are continua. For each non-negative integer n, let M_n be the union of A_n with the closure of the union of all of the components of $M - A$ whose boundaries intersect A_n. Then each term of $M_0 \cup M_1, M_2, M_3, ...$ is a continuum with an interior point that fails to lie in the closure of the union of the others, and $M = M_0 \cup M_1 \cup M_2 \cup ....$ □

Theorem 2.7. *If a continuum M has infinitely many subcontinua with a common point each of which has an interior point that fails to lie in the closure of the union of the others, then M is the union of infinitely many weakly nonseparating subcontinua each of which has an interior point that fails to lie in the closure of the union of the others.*

PROOF. By Theorem 2.6, M is the union of a countable collection of its subcontinua with a common point, each member of which has an interior point that fails to lie in the closure of the union of the others. Denote the terms of such a collection by $M_1, M_2, M_3, ...$ Then $M - \overline{M_2 \cup M_3 \cup ...}$ is an open subset of M_1; furthermore it is nonempty since M_1 has an interior point that does not belong to the closure of the union of the terms of $M_2, M_3, M_4, ...$ The complement of $M - \overline{M_2 \cup M_3 \cup ...}$ is the continuum $\overline{M_2 \cup M_3 \cup ...}$, so $M - \overline{M_2 \cup M_3 \cup ...}$ is a nonseparating open subset of M_1. Similarly, M_n is a weakly nonseparating subcontinuum of M for each n not less than 2. The conclusion of the theorem follows. □

Lemma 2.8. *Suppose M is a continuum, and suppose I and J are distinct indecomposable subcontinua of M with nonempty interior. Every composant of I that contains a point of J also contains a point of the boundary of I.*

PROOF. Suppose C is a composant of I that contains a point p of J. If p is a boundary point of I, then the conclusion of the lemma holds. Suppose p is an interior point of I. Let $\text{int}(I)$ denote the interior of I with respect to $I \cup J$. Then J is a continuum that contains both p and $(I \cup J) - \text{int}(I)$. There is a subcontinuum H of J that is irreducible from p to $(I \cup J) - \text{int}(I)$. Every point of H that belongs to $(I \cup J) - \text{int}(I)$ is a limit point of $H - [(I \cup J) - \text{int}(I)]$, and every point of H that belongs to $H - [(I \cup J) - \text{int}(I)]$ is a point of $\text{int}(I)$. It follows that every point of H is a point of I. Hence H is a subcontinuum of I; furthermore, it is a proper subcontinuum of I – for if it were not, then J would contain I, and, since J and I are not equal, J would properly contain I, thus contradicting the assumption that I has nonempty interior. Note that H contains a boundary point of I. Since H is a proper subcontinuum of I, H contains p, and p is a point of C, it follows that H is a subset of C. Hence, C contains a boundary point of I. □

Theorem 2.9. *If a continuum M does not have infinitely many weakly-nonseparating subcontinua each of which has an interior point that fails to lie in the closure of the union of the others, then M is irreducible about a finite set of points.*

Proof. If M is indecomposable, then it follows that M is irreducible about a finite set of points. Suppose M is decomposable. Denote the collection of all weakly nonseparating indecomposable subcontinua of M by $\mathcal{I}$. It will be assumed that $\mathcal{I}$ is nonempty. The proof is similar if $\mathcal{I}$ is empty. It will be shown that $\mathcal{I}$ is finite, which is trivial if it contains only one element. Let J and I denote two elements of $\mathcal{I}$. Suppose J intersects every composant of I. Then by Lemma 2.8, the boundary of I intersects every composant of I, and, by Theorem 2.5, there is a subcontinuum of M that separates M into infinitely many mutually separated sets. But this contradicts Theorem 2.7. Hence, J does not intersect every composant of I. Consequently, every point of $I \cap J$ is a limit point of $M - J$. Thus, no point of I lies in the interior of J. It follows that the interior of each term of $\mathcal{I}$ fails to lie in the closure of the union of the other terms of $\mathcal{I}$. Then by hypothesis, $\mathcal{I}$ is finite. Denote the terms of $\mathcal{I}$ by $I_1, I_2, ..., I_N$. Each I_k has a composant that does not intersect the boundary of I_k by Theorem 2.5. For each positive integer k not greater than N, let p_k denote a point of I_k that belongs to a composant of I_k that does not intersect the boundary of I_k.

