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Simplicial maps of graphs that factor through an arc

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Abstract

In this paper it is shown that a simplicial map φ from a connected graph into a graph can be factored through an arc if and only if there are a monotone map μ, a sequence $\pi_1, \pi_2, \ldots, \pi_n$ of folds, and an irreducible map ψ whose domain is an arc such that $\varphi = \psi \circ \pi_n \circ \cdots \circ \pi_2 \circ \pi_1 \circ \mu$.

Implicit in this result is a procedure that can be useful for determining whether a simplicial map factors through an arc. For those that do, the procedure produces a factorization. This is demonstrated by means of an example.

On the way to the main results, quotient graphs are defined and fundamental results relating them to simplicial maps are proved. Folds are then defined as a specific type of projection onto a quotient graph, generalizing previous definitions.

MSC: 37B45; 54F15

Keywords: Folds; Factoring through an arc; Simplicial maps; Quotient graphs

1. Introduction

In 1972 Ingram [1] constructed an atriodic tree-like continuum that is not chainable. He defined it as the inverse limit of a carefully chosen map from a simple triod onto itself. Recently, Ingram [2] looked at five natural variations on that map and showed that four of them produce chainable inverse limits. The means by which he did so was to show that the square of each of the four maps can be factored through an arc. This exemplifies the well-known rule of thumb that identifying maps between graphs that can be factored through an arc can be useful when considering whether a graph-like continuum is chainable.

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This paper addresses the question of which piecewise linear maps between graphs can be factored through an arc, primarily by describing those that can. It is useful to think about a piecewise linear map between graphs as a simplicial map between graphs. Piotr Minc [3] gave necessary and sufficient conditions for a simplicial map between graphs to factor through an arc. Minc's result is particularly nice because his conditions so effectively exploit a simple process involving dual graphs.

Other important results about factoring piecewise-linear graph maps through an arc are due to Lex G. Oversteegen [5] and Lee Mohler and Oversteegen [4]. Every simplicial map between graphs can be described by a graph-word. Mohler and Oversteegen gave sufficient conditions for a graph-word to reduce to a chain-word. The existence of such a reduction is equivalent to the existence of a factorization of the map through an arc. Their primary means of reduction is to replace folds in the word with natural simplifications; for example, $\ldots ABCDCBCDE \ldots$ reduces to $\ldots ABCDE \ldots$. Oversteegen [6] later used the results of [4] and [5] to show that every piecewise linear map from a tree to a graph with (surjective semi-) span zero factors through an arc.

One advantage of reduction by means of folds is that the resulting factorization is easy to track through the folds. An advantage of Minc's approach with dual graphs is that it provides conditions that are both necessary and sufficient for factorization through an arc.

In this paper, folds are used to give necessary and sufficient conditions for a simplicial map from a connected graph into a graph to admit a factorization through an arc. Thus the main results of this paper offer some (though not all) of the advantages of both previous approaches. The example in Section 2 motivates the main results of the paper in a way that suggests a procedure for determining whether such simplicial maps factor through an arc. While this procedure is effective for simple maps, it is not practical in general. The remark at the end of the paper highlights this shortcoming.

A *graph* is a finite collection of vertices and edges such that each edge connects two distinct vertices, and each pair of vertices is connected by at most one edge. Vertices that are connected by an edge are said to be *adjacent*. The *valence* of a vertex v is the number of edges having v as an endpoint. A *terminal vertex* of a graph is a vertex with valence one. A graph G is *connected* provided it is not the union of two mutually exclusive subgraphs.

A *simplicial map* from a graph G into a graph G' is a function φ from the vertices of G into the vertices of G' with the property that if x and y are adjacent vertices in G, then either $\varphi(x)$ and $\varphi(y)$ are adjacent in G', or $\varphi(x) = \varphi(y)$. If $\varphi(x)$ and $\varphi(y)$ are adjacent for each such x and y, then the simplicial map φ is said to be *light*. If, for each y in the range of φ, there is a connected subgraph of G whose set of vertices is $\varphi^{-1}(y)$, then φ is said to be *monotone*.

If φ is a simplicial map from a graph G into a graph G', and H is a subgraph of G, then $\varphi[H]$ denotes the subgraph of G' such that a vertex of G' is a vertex of $\varphi[H]$ if and only if it is the image of a vertex of G, and an edge of G' is an edge of $\varphi[H]$ if and only if it connects the images of two adjacent vertices of H. If $\varphi[G] = G'$, φ is said to be *surjective*. An *isomorphism* is a surjective one-to-one simplicial map whose inverse is simplicial.

An *arc* is a connected graph with exactly two terminal vertices and all other vertices of valence two. The notation $\langle a_0, a_1, \ldots, a_n \rangle$ denotes the arc with vertices $a_0, a_1, \ldots, a_n$ such that a_i and a_j are joined by an edge if and only if $|i - j| = 1$.

A *path* P in a graph G is a finite sequence $p_0, p_1, \ldots, p_n$ of vertices of G such that p_{i-1} and p_i are joined by an edge for $i = 1, 2, \ldots, n$, and $p_{i-1} \neq p_{i+1}$ for $i = 1, 2, \ldots, n-1$. If either p_0 or p_n is a terminal vertex of G, then $p_0, p_1, \ldots, p_n$ is a *terminal path* in G.

A *simple closed curve* is a connected graph such that every vertex has valence two. The notation $\langle c_0, c_1, \ldots, c_n = c_0\rangle$ denotes the simple closed curve with vertices $c_1, c_2, \ldots, c_n$ and edges joining c_n to c_1 and c_i to c_{i+1} for $i = 1, 2, \ldots, n-1$. The vertex c_n is also denoted by c_0. A *tree* is a connected graph in which there is no simple closed curve. Note that for any path $p_0, p_1, \ldots, p_n$ in a tree, $p_i \neq p_j$ for $i \neq j$.

2. A preliminary example

Consider the two graphs in Fig. 1(a). The schematic in Fig. 1(b) describes a simplicial map—denote it by φ_b—from the triod on the left of Fig. 1(a) to the graph on its right in the following way. Fig. 1(b) shows an inflated version of the range graph with the domain graph winding around inside. The placement of the vertex c_2 inside the inflated vertex A indicates that $\varphi_b(c_2) = A$. Similarly, $\varphi_b(o) = C$, $\varphi_b(a_3) = F$, $\varphi_b(b_4) = E$, etc.

Denote the simplicial maps in Fig. 1(c) and (d) by φ_c and φ_d, respectively. Notice that the path $b_1, b_2, b_3, b_4, b_5, b_6, b_7$ is folded by φ_b back and then forth through the vertices D, E, F. The map φ_c is obtained from φ_b as follows: the domain of φ_c is obtained from that of φ_b by replacing the path $b_1, b_2, b_3, b_4, b_5, b_6, b_7$ with the path $\{b_1, b_5\}, \{b_2, b_4, b_6\}, \{b_3, b_7\}$, and φ_c is defined so that $\varphi_c\{b_1, b_5\} = \varphi_b(b_1) = \varphi_b(b_5)$, $\varphi_c\{b_2, b_4, b_6\} = \varphi_b(b_2) = \varphi_b(b_4) = \varphi_b(b_6)$, and $\varphi_c\{b_3, b_7\} = \varphi_b(b_3) = \varphi_b(b_7)$. Using the word language of Mohler and Oversteegen [4], the graph-word corresponding to the map φ_b contains the word $DEFEDEF$, which reduces to the word DEF.

