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The Sarkovskii order for periodic continua

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Abstract

Suppose f is a map of a continuum X onto itself. A *periodic continuum* of f is a subcontinuum K of X such that $f^n[K] = K$ for some positive integer n. A *proper periodic continuum* of f is a periodic continuum of f that is a proper subcontinuum of X. A proper periodic continuum of f is *maximal* if and only if X is the only periodic continuum that properly contains it. In this paper it is shown that the maximal proper periodic continua of a map of a hereditarily decomposable chainable continuum onto itself follow the Sarkovskii order, provided the maximal proper periodic continua are disjoint. The case in which the Sarkovskii order does not hold reduces to the scenario in which the map's domain is the union of two overlapping period-two continua, each of which is maximal.

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1. Introduction

In 1964 Sarkovskii [12,13] proved that if $p \prec q$ according to the following ordering of the positive integers, then, for any mapping of the real line into itself with a periodic point of period p, there is a periodic point of period q.

$$3 \prec 5 \prec 7 \prec 9 \prec 11 \prec \cdots \prec 3 \cdot 2 \prec 5 \cdot 2 \prec \cdots \prec 3 \cdot 2^2 \prec 5 \cdot 2^2 \prec \cdots \prec 2^3 \prec 2^2 \prec 2 \prec 1.$$

Piotr Minc and W.R.R. Transue [8] extended Sarkovskii's theorem by showing that if $p \prec q$ and f is a mapping from a subcontinuum of a hereditarily decomposable chainable continuum Y into Y with a periodic point of period p, there is a periodic point of period q. They also give a simple example of an indecomposable continuum that admits a homeomorphism with a point of period three but none of period two. Hence the assumption of hereditary decomposability cannot be dropped. Indeed, exception to Sarkovskii order for periodic points of maps of indecomposable chainable continua may be striking. Wayne Lewis [7] has shown that, for each positive integer n, the pseudo-arc admits a homeomorphism with periodic points of periods 1 and n only.

For reasons unrelated to Sarkovskii's theorem, the author became interested, for a map of an interval I onto itself, in proper subintervals K of I that are periodic in the sense that $f^p[K] = K$ for some positive integer p. Of particular interest were proper subintervals that are maximal with respect to this property. Such maximal proper periodic continua are interesting at least for the contrast between the following two results.

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(1) (Barge and Martin; [1]) If $f : I \to I$ has a period-three point, then the inverse limit of $\{I, f\}$ contains an indecomposable continuum.
(2) [11] If $f : I \to I$ has at least three maximal proper periodic continua, then the inverse limit of $\{I, f\}$ is an indecomposable continuum.

Thus maximal proper periodic continua provide a global perspective on the topology of the inverse limit of f as well as on the dynamics of f.

When the author noticed that maximal proper periodic continua of interval maps for which such continua are disjoint follow the Sarkovskii order, W.T. Ingram pointed out the theorem of Minc and Transue, according to which the Sarkovskii order holds for maps of hereditarily decomposable chainable continua. Since such continua occur as global attractors in the plane [2], they are of dynamical interest as well as topological interest.

In this paper it will be shown that the maximal proper periodic continua of a map of a hereditarily decomposable chainable continuum onto itself follow the Sarkovskii order, provided the maximal proper periodic continua are disjoint. Since the maximal proper periodic continua are disjoint for any map with at least three of them, the case in which the Sarkovskii order does not hold reduces to the scenario in which the map's domain is the union of two overlapping period-two continua, each of which is maximal. For example, the only maximal proper periodic continua for the map $f(x) = -x^3$ restricted to the interval $[-1, 1]$ are the intervals $[-1, 0]$ and $[0, 1]$, each of which has period two.

The main result appears in Section 4. The idea of the proof is to fix a pair of positive integers p and q with $p \prec q$ and shrink all maximal proper periodic continua of f with period $2\max\{p, q\}$ or smaller to points. Then f is semiconjugate to a map g such that each maximal proper periodic continuum of f with such a period corresponds to a periodic point of g with the same period. If, conversely, each periodic point of g with period not larger than $2\max\{p, q\}$ corresponds to a maximal proper periodic continuum of f with the same period, then the maximal proper periodic continua of f inherit the Sarkovskii order from the periodic points of g. Whether or not every periodic point of g corresponds to a maximal proper periodic continuum of f turns out to be related to whether or not every periodic point of f belongs to a maximal proper periodic continuum of f.

It is the task of Section 3 to give sufficient conditions for this to be true and describe the behavior of functions for which it might be false. The proof of the main theorem in Section 4 is divided into cases accordingly. Section 3 also contains some basic results on maximal proper periodic continua.