Let $\mathcal{G}$ be the collection of all closed subsets of M containing $\{p_1, p_2, ..., p_N\}$ about which M is irreducible. Suppose $\mathcal{H}$ is a monotonic subcollection of $\mathcal{G}$, and denote by H the set of points that belong to each term of $\mathcal{H}$. Every subcontinuum of M that contains H in its interior also contains a term of $\mathcal{H}$ and is, therefore, equal to M. Some subcontinuum of M is irreducible about H; denote it by M_H. By Theorem 2.3, every point of $M - M_H$ belongs to one of the terms of $\mathcal{I}$, which is to say, to one of $I_1, I_2, ..., I_N$. Denote by m_H the closure of the component of $M_H \cap \text{int}(I_1)$ that contains p_1. Then m_H is a subcontinuum of M_H that is also a subcontinuum of I_1. Since M is decomposable, $M - \text{int}(I_1)$ is nonempty. Hence, each term of $\mathcal{H}$, and therefore M_H, contains a point of $M - \text{int}(I_1)$. It follows that m_H contains a point of the boundary of I_1, which does not intersect the composant in which p_1 lies. Consequently, $m_H = I_1$, so M_H contains I_1. Similarly, M_H contains each of $I_1, I_2, ..., I_N$. Since every point of $M - M_H$ belongs to one of $I_1, I_2, ..., I_N$, it follows that $M = M_H$. Therefore, $H \in \mathcal{G}$. By Zorn's Lemma, $\mathcal{G}$ has a minimal element G.

If G is finite, then the conclusion of the theorem follows. Suppose to the contrary that G is infinite. Then there is a pairwise disjoint sequence $G_1, G_2, G_3, ...$ of nonempty subsets of $G - \{p_1, p_2, ..., p_N\}$ that are open relative to G. For each positive integer n, let M_n denote a subcontinuum of M that is irreducible about $G - G_n$. Then, for each n, M_n is a proper subcontinuum of M since G is minimal in $\mathcal{G}$. If n and k are distinct integers, then $M_n \cup M_k = M$. It follows that $M - M_n$ and $M - M_k$ are disjoint for $n \neq k$. For each positive integer n, $M - M_n$ has finitely many components by Theorem 2.7; hence, each of them is open. For each n, denote one such component by g_n. Note that $g_1, g_2, g_3, ...$ is pairwise disjoint. Then $\overline{g_1}, \overline{g_2}, \overline{g_3}, ...$ is an infinite collection of subcontinua of M with nonseparating interior, the interior of each term of which fails to lie in the closure of the union of the others. This is a contradiction with the hypothesis of the theorem. □

Theorem 2.10. *If M is a continuum, and every pairwise disjoint collection of nonseparating open subsets of M is finite, then M is irreducible about a finite set of points.*

The proof of Theorem 2.10 is similar to that of Theorem 2.9.

2.2. **Unicoherence and Finitely Irreducibility.** In this section, the conclusion of the proof of Theorem 2.1 is given. Specifically, conditions (5) through (8) are shown to be equivalent to (1) through (4) for a unicoherent continuum M. Notice that the hypothesis of unicoherence is invoked only to prove (8) $\Rightarrow$ (2).

Conclusion of proof of Theorem 2.1. Conditions (5) and (6) are equivalent by Theorem 2.6. The remainder of the proof goes (1) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (2), (1) $\Rightarrow$ (6), and (5) $\Rightarrow$ (8). The first of these implications follows from the observation that a continuum that is irreducible about n points fails to be an $n+1$-od; the second is trivial. The contrapositive of (8) $\Rightarrow$ (2) may be easily verified via the observation that if $D_1, D_2, D_3, ...$ is a pairwise disjoint sequence of nonseparating open subsets of a unicoherent continuum M, then $M - (D_1 \cup D_2 \cup D_3 \cup ...)$ is a subcontinuum of M whose complement has infinitely many components.

Consider (1) $\Rightarrow$ (6), and suppose (6) is false. Then there is an infinite collection $\mathcal{A}$ of subcontinua of M with a common point whose union is M, each term of which has an interior point that fails to lie in the closure of the union of the others. It follows that each term of $\mathcal{A}$ contains a nonempty open set that fails to intersect any other term of $\mathcal{A}$. Since no second countable space contains an uncountable collection of mutually exclusive open sets, it follows that $\mathcal{A}$ is a countable collection. Denote its terms by $A_1, A_2, A_3, ...$ Then $A_1, A_1 \cup A_2, A_1 \cup A_2 \cup A_3, ...$ is a monotonic sequence of subcontinua whose union is M. It follows that M fails to be irreducible about finitely many points.