A similar reduction of φ_c produces φ_d: the terminal path $a_4, a_3, a_2, a_1, o, \{b_1, b_5\}, \{b_2, b_4, b_6\}, \{b_3, b_7\}, b_8$ is reduced to $\{a_4, b_8\}, \{a_3, b_3, b_7\}, \{a_2, b_2, b_4, b_6\}, \{a_1, b_1, b_5\}, o$. Notice that the former of these paths is a terminal path and is folded by φ_c into two passes through the vertices G, F, E, D, C, whereas φ_b folds the path $b_1, b_2, b_3, b_4, b_5, b_6, b_7$, which is not terminal, into three passes through D, E, F.

These two types of reductions motivate the definitions of terminal fold and interior fold, which appear in Section 4. More importantly, the example as a whole motivates the entire paper for the following reason. Notice that the domain of φ_d, the map resulting from the reductions, is an arc. It follows that φ_b can be factored through this arc: $\varphi_b = \varphi_d \circ \pi$ where π is the map that projects the domain of φ_b onto the arc. One of the main results of the paper may be crudely stated: a light simplicial map from a connected graph into a graph can be factored through an arc if and only if there is a sequence of reductions, like the two in this example, that produce a map whose domain is an arc.

The reductions used in this example are put to rigor with quotient graphs in Sections 3 and 4. Some basic results about quotient graphs are given in Section 3, and they are used in Section 4 to define folds. Theorem 6, which is also in Section 4, is the crux for the main results of the paper, Theorems 14, 15, and 16. These results appear in Section 6. The proofs given there rely on implications of Theorem 6 in certain special cases, which are addressed in Section 5.

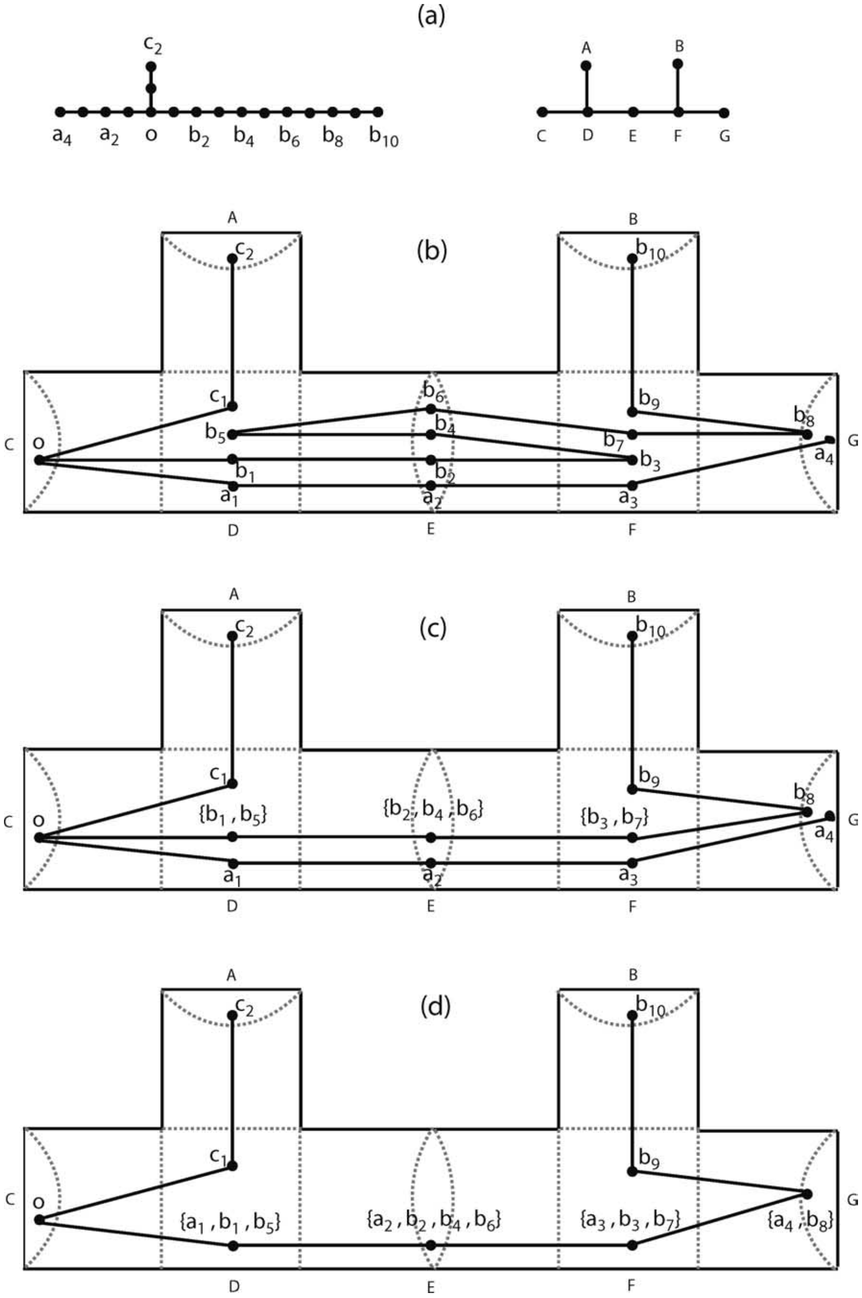
(a)
c_2
a_4 a_2 o b_2 b_4 b_6 b_8 b_{10}
A B
C D E F G
(b)
A B
c_2 b_{10}
c_1 b_6 b_9
b_5 b_4 b_8
C o b_7 b_3 a_4 G
b_1 b_2
a_1 a_2 a_3
D E F
(c)
A B
c_2 b_{10}
c_1 b_9 b_8
C o $\{b_1, b_5\}$ $\{b_2, b_4, b_6\}$ $\{b_3, b_7\}$ a_4 G
a_1 a_2 a_3
D E F
(d)
A B
c_2 b_{10}
c_1 b_9
C o G
$\{a_1, b_1, b_5\}$ $\{a_2, b_2, b_4, b_6\}$ $\{a_3, b_3, b_7\}$ $\{a_4, b_8\}$
D E F

Fig. 1.

3. Quotient graphs

In this section quotient graphs are defined, and three fundamental results are proved. Each result has a well-known analogue in the more general theory of quotient spaces. The combinatorial versions given here are less prominent, but they are predictable to anyone familiar with their classical cousins.

Definitions. A *partition on a graph* G is a partition $\mathcal{P}$ of the collection of vertices of G into equivalence classes. If x is a vertex of G, then the equivalence class of x determined by $\mathcal{P}$ is denoted by $[x]_{\mathcal{P}}$. The *quotient graph* $G/\mathcal{P}$ is the graph whose vertices are the equivalence classes determined by $\mathcal{P}$ such that two distinct equivalence classes, $[x]_{\mathcal{P}}$ and $[y]_{\mathcal{P}}$, are joined by an edge if and only if there are two vertices of G belonging to $[x]_{\mathcal{P}}$ and $[y]_{\mathcal{P}}$ respectively that are joined by an edge of G. Suppose G is a graph and $\mathcal{P}$ is a partition on G. The map $\pi_{\mathcal{P}}$ from the vertices of G into the vertices of $G/\mathcal{P}$ defined by $\pi_{\mathcal{P}}(x) = [x]_{\mathcal{P}}$ is the *projection mapping of* G *onto* $G/\mathcal{P}$.