Section 2 explores the relationship between the periods of periodic continua that overlap for maps of atriodic and hereditarily unicoherent continua, which leads to an easy proof that the closure of the union of the collection of all proper periodic continua that intersect a given periodic continuum is a periodic continuum. This reduces the question as to whether or not every periodic point of a map $f : X \to X$ belongs to a maximal proper periodic continuum to the question as to whether or not the closure of the union of all proper periodic continua of f that contain a periodic point is a proper subcontinuum of X.

A *continuum* is a compact connected metrizable space. A continuum is said to be *decomposable* if and only if it is the union of two of its proper subcontinua. A *hereditarily decomposable* continuum is a continuum for which every nondegenerate subcontinuum is decomposable.

A continuum is a *triod* if and only if it has a subcontinuum N whose complement has at least three components. A continuum that contains no triod is *atriodic*. It is well known that a continuum is atriodic if and only if, for every three of its subcontinua with common point, some one of them is a subset of the union of the other two [14, Theorem 1.8].

A *unicoherent continuum* is a continuum M with the property that if H and K are two subcontinua of M whose union is M, then their intersection is a continuum. A continuum is *hereditarily unicoherent* if and only if each of its proper subcontinua is unicoherent.

A *chain* is a sequence $C_1, C_2, \ldots, C_n$ of open sets such that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leqslant 1$. An *ε-chain* is a chain such that the diameter$(C_i) \leqslant \varepsilon$ for each i. A continuum is *chainable* if and only if it can be covered by an ε-chain for each positive ε. It is well known that a chainable continuum is atriodic and hereditarily unicoherent, and that a hereditarily decomposable continuum is chainable if and only if it is atriodic and hereditarily unicoherent [3, Theorem 11].

A continuum is *irreducible between* two closed sets H and K if and only if it intersects both H and K, but none of its proper subcontinua does. A continuum is *irreducible* if and only if it is irreducible between some two of its points. Atriodic and hereditarily unicoherent continua are irreducible by Sorgenfrey's Theorem [9, Theorem 11.34], [14, Theorem 3.2]. Hence chainable continua are irreducible.

A *map* is a continuous function. A *monotone map* is a map for which point inverses are connected. Inverse images of continua under monotone maps are continua.

Suppose f is a mapping of a continuum X onto itself. A subcontinuum K of X is said to be *periodic* under f if and only if there is a positive integer n such that $f^n[K] = K$; the *period* of K is the smallest such positive integer. A *periodic point* is a degenerate periodic continuum. A *proper periodic continuum* is a proper subcontinuum of X that is periodic. A *maximal proper periodic continuum* is a periodic continuum K with the property that the only periodic continuum that properly contains K is X.

If $\mathcal{K}$ is a collection of sets, then $\mathcal{K}^*$ denotes the union of all elements of $\mathcal{K}$.

2. Periodic continua

2.1. End continua

A subcontinuum K of a continuum X is an *end continuum* of X provided that for each pair of subcontinua of X that contain K, one is a subset of the other. Theorems 2.1 and 2.2 are proved in [10].

Theorem 2.1. *Suppose X is a continuum. Each of the following statements is equivalent to the condition that X is unicoherent and contains no triod with interior.*

(1) *If A is a subcontinuum of X, and C is a component of $X - A$, then $\overline{C}$ is an end continuum of X.*
(2) *If A is a proper subcontinuum of X, and B is a subcontinuum of X such that $A \cup B = X$, then B is an end continuum of X.*

Theorem 2.2. *Suppose X is a unicoherent continuum that contains no triod with interior. If each of A, B, and C is an end continuum of X with interior, then one of A, B, and C is a subset of some other one of A, B, and C.*

Lemma 2.3. *Suppose X is a unicoherent continuum that contains no triod with interior. If $\mathcal{K}$ is a finite nondegenerate collection of proper subcontinua of X whose union is X, then at least two terms of $\mathcal{K}$ are end continua of X with interior.*

Proof. Denote the terms of $\mathcal{K}$ by $K_1, K_2, \ldots, K_n$. First it will be shown that some term other than K_1 is an end continuum of X. Since X is connected, there is an arrangement $J_2, J_3, \ldots, J_n$ of the terms of $K_2, K_3, \ldots, K_n$ such that, for each i, $K_1 \cup J_2 \cup \cdots \cup J_i$ is a continuum. Denote by k the smallest of all integers i such that $K_1 \cup J_2 \cup \cdots \cup J_i = X$. Then $k \geqslant 2$ and J_k is an end continuum of X by (2) of Theorem 2.1. Furthermore, J_k contains the nonempty open set $X - (K_1 \cup J_2 \cup \cdots \cup J_{k-1})$.

