

Indecomposability and the structure of periodic orbits for interval maps

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Abstract

Suppose $f : [a, b] \rightarrow [a, b]$ is continuous. Barge and Martin, and Ingram have shown that if the inverse limit of $\{[a, b], f\}$ is hereditarily decomposable, then the period of every periodic orbit of f is a power of two. We will elaborate on the structure of these orbits, and, assuming f is a Markov map whose partition is a single periodic orbit, give necessary and sufficient conditions for the inverse limit to be (1) decomposable and (2) hereditarily decomposable.

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1. Introduction

Suppose f is a map from an interval $[a, b]$ into itself. Marcy Barge and Joe Martin elucidated the relationship between the dynamics of f and the topology of $\varprojlim\{[a, b], f\}$ in [2]. Among their results is the following: if f has a periodic point whose period is not a power of two, then $\varprojlim\{[a, b], f\}$ contains an indecomposable continuum. Ingram [5] discovered the same theorem in connection with a similar result for inverse limits of atriodic and hereditarily unicoherent continua. Barge and Roe [3] obtained a comparable theorem about maps of circles, which was generalized by Roe [7] to include maps of finite graphs.

The author proved a structural theorem for periodic orbits of maps of intervals and used it to study the family of inverse limits of intervals that are generated by a single

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Markov bonding map whose partition consists of a single periodic orbit [8]; of particular interest was the composant structure of indecomposable continua in this family. This paper explores the utility of the same structural theorem as it relates the dynamics of an interval map to the presence of indecomposable continua in its inverse limit.

Section 2 recalls some results of [8]. In Section 3, necessary dynamical conditions are given for an interval map to generate (1) a decomposable inverse limit and (2) a hereditarily decomposable inverse limit. Theorem 5, which is concerned with the latter case, strengthens the above result of Barge, Martin, and Ingram; and its proof involves incidentally an alternate proof of their result. Section 4 returns to the family that is of primary interest in [8]—the family of inverse limits of intervals that are generated with a single Markov bonding map whose partition consists of a single periodic orbit. The results of Section 3 motivate conditions that are shown in Section 4, within its more restrictive context, to characterize (1) decomposability and (2) hereditary decomposability.

Suppose f is a function from $[a, b]$ into itself. The *orbit* of a point p of $[a, b]$, denoted by $\text{orbit}(p)$, is the set $\{y: y = f^i(p) \text{ for some } i \in \mathbb{N}\}$. A point p of $[a, b]$ is said to be *periodic* provided there is a positive integer n such that $f^n(p) = p$. The *period* of a periodic point p is the smallest positive integer n such that $f^n(p) = p$.

A sequence p_1, p_2, \dots, p_n of points of $[a, b]$ is said to be an n -*cycle* of f provided

- (1) $a \leq p_1 < p_2 < \dots < p_n \leq b$,
- (2) p_1 is periodic with period n , and
- (3) $\text{orbit}(p_1) = \{p_1, p_2, \dots, p_n\}$.

A set $\{p_{l+1}, p_{l+2}, \dots, p_{l+k}\}$ of consecutive terms of an n -cycle p_1, p_2, \dots, p_n is called a *block* of p_1, p_2, \dots, p_n with respect to f if and only if, for each positive integer i , there is a set $\{p_{m+1}, p_{m+2}, \dots, p_{m+k}\}$ of consecutive terms of p_1, p_2, \dots, p_n such that $f^i\{p_{l+1}, p_{l+2}, \dots, p_{l+k}\} = \{p_{m+1}, p_{m+2}, \dots, p_{m+k}\}$. Either or both of the phrases “of p_1, p_2, \dots, p_n ” and “with respect to f ” may be dropped when context permits doing so without diminishing clarity.

A block $\{p_{l+1}, p_{l+2}, \dots, p_{l+k}\}$ is called a *maximal block* provided it is a block, and every set of consecutive terms of p_1, p_2, \dots, p_n that properly contains $\{p_{l+1}, p_{l+2}, \dots, p_{l+k}\}$ and is properly contained by $\{p_1, p_2, \dots, p_n\}$ fails to be a block.

A block B is said to be *periodic* provided there is a positive integer j such that $f^j[B] = B$. The *period* of a periodic block B is the smallest positive integer j such that $f^j[B] = B$.

A sequence, B_1, B_2, \dots, B_j of blocks is said to be a *block cycle* of f provided

- (1) for $i_1 < i_2$, every point of B_{i_1} is less than every point of B_{i_2} ,
- (2) B_1 is periodic with period j , and
- (3) $\text{orbit}(B_1) = \{B_1, B_2, \dots, B_j\}$.

A *map* is a continuous function. Suppose f is a map from an interval $[a, b]$ into itself. A partition $a = x_0 < x_1 < \dots < x_n = b$ of $[a, b]$ is said to be a *Markov partition* for f provided $\{x_0, x_1, \dots, x_n\}$ is invariant under f , and f is monotone on $[x_{i-1}, x_i]$ for each

positive integer i not larger than n . A map possessing a Markov partition is called a *Markov map*.

A *continuum* is a compact connected subset of a metric space. A continuum is said to be *indecomposable* if and only if it is not the union of two of its proper subcontinua; otherwise, it is said to be *decomposable*.

Suppose X_1, X_2, X_3, \dots is a sequence of metric spaces, and, for each positive integer n , f_n is a continuous function from X_{n+1} into X_n . The sequence $\{X_n, f_n\}$ is called an *inverse sequence*, the spaces X_n are called *factor spaces*, and the functions f_n are called *bonding maps*. The *inverse limit* of the inverse sequence $\{X_n, f_n\}$, denoted by $\varprojlim \{X_n, f_n\}$, is the subset of the product space $\prod X_n$ to which x belongs if and only if $f_n(x_{n+1}) = x_n$ for each positive integer, n . It is well known that $\varprojlim \{X_n, f_n\}$ is a continuum if each of the factor spaces is a continuum. The *projection* of the product space $\prod X_n$ into X_n , denoted by π_n , is the function from $\prod X_n$ into X_n that satisfies $\pi_n(x) = x_n$ for each x in $\prod X_n$.

The factor spaces for all of the inverse limits in this paper are intervals. If there is a map $f : [a, b] \rightarrow [a, b]$ such that $f_i = f$ for each positive integer i , then $\varprojlim \{X_i, f_i\}$ may be denoted by $\varprojlim \{[a, b], f\}$. If $[c, d]$ is a subinterval of $[a, b]$ such that $f[c, d] = [c, d]$, then $\varprojlim \{[c, d], f\}$ denotes the subcontinuum $\varprojlim \{[c, d], f|_{[c, d]}\}$ of $\varprojlim \{[a, b], f\}$.

2. Some preliminary examples and results

From this point forward, it will be assumed that f is a map from $[a, b]$ into itself with an n -cycle p_1, p_2, \dots, p_n . Both of the theorems in this section are proved in [8]. However, the example that follows is helpful for understanding Theorem 1.