Theorem 1. *If* G *is a graph and* $\mathcal{P}$ *is a partition on* G*, then the projection* $\pi_{\mathcal{P}}$ *is a surjective simplicial map.*

Proof. Suppose x and y are vertices of G that are adjacent in G. If $[x]_{\mathcal{P}} = [y]_{\mathcal{P}}$, then $\pi_{\mathcal{P}}(x) = \pi_{\mathcal{P}}(y)$. If $[x]_{\mathcal{P}}$ and $[y]_{\mathcal{P}}$ are distinct, then they are adjacent in $G/\mathcal{P}$ because x and y are adjacent in G; hence $\pi_{\mathcal{P}}(x)$ and $\pi_{\mathcal{P}}(y)$ are adjacent in $G/\mathcal{P}$. Consequently, $\pi_{\mathcal{P}}$ is simplicial.

Given adjacent vertices, y_1 and y_2, of $G/\mathcal{P}$, there are adjacent vertices x_1 and x_2 of G such that $y_1 = [x_1]_{\mathcal{P}}$ and $y_2 = [x_2]_{\mathcal{P}}$. In other words, there are adjacent vertices of G whose images under $\pi_{\mathcal{P}}$ are y_1 and y_2. Thus $\pi_{\mathcal{P}}$ is surjective. □

Theorem 2. *If* φ *is a simplicial map from a graph* G *into a graph* G', $\mathcal{P}$ *is a partition on* G*, and* φ *is constant on each equivalence class of* $\mathcal{P}$*, then there is a unique simplicial map* ψ *from* $G/\mathcal{P}$ *into* G' *such that* $\varphi = \psi \circ \pi_{\mathcal{P}}$.

Proof. For each vertex $[x]_{\mathcal{P}}$ of $G/\mathcal{P}$, define $\psi([x]_{\mathcal{P}})$ to be $\varphi(x)$. Since φ is constant on each equivalence class of $\mathcal{P}$, ψ is well defined. Furthermore, $\varphi = \psi \circ \pi_{\mathcal{P}}$. To see that ψ is simplicial, suppose $[x]_{\mathcal{P}}$ and $[y]_{\mathcal{P}}$ are vertices of $G/\mathcal{P}$ that are joined by an edge. Then there are vertices x' and y' of G belonging to $[x]_{\mathcal{P}}$ and $[y]_{\mathcal{P}}$, respectively, that are joined by an edge of G. Then $[x']_{\mathcal{P}} = [x]_{\mathcal{P}}$ and $[y']_{\mathcal{P}} = [y]_{\mathcal{P}}$. Hence, $\psi([x]_{\mathcal{P}}) = \psi([x']_{\mathcal{P}}) = \psi \circ \pi_{\mathcal{P}}(x') = \varphi(x')$, and, similarly, $\psi([y]_{\mathcal{P}}) = \varphi(y')$. Since φ is simplicial and x' and y' are joined by an edge, it follows that either $\varphi(x') = \varphi(y')$ or $\varphi(x')$ and $\varphi(y')$ are adjacent in G'. Consequently, either $\psi([x]_{\mathcal{P}}) = \psi([y]_{\mathcal{P}})$ or $\psi([x]_{\mathcal{P}})$ and $\psi([y]_{\mathcal{P}})$ are adjacent. It remains only to show that ψ is unique. Suppose ψ_1 and ψ_2 are simplicial maps from $G/\mathcal{P}$ into G' such that $\varphi = \psi_1 \circ \pi_{\mathcal{P}} = \psi_2 \circ \pi_{\mathcal{P}}$. Then for each vertex $[x]_{\mathcal{P}}$ of $G/\mathcal{P}$, $\psi_1([x]_{\mathcal{P}}) = \psi_1 \circ \pi_{\mathcal{P}}(x) = \psi_2 \circ \pi_{\mathcal{P}}(x) = \psi_2([x]_{\mathcal{P}})$, which is to say $\psi_1 = \psi_2$. Hence ψ is unique. □

Theorem 3. *If φ is a simplicial map from a graph G into a graph G', then there are a light simplicial map ψ and a surjective monotone simplicial map μ such that $\varphi = \psi \circ \mu$.*

Proof. Let $\mathcal{P}$ denote the partition on G according to which two vertices, x and y, of G are equivalent if and only if there is a connected subgraph of G containing x and y on which φ is constant. Then, for each x in G, there is connected subgraph H_x that has for its set of vertices all vertices of $[x]_{\mathcal{P}}$. It follows that $\pi_{\mathcal{P}}$ is monotone. By Theorem 1, it is surjective.

Since φ is constant on each equivalence class of $\mathcal{P}$, it follows by Theorem 2 that there is a simplicial map ψ from $G/\mathcal{P}$ into G' such that $\varphi = \psi \circ \pi_{\mathcal{P}}$.

To see that ψ is light, suppose $[p]_{\mathcal{P}}$ and $[q]_{\mathcal{P}}$ are adjacent vertices of $G/\mathcal{P}$. Then H_p and H_q are joined by an edge of G. Consequently, there is a connected subgraph H of G whose set of vertices is $[p]_{\mathcal{P}} \cup [q]_{\mathcal{P}}$. Then φ is not constant on $[p]_{\mathcal{P}} \cup [q]_{\mathcal{P}}$; otherwise, p and q would belong to the same equivalence class. Since φ is constant on each of $[p]_{\mathcal{P}}$ and $[q]_{\mathcal{P}}$, it follows that $\varphi(p) \neq \varphi(q)$. But $\varphi(p) = \psi([p]_{\mathcal{P}})$ and $\varphi(q) = \psi([q]_{\mathcal{P}})$, so $\psi([p]_{\mathcal{P}}) \neq \psi([q]_{\mathcal{P}})$. Hence, $\psi([p]_{\mathcal{P}})$ and $\psi([q]_{\mathcal{P}})$ are adjacent in G'. Consequently, ψ is light. □

4. Folds and irreducible maps

Mohler and Oversteegen [4] define folds for the class of simplicial maps from $\langle 1, 2, \ldots, n\rangle$ onto $\langle 1, 2, \ldots, m\rangle$. In this section folds are defined within the larger context of simplicial maps from a graph into a graph. In that sense, this definition is more general. However, Mohler and Oversteegen do incorporate the more general fold implicitly via their treatment of reduction functions on graph-words.

Definitions. A *terminal-fold partition* on a graph G is a partition $\mathcal{F}$ on G for which there is a terminal path $p_0, p_1, \ldots, p_{2n}$ in G such that the equivalence classes of $\mathcal{F}$ are sets of the form $\{p_k, p_{2n-k}\}$ for some nonnegative integer k not larger than n and sets of the form $\{x\}$ for some vertex x of G that is not in $\{p_0, p_1, \ldots, p_{2n}\}$. The projection mapping $\pi_{\mathcal{F}}$ from G onto $G/\mathcal{F}$ is called a *terminal fold*.