To see that $K_1, K_2, \ldots, K_n$ contains an end continuum of X different from J_k, apply the argument of the previous paragraph to any rearrangement, $L_1, L_2, \ldots, L_n$, of the terms of $K_1, K_2, \ldots, K_n$ for which $L_1 = J_k$. □

2.2. Periodicity and overlapping periodic continua

Lemma 2.4. *Suppose f is a map of an atriodic and hereditarily unicoherent continuum X onto itself. If K is a periodic continuum of f, then no three distinct terms of* Orbit(K) *have a common point.*

Proof. Denote the period of K by p. Suppose $p \geqslant 3$, noting that the lemma is trivial otherwise. For the purpose of establishing a contradiction, suppose, contrary to the lemma, that there are distinct integers q, r, and s from $\{0, 1, \ldots, p-1\}$ such that $f^q[K]$, $f^r[K]$, and $f^s[K]$ have a common point. Since X is atriodic, some one of them, say $f^q[K]$, is a subset of the union of the other two. Thus $f^q[K]$ is a subset of $\bigcup\{f^i[K]\colon\ 0 \leqslant i < p \text{ and } i \neq q\}$.

Denote by $\mathcal{L}$ the largest of all subcollections $\mathcal{K}$ of Orbit(K) containing $f^q[K]$ such that $\mathcal{K}^*$ is connected. Then $\mathcal{L}^*$ is an atriodic and hereditarily unicoherent continuum. Note that $\mathcal{L}$ contains each of $f^q[K]$, $f^r[K]$, and $f^s[K]$.

By Lemma 2.3, some term of $\mathcal{L}$ is and end continuum of $\mathcal{L}^*$ and has interior relative to $\mathcal{L}^*$. Denote it by L, and let n denote a positive integer such that $f^{q+n}[K] = L$.

Applying f^n to $\bigcup\{f^i[K]\colon 0 \leqslant i < p \text{ and } i \neq q\}$ gives $\bigcup\{f^{n+i}[K]\colon 0 \leqslant i < p \text{ and } i \neq q\}$, which equals $\bigcup\{f^i[K]\colon 0 \leqslant i < p \text{ and } i \neq (n+q)_{\mathrm{mod}(p)}\}$. Hence $f^{q+n}[K]$, which is to say L, is a subset of $\bigcup\{f^i[K]\colon 0 \leqslant i < p \text{ and } i \neq (n+q)_{\mathrm{mod}(p)}\}$. Consequently $\{\mathcal{L} - \{L\}\}^* = \mathcal{L}^*$. Then it follows from Lemma 2.3, that there are two terms, L_1 and L_2, of $\mathcal{L} - \{L\}$ that are end continua of $\mathcal{L}^*$ and have interior relative to $\mathcal{L}^*$. By Theorem 2.2, some one of L, L_1, and L_2 is a subset of some other one of L, L_1, and L_2. But it is impossible for two distinct terms of Orbit(K) to be nested. Thus the assumption that some three terms of Orbit(K) have a common point is false. □

Lemma 2.5. *Suppose f is a map of an atriodic and hereditarily unicoherent continuum onto itself. If K is a periodic continuum of f, then each term of* Orbit(K) *intersects at most one other term of* Orbit(K).

Proof. Denote the period of K by p, and let $\mathcal{L}$ denote the largest of all subcollections $\mathcal{K}$ of Orbit(K) containing K such that $\mathcal{K}^*$ is connected. If $\mathcal{L}$ is nondegenerate, then there is a term L of $\mathcal{L}$ that is an end continuum of $\mathcal{L}^*$ by Lemma 2.3. If $\mathcal{L}$ is degenerate, then the existence of such a term L is trivial.

Note that A and B are distinct terms of Orbit(K) with a common point if and only if, for each positive integer n, $f^n[A]$ and $f^n[B]$ are distinct terms of Orbit(K) with a common point. It follows that all terms of Orbit(K) intersect the same number of other terms of Orbit(K). Thus, to complete the proof of the lemma, it suffices to show that L intersects at most one other term of Orbit(K).