Notation. Suppose p_1, p_2, \dots, p_n is an n -cycle of f , and suppose k is a positive integer that divides n . For each positive integer j not larger than $\frac{n}{k}$, $B_{k,j}$ denotes the set $\{p_{k(j-1)+1}, p_{k(j-1)+2}, \dots, p_{k(j-1)+k} = p_{kj}\}$. In particular, $B_{k,1}$ denotes $\{p_1, p_2, \dots, p_k\}$, and $B_{k,\frac{n}{k}}$ denotes $\{p_{n-k+1}, p_{n-k+2}, \dots, p_n\}$. For real numbers x and y , \overline{xy} denotes $[x, y]$ if $x < y$, and $[y, x]$ otherwise.

Theorem 1. *Suppose f is a map from $[a, b]$ into itself with an n -cycle p_1, p_2, \dots, p_n . The following are equivalent.*

- (1) $\{p_1, p_2, \dots, p_k\}$ is a block.
- (2) There are $\frac{n}{k}$ blocks of length k .
- (3) $B_{k,1}, B_{k,2}, \dots, B_{k,\frac{n}{k}}$ are the blocks of length k .
- (4) $B_{k,1}, B_{k,2}, \dots, B_{k,\frac{n}{k}}$ is a block cycle.

Theorem 2. *Suppose f is a map from $[a, b]$ into itself with an n -cycle p_1, p_2, \dots, p_n . There is a positive integer N such that for each pair, x and y , of points belonging to different maximal blocks, $[p_1, p_n] \subset f^N[\overline{xy}]$.*

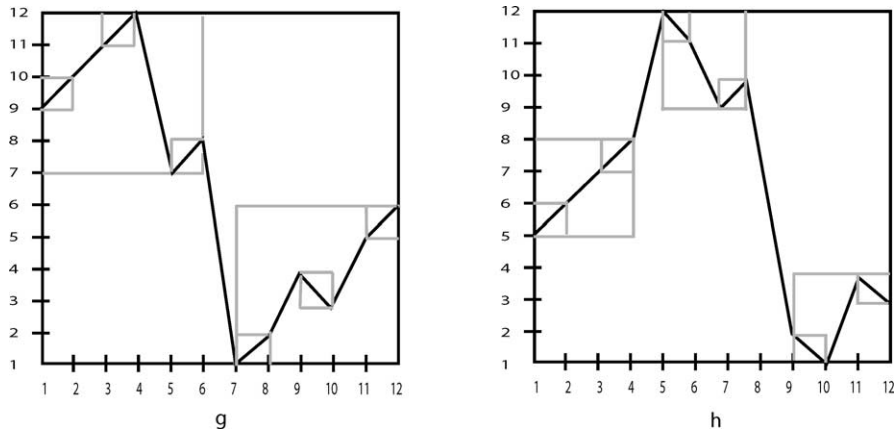


Fig. 1.

Example 3. Consider the maps g and h whose graphs appear in Fig. 1. Note that $1, 2, \dots, 12$ is a 12-cycle of both g and h . There are blocks of length 1, 2, and 12 with respect to both g and h , whereas only with respect to g are there blocks of length 6, and only with respect to h are there blocks of length 4. The maximal blocks of $1, 2, \dots, 12$ with respect to g are $\{1, 2, 3, 4, 5, 6\}$ and $\{7, 8, 9, 10, 11, 12\}$; those with respect to h are $\{1, 2, 3, 4\}$, $\{5, 6, 7, 8\}$, and $\{9, 10, 11, 12\}$. Since $1, 2, \dots, 12$ is a Markov partition for both g and h , the structure of periodic orbits that is guaranteed by Theorem 1 is easily detected in both.

3. Consequences of decomposability

It has been noted that if f contains a periodic point whose period is not a power of two, then $\varprojlim\{[a, b], f\}$ contains an indecomposable continuum [2], [5]. Thus there are considerable restrictions for a periodic orbit of a map whose inverse limit is hereditarily decomposable—the period of the orbit must be a power of two. It follows from the results of this section that many power-of-two orbits are also impermissible for such maps. In particular, Theorem 5 describes via blocks certain structure that is necessary for periodic orbits of maps with hereditarily decomposable inverse limits.

Theorem 4. Suppose f is a map from $[a, b]$ into itself with an n -cycle p_1, p_2, \dots, p_n . If the smallest of all subcontinua H of $\varprojlim\{[a, b], f\}$ such that $[p_1, p_n] \subset \pi_i[H]$ for each i is decomposable, then

- (1) n is even,
- (2) $\{p_1, p_2, \dots, p_{\frac{n}{2}}\}$ is a block of p_1, p_2, \dots, p_n , and
- (3) there is an interval I such that $f^2[I] = I$ and $I \cap \{p_1, p_2, \dots, p_n\} = \{p_1, p_2, \dots, p_{\frac{n}{2}}\}$.

Proof. Denote the points $(p_1, f^{n-1}(p_1), f^{n-2}(p_1), \dots), (p_2, f^{n-1}(p_2), f^{n-2}(p_2), \dots), \dots, (p_n, f^{n-1}(p_n), f^{n-2}(p_n), \dots)$ by $\overline{p_1}, \overline{p_2}, \dots, \overline{p_n}$, respectively, and denote by H the

unique subcontinuum of $\varprojlim\{[a, b], f\}$ that is irreducible about $\{\overline{p_1}, \overline{p_2}, \dots, \overline{p_n}\}$. Note that H is decomposable. Then there are sequences I_1, I_2, I_3, \dots and J_1, J_2, J_3, \dots such that $\varprojlim\{I_k, f|I_{k+1}\}$ and $\varprojlim\{J_k, f|J_{k+1}\}$ are proper subcontinua of H whose union is H . Then there is a positive integer K such that both I_k and J_k fail to contain $\{p_1, p_2, \dots, p_n\}$ for k not less than K . By Theorem 2, there is a positive integer N such that if x and y are points of $\{p_1, p_2, \dots, p_n\}$ that belong to different maximal blocks, then $\{p_1, p_2, \dots, p_n\} \subset f^N[\overline{xy}]$. Consider I_{N+K} and J_{N+K} . If I_{N+K} intersects more than one maximal block, then $\{p_1, p_2, \dots, p_n\} \subset f^N[I_{N+K}] = I_K$, but this is not true. Consequently, I_{N+K} intersects at most one maximal block. Similarly, J_{N+K} intersects at most one maximal block. Since $I_{N+K} \cup J_{N+K}$ contains $\{p_1, p_2, \dots, p_n\}$, and p_1, p_2, \dots, p_n contains at least two maximal blocks, it follows that p_1, p_2, \dots, p_n has exactly two maximal blocks, one lying in I_{N+K} and one lying in J_{N+K} . The blocks are $\{p_1, p_2, \dots, p_{\frac{n}{2}}\}$ and $\{p_{\frac{n}{2}+1}, p_{\frac{n}{2}+2}, \dots, p_n\}$ by (2) \rightarrow (3) of Theorem 1, and each has period two by (3) \rightarrow (4) of Theorem 1. Parts (1) and (2) of the conclusion of the present theorem follow.