An *interior-fold partition* on a graph G is a partition $\mathcal{F}$ on G for which there is a path $p_0, p_1, \ldots, p_{3n}$ in G such that the equivalence classes of $\mathcal{F}$ are sets of the form $\{p_k, p_{2n-k}, p_{2n+k}\}$ for some nonnegative integer k not larger than n and sets of the form $\{x\}$ for some $x \notin \{p_0, p_1, \ldots, p_{3n}\}$. In this case, the projection mapping $\pi_{\mathcal{F}}$ from G onto $G/\mathcal{F}$ is called an *interior fold*.

The *trivial-fold partition* on a graph G is the partition $\mathcal{F}$ on G whose equivalence classes are sets of the form $\{x\}$ for some vertex x of G. The projection mapping $\pi_{\mathcal{F}}$ from G onto $G/\mathcal{F}$ is called the *trivial fold*. Note that the trivial fold is an isomorphism. Interior folds and terminal folds are *nontrivial folds*.

A *fold partition* on a graph is a partition that is either a terminal-fold partition, an interior-fold partition, or the trivial-fold partition. A *fold* is a projection map that is either a terminal fold, an interior fold, or the trivial fold.

For example, if G is the simple closed curve $\langle c_0, c_1, c_2, c_3, c_4 = c_0\rangle$, then the path $p_0, p_1, \ldots, p_6$ such that $p_i = c_{i \bmod 4}$ for $0 \leqslant i \leqslant 6$ determines an interior-fold partition for which the corresponding fold maps G onto an arc by identifying the vertices c_1 and c_3.

Theorem 4. *If φ is a simplicial map from a graph G into a graph G', then the following are equivalent*:

(1) *There is a nontrivial fold π and a simplicial map ψ such that $\varphi = \psi \circ \pi$.*
(2) *Either there is a path $p_0, p_1, \ldots, p_{3n}$ such that $\varphi(p_k) = \varphi(p_{2n-k}) = \varphi(p_{2n+k})$ for $k = 0, 1, \ldots, n$ or there is a terminal path $p_0, p_1, \ldots, p_{2n}$ such that $\varphi(p_k) = \varphi(p_{2n-k})$ for $k = 0, 1, \ldots, n$.*

Proof. Suppose (1) is true. Let $\mathcal{F}$ denote the fold partition on G such that $\pi = \pi_{\mathcal{F}}$. Either $\mathcal{F}$ is a interior-fold partition or $\mathcal{F}$ is an terminal-fold partition. First suppose the former. Then there is a path $p_0, p_1, \ldots, p_{3n}$ in G such that the sets $\{p_k, p_{2n-k}, p_{2n+k}\}$ for $k = 0, 1, \ldots, n$ are among the equivalence classes of $\mathcal{F}$. Then $\pi_{\mathcal{F}}(p_k) = \pi_{\mathcal{F}}(p_{2n-k}) = \pi_{\mathcal{F}}(p_{2n+k})$ for $k = 0, 1, \ldots, n$. Since $\varphi = \psi \circ \pi_{\mathcal{F}}$, it follows that $\varphi(p_k) = \varphi(p_{2n-k}) = \varphi(p_{2n+k})$ for $k = 0, 1, \ldots, n$. By a similar argument, (2) holds if $\mathcal{F}$ is a terminal-fold partition.

Conversely, suppose (2) is true. In particular, suppose there is a terminal path $p_0, p_1, \ldots, p_{2n}$ such that $\varphi(p_k) = \varphi(p_{2n-k})$ for $k = 0, 1, \ldots, n$. Denote by $\mathcal{F}$ the partition whose equivalence classes are the sets $\{p_k, p_{2n-k}\}$ for some $k = 0, 1, \ldots, n$ and the sets $\{x\}$ for $x \notin \{p_1, p_2, \ldots, p_n\}$. Then φ is constant on each equivalence class of $\mathcal{F}$. By Theorem 2, there is a simplicial map ψ from $G/\mathcal{F}$ into G' such that $\varphi = \psi \circ \pi_{\mathcal{F}}$. Similarly, if there is a path $p_0, p_1, \ldots, p_{3n}$ such that $\varphi(p_k) = \varphi(p_{2n-k}) = \varphi(p_{2n+k})$ for $k = 0, 1, \ldots, n$, then (1) holds. □

Definitions. A simplicial map φ is said to be *irreducible* if and only if it is light and fails to satisfy either, and hence both, of the equivalent conditions in Theorem 4. A simplicial map that is not irreducible is said to be *reducible*.

Lemma 5. *Suppose $\pi_{\mathcal{P}}$ is a projection and φ and ψ are simplicial maps such that $\varphi = \psi \circ \pi_{\mathcal{P}}$. If φ is light, then ψ is light.*

Proof. Suppose ψ is not light. Then there are adjacent vertices $[x]_{\mathcal{P}}$ and $[y]_{\mathcal{P}}$ of $G/\mathcal{P}$ such that $\psi([x]_{\mathcal{P}}) = \psi([y]_{\mathcal{P}})$. Since $[x]_{\mathcal{P}}$ and $[y]_{\mathcal{P}}$ are adjacent in $G/\mathcal{P}$, there are adjacent vertices x' and y' of G that belong to $[x]_{\mathcal{P}}$ and $[y]_{\mathcal{P}}$, respectively. Then $\varphi(x') = \psi \circ \pi_{\mathcal{P}}(x') = \psi([x]_{\mathcal{P}}) = \psi([y]_{\mathcal{P}}) = \psi \circ \pi_{\mathcal{P}}(y') = \varphi(y')$. Thus φ is not light if ψ is not light. □

Theorem 6. *If φ is a light simplicial map from a graph G into a graph G', then there are an irreducible map ψ and a sequence $\pi_1, \pi_2, \ldots, \pi_n$ of folds such that $\varphi = \psi \circ \pi_n \circ \cdots \circ \pi_2 \circ \pi_1$.*

Proof. If φ is irreducible, then $\varphi = \psi \circ \pi$ where π is the trivial fold and $\psi = \varphi \circ \pi^{-1}$. Suppose, for the purpose of establishing a contradiction, that ψ is reducible. Then, because ψ is light by Lemma 5, either there is a path $p_0, p_1, \ldots, p_{3m}$ such that $\varphi \circ \pi^{-1}(p_k) = \varphi \circ \pi^{-1}(p_{2m-k}) = \varphi \circ \pi^{-1}(p_{2m+k})$ for $k = 0, 1, \ldots, m$ or there is a terminal path $q_0, q_1, \ldots, q_{2m}$ such that $\varphi \circ \pi^{-1}(q_k) = \varphi \circ \pi^{-1}(q_{2m-k})$ for $k = 0, 1, \ldots, m$. In the former case, since π^{-1} is an isomorphism, $\pi^{-1}(p_0), \pi^{-1}(p_1), \ldots, \pi^{-1}(p_{3m})$ is a path in G such that $\varphi(\pi^{-1}(p_k)) = \varphi(\pi^{-1}(p_{2m-k})) = \varphi(\pi^{-1}(p_{2m+k}))$ for $k = 0, 1, \ldots, m$. Consequently, φ is reducible, which is a contradiction. Similarly, the latter case involves a contradiction. Thus ψ is irreducible, and the conclusion of the theorem holds.