Suppose, for the purpose of establishing a contradiction, that L intersects two or more terms of Orbit(K). Denote two such terms by L_1 and L_2. Then L_1 and L_2 both belong to $\mathcal{L}$. Since L is an end continuum of $\mathcal{L}^*$, one of $L \cup L_1$ and $L \cup L_2$ contains the other, say $L \cup L_1 \subset L \cup L_2$. Either L_1 contains a point of $L \cap L_2$ or it is a proper subset of one of L and L_2. The former case is precluded by Lemma 2.4, and the latter is impossible because no term of Orbit(K) is a proper subset of another. Hence L intersects at most one other term of Orbit(K). □

Lemma 2.6. *Suppose f is a map of an atriodic and hereditarily unicoherent continuum onto itself. If K is a periodic continuum of f, and n is the smallest of all positive integers i such that $f^i[K] \cap K \neq \emptyset$, then the period of K is either n or $2n$.*

Proof. First note that $f^{2n}[K] \cap f^n[K] \neq \emptyset$. By Lemma 2.5, $f^n[K]$ intersects at most one term of Orbit(K). Hence, some two of K, $f^n[K]$, and $f^{2n}[K]$ are equal. If $f^n[K] = K$, then the period of K is n. If $f^{2n}[K] = f^n[K]$, then $f^n[K]$ has period not greater than n. Since the periods of K and $f^n[K]$ are equal, it follows that the period of K is n. Finally, suppose $f^{2n}[K] = K$. Then the period of K is at least n and at most $2n$. Denote the period of K by p. Then $f^{2n-p}[K] = f^{2n-p}[f^p[K]] = f^{2n}[K] = K$. Either $2n - p = 0$ or $2n - p \geqslant n$. Equivalently, either $p = 2n$ or $p \leqslant n$. It has been noted that $p \geqslant n$; hence, the conclusion of the lemma follows. □

Lemma 2.7. *Suppose f is a map of an atriodic and hereditarily unicoherent continuum onto itself. If K is a periodic continuum of f, and n is the smallest of all positive integers i such that $f^i[K] \cap K \neq \emptyset$, then n divides a positive integer m if and only if $f^m[K] \cap K \neq \emptyset$.*

Proof. First suppose $m = kn$ for some positive integer k. Then, since $f^{2n}[K] = K$ by Lemma 2.6, either $f^m[K] = K$ or $f^m[K] = f^n[K]$. In either case, $f^m[K] \cap K \neq \emptyset$.

Suppose m is a positive integer such that $f^m[K] \cap K \neq \emptyset$. There is a multiple N of n such that $N \leqslant m < N + n$. Either N is a multiple of $2n$, or $N + n$ is. These cases will be treated separately.

First consider the case in which N is a multiple of $2n$. Then $f^N[K] = K$ by Lemma 2.6. From the inequality $N \leqslant m < N + n$ it follows that $0 \leqslant m - N < n$. Notice that $f^{m-N}[K] = f^{m-N}[f^N[K]] = f^m[K]$. Consequently $f^{m-N}[K] \cap K \neq \emptyset$, and $m - N = 0$. Hence m is a multiple of $2n$, and n divides m.

Finally, suppose $N + n$ is a multiple of $2n$. Then $f^{N+n}[K] = K$ by Lemma 2.6. It follows from the inequalities $m < N + n$ and $N \leqslant m$, respectively, that $N + n - m > 0$ and $N + n - m \leqslant n$. Applying f^{N+n-m} to $f^m[K] \cap K \neq \emptyset$ gives $K \cap f^{N+n-m}[K] \neq \emptyset$. Consequently, $N + n - m = n$, $m = N$, m is a multiple of n, and n divides m. □

Theorem 2.8. *Suppose f is a map of an atriodic and hereditarily unicoherent continuum onto itself. Suppose further that K and L are periodic continua with a common point and periods k and l, respectively.*

(1) *If K contains L, then k divides $2l$.*
(2) *If neither K nor L is a subset of the other, then $k = l$, $k = 2l$, or $l = 2k$.*

Proof. Consider (1), and suppose K contains L. Denote by n the smallest of all positive integers i such that $f^i[K] \cap K$ is nonempty. Since $f^l[K] \cap K$ contains L, it follows from Lemma 2.7 that n divides l. By Lemma 2.6, either $n = k$ or $2n = k$. In either case, k divides $2l$.