To see that (3) holds, first note that either $\{p_1, p_2, \dots, p_{\frac{n}{2}}\} \subset I_{2i}$ for each i , or $\{p_1, p_2, \dots, p_{\frac{n}{2}}\} \subset J_{2i}$ for each i . The two cases are similar, so a proof of (3) will be given only in the former. Consider the interval $\tilde{I} = \bigcap_{i \geq 2} I_{2i}$. Then

$$f^2[\tilde{I}] = f^2\left[\bigcap_{i \geq 2} I_{2i}\right] \subset \bigcap_{i \geq 2} f^2[I_{2i}] = \bigcap_{i \geq 2} I_{2(i-1)} = I_2 \cap \left(\bigcap_{i \geq 2} I_{2i}\right) \subset \tilde{I}.$$

It follows that $f^{2(i+1)}[\tilde{I}] \subset f^{2i}[\tilde{I}]$ for each i . Denote by I the interval $\bigcap_{i \geq 0} f^{2i}[\tilde{I}]$. Since $\tilde{I}, f^2[\tilde{I}], f^4[\tilde{I}], \dots$ is nonincreasing, it follows that

$$f^2\left[\bigcap_{i \geq 0} f^{2i}[\tilde{I}]\right] = \bigcap_{i \geq 0} f^2 \circ f^{2i}[\tilde{I}];$$

consequently,

$$f^2[I] = \bigcap_{i \geq 0} f^2 \circ f^{2i}[\tilde{I}] = \bigcap_{i \geq 0} f^{2(i+1)}[\tilde{I}] = \bigcap_{i \geq 1} f^{2i}[\tilde{I}].$$

But $f^2[\tilde{I}] \subset \tilde{I}$, so

$$\bigcap_{i \geq 1} f^{2i}[\tilde{I}] = \bigcap_{i \geq 0} f^{2i}[\tilde{I}] = I.$$

Thus I is invariant under f^2 . Since $\{p_1, p_2, \dots, p_{\frac{n}{2}}\} \subset I_{2i}$ for each i , it follows that $\{p_1, p_2, \dots, p_{\frac{n}{2}}\} \subset \tilde{I}$. Hence $\{p_1, p_2, \dots, p_{\frac{n}{2}}\} \subset I$.

To establish (3), it remains only to show that $I \cap \{p_{\frac{n}{2}+1}, p_{\frac{n}{2}+2}, \dots, p_n\}$ is empty. Recall that I_{N+K} intersects at most one of $\{p_1, p_2, \dots, p_{\frac{n}{2}}\}$ and $\{p_{\frac{n}{2}+1}, p_{\frac{n}{2}+2}, \dots, p_n\}$. Then the same is true of I_{N+K+1} . Whichever of the two has an even subscript fails to intersect $\{p_{\frac{n}{2}+1}, p_{\frac{n}{2}+2}, \dots, p_n\}$. Hence \tilde{I} , defined to be $\bigcap_{i \geq 2} I_{2i}$, fails to intersect $\{p_{\frac{n}{2}+1}, p_{\frac{n}{2}+2}, \dots, p_n\}$. Consequently I , defined to be $\bigcap_{i \geq 0} f^{2i}[\tilde{I}]$, also fails to intersect $\{p_{\frac{n}{2}+1}, p_{\frac{n}{2}+2}, \dots, p_n\}$. \square

Theorem 5. *Suppose f is a map from $[a, b]$ into itself with an n -cycle p_1, p_2, \dots, p_n . If $\varprojlim\{[a, b], f\}$ is hereditarily decomposable, then*

- (1) (Barge, Martin; Ingram) n is a power of two,
- (2) for each positive integer k not larger than $\log_2(n)$, $\{p_1, p_2, \dots, p_{n2^{-k}}\}$ is a block, and
- (3) for each positive integer k not larger than $\log_2(n)$, there is an interval I_k such that $f^{2^k}[I_k] = I_k$ and $I_k \cap \{p_1, p_2, \dots, p_n\} = \{p_1, p_2, \dots, p_{n2^{-k}}\}$.

Proof. Denote by $P(m)$ the proposition that the theorem is true provided n does not exceed 2^m . To prove the theorem it suffices to show that $P(m)$ is true for every positive integer m . The only value of n that satisfies the hypothesis of $P(1)$ is $n = 2$; thus the conclusion of $P(1)$ follows from choosing $I_1 = \{p_1\}$ and noting that $\{p_1\}$ is a trivial block and $f^2(p_1) = p_1$. Suppose $P(m_0)$ is true for some positive integer m_0 , and suppose n does not exceed 2^{m_0+1} . If n fails to exceed 2^{m_0} , then the conclusion of $P(m_0 + 1)$ follows by $P(m_0)$. Suppose $2^{m_0} < n \leq 2^{m_0+1}$. Then by Theorem 4, there is a subinterval I_1 of $[a, b]$ such that $f^2[I_1] = I_1$ and $I_1 \cap \{p_1, p_2, \dots, p_n\} = \{p_1, p_2, \dots, p_{\frac{n}{2}}\}$; furthermore, $\{p_1, p_2, \dots, p_{\frac{n}{2}}\}$ is a block of p_1, p_2, \dots, p_n with respect to f . Note that $\varprojlim\{I_1, f^2\}$ is hereditarily decomposable, and that $p_1, p_2, \dots, p_{\frac{n}{2}}$ is a cycle with respect to f^2 . Applying $P(m_0)$ to $f^2|_{I_1}$ and $p_1, p_2, \dots, p_{\frac{n}{2}}$ gives that $\frac{n}{2}$ is a power of two, and, for each positive integer k not greater than $\log_2(\frac{n}{2}) = \log_2(n) - 1$, that there is an interval J_k such that $f^{2^{k+1}}[J_k] = (f^2)^{2^k}[J_k] = J_k$ and $J_k \cap \{p_1, p_2, \dots, p_{\frac{n}{2}}\} = \{p_1, p_2, \dots, p_{\frac{n}{2}2^{-k}}\}$. For each integer k , such that $2 \leq k \leq \log_2(n)$, let $I_k = J_{k-1}$. Then, for $2 \leq k \leq \log_2(n)$, $f^{2^k}[I_k] = I_k$ and $I_k \cap \{p_1, p_2, \dots, p_{\frac{n}{2}}\} = \{p_1, p_2, \dots, p_{\frac{n}{2}2^{-(k-1)}}\} = \{p_1, p_2, \dots, p_{n2^{-k}}\}$. Since, for such values of k , I_k is connected and fails to contain $p_{\frac{n}{2}}$, it follows that

$$I_k \cap \{p_1, p_2, \dots, p_n\} = (I_k \cap \{p_1, p_2, \dots, p_{\frac{n}{2}}\}) \cup (I_k \cap \{p_{\frac{n}{2}+1}, p_{\frac{n}{2}+2}, \dots, p_n\}) \\ = \{p_1, p_2, \dots, p_{n2^{-k}}\} \cup \emptyset = \{p_1, p_2, \dots, p_{n2^{-k}}\}$$

for $2 \leq k \leq \log_2(n)$. That $I_k \cap \{p_1, p_2, \dots, p_n\} = \{p_1, p_2, \dots, p_{n2^{-k}}\}$ holds for $k = 1$ has already been established. Hence, part (3) of the conclusion of $P(m_0 + 1)$ is true. Since $\frac{n}{2}$ is a power of 2, (1) holds. To see that (2) holds, suppose there is a positive integer k not greater than $\log_2(n)$ such that $\{p_1, p_2, \dots, p_{n2^{-k}}\}$ fails to be a block. Then there is a positive integer N such that $f^i[p_1, p_{n2^{-k}}]$ contains more than $n2^{-k}$ points of $\{p_1, p_2, \dots, p_n\}$ for integers i larger than N . But this is inconsistent with the fact that $f^{i2^k}[I_k] = I_k$ for every positive integer i , because I_k contains only $n2^{-k}$ points of $\{p_1, p_2, \dots, p_n\}$. Thus (2) holds, and, therefore, $P(m_0 + 1)$ is true. \square