Suppose φ is reducible. Since φ is also light, there is a nontrivial fold π_1 and a simplicial map ψ_1 such that $\varphi = \psi_1 \circ \pi_1$; furthermore, ψ_1 is light by Lemma 5. For convenience, denote G by G_0. Let G_1 denote the quotient graph that is both the image of G_0 under π_1 and the domain of ψ_1.

Suppose k is a positive integer for which there are nontrivial folds $\pi_1, \pi_2, \ldots, \pi_k$ and a light simplicial map ψ_k such that $\varphi = \psi_k \circ \pi_k \circ \cdots \circ \pi_2 \circ \pi_1$. If ψ_k is irreducible, then the conclusion of the theorem follows. Suppose ψ_k is reducible, and let G_k denote the quotient graph that is both the image of G_{k-1} under π_k and the domain of ψ_k. Since ψ_k is reducible and light, there is a nontrivial fold π_{k+1} and a light simplicial map ψ_{k+1} such that $\psi_k = \psi_{k+1} \circ \pi_{k+1}$. Note that the image of G_k under π_{k+1} has fewer vertices than G_k, and denote it by G_{k+1}.

Proceeding inductively yields that either there is a positive integer n such that ψ_n is irreducible, from which the conclusion of the theorem follows, or there is an infinite sequence $G_0, G_1, G_2, \ldots$ of graphs, each term of which has fewer vertices than its predecessor. But the latter is impossible since G_0 has only finitely many vertices. □

5. Simplicial maps from graphs into arcs and from arcs into graphs

It has been noted that Theorem 6 is pivotal to this paper. Theorem 6 describes the composition of light simplicial maps of graphs in terms of folds and irreducible maps. If φ is a simplicial map that can be factored through an arc, then there are simplicial maps φ_1 and φ_2 such that $\varphi = \varphi_2 \circ \varphi_1$, the domain of φ_2 is an arc, and φ_1 is surjective. Consequently, in order to describe maps that factor through an arc, it is useful to consider first the implications of Theorem 6 for maps whose domain is an arc and for maps whose range is an arc.

5.1. Simplicial maps from arcs into graphs

Lemma 7. *If A is an arc, and $\mathcal{F}$ is a fold partition on A, then $A/\mathcal{F}$ is an arc.*

Proof. The proof is straightforward but tedious. An indication of the proof in case $\mathcal{F}$ is an interior-fold partition is given here. Denote the path corresponding to $\mathcal{F}$ by $p_0, p_1, \ldots, p_{3n}$, and denote A by $\langle p_k, p_{k+1}, \ldots, p_m \rangle$. Then the distinct equivalence classes of $\mathcal{F}$ are $[p_k]_{\mathcal{F}}, [p_{k+1}]_{\mathcal{F}}, \ldots, [p_n]_{\mathcal{F}}$ together with $[p_{3n+1}]_{\mathcal{F}}, [p_{3n+2}]_{\mathcal{F}}, \ldots, [p_m]_{\mathcal{F}}$. Each of $[p_k]_{\mathcal{F}}$

and $[p_m]_{\mathcal{F}}$ is a terminal vertex of $A/\mathcal{F}$, and each of the remaining vertices has valence two. Thus $A/\mathcal{F}$ is an arc if $\mathcal{F}$ is a interior-fold partition. □

Theorem 8. *If φ is a light simplicial map from an arc A into a graph G', then there are an arc A', an irreducible map ψ from A' into G', and a sequence $\pi_1, \pi_2, \ldots, \pi_n$ of folds such that $\varphi = \psi \circ \pi_n \circ \cdots \circ \pi_2 \circ \pi_1$.*

By Theorem 6, there is an irreducible map ψ and a sequence $\pi_1, \pi_2, \ldots, \pi_n$ of folds such that $\varphi = \psi \circ \pi_n \circ \cdots \circ \pi_2 \circ \pi_1$. It remains only to show that the quotient graph onto which π_n projects is an arc. It follows from Lemma 7 that if the domain of a fold is an arc, then so is its range. Since the domain of π_1 is an arc, applying Lemma 7 inductively yields that the range of π_n is an arc.

5.2. Simplicial maps from graphs into arcs

For convenience, Lemma 9 is stated and proved under the assumption that φ is irreducible. However, it also holds under the weaker assumption that φ is light and does not admit an interior fold.

Lemma 9. *Suppose φ is an irreducible map from a graph G onto an arc $A' = \langle a_0, a_1, \ldots, a_n \rangle$, and suppose $p_0, p_1, \ldots, p_m$ is a path in G. If $\varphi(p_0) = a_0$ and $\varphi(p_m) = a_n$, then $m = n$ and $\varphi(p_i) = a_i$ for $i = 0, 1, \ldots, n$.*

Proof. It suffices to show that the following proposition is true for $k = n$.

$P(k)$ If φ is an irreducible map from a graph G onto an arc A', $p_0, p_1, \ldots, p_m$ is a path in G and $\langle a_0, a_1, \ldots, a_k \rangle$ is a subarc of A' such that $\varphi(p_0) = a_0$, $\varphi(p_m) = a_k$, and $\varphi(p_i) \in \{a_0, a_1, \ldots, a_k\}$ for $i = 0, 1, \ldots, m$, then $m = k$ and $\varphi(p_i) = a_i$ for $i = 0, 1, \ldots, k$.

First consider $P(1)$. Since $\varphi(p_0) = a_0$ and φ is light, $\varphi(p_1) = a_1$. If $m = 1$, then $P(1)$ follows. Suppose $m \neq 1$. Then $\varphi(p_2) = a_0$. But $\varphi(p_m) = a_1$, so $m \neq 2$. Then $\varphi(p_3) = a_1$. Consequently, $\varphi(p_k) = \varphi(p_{2-k}) = \varphi(p_{2+k})$ for $k = 0, 1$, contrary to the assumption that φ is irreducible. Thus $m = 1$, and $P(1)$ is true.

Suppose k is a positive integer such that $P(k)$ is true. If $k = n$, then the Lemma follows. Suppose $k < n$, and consider $P(k+1)$. Before dealing with the path $p_0, p_1, \ldots, p_m$, suppose $q_l, q_{l+1}, \ldots, q_{l+j}$ is any path in G such that $\varphi(q_l) = a_0$, $\varphi(q_{l+j}) = a_{k+1}$ and $\varphi(q_{l+i}) \in \{a_0, a_1, \ldots, a_k\}$ for $i = 1, 2, \ldots, j-1$. Since φ is light, $\varphi(q_{l+(j-1)}) = a_k$. Applying $P(k)$ to the path $q_l, q_{l+1}, \ldots, q_{l+(j-1)}$ gives that $j - 1 = k$ and $\varphi(q_{l+i}) = a_i$ for $i = 0, 1, \ldots, k$. Since $j = k+1$, $\varphi(q_{l+(k+1)}) = a_{k+1}$. Thus,

(1) if $q_l, q_{l+1}, \ldots, q_{l+j}$ is a path in G such that $\varphi(q_l) = a_0$, $\varphi(q_{l+j}) = a_{k+1}$, and $\varphi(q_{l+i}) \in \{a_0, a_1, \ldots, a_k\}$ for $i = 1, 2, \ldots, j-1$, then $j = k+1$ and $\varphi(q_{l+i}) = a_i$ for $i = 0, 1, \ldots, k+1$

and, similarly,

(2) if $q_l, q_{l+1}, \ldots, q_{l+j}$ is a path in G such that $\varphi(q_l) = a_{k+1}$, $\varphi(q_{l+j}) = a_0$, and $\varphi(q_{l+i}) \in \{a_1, a_2, \ldots, a_{k+1}\}$ for $i = 1, 2, \ldots, j-1$, then $j = k+1$ and $\varphi(q_{l+(k+1)-i}) = a_i$ for $i = 0, 1, \ldots, k+1$.