Finally, consider (5) $\Rightarrow$ (8), and suppose M is an ∞-od. Let H be a subcontinuum of M such that $M - H$ has infinitely many components. Then $M - H$ is the union of two mutually exclusive open sets, A_1 and B_1. Note that $H \cup A_1$ and $H \cup B_1$ are both continua. One of A_1 and B_1, say B_1, contains infinitely many components of $M - H$. Then B_1 is the union of two mutually exclusive open sets, A_2 and B_2, one of which, say B_2, contains infinitely many components of $M - H$. The set $H \cup A_2$ is a continuum. Continuing inductively yields a sequence $A_1, A_2, A_3, \ldots$ of mutually exclusive open subsets of $M - H$ such that $H \cup A_n$ is a continuum for each positive integer n. Then $H \cup A_1, H \cup A_2, H \cup A_3, \ldots$ is a sequence of subcontinua of M with a common point each term of which has an interior point that fails to lie in the closure of the union of the other terms. $\square$

2.3. **Almost Finitely Irreducible Continua.** Consider again the eight conditions that, according to Theorem 2.1, are equivalent for a unicoherent continuum M; but consider them now apart from the assumption of unicoherence. It follows from the proof of Theorem 2.1 that the following relationships between these conditions hold.

$$\begin{array}{ccccccccccc} (4) & \Leftrightarrow & (3) & \Leftrightarrow & (2) & \Leftrightarrow & (1) & \Rightarrow & (6) & \Leftrightarrow & (5) \\ & & & & & & \Downarrow & & \Downarrow & & \\ & & & & & & (7) & \Rightarrow & (8) & & \end{array}$$

Each of the implications (1) $\Rightarrow$ (6), (1) $\Rightarrow$ (7), (7) $\Rightarrow$ (8), and (6) $\Rightarrow$ (8) is sharp. A simple closed curve satisfies both (6) and (7), but fails to satisfy (1). Jo Heath [2] constructed an example of a continuum that is an n-od for each positive integer n, but fails to be an ∞-od. Her example satisfies (8), but does not satisfy either (6) or (7). Consequently, a complete determination of the relationship between these eight properties reduces to the following question.

Question. How are the following related?

(6) The continuum M is not the union of infinitely many subcontinua with a common point, each of which has an interior point that fails to lie in the closure of the union of the others.

(7) For some positive integer n, M fails to be an n-od.

In connection with (6) and (8), it is interesting to note that although a continuum that is the union of a countable collection of subcontinua with a common point, each of which contains a point that is not in the closure of the union of the others, need not be an ∞-od (Heath's continuum for example), such a continuum does contain an ∞-od by a theorem of Van C. Nall [4].

3. Continua Irreducible about n Points

This section concerns continua irreducible about a given finite number of points. Theorem 2.1 can be used to deduce a number of interesting characterizations of such continua relatively easily, including the classic criteria of Sorgenfrey. Theorem 3.1 summarizes these results.

3.1. **Irreducibility about n points.**

Theorem 3.1. *Suppose M is a continuum, and suppose n is a positive integer not less than two. The following are equivalent.*

(1) *The continuum M is irreducible about n points.*
(2) *Every pairwise disjoint collection of nonseparating open subsets of M has at most n terms.*
(3) *The continuum M is not an ∞-od, and every collection of subcontinua of M, each term of which both fails to lie in the union of the others and has a dense subset that is open and nonseparating in M, has at most n terms.*
(4) *(Sorgenfrey) For each proper decomposition of M into $n+1$ continua, the union of some n of them fails to be connected.*

Furthermore, if M is unicoherent, then the previous conditions are also equivalent to each of the following.

(5) *For each proper decomposition of M into $n+1$ continua, there is no point common to all $n+1$ of them.*
(6) *The continuum M does not have $n+1$ subcontinua with a common point, each of which has an interior point not in the union of the others.*
(7) *(Sorgenfrey) The continuum M fails to be an $n+1$-od.*

Proof. Theorem 2.1 is used to show that the first three are equivalent in [5]. They are equivalent to the fourth condition by Sorgenfrey's Theorem [7], for which an alternate proof is given in Section 3.2. The remainder of the proof goes (4) $\Rightarrow$ (5) $\Rightarrow$ (7) $\Rightarrow$ (1) and (5) $\Leftrightarrow$ (6). The first two of these implications are trivial. Another theorem of Sorgenfrey [6] gives (7) $\Rightarrow$ (1). (Alternatively, one can easily verify (7)$\Rightarrow$ (2) by noting that if $\{D_1, D_2, ..., D_k\}$ is a pairwise disjoint collection of nonseparating open sets of the unicoherent continuum M, then $M - (D_1 \cup D_2 \cup ... \cup D_k)$ is a subcontinuum of M whose complement has at least k components.) It is trivial that (6) $\Rightarrow$ (5), so it remains only to show that (5) $\Rightarrow$ (6).