Example 6. Consider the maps g and h of Fig. 1, and k and l of Fig. 2. Since g and h both have periodic points of period 12, it follows from (1) of Theorem 5 that both $\varprojlim\{[1, 12], g\}$ and $\varprojlim\{[1, 12], h\}$ contain an indecomposable continuum.

While $1, 2, \dots, 8$ is a cycle of both k and l , and $\{1, 2, 3, 4\}$ is a block with respect to both k and l , the set $\{1, 2\}$ is a block with respect to k , but fails to be so with respect to l . Consequently, $\varprojlim\{[1, 8], l\}$ contains an indecomposable continuum by (2) of Theorem 5.

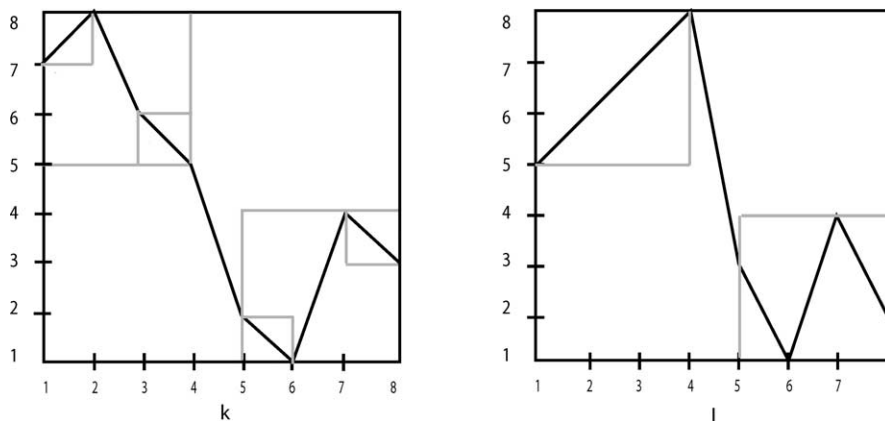


Fig. 2.

The continuum $\overleftarrow{\lim}\{[1, 12], h\}$ is indecomposable by (2) of Theorem 4. Note, however, that one cannot determine from either Theorem 4 or Theorem 5 that $\overleftarrow{\lim}\{[1, 12], g\}$ and $\overleftarrow{\lim}\{[1, 8], l\}$ are decomposable or that $\overleftarrow{\lim}\{[1, 8], k\}$ is hereditarily decomposable. The results of the next section justify such conclusions.

4. Markov maps and decomposability

The standing assumption of the previous two sections has been that f is a map from an interval $[a, b]$ into itself with an n -cycle p_1, p_2, \dots, p_n . Henceforth, this will be accompanied by the additional assumption that p_1, p_2, \dots, p_n is a Markov partition for f . It follows that f is a mapping of $[p_1, p_n]$ onto itself.

With this additional assumption, a sort of converse for each of Theorems 4 and 5 can be proved. These converses, Theorems 7 and 11, are the main results of this section. On the way to Theorem 11, the basic structure of hereditarily decomposable continua arising from such Markov maps is uncovered.

Theorem 7. *Suppose f is a Markov map whose partition, p_1, p_2, \dots, p_n , is an n -cycle. Then $\overleftarrow{\lim}\{[p_1, p_n], f\}$ is decomposable if and only if n is even and $\{p_1, p_2, \dots, p_{\frac{n}{2}}\}$ is a block.*

Proof. If $\overleftarrow{\lim}\{[p_1, p_n], f\}$ is decomposable, then, by Theorem 4, n is even and $\{p_1, p_2, \dots, p_{\frac{n}{2}}\}$ is a block. To see the converse, suppose n is even and $\{p_1, p_2, \dots, p_{\frac{n}{2}}\}$ is a block. Denote by B_1 and B_2 , respectively, the sets $\{p_1, p_2, \dots, p_{\frac{n}{2}}\}$ and $\{p_{\frac{n}{2}+1}, p_{\frac{n}{2}+2}, \dots, p_n\}$. By (1) \rightarrow (4) of Theorem 1, $f[B_1] = B_2$ and $f[B_2] = B_1$, it follows that $f(p_{\frac{n}{2}}) \geq p_{\frac{n}{2}+1}$ and $f(p_{\frac{n}{2}+1}) \leq p_{\frac{n}{2}}$. Then by the Intermediate Value Theorem, there is a fixed point, q , between $p_{\frac{n}{2}}$ and $p_{\frac{n}{2}+1}$.

The map f is monotone on $[p_{\frac{n}{2}}, p_{\frac{n}{2}+1}]$, and $f(p_{\frac{n}{2}}) \geq p_{\frac{n}{2}+1} \geq q$, so $f[p_{\frac{n}{2}}, q] \subset [q, p_n]$. Recall that $f[p_1, p_{\frac{n}{2}}] = [p_{\frac{n}{2}+1}, p_n] \subset [q, p_n]$. Consequently $f[p_1, q] \subset [q, p_n]$. Since q is fixed, and some point of B_1 is mapped to p_n , it follows that $f[p_1, q] = [q, p_n]$. Similarly $f[q, p_n] = [p_1, q]$. Consequently, $\varprojlim\{[p_1, p_n], f\}$ is decomposable. \square

The following theorem is a generalization of a result proved by Ralph Bennett in his Masters thesis [4]. The theorem, as it is stated here, was proved by Ingram [6].

Theorem 8 (Bennett). *Suppose g is a mapping of the interval $[a, b]$ onto itself and d is a number between a and b such that (1) $g[d, b]$ is a subset of $[d, b]$, (2) $g|[a, d]$ is monotone, and (3) there is a positive integer N such that $g^N[a, d] = \overline{[a, b]}$. Then $\varprojlim\{[a, b], g\}$ is the union of a topological ray R and a continuum K such that $\overline{R} - R = K$ and $K = \varprojlim\{[d, b], g\}$.*

Lemma 9. *Suppose f is a Markov map whose partition, p_1, p_2, \dots, p_n , is an n -cycle. If $B_1 = \{p_{j+1}, p_{j+2}, \dots, p_{j+k}\}$, $B_2 = \{p_{j+k+1}, p_{j+k+2}, \dots, p_{j+2k}\}$, and $B_1 \cup B_2$ are blocks, then for each nonnegative integer i , there are points α_i and β_i such that*

- (1) $p_{j+k} \leq \alpha_i < \beta_i \leq p_{j+k+1}$,
- (2) $f^i\{\alpha_i, \beta_i\} \subset f^i[B_1 \cup B_2]$,
- (3) $f^i[p_{j+1}, \alpha_i] = f^i[p_{j+1}, p_{j+k}]$,
- (4) $f^i[\beta_i, p_{j+2k}] = f^i[p_{j+k+1}, p_{j+2k}]$, and
- (5) f^i is monotone on $[\alpha_i, \beta_i]$.