Now consider the path $p_0, p_1, \ldots, p_m$. Let x denote the smallest of all positive integers i for which $\varphi(p_i) = a_{k+1}$. Applying (1) to the path $p_0, p_1, \ldots, p_x$ gives that $x = k+1$ and $\varphi(p_i) = a_i$ for $i = 0, 1, \ldots, k+1$. For the purpose of establishing a contradiction, suppose that p_0 is not the only vertex of the path $p_0, p_1, \ldots, p_m$ whose image under φ is a_0, and let y denote the smallest of all positive integers i for which $\varphi(p_{(k+1)+i}) = a_0$. Applying (2) to the path $p_{k+1}, p_{(k+1)+1}, \ldots, p_{(k+1)+y}$ gives that $y = k+1$ and $\varphi(p_{2(k+1)-i}) = a_i$ for $i = 0, 1, \ldots, k+1$. Finally, let z denote the smallest of all positive integers i for which $\varphi(p_{2(k+1)+i}) = a_{k+1}$—such an integer exists because $\varphi(p_m) = a_{k+1}$. Applying (1) to the path $p_{2(k+1)}, p_{2(k+1)+1}, \ldots, p_z$ gives that $z = k+1$ and $\varphi(p_{2(k+1)+i}) = a_i$ for $i = 0, 1, \ldots, k+1$. It follows that $\varphi(p_i) = \varphi(p_{2(k+1)-i}) = \varphi(p_{2(k+1)+i})$ for $i = 0, 1, \ldots, k+1$, contrary to the hypothesis that φ is irreducible. Thus the assumption that a_0 is the image under φ of some term of $p_1, p_2, \ldots, p_m$ is false.

The hypothesis of $P(k+1)$ gives that $\varphi(p_0) = a_0$ and $\varphi(p_m) = a_{k+1}$. Since φ is light, $\varphi(p_1) = a_1$. By the conclusion of the previous paragraph, $\varphi(p_i) \in \{a_1, a_2, \ldots, a_{k+1}\}$ for $i = 1, 2, \ldots, m$. Then, by $P(k)$, $m = k+1$ and $\varphi(p_i) = a_i$ for $i = 1, 2, \ldots, k+1$. Consequently $P(k+1)$ follows from $P(k)$. Proceeding inductively yields that $P(n)$, and hence the lemma, is true. □

Theorem 10 can be proved using Lemma 9; however, similar results due to Young [8] and Oversteegen and Tymchatyn [7] appear in the literature with proofs.

Theorem 10 (Young; Oversteegen, Tymchatyn). *If φ is an irreducible map from an arc onto an arc, then φ is an isomorphism.*

Theorem 11. *If φ is an irreducible map from a connected graph G onto an arc A', then φ is an isomorphism.*

Proof. By Theorem 10, it suffices to show that G is an arc. To prove that this is so, it will be shown first that G is a tree. Suppose, to the contrary, that G contains a simple closed curve C. Note that $\varphi[C]$ is an arc in A'. Denote $\varphi[C]$ by $\langle a_0, a_1, \ldots, a_m \rangle$. Then there is a vertex c_0 of C such that $\varphi(c_0) = a_0$. Denote the vertices of C by $c_0, c_1, \ldots, c_k = c_0$. Some vertex, c_l of C is mapped by φ to a_m. Then φ is one-to-one on both $\langle c_0, c_1, \ldots, c_l \rangle$ and $\langle c_l, c_{l+1}, \ldots, c_k = c_0 \rangle$ by Lemma 9. It follows that $k = 2l$ and that $\varphi(c_i) = \varphi(c_{2l-i})$ for $i = 0, 1, \ldots, l$. Consider the path $q_0, q_1, \ldots, q_{3l}$ where $q_i = c_i$ for $i = 0, 1, \ldots, 2l$ and $q_{2l+i} = c_i$ for $i = 0, 1, \ldots, l$. Then $\varphi(q_i) = \varphi(q_{2l-i}) = \varphi(q_{2l+i})$ for $i = 0, 1, \ldots, l$. Since φ was assumed to be irreducible, this is a contradiction. Consequently, G contains no simple closed curves, and, hence, G is a tree.

Suppose s and t are distinct terminal vertices of G. Then there is a unique arc $\overline{st}$ in G whose two terminal vertices are s and t. Note that if $p_0, p_1, \ldots, p_{2n}$ is a terminal path in

$\overline{st}$ such that $\varphi|\overline{st}(p_k) = \varphi|\overline{st}(p_{2n-k})$ for $k = 0, 1, \ldots, n$, then $p_0, p_1, \ldots, p_{2n}$ is a terminal path in G such that $\varphi(p_k) = \varphi(p_{2n-k})$ for $k = 0, 1, \ldots, n$. Similarly, if $p_0, p_1, \ldots, p_{3n}$ is a path in $\overline{st}$ such that $\varphi|\overline{st}(p_k) = \varphi|\overline{st}(p_{2n-k}) = \varphi|\overline{st}(p_{2n+k})$ for $k = 0, 1, \ldots, n$, then $p_0, p_1, \ldots, p_{3n}$ is a path in G such that $\varphi(p_k) = \varphi(p_{2n-k}) = \varphi(p_{2n+k})$ for $k = 0, 1, \ldots, n$. Since φ is irreducible, it follows that $\varphi|\overline{st}$ is irreducible. Then φ is one-to-one on $\overline{st}$ by Theorem 10. Consequently, for each pair, s and t, of terminal vertices of G, φ is one-to-one on $\overline{st}$.