For the purpose of demonstrating (2), suppose neither K nor L is a subset of the other. Denote the period of $K \cup L$ by p. By (1), p divides $2k$. Hence $f^{2k}[K \cup L] = K \cup L$. Since $f^{2k}[K] = K$, it follows that $K \cup f^{2k}[L] = K \cup L$. Consequently L and $f^{2k}[L]$ both contain $\overline{(K \cup L) - K}$. By (1) of Theorem 2.1, $\overline{(K \cup L) - K}$ contains an end continuum of $K \cup L$. Hence one of L and $f^{2k}[L]$ contains the other. Since L is periodic, $f^{2k}[L] = L$. Consequently $2k$ is a multiple of l, which is to say $2k = ml$ for some positive integer m. Similarly, $2l = m'k$ for some positive integer m'. Multiplying gives $4kl = mm'kl$; thus $4 = mm'$. It follows that m is either 1, 2, or 4. Equivalently, either $2k = l$, $k = l$, or $k = 2l$. □

Theorem 2.9. *Suppose f is a map of an atriodic and hereditarily unicoherent continuum onto itself, and suppose K is a periodic continuum of f. The closure of the union of all proper periodic continua that intersect K is a periodic continuum.*

Proof. Denote by p the period of K, and let $\mathcal{K}$ denote the collection of all proper periodic continua that intersect both K and its complement together with K itself. Note that the union of all periodic continua intersecting K is equal to $\mathcal{K}^*$. According to Theorem 2.8, each continuum in $\mathcal{K}$ has a period of either p, $2p$, or $\frac{p}{2}$. In any case, $f^{2p}[K] = K$ for each $K \in \mathcal{K}$. Consequently, $f^{2p}[\mathcal{K}^*] = \mathcal{K}^*$, from which it follows that $f^{2p}[\overline{\mathcal{K}^*}] = \overline{\mathcal{K}^*}$. □

3. Maximal proper periodic continua

3.1. Properties of maximal proper periodic continua

Lemma 3.1. *Suppose f is a map of a continuum X onto itself. If K is a maximal proper periodic continuum of f, and L is a proper periodic continuum of f that is not a subset of K, then either $K \cup L = X$ or $K \cap L = \emptyset$.*

Proof. Suppose $K \cap L \neq \emptyset$. Then $K \cup L$ is a continuum. Furthermore, for p equal to the product of the periods of K and L, $f^p[K \cup L] = K \cup L$. Hence $K \cup L$ is periodic. Since $K \cup L$ properly contains the maximal proper periodic continuum K, it follows that $K \cup L = X$. □

Lemma 3.2. *Suppose f is a map of an atriodic and hereditarily unicoherent continuum X onto itself. If f has at least three maximal proper periodic continua, then the collection of all maximal proper periodic continua is pairwise disjoint.*

Proof. Suppose to the contrary that f has two maximal proper periodic continua, K and L, with a common point. Note that pairs of distinct maximal proper periodic continua are not nested. Then, by Lemma 3.1, $K \cup L = X$. By hypothesis, there is a third maximal proper periodic continuum H. Since $K \cup L = X$ it follows that H intersects at least one of K and L. Switching the names of K and L if necessary, suppose H intersects K. Then $K \cup H = X$ by Lemma 3.1. By (2) of Theorem 2.1, each of H, K, and L is an end continuum of X; furthermore, each has interior. Consequently, by Theorem 2.2, one of H and L is a subset of the other. But, as has been noted, it is not possible for two distinct maximal proper periodic continua to be nested. Thus the assumption that there are two maximal proper periodic continua of f with a common point is false. □

Lemma 3.3. *Suppose f is a map of a continuum X onto itself. If K is a maximal proper periodic continuum of f, then, for every positive integer n, $f^n[K]$ is a maximal proper periodic continuum of f.*

Proof. It suffices to show that $f^n[K]$ is a maximal proper periodic continuum for $n = 1, 2 \dots, p - 1$ where p is the period of K. Suppose such an n is given, and suppose L is a periodic continuum containing $f^n[K]$. Let q denote the period of L. Then $f^{pq-n}[L]$ is a periodic continuum that contains $f^{pq-n}[f^n[K]] = f^{pq}[K] = K$. Since K is maximal, $f^{pq-n}[L]$ is either X or K. Applying f^n to $f^{pq-n}[L]$ gives that $f^{pq}[L]$, which is equal to L, is either X or $f^n[K]$. □

3.2. Kuratowski maps

K. Kuratowski's work on monotone decompositions of irreducible continua can be found in Section 48 of [6]. Some of the following results are explicitly stated there, and others are alternate statements or easy consequences of the material therein.