Proof. Let $P(m)$ denote the proposition obtained from Lemma 9 by replacing the phrase “for each nonnegative integer i ” by “for $i = m$ ”. To prove the lemma, it suffices to show that $P(m)$ is true for each nonnegative integer m . First consider $P(0)$. Choose α_0 and β_0 to be p_{j+k} and p_{j+k+1} , respectively. Then the conclusion of $P(0)$ is a triviality.

Suppose m_0 is a nonnegative integer such that $P(m_0)$ is true, and consider $P(m_0 + 1)$. Since B_1 , B_2 , and $B_1 \cup B_2$ are blocks, it follows that $f[B_1]$, $f[B_2]$, and $f[B_1 \cup B_2]$ are blocks. Either $f(p) < f(q)$ for all p in B_1 and all q in B_2 , or $f(p) > f(q)$ for all p in B_1 and all q in B_2 . The two cases are similar, so the inductive step will be demonstrated only for the latter. In this case, there is an integer l such that $f[B_2] = \{p_{l+1}, p_{l+2}, \dots, p_{l+k}\}$ and $f[B_1] = \{p_{l+k+1}, p_{l+k+2}, \dots, p_{l+2k}\}$. By $P(m_0)$, there are points α_{m_0} and β_{m_0} such that $p_{l+k} \leq \alpha_{m_0} < \beta_{m_0} \leq p_{l+k+1}$, $f^{m_0}\{\alpha_{m_0}, \beta_{m_0}\} \subset f^{m_0}[f[B_1] \cup f[B_2]]$, $f^{m_0}[p_{l+1}, \alpha_{m_0}] = f^{m_0}[p_{l+1}, p_{l+k}]$, $f^{m_0}[\beta_{m_0}, p_{l+2k}] = f^{m_0}[p_{l+k+1}, p_{l+2k}]$, and f^{m_0} is monotone on $[\alpha_{m_0}, \beta_{m_0}]$. Note that $[p_{l+k}, p_{l+k+1}] \subset f[p_{j+k}, p_{j+k+1}]$. Then there are points α_{m_0+1} and β_{m_0+1} belonging to $f^{-1}(\beta_{m_0}) \cap [p_{j+k}, p_{j+k+1}]$ and $f^{-1}(\alpha_{m_0}) \cap [p_{j+k}, p_{j+k+1}]$, respectively.

Clearly $p_{j+k} \leq \alpha_{m_0+1}, \beta_{m_0+1} \leq p_{j+k+1}$. Since f is monotone on $[p_{j+k}, p_{j+k+1}]$ and $f(p_{j+k}) > f(p_{j+k+1})$, it follows that f is nonincreasing on $[p_{j+k}, p_{j+k+1}]$. Then $\alpha_{m_0+1} < \beta_{m_0+1}$ because $f(\alpha_{m_0+1}) = \beta_{m_0} > \alpha_{m_0} = f(\beta_{m_0+1})$. Thus (1) in the conclusion of $P(m_0 + 1)$ holds.

Part (2) also holds:

$$\begin{aligned} f^{m_0+1}\{\alpha_{m_0+1}, \beta_{m_0+1}\} &= f^{m_0}\{f(\alpha_{m_0+1}), f(\beta_{m_0+1})\} \\ &= f^{m_0}\{\beta_{m_0}, \alpha_{m_0}\} \subset f^{m_0}[f[B_1] \cup f[B_2]] \\ &= f^{m_0+1}[B_1 \cup B_2]. \end{aligned}$$

Now consider (3). Note that $f[p_{j+1}, \alpha_{m_0+1}] = f[p_{j+1}, p_{j+k}] \cup f[p_{j+k}, \alpha_{m_0+1}]$. The map f is monotone on $[p_{j+k}, \alpha_{m_0+1}]$, and $f(p_{j+k})$ and $f(\alpha_{m_0+1})$ both belong to $[\beta_{m_0}, p_{l+2k}]$; so $f[p_{j+k}, \alpha_{m_0+1}] \subset [\beta_{m_0}, p_{l+2k}]$. But $f[p_{j+1}, p_{j+k}] = [p_{l+k+1}, p_{l+2k}]$, which is also a subset of $[\beta_{m_0}, p_{l+2k}]$, so $f[p_{j+1}, \alpha_{m_0+1}] \subset [\beta_{m_0}, p_{l+2k}]$. Consequently,

$$\begin{aligned} f^{m_0+1}[p_{j+1}, \alpha_{m_0+1}] &\subset f^{m_0}[\beta_{m_0}, p_{l+2k}] = f^{m_0}[p_{l+k+1}, p_{l+2k}] \\ &= f^{m_0+1}[p_{j+1}, p_{j+k}]. \end{aligned}$$

The reverse containment is trivial, so $f^{m_0+1}[p_{j+1}, \alpha_{m_0+1}] = f^{m_0+1}[p_{j+1}, p_{j+k}]$. Thus (3) is true. It can be shown by a similar argument that (4) holds.

Finally, consider (5). The map f is monotone on $[\alpha_{m_0+1}, \beta_{m_0+1}]$ because $[\alpha_{m_0+1}, \beta_{m_0+1}] \subset [p_{j+k}, p_{j+k+1}]$; therefore, $f[\alpha_{m_0+1}, \beta_{m_0+1}] = [\alpha_{m_0}, \beta_{m_0}]$. Since f^{m_0} is monotone on $[\alpha_{m_0}, \beta_{m_0}]$, it follows that f^{m_0+1} is monotone on $[\alpha_{m_0+1}, \beta_{m_0+1}]$. \square

Notation. For each block $B = \{p_{l+1}, p_{l+2}, \dots, p_{l+k}\}$, B^* refers to $[p_{l+1}, p_{l+k}]$, the smallest interval containing B . Note that if f is monotone between each two consecutive points of B , then $f[B^*] = (f[B])^*$. For convenience, $(f[B])^*$ will be denoted by $f[B]^*$.