For the purpose of establishing contradiction, suppose G has three distinct terminal vertices, t_1, t_2, and t_3. Denote by $\overline{t_1t_2}$, $\overline{t_1t_3}$, and $\overline{t_2t_3}$ the arcs in G with terminal vertices t_1 and t_2, t_1 and t_3, and t_2 and t_3, respectively. Then φ is one-to-one on $\overline{t_1t_2}$, $\overline{t_1t_3}$, and $\overline{t_2t_3}$. Consequently, $\varphi(t_1)$, $\varphi(t_2)$, and $\varphi(t_3)$ are distinct. Since all belong to A', one separates the other two in A'. Renaming if necessary, suppose it is $\varphi(t_2)$ that separates the other two. Denote by p the junction vertex of the simple triod whose terminal vertices are t_1, t_2, and t_3. Note that $p \neq t_i$ for $i = 1, 2, 3$. Since φ is one-to-one on $\overline{t_1t_2}$ and $\overline{t_2t_3}$, $\varphi(p)$ separates both $\varphi(t_1)$ from $\varphi(t_2)$, and $\varphi(t_2)$ from $\varphi(t_3)$. Since this is not possible, the assumption that G has at least three terminal vertices is false. Consequently, G is an arc. The conclusion of the theorem follows from Theorem 10. □

Theorem 12. *If φ is a light map from a connected graph G onto an arc A', then there are an isomorphism ζ and a sequence $\pi_1, \pi_2, \ldots, \pi_n$ of folds such that $\varphi = \zeta \circ \pi_n \circ \cdots \circ \pi_2 \circ \pi_1$.*

Proof. By Theorem 6, there is an irreducible map ζ and a sequence $\pi_1, \pi_2, \ldots, \pi_n$ of folds such that $\varphi = \zeta \circ \pi_n \circ \cdots \circ \pi_2 \circ \pi_1$. Since the simplicial image of a connected graph is connected, it follows that $\pi_n \circ \cdots \circ \pi_2 \circ \pi_1[G]$, the domain of ζ, is connected. Since φ is surjective, so is ζ. Consequently, ζ is an isomorphism by Theorem 11. □

6. Factoring through an arc

In this section the main results of the paper are stated and proved. Although there are several different theorems, they are all variations on the same theme. Theorem 15 is the version that the example in Section 2 anticipates most naturally.

Lemma 13. *If ζ is an isomorphism from G_1 onto G_2 and π is a fold that projects G_2 onto G_3, then there is a fold π' and an isomorphism ζ' such that $\zeta' \circ \pi' = \pi \circ \zeta$.*

Proof. A proof will be given only for the case where π is an interior fold. If π is a terminal fold, then a similar argument will suffice, and if π is the trivial fold, then the result is trivial. Suppose π is an interior fold. Then there is a partition $\mathcal{F}$ and a path $p_0, p_1, \ldots, p_{3n}$ in G_2 such that $\pi = \pi_{\mathcal{F}}$ and such that the equivalence classes of $\mathcal{F}$ are sets of the form $\{p_k, p_{2n-k}, p_{2n+k}\}$ for some $k = 0, 1, \ldots, n$ and sets of the form $\{x\}$ for some vertex x of G_2 that is not in $\{p_0, p_1, \ldots, p_{3n}\}$.

Consider the path $\zeta^{-1}(p_0), \zeta^{-1}(p_1), \ldots, \zeta^{-1}(p_{3n})$ in G_1, and denote its terms by q_0, $q_1, \ldots, q_{3n}$, respectively. Then $\pi \circ \zeta(q_k) = \pi(p_k) = [p_k]_{\mathcal{F}} = [p_{2n-k}]_{\mathcal{F}} = \pi(p_{2n-k}) = \pi \circ \zeta(q_{2n-k})$, and, similarly, $\pi \circ \zeta(q_k) = \pi \circ \zeta(q_{2n+k})$ for $k = 0, 1, \ldots, n$. Thus $\pi \circ \zeta$

is constant on $\{q_k, q_{2n-k}, q_{2n+k}\}$ for $k = 0, 1, \dots, n$. Let $\mathcal{F}'$ denote the partition on the vertices of G whose equivalence classes are sets of the form $\{q_k, q_{2n-k}, q_{2n+k}\}$ for $k = 0, 1, \dots, n$ and sets of the form $\{x\}$ for $x \notin \{q_0, q_1, \dots, q_{3n}\}$, and let π' denote the fold $\pi_{\mathcal{F}'}$. By Theorem 2, there is a simplicial map ζ' such that $\pi \circ \zeta = \zeta' \circ \pi'$.

It remains only to show that ζ' is an isomorphism. Suppose x and y are vertices of G_1 such that $\pi \circ \zeta(x) = \pi \circ \zeta(y)$. Either $\zeta(x) = \zeta(y)$ or $\zeta(x) \neq \zeta(y)$. In the latter case, there is a nonnegative integer k not larger than n such that $\{\zeta(x), \zeta(y)\} \subset \{p_k, p_{2n-k}, p_{2n+k}\}$. It follows that $\{x, y\} \subset \{\zeta^{-1}(p_k), \zeta^{-1}(p_{2n-k}), \zeta^{-1}(p_{2n+k})\} = \{q_k, q_{2n-k}, q_{2n+k}\}$. Consequently, $\pi'(x) = \pi'(y)$. If $\zeta(x) = \zeta(y)$, then $x = y$ from which it also follows that $\pi'(x) = \pi'(y)$. In either case, if $\pi \circ \zeta(x) = \pi \circ \zeta(y)$, then $\pi'(x) = \pi'(y)$. Since $\pi \circ \zeta = \zeta' \circ \pi'$, it follows that if $\pi'(x) \neq \pi'(y)$, then $\zeta' \circ \pi'(x) \neq \zeta' \circ \pi'(y)$. Equivalently, if $[x]_{\mathcal{F}'} \neq [y]_{\mathcal{F}'}$, then $\zeta'([x]_{\mathcal{F}'}) \neq \zeta'([y]_{\mathcal{F}'})$. Consequently, ζ' is one-to-one.

To see that ζ' is surjective, suppose x and y are adjacent vertices of G_3. Note that since π and ζ are both surjective, $\pi \circ \zeta$ and, hence, $\zeta' \circ \pi'$ are both surjective. Then there are adjacent vertices a and b of G_1 such that $\zeta' \circ \pi'(a) = x$ and $\zeta' \circ \pi'(b) = y$. By Theorem 1 π' is simplicial , so $\pi'(a)$ and $\pi'(b)$ are either adjacent or equal. But the latter would imply that $x = y$ contrary to their adjacency. Consequently, $[a]_{\mathcal{F}'}$ and $[b]_{\mathcal{F}'}$ are adjacent vertices of $G/\mathcal{F}'$ such that $\zeta'([a]_{\mathcal{F}'}) = x$ and $\zeta'([b]_{\mathcal{F}'}) = y$.

Finally, since ζ' is both one-to-one and surjective, it follows that vertices a and b of the domain of ζ' are adjacent if and only if $\zeta'(a)$ and $\zeta'(b)$ are adjacent. Consequently, ζ'^{-1} is simplicial. □

Theorem 14. *Suppose φ is a light simplicial map from a connected graph into a graph. Then φ can be factored through an arc if and only if there are a sequence $\pi_1, \pi_2, \dots, \pi_n$ of folds and an irreducible map ψ whose domain is an arc such that $\varphi = \psi \circ \pi_n \circ \dots \circ \pi_2 \circ \pi_1$.*

Proof. If the latter holds, then φ clearly factors through an arc. To see the converse, suppose φ factors through an arc A'. Then there are an arc A', a surjective simplicial map $\varphi_1 : G \to A'$, and a simplicial map $\varphi_2 : A' \to G'$ such that $\varphi = \varphi_2 \circ \varphi_1$. Because φ is light, it follows that φ_1 is light. To see that φ_2 is light, suppose x and y are adjacent vertices of A'. Since φ_1 is surjective, there are adjacent vertices, a and b, of G such that $\varphi_1(a) = x$ and $\varphi_1(b) = y$. Since φ is light, $\varphi(a)$ and $\varphi(b)$ are adjacent in G'. But, $\varphi(a) = \varphi_2(x)$ and $\varphi(b) = \varphi_2(y)$, so $\varphi_2(x)$ and $\varphi_2(y)$ are adjacent. Therefore, φ_2 is light.