Theorem 3.4 *(Kuratowski). Suppose X is a hereditarily decomposable irreducible continuum. Then there is a monotone map g from X onto* [0, 1] *that satisfies each of the following.*

(1) *A subcontinuum M of X is nowhere dense in X if and only if there is a $t \in [0, 1]$ such that $M \subset g^{-1}(t)$.*
(2) *X is irreducible between two points if and only if they belong to $g^{-1}(0)$ and $g^{-1}(1)$, respectively.*
(3) *If K is a subcontinuum of X such that $g[K]$ contains s in its interior, then K contains $g^{-1}(s)$.*

A *Kuratowski map* for a hereditarily decomposable irreducible continuum is any map that satisfies the conclusion of Theorem 3.4. If g is a Kuratowski map for X, and h is a homeomorphism of [0, 1] onto itself, then $h \circ g$ is also a Kuratowski map for X.

A *layer* of X is a nowhere dense subcontinuum A of X such that every subcontinuum of X that properly contains A has nonempty interior. Equivalently, A is a layer of X if and only if, for every Kuratowski map g of X, there is a t in [0, 1] such that $A = g^{-1}(t)$.

A layer A of X is an *end layer* if and only if X is irreducible between two points, one of which belongs to A. Equivalently, A is an end layer of X if and only if, for every Kuratowski map g of X, $A = g^{-1}(0)$ or $A = g^{-1}(1)$. All other layers of X are called *interior layers*.

3.3. Periodic points and maximal proper periodic continua

Theorem 3.5. *Suppose f is a map of a hereditarily decomposable chainable continuum X onto itself with disjoint maximal proper periodic continua.*

(1) *If f has two maximal proper periodic continua, one of which fails to lie in an end layer of X, then every proper periodic continuum of f is contained in a maximal proper periodic continuum.*
(2) *If A is an end layer of X that contains a maximal proper periodic continuum of f, then every periodic continuum of f lying in A is contained in a maximal proper periodic continuum.*

Proof. First consider (1). Let g denote a Kuratowski map for X. By hypothesis, there is a maximal proper periodic continuum K such that $g[K]$ contains a point s of (0, 1) and another maximal proper periodic continuum L that is disjoint from K. Suppose H is a proper periodic continuum of f. If H intersects either K or L, then either H is contained by one of them and the conclusion of (1) follows, or the union of H with one of K and L is X, by Lemma 3.1. But, since K and L are disjoint, the latter case implies that H properly contains one of them, which is not possible because K and L are maximal.

Suppose H fails to intersect both K and L. Denote the collection of all proper periodic continua that intersect H by $\mathcal{J}$. Then $\overline{\mathcal{J}^*}$ is a periodic continuum by Theorem 2.9. Thus, to complete the proof of the theorem, it suffices to show that $\overline{\mathcal{J}^*}$ is a proper subcontinuum of X.

Since H, K, and L are disjoint, and every term of $\mathcal{J}$ contains a point of H, it follows from Lemma 3.1 that any term of $\mathcal{J}$ that intersects one of K and L properly contains the other. But it is impossible for a proper periodic continuum to properly contain a maximal proper periodic continuum, so no term of $\mathcal{J}$ intersects K or L.

If J is a continuum such that s belongs to the interior of $g[J]$, then, by (3) of Theorem 3.4, J contains $g^{-1}(s)$. But $g^{-1}(s)$ intersects K, and it has been established that no term of $\mathcal{J}$ intersects K. Consequently, either $g[J] \subset [0, s]$ for all J in $\mathcal{J}$, or $g[J] \subset [s, 1]$ for all J in $\mathcal{J}$. Since g is continuous, it follows that either $g[\overline{\mathcal{J}^*}] \subset [0, s]$ or $g[\overline{\mathcal{J}^*}] \subset [s, 1]$. In either case, $\overline{\mathcal{J}^*}$ is a proper subcontinuum of X.

The proof of (2) is similar. Suppose H is a proper periodic continuum that lies in A, and, as before, let $\mathcal{J}$ denote the collection of all proper periodic continua that intersect H. It follows from (3) of Theorem 3.4 that any continuum that intersects both A and $X - A$ contains A. Since A contains a maximal proper periodic continuum and each term of $\mathcal{J}$ intersects A, it follows that no term of $\mathcal{J}$ intersects $X - A$. Consequently, $\overline{\mathcal{J}^*}$ is a proper subcontinuum of X. □

Theorem 3.6. *Suppose f is a map of a hereditarily decomposable chainable continuum X onto itself. If f has at least two maximal proper periodic continua, then at least one of the following holds.*

(1) *Every periodic point of f belongs to a maximal proper periodic continuum of f.*
(2) *There is a positive integer n such that f^n maps each end layer of X onto X, and each maximal proper periodic continuum of f is contained in an end layer of X.*
(3) *All maximal proper periodic continua lie in a single end layer of X.*