Theorem 10. Suppose f is a Markov map whose partition, p_1, p_2, \dots, p_n , is an n -cycle. If $B_1 = \{p_1, p_2, \dots, p_k\}$, $B_2 = \{p_{k+1}, p_{k+2}, \dots, p_{2k}\}$, and $B_1 \cup B_2$ are blocks, then

- (1) $\{\{p_1, p_k\}, f^{\frac{n}{k}}\}$, $\{\{p_{k+1}, p_{2k}\}, f^{\frac{n}{k}}\}$, and $\{\{p_1, p_{2k}\}, f^{\frac{n}{2k}}\}$ are inverse sequences,
- (2) $\varprojlim \{\{p_1, p_k\}, f^{\frac{n}{k}}\}$ and $\varprojlim \{\{p_{k+1}, p_{2k}\}, f^{\frac{n}{k}}\}$ are homeomorphic;

and there is a point p in (p_k, p_{k+1}) such that

- (3) $\{\{p_1, p\}, f^{\frac{n}{k}}\}$, and $\{\{p, p_{2k}\}, f^{\frac{n}{k}}\}$, are inverse sequences,
- (4) $\varprojlim \{\{p_1, p\}, f^{\frac{n}{k}}\}$ and $\varprojlim \{\{p, p_{2k}\}, f^{\frac{n}{k}}\}$ are homeomorphic,
- (5) $\varprojlim \{\{p, p_{2k}\}, f^{\frac{n}{k}}\} = R \cup K$ where R is a ray, $\bar{R} - R = K$, and K is homeomorphic to $\varprojlim \{\{p_{k+1}, p_{2k}\}, f^{\frac{n}{k}}\}$, and
- (6) $\varprojlim \{\{p_1, p_{2k}\}, f^{\frac{n}{2k}}\} = M_1 \cup M_2$ where M_1 and M_2 are both homeomorphic to $\varprojlim \{\{p, p_{2k}\}, f^{\frac{n}{k}}\}$, and $M_1 \cap M_2 = \{(p, p, p, \dots)\}$.

Proof. Since p_1, p_2, \dots, p_n is a Markov partition for f , it follows that if B is a block of p_1, p_2, \dots, p_n , then $f[B]^* = f[B^*]$. Hence $f^2[B]^* = f[f[B]^*] = f[f[B^*]] = f^2[B^*]$. Proceeding inductively gives that $f^m[B]^* = f^m[B^*]$ for every positive integer m . Then, in particular,

$$f^{\frac{n}{k}}[B_1]^* = f^{\frac{n}{k}}[B_1^*], \quad f^{\frac{n}{k}}[B_2]^* = f^{\frac{n}{k}}[B_2^*],$$

and

$$f^{\frac{n}{2k}}[B_1 \cup B_2]^* = f^{\frac{n}{2k}}[[B_1 \cup B_2]^*].$$

But B_1 , B_2 , and $B_1 \cup B_2$ have periods $\frac{n}{k}$, $\frac{n}{k}$, and $\frac{n}{2k}$, respectively, by (1) \rightarrow (4) of Theorem 1, so $f^{\frac{n}{k}}[B_1] = B_1$, $f^{\frac{n}{k}}[B_2] = B_2$, and $f^{\frac{n}{2k}}[B_1 \cup B_2] = B_1 \cup B_2$. Hence, $B_1^* = f^{\frac{n}{k}}[B_1^*]$, $B_2^* = f^{\frac{n}{k}}[B_2^*]$, and $(B_1 \cup B_2)^* = f^{\frac{n}{2k}}[[B_1 \cup B_2]^*]$. Consequently,

$$[p_1, p_k] = f^{\frac{n}{k}}[p_1, p_k], \quad [p_{k+1}, p_{2k}] = f^{\frac{n}{k}}[p_{k+1}, p_{2k}],$$

and

$$[p_1, p_{2k}] = f^{\frac{n}{2k}}[p_1, p_{2k}].$$

Part (1) of the theorem follows.

To see that (2) holds, first note that B_1 , B_2 , and $B_1 \cup B_2$ are $B_{k,1}$, $B_{k,2}$, and $B_{2k,1}$, respectively, in the notation of Theorem 1. It follows from (1) \rightarrow (4) of Theorem 1 that B_1 , B_2 , and $B_1 \cup B_2$ have periods $\frac{n}{k}$, $\frac{n}{k}$, and $\frac{n}{2k}$, respectively, and that $f^{\frac{n}{2k}}[B_1]$ is one of the blocks $B_{k,1}, B_{k,2}, \dots, B_{k,\frac{n}{k}}$. Since $f^{\frac{n}{2k}}[B_1 \cup B_2] = B_1 \cup B_2$, $f^{\frac{n}{2k}}[B_1]$ is either B_1 or B_2 . However, the period of B_1 is $\frac{n}{k}$, so $f^{\frac{n}{2k}}[B_1] = B_2$. Similarly, $f^{\frac{n}{2k}}[B_2] = B_1$. Consequently, $f^{\frac{n}{2k}}[p_1, p_k] = [p_{k+1}, p_{2k}]$ and $f^{\frac{n}{2k}}[p_{k+1}, p_{2k}] = [p_1, p_k]$.

Consider the inverse sequence

$$[p_1, p_k] \xleftarrow{g_1} [p_{k+1}, p_{2k}] \xleftarrow{g_2} [p_1, p_k] \xleftarrow{g_3} [p_{k+1}, p_{2k}] \xleftarrow{g_4} \dots$$

where $g_{2i} = f^{\frac{n}{2k}}|[p_1, p_k]$ and $g_{2i+1} = f^{\frac{n}{2k}}|[p_{k+1}, p_{2k}]$ for each positive integer i . Note that $g_{2i} \circ g_{2i+1} = f^{\frac{n}{k}}|[p_{k+1}, p_{2k}]$ and $g_{2i-1} \circ g_{2i} = f^{\frac{n}{k}}|[p_1, p_k]$ are both true for every positive integer i . It follows that $\varprojlim\{[p_1, p_k], f^{\frac{n}{k}}\}$ and $\varprojlim\{[p_{k+1}, p_{2k}], f^{\frac{n}{k}}\}$ are both homeomorphic to $\varprojlim\{X_i, g_i\}$ where X_i is $[p_1, p_k]$ or $[p_{k+1}, p_{2k}]$ if i is odd or even respectively. Thus (2) holds.

By Lemma 9, there are points α and β such that $p_k \leq \alpha < \beta \leq p_{k+1}$,

$$f^{\frac{n}{2k}}[p_1, \alpha] = f^{\frac{n}{2k}}[p_1, p_k] = [p_{k+1}, p_{2k}],$$

$$f^{\frac{n}{2k}}[\beta, p_{2k}] = f^{\frac{n}{2k}}[p_{k+1}, p_{2k}] = [p_1, p_k],$$

and $f^{\frac{n}{2k}}$ is monotone on $[\alpha, \beta]$. It follows that $f^{\frac{n}{2k}}(\alpha) \geq p_{k+1} \geq \beta$ and $f^{\frac{n}{2k}}(\beta) \leq p_k \leq \alpha$. Hence $f^{\frac{n}{2k}}$ has a fixed point p in (α, β) . Note that $f^{\frac{n}{2k}}$ is monotone on both $[\alpha, p]$ and $[p, \beta]$.