For the remainder of the proof, it will be assumed that φ_1 and φ_2 are both reducible. For the cases where either or both of them is irreducible, similar but simpler arguments will suffice. By Theorem 12, there are an isomorphism ζ and a sequence of folds, $\pi_1, \pi_2, \dots, \pi_k$ such that $\varphi_1 = \zeta \circ \pi_k \circ \dots \circ \pi_2 \circ \pi_1$, and by Theorem 8, there are an arc A^*, an irreducible map $\psi : A^* \to G'$, and a sequence $\pi_{k+1}, \pi_{k+2}, \dots, \pi_n$ of folds such that $\varphi_2 = \psi \circ \pi_n \circ \dots \circ \pi_{k+2} \circ \pi_{k+1}$. Consequently, $\varphi = \psi \circ \pi_n \circ \dots \circ \pi_{k+1} \circ \zeta \circ \pi_k \circ \dots \circ \pi_2 \circ \pi_1$.

By Lemma 13, there is a fold π'_{k+1} and an isomorphism ζ' such that $\pi_{k+1} \circ \zeta = \zeta' \circ \pi'_{k+1}$. Hence $\varphi = \psi \circ \pi_n \circ \dots \circ \pi_{k+2} \circ \zeta' \circ \pi'_{k+1} \circ \pi_k \circ \dots \circ \pi_2 \circ \pi_1$. Applying Lemma 13 inductively gives folds $\pi'_{k+1}, \pi'_{k+2}, \dots, \pi'_n$ and an isomorphism ζ^* such that $\varphi = \psi \circ \zeta^* \circ \pi'_n \circ \dots \circ \pi'_{k+1} \circ \pi_k \circ \dots \circ \pi_2 \circ \pi_1$. Denote $\psi \circ \zeta^*$ by ψ' and $\zeta^{*-1}[A^*]$ by A. Then A is an arc, and ψ' is a light simplicial map from A into G'.

It remains only to show that ψ' is irreducible. Suppose, to the contrary, that it is not. Then, because ψ' is light, either there is a path $p_0, p_1, \dots, p_{3m}$ such that $\varphi'(p_k) = \varphi'(p_{2m-k}) = \varphi'(p_{2m+k})$ for $k = 0, 1, \dots, m$ or there is a terminal path $q_0, q_1, \dots, q_{2m}$ such that $\varphi'(q_k) = \varphi'(q_{2m-k})$ for $k = 0, 1, \dots, m$. In the former case, since ζ^* is an isomorphism, $\zeta^*(p_0), \zeta^*(p_1), \dots, \zeta^*(p_{3m})$ is a path in A^* such that $\psi(\zeta^*(p_k)) = \psi(\zeta^*(p_{2m-k})) = \psi(\zeta^*(p_{2m+k}))$ for $k = 0, 1, \dots, m$. Consequently, ψ is not irreducible, which is a contradiction. Similarly, the latter case involves a contradiction. Thus the assumption that ψ' is reducible is false. □

Theorem 15. *Suppose φ is a light simplicial map from a connected graph G into a graph. For φ to factor through an arc it is both necessary and sufficient that there exist a simplicial map ψ and a sequence $\pi_1, \pi_2, \dots, \pi_n$ of folds such that $\varphi = \psi \circ \pi_n \circ \cdots \circ \pi_2 \circ \pi_1$ and $\pi_n \circ \cdots \circ \pi_2 \circ \pi_1[G]$ is an arc.*

Proof. The sufficiency is trivial, and the necessity follows from Theorem 14. □

Theorem 16. *Suppose φ is a simplicial map from a connected graph into a graph. Then φ factors through an arc if and only if there exist a monotone map μ, a sequence $\pi_1, \pi_2, \dots, \pi_n$ of folds, and an irreducible map ψ whose domain is an arc such that $\varphi = \psi \circ \pi_n \circ \cdots \circ \pi_2 \circ \pi_1 \circ \mu$.*

Proof. If $\varphi = \psi \circ \pi_n \circ \cdots \circ \pi_2 \circ \pi_1 \circ \mu$ and the domain of ψ is an arc, then clearly φ factors through an arc. Conversely, suppose φ factors through an arc. Then there are simplicial maps φ_1 and φ_2 such that the domain of φ_2 is an arc and $\varphi = \varphi_2 \circ \varphi_1$. Denote by A the domain of φ_2. By Theorem 3, there is a light simplicial map ψ_2 and a surjective monotone simplicial map μ_2 such that $\varphi = \psi_2 \circ \mu_2 \circ \varphi_1$. Note that $\mu_2[A]$, the domain of ψ_2, is an arc. Applying Theorem 3 to the map $\mu_2 \circ \varphi_1$ yields a light simplicial map ψ_1 and a monotone simplicial map μ such that $\varphi = \psi_2 \circ \psi_1 \circ \mu$. Note that $\psi_2 \circ \psi_1$ is light and factors through an arc. Then applying Theorem 14 to $\psi_2 \circ \psi_1$ gives that there is a sequence $\pi_1, \pi_2, \dots, \pi_n$ of folds and an irreducible map ψ whose domain is an arc such that $\varphi = \psi \circ \pi_n \circ \cdots \circ \pi_2 \circ \pi_1 \circ \mu$. □

Remark. The reader is owed one final remark relating the results of this section, particularly Theorem 15, to the example given in Section 2. The procedure used in Section 2 to factor the map in Fig. 1(b) through an arc is shown in Theorem 15 to be a valid procedure for determining whether any light simplicial map from a connected graph into a graph can be factored through an arc. What should be noted is that it is possible to reduce a map that can be factored through an arc to one that cannot by factoring out the wrong fold when there is more than one alternative. According to Theorem 14, however, there is always a correct choice. For each of the maps in Fig. 1(b) and (c), there is only one way to factor out a fold. Consequently, the example is insidiously cooperative in demonstrating the procedure.

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References

[1] W.T. Ingram, An atriodic tree-like continuum with positive span, Fund. Math. 77 (1972) 99–107.
[2] W.T. Ingram, A tale of six atriodic tree-like continua, in: A. Illanes, S. Macias, W. Lewis (Eds.), Continuum Theory, Marcel Dekker, New York, 2002, pp. 163–172.
[3] P. Minc, On simplicial maps and chainable continua, Topology Appl. 57 (1994) 1–21.
[4] L. Mohler, L.G. Oversteegen, Reduction and irreducibility for words and tree-words, Fund. Math. 126 (1986) 107–121.
[5] L.G. Oversteegen, On reductions of tree-words to (chain-) words, Houston J. Math. 15 (1) (1989) 121–135.
[6] L.G. Oversteegen, Factorizations of mappings through arcs, Houston J. Math. 17 (3) (1991) 443–451.
[7] L.G. Oversteegen, E.D. Tymchatyn, On hereditarily indecomposable compacta, in: Geometric and Algebraic Topology, in: Banach Center Publ., vol. 18, PWN, Warsaw, 1986, pp. 407–417.
[8] S.W. Young, Finitely generated semigroups of continuous functions on $[0,1]$, Fund. Math. 68 (1970) 297–305.