Proof. First note that if f has two maximal proper periodic continua that are not disjoint, then their union is X by Lemma 3.1, and (1) holds trivially. Suppose f has disjoint maximal proper periodic continua. If there is a maximal proper periodic continuum that fails to lie in an end layer of X, then (1) holds by Theorem 3.5. Suppose that each maximal proper periodic continuum lies in an end layer of X. If one of the end layers of X contains all the maximal proper periodic continua, then (3) holds. Suppose both end layers of X contain maximal proper periodic continua. If there is a positive integer n such that f^n maps each end layer of X onto X, then (2) holds. Suppose there is an end layer of X that is not mapped onto X by f^n for any positive integer n. The remainder of the proof is concerned with demonstrating that, under these assumptions, (1) holds.

Let K and L denote two maximal proper periodic continua that belong to opposite end layers of X. (A) One of K and L, say K, belongs to an end layer of X that is not mapped onto X by f^n for any positive integer n. Let g be a Kuratowski map for X. If $L \subset g^{-1}(0)$ and $K \subset g^{-1}(1)$, then g may be composed with $h(t) = 1 - t$ to obtain a Kuratowski map for which K and L lie in the layers associated with 0 and 1, respectively. Thus it will be assumed that (B) $K \subset g^{-1}(0)$ and $L \subset g^{-1}(1)$.

Suppose x is a periodic point of f. If x belongs to either K or L, then x certainly belongs to a maximal proper periodic continuum of f. Suppose x is in neither K nor L. Denote by $\mathcal{J}$ the collection of all proper periodic continua that contain x. By Theorem 2.9, $\overline{\mathcal{J}^*}$ is a periodic continuum, so it remains only to show that $\overline{\mathcal{J}^*}$ is a proper subcontinuum of X. Denote $2\,\mathrm{period}(x)\mathrm{period}(K)\mathrm{period}(L)$ by N. Then $f^N(x) = x$, $f^N[K] = K$, $f^N[L] = L$, and, by Theorem 2.8, $f^N[J] = J$ for every J in $\mathcal{J}$.

Suppose, for the sake of establishing contradiction, that every neighborhood of the form $g^{-1}[0, \frac{1}{n})$ for $n \in \mathbb{N}$ contains a point z_n such that $f^N(z_n) \in g^{-1}(1)$. Then there is a subsequence of $z_1, z_2, z_3, \ldots$ that converges to some point z in $g^{-1}(0)$. Since f^N is continuous and $g^{-1}(1)$ is closed, it follows that $f^N(z) \in g^{-1}(1)$. Thus K and z both lie in $g^{-1}(0)$, but $f^N[K]$ lies in $g^{-1}(0)$ whereas $f^N(z)$ lies in $g^{-1}(1)$. Since X is irreducible between any point of $g^{-1}(0)$ and any point of $g^{-1}(1)$, it follows that $f^N[g^{-1}(0)] = X$. Since, by (B), $K \subset g^{-1}(0)$, this is contrary to the assumption in (A) that K belongs to an end layer of X that is not mapped onto X by f^n for any positive integer n. Thus there is a neighborhood of the form $g^{-1}[0, \frac{1}{n})$ whose image under f^N fails to intersect $g^{-1}(1)$. It follows that there is a closed connected neighborhood D of $g^{-1}(0)$ such that neither D nor $f^N[D]$ intersect $g^{-1}(1)$. Notice that D is also a neighborhood of K.

Suppose, for the purpose of establishing a contradiction, that $\overline{\mathcal{J}^*}$ is not a proper subcontinuum of X. Denote by s the maximum value of $g[D \cup f^N[D]]$. Then there is a term J of $\mathcal{J}$ that intersects both D and $g^{-1}(s, 1]$. Since $D \cup J$ contains the maximal proper periodic continuum K, a contradiction can be reached by showing that $D \cup J$ is a proper periodic continuum.

First it will be shown that $D \cup J$ is a proper subcontinuum of X. Any periodic continuum that intersects the maximal proper periodic continuum L contains $X - L$ by Lemma 3.1 and, since L lies in the nowhere dense set $g^{-1}(1)$, is therefore equal to X. But J is a proper periodic continuum; consequently J fails to intersect L. Notice that

D also fails to intersect L since it fails even to intersect $g^{-1}(1)$, of which L is a subset. It follows that $D \cup J$ is a proper subcontinuum of X.