To that end, suppose $A_1, A_2, ..., A_n$ are subcontinua of M with a common point, each of which contains an interior point that is not in the union of the others. For each i, let B_i denote the union of A_i with all of the components of $M - (A_1 \cup A_2 \cup ... \cup A_n)$ whose closures intersect A_i. Then each B_i is a continuum, each B_i contains a point that is not in the union of the others, $M = B_1 \cup B_2 \cup ... \cup B_n$, and $B_1, B_2, ..., B_n$ have a common point. □

3.2. **Sorgenfrey's Theorem.**

Lemma 3.2. *Suppose $\mathcal{A}$ is a nondegenerate finite collection of nonseparating open subsets of a continuum M. Then every component of $M - \mathcal{A}^*$ intersects the boundary of at least two terms of $\mathcal{A}$.*

PROOF. Suppose K is a component of $M - \mathcal{A}^*$. By the Boundary Bumping Theorem, K intersects the boundary of $\mathcal{A}^*$, and, hence, the boundary of some term D of $\mathcal{A}$. Note that $M - D$ is a continuum and that K is a component of $(M - D) - \{\mathcal{A} - \{D\}\}^*$. Then, again by the Boundary Bumping Theorem, K intersects the boundary of some term of $\mathcal{A} - \{D\}$, and, therefore, two terms of $\mathcal{A}$. □

Theorem 3.3 (Sorgenfrey [7]). *If, for each proper decomposition of a continuum M into $n+1$ subcontinua, the union of some n of them fails to be connected, then M is irreducible about n points.*

PROOF. By (2) ⇒ (1) of Theorem 3.1, it suffices to show that every pairwise disjoint collection of nonseparating open subsets of M has at most n terms. First note that the hypothesis implies that M is not an $n+1$-od, or, equivalently, that every nonseparating open subset of M has at most n components. Thus each nonseparating open set is the union of finitely many connected nonseparating open sets, and it suffices to show that every pairwise disjoint collection of connected nonseparating open subsets of M has at most n terms.

Suppose, for the purpose of establishing a contradiction, that there is a pairwise disjoint collection of connected nonseparating open subsets of M with $n+1$ terms. Denote the terms of this collection by $D_1, D_2, ..., D_{n+1}$. For each i, let $\mathcal{D}_i$ denote the union of D_i with every component of $M - (D_1 \cup D_2 \cup ... \cup D_{n+1})$ that intersects its boundary. Note that each $\mathcal{D}_i$ is a continuum. By Lemma 3.2, each component of $M - (D_1 \cup D_2 \cup ... \cup D_{n+1})$ intersects the boundary of at least two terms of $D_1, D_2, ..., D_{n+1}$. It follows that $\cup_{i \neq j} \mathcal{D}_i = M - D_j$ for each j, and that $\mathcal{D}_1 \cup \mathcal{D}_2 \cup ... \cup \mathcal{D}_{n+1}$ is a proper decomposition of M. Then, by hypothesis, the union of some n terms of $\mathcal{D}_1, \mathcal{D}_2, ..., \mathcal{D}_{n+1}$ fails to be connected. But each such union is the complement of some nonseparating set, which is not possible. Thus the assumption that there is a pairwise disjoint collection of connected nonseparating open subsets of M with $n+1$ terms is false. □

3.3. **Continua Almost Irreducible about n Points.** Without the assumption of unicoherence, the relationship between the seven conditions listed in Theorem 3.1 is as follows.

$$\begin{array}{ccccccccccc} (4) & \Leftrightarrow & (3) & \Leftrightarrow & (2) & \Leftrightarrow & (1) & \Rightarrow & (5) & \Leftrightarrow & (6) \\ & & & & & & & & \Downarrow & & \\ & & & & & & & & (7) & & \end{array}$$

Each of the above implications follows from the proof of Theorem 3.1, which puts unicoherence into play only to establish (7) $\Rightarrow$ (1). Each of (1) $\Rightarrow$ (5) and (5) $\Rightarrow$ (7) is sharp, as the following examples demonstrate. A simple closed curve satisfies (5) but not (1), and a continuum that satisfies (7) but not (5) may be constructed by identifying an endpoint of each of $n-1$ arcs with a single point of a simple closed curve. Hence the above diagram completely describes the relationship between these seven properties.

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