Now consider (3). Since $f^{\frac{n}{2k}}$ is monotone on $[\alpha, p]$, $f^{\frac{n}{2k}}(\alpha) \in [p_{k+1}, p_{2k}] \subset [p, p_{2k}]$, and $f^{\frac{n}{2k}}(p) = p \in [p, p_{2k}]$, it follows that $f^{\frac{n}{2k}}[\alpha, p] \subset [p, p_{2k}]$. It has been noted that $f^{\frac{n}{2k}}[p_1, \alpha] = [p_{k+1}, p_{2k}]$, which is a subset of $[p, p_{2k}]$. Hence $f^{\frac{n}{2k}}[p_1, p] \subset [p, p_{2k}]$. Then $f^{\frac{n}{2k}}[p_1, p] = [p, p_{2k}]$ because $p = f^{\frac{n}{2k}}(p)$ and $p_{2k} \in f^{\frac{n}{2k}}[p_1, \alpha]$. Similarly $f^{\frac{n}{2k}}[p, p_{2k}] = [p_1, p]$. Consequently, $f^{\frac{n}{k}}[p_1, p] = [p_1, p]$ and $f^{\frac{n}{k}}[p, p_{2k}] = [p, p_{2k}]$. It follows that (3) is true.

To complete the proof of (4), an argument similar to that given in the second paragraph of the proof of (2) will suffice.

By Lemma 9, there are points γ and δ such that $p_k \leq \gamma < \delta \leq p_{k+1}$,

$$f^{\frac{n}{k}}[p_1, \gamma] = f^{\frac{n}{k}}[p_1, p_k] = [p_1, p_k],$$

$$f^{\frac{n}{k}}[\delta, p_{2k}] = f^{\frac{n}{k}}[p_{k+1}, p_{2k}] = [p_{k+1}, p_{2k}],$$

and $f^{\frac{n}{k}}$ is monotone on $[\gamma, \delta]$. Recall that $p \in (\alpha, \beta) \subset (p_k, p_{k+1})$. Since p is fixed by $f^{\frac{n}{k}}$ it remains fixed under $f^{\frac{n}{k}}$. Consequently, $f^{\frac{n}{k}}(p) \notin f^{\frac{n}{k}}[p_1, \gamma]$ and $f^{\frac{n}{k}}(p) \notin f^{\frac{n}{k}}[\delta, p_{2k}]$, from which it follows that $p \notin [p_1, \gamma]$ and $p \notin [\delta, p_{2k}]$. Equivalently, $p \in (\gamma, \delta)$.

In the proof of (3), it was established that $f^{\frac{n}{k}}[p, p_{2k}] = [p, p_{2k}]$. It has now been shown that $\delta \in (p, p_{2k})$; $f^{\frac{n}{k}}$ is monotone on $[\gamma, \delta]$ and, hence, on $[p, \delta]$; and $f^{\frac{n}{k}}[\delta, p_{2k}] = [p_{k+1}, p_{2k}] \subset [\delta, p_{2k}]$. In order to apply Bennett’s Theorem to $f^{\frac{n}{k}}|[p, \delta]$, it remains only to show that there is a positive integer N such that $(f^{\frac{n}{k}})^N[p, \delta] = [p, p_{2k}]$. Since p is fixed by $f^{\frac{n}{k}}$, it suffices to show that for some positive integer N , $p_{2k} \in (f^{\frac{n}{k}})^N[p, \delta]$. Note that p_{k+1} is in $f^{\frac{n}{k}}[p, \delta]$, because $f^{\frac{n}{k}}(p) = p$ and $f^{\frac{n}{k}}(\delta) \geq p_{k+1}$. Each point of $B_2 = \{p_{k+1}, p_{k+2}, \dots, p_{2k}\}$ has period k under $f^{\frac{n}{k}}$, and B_2 is fixed under $f^{\frac{n}{k}}$ by Theorem 1. Consequently, p_{2k} lies in the orbit of p_{k+1} under $f^{\frac{n}{k}}$. Choose N so that $(f^{\frac{n}{k}})^{N-1}(p_{k+1}) = p_{2k}$. Then

$$p_{2k} = (f^{\frac{n}{k}})^{N-1}(p_{k+1}) \in (f^{\frac{n}{k}})^{N-1} \circ f^{\frac{n}{k}}[p, \delta] = (f^{\frac{n}{k}})^N[p, \delta].$$

Thus, $(f^{\frac{n}{k}})^N[p, \delta] = [p, p_{2k}]$. Applying Bennett’s Theorem to $f^{\frac{n}{k}}|[p, p_{2k}]$ gives that

$$\varprojlim \{ [p, p_{2k}], f^{\frac{n}{k}} \} = R \cup K$$

where R is a topological ray, $\bar{R} - R = K$, and $K = \varprojlim \{ [\delta, p_{2k}], f^{\frac{n}{k}} \}$. Since $f^{\frac{n}{k}}[\delta, p_{2k}] = [p_{k+1}, p_{2k}]$, $K = \varprojlim \{ [p_{k+1}, p_{2k}], f^{\frac{n}{k}} \}$. This establishes (5).

Finally, consider (6). Denote $\varprojlim \{ [p_1, p_{2k}], f^{\frac{n}{2k}} \}$ by M . Let M_1 and M_2 denote $\{x \in M: x_{2i} \in [p, p_{2k}] \text{ for } i \in \mathbb{N}\}$ and $\{x \in M: x_{2i-1} \in [p, p_{2k}] \text{ for } i \in \mathbb{N}\}$, respectively. Since $f^{\frac{n}{2k}}[p, p_{2k}] = [p_1, p]$ and $f^{\frac{n}{2k}}[p_1, p] = [p, p_{2k}]$, it follows that M_1 and M_2 are both homeomorphic to $\varprojlim \{ [p, p_{2k}], f^{\frac{n}{k}} \}$.

To see that $M = M_1 \cup M_2$, suppose x is a point of M . Either $x_{2i} \in [p_1, p]$ for infinitely many i , or $x_{2i} \in [p, p_{2k}]$ for infinitely many i . Since $[p_1, p]$ and $[p, p_{2k}]$ are both invariant under $(f^{\frac{n}{2k}})^2 = f^{\frac{n}{k}}$, it follows that either $x_{2i} \in [p_1, p]$ for every i in \mathbb{N} , or $x_{2i} \in [p, p_{2k}]$ for every i in \mathbb{N} . In the latter case, $x \in M_1$, and in the former case, since $f^{\frac{n}{2k}}[p_1, p] = [p, p_{2k}]$, $x \in M_2$. Thus $M = M_1 \cup M_2$.

To finish the proof, it remains only to show that $M_1 \cap M_2 = \{(p, p, p, \dots)\}$. First note that $M_2 = \{x \in M: x_{2i} \in [p_1, p] \text{ for } i \in \mathbb{N}\}$. Hence, $M_1 \cap M_2 = \{x \in M: x_{2i} = p \text{ for } i \in \mathbb{N}\}$. Since $f^{\frac{n}{2k}}(p) = p$, it follows that $M_1 \cap M_2 = \{(p, p, p, \dots)\}$. \square

Theorem 11. *Suppose f is a Markov map whose partition, p_1, p_2, \dots, p_n , is an n -cycle. Then $\varprojlim \{ [p_1, p_n], f \}$ is hereditarily decomposable if and only if n is a power of two and $\{p_1, p_2, \dots, p_{2^k}\}$ is a block for each positive integer k not greater than $\log_2(n)$.*

Proof. The latter follows from the former by Theorem 5. To see the converse, suppose n is a power of two and $\{p_1, p_2, \dots, p_{2^m}\}$ is a block for each positive integer m not larger than $\log_2(n)$. For each such m , denote by $P(m)$ the proposition that $\varprojlim \{ [p_1, p_{2^m}], f^{n^{2^{-m}}} \}$

is hereditarily decomposable. The theorem will be proved if it is shown that $P(\log_2(n))$ is true. First consider $P(1)$. By hypothesis, $\{p_1, p_2\}$ is a block, so $f^i(p_1)$ and $f^i(p_2)$ are consecutive terms of p_1, p_2, \dots, p_n for each positive integer i . Hence f is monotone between $f^i(p_1)$ and $f^i(p_2)$ for each such i . It follows that f^i is monotone on $[p_1, p_2]$ for each i . By (1) \rightarrow (4) of Theorem 1, $f^{\frac{n}{2}}$ maps $\{p_1, p_2\}$ onto itself; hence $f^{\frac{n}{2}}$ maps $[p_1, p_2]$ monotonically onto itself. Consequently, $\varprojlim\{[p_1, p_2], f^{\frac{n}{2}}\}$ is an arc. Thus $P(1)$ is true.

Suppose m_0 is a positive integer not greater than $\log_2(n)$ for which $P(m_0)$ is true. If $m_0 = \log_2(n)$, the theorem follows. Suppose $m_0 < \log_2(n)$. Denote $\{p_1, p_2, \dots, p_{2^{m_0}}\}$ and $\{p_{2^{m_0}+1}, p_{2^{m_0}+2}, \dots, p_{2^{m_0+1}}\}$ by B_1 and B_2 , respectively. Note that $B_1 \cup B_2$ is a block, also by hypothesis. Then by (6) of Theorem 10, there is a point $p \in (p_{2^{m_0}}, p_{2^{m_0}+1})$ such that

$$\varprojlim\{[p_1, p_{2^{m_0+1}}], f^{n2^{-(m_0+1)}}\} = M_1 \cup M_2$$

where $M_1 \cap M_2 = (p, p, p, \dots)$, and M_1 and M_2 are both homeomorphic to $\varprojlim\{[p, p_{2^{m_0+1}}], f^{n2^{-m_0}}\}$. Since M_1 and M_2 intersect only at a point, in order to show that $\varprojlim\{[p_1, p_{2^{m_0+1}}], f^{n2^{-(m_0+1)}}\}$ is hereditarily decomposable, and thus demonstrate $P(m_0 + 1)$, it suffices to show that $\varprojlim\{[p, p_{2^{m_0+1}}], f^{n2^{-m_0}}\}$ is hereditarily decomposable. By (5) of Theorem 10,

$$\varprojlim\{[p, p_{2^{m_0+1}}], f^{n2^{-m_0}}\} = R \cup K$$

where R is a topological ray, $\overline{R} - R = K$, and

$$K = \varprojlim\{[p_{2^{m_0}+1}, p_{2^{m_0+1}}], f^{n2^{-m_0}}\}.$$

By $P(m_0)$, K is hereditarily decomposable; hence, $R \cup K$ is hereditarily decomposable. This completes the proof of $P(m_0 + 1)$. Proceeding inductively yields that $P(\log_2(n))$ is true. \square

Example 12. Consider once again the maps g, h, k , and l , whose graphs appear in Figs. 1 and 2. It has already been noted that $\varprojlim\{[1, 12], h\}$ is indecomposable. By Theorem 7, $\varprojlim\{[1, 12], g\}$ and $\varprojlim\{[1, 8], l\}$ are both decomposable, and, by Theorem 11, $\varprojlim\{[1, 8], k\}$ is hereditarily decomposable.

Example 13. By inductively applying Theorem 10, one can obtain a rough sketch of any hereditarily decomposable continuum that arises from the inverse limit of a single Markov bonding map whose partition consists of a single periodic orbit. This process will be indicated by means of an example.

One last time, consider the indefatigable mapping k of Fig. 2. Since $\{1, 2\}, \{1, 2, 3, 4\}$, and, by (1) implies (3) of Theorem 1, $\{3, 4\}$ are blocks with respect to k , it follows from (1) and (2) of Theorem 10 that $\{[1, 2], k^4\}$ and $\{[3, 4], k^4\}$ are inverse sequences with homeomorphic inverse limits. Note that k^4 is monotone on $[1, 2]$ because $\{1, 2\}$ is a block. Hence, $\varprojlim\{[1, 2], k^4\}$ and $\varprojlim\{[3, 4], k^4\}$ are both arcs.

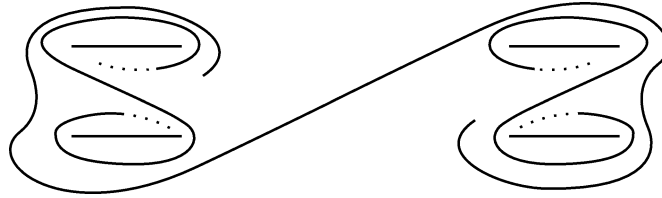


Fig. 3.

By (5) of Theorem 10, there is a point $p \in (2, 3)$ such that $\varprojlim\{[p, 4], k^4\}$ is a topological $\sin(\frac{1}{x})$ curve, whose limit bar corresponds to $\varprojlim\{[3, 4], k^4\}$. Hence, the point (p, p, p, \dots) , which is an endpoint of $\varprojlim\{[p, 4], k^4\}$, is the endpoint of the ray in $\varprojlim\{[p, 4], k^4\}$ whose limit bar is $\varprojlim\{[3, 4], k^4\}$. By (6) of Theorem 10, $\varprojlim\{[1, 4], k^2\}$ is the union of two topological $\sin(\frac{1}{x})$ curves joined at the endpoint of their rays.

By a similar argument, $\varprojlim\{[1, 8], k\}$ is the union of two rays joined at their endpoint, each limiting onto a copy of $\varprojlim\{[1, 4], k^2\}$. A rough sketch of $\varprojlim\{[1, 8], k\}$ appears in Fig. 3.

The property of the mapping k that allows Example 13 to proceed as it does with the aid of Theorem 10 is that it contains blocks of length 2^i for $i = 1, 2, 3$. The properties of k at any higher level of discrimination do not contribute to the solution. It is tempting to wonder whether a map k' that is different from k , but still has blocks of length 2^i for $i = 1, 2, 3$, would generate an inverse limit homeomorphic to $\varprojlim\{[1, 8], k\}$. Theorem 10 does not warrant such a conclusion.

For certain parameter values of the logistic mapping, $f_\lambda(x) = 4\lambda x(1-x)$, the restriction of the mapping to its core is an example of a Markov map whose partition consists of a single periodic orbit; however, such examples are always unimodal. This was noted by Barge and Ingram [1]. They describe how, for a few of these parameter values, continua like the one shown in Fig. 3 are embedded in the inverse limit of such logistic maps.

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