Finally it will be shown that $f^N[D \cup J] = D \cup J$. Recall that $D \cup J$ intersects both K and $g^{-1}(s, 1]$. Note that the same is true of $f^N[D \cup J]$, which contains both $f^N[K] = K$ and $f^N[J] = J$. Hence $g[D \cup J]$ and $g[f^N[D \cup J]]$ both contain $[0, s]$ in their interior. It follows from (3) of Lemma 3.4 that $g^{-1}[0, s] \subset D \cup J$ and that $g^{-1}[0, s] \subset f^N[D \cup J]$. Also recall that $D \cup f^N[D] \subset g^{-1}[0, s]$. Consequently, $D \subset g^{-1}[0, s] \subset f^N[D \cup J]$, and $f^N[D] \subset g^{-1}[0, s] \subset D \cup J$. Since $J = f^N[J]$, it follows, respectively, that $D \cup J \subset f^N[D \cup J]$ and $f^N[D \cup J] \subset D \cup J$. This establishes contradiction with the maximality of K. Consequently $\overline{\mathcal{J}^*}$ is a proper subcontinuum of X and, hence, a maximal proper periodic continuum containing x. □

Corollary 3.7. *Suppose f is a continuous map of an interval onto itself. If f has at least two disjoint maximal proper periodic continua, then every periodic point of f belongs to a maximal proper periodic continuum of f.*

4. A variation on Sarkovskii's theorem

4.1. Sufficient conditions for maximal proper periodic continua of all periods

The following theorem is a consequence of a more general theorem by W.T. Ingram.

Theorem 4.1. *(See Ingram, [4].) Suppose f is a map of a chainable continuum X into itself. If K is a subcontinuum of X such that $K \subset f[K]$, then f has a fixed point in K.*

Theorem 4.2. *Suppose f is a map of a chainable continuum X into itself, and suppose there are mutually exclusive subcontinua, A and B, of X such that $A \cup B \subset f[A]$ and $A \subset f[B]$.*

(1) *(Ingram, [4]) For each positive integer p, f has a periodic point x of period p such that* orbit(x) *intersects each of A and B, and* orbit$(x) \subset A \cup B$.
(2) *If the collection of proper periodic continua of f that intersect A and whose forward iterates all intersect $A \cup B$ is bounded away from B, then, for each positive integer p, A intersects a maximal proper periodic continuum C of period p such that, for $p > 1$, $f^{p-1}[C]$ intersects B.*

Proof. Since $A \subset f[A]$, f has a fixed point in A by Theorem 4.1. Suppose p is a positive integer larger than one. Every map of a continuum onto a chainable continuum has the property that every subcontinuum of the range is the image of some subcontinuum of the domain [9, Theorem 12.46]. Hence there are subcontinua K_0 and K_1 of B and A, respectively, such that $f[K_0] = A$ and $f[K_1] = K_0$. Proceeding inductively, there are continua $K_1, K_2, \dots, K_{p-1}$, all subcontinua of A, such that $f[K_n] = K_{n-1}$ for $n = 1, 2, \dots, p-1$. It follows that $f^{p-1}[K_{p-1}] = K_0$ and $f^p[K_{p-1}] = A$. Consequently f^p has a fixed point c in K_{p-1} by Theorem 4.1. Then c is periodic under f, and its period does not exceed p. Since $f^n(c) \in A$ for $n = 0, 1, \dots, p-2$ and $f^{p-1}(c) \in B$, it follows that the period of c is not less than p. Thus (1) is true.

Now consider (2). Suppose that the collection $\mathcal{C}$ of proper periodic continua of f that intersect A and all of whose forward iterates intersect $A \cup B$ is bounded away from B, and suppose p is a positive integer larger than one. As in the previous paragraph, there is a periodic point c in A whose period is p such that $f^n(c) \in A$ for $n = 0, 1, \dots, p-2$ and $f^{p-1}(c) \in B$, so $\mathcal{C}$ is nonempty. Denote by C the closure of the union of all terms of $\mathcal{C}$ that contain c. Then C is a periodic continuum by Theorem 2.9, and C is a proper subcontinuum of X because it fails to intersect B. Hence C is a maximal proper periodic continuum. For $n = 0, 1, \dots, p-2$, $f^n[C]$ intersects A and, therefore, fails to intersect B. Since $f^{p-1}[C]$ intersects B, it follows that the period of C is at least p. Both C and $f^p[C]$ contain the point c, so, by Lemma 3.1, either $f^p[C] = C$ or $f^p[C] \cup C = X$. But the latter is impossible because neither C nor $f^p[C]$ intersects B. Consequently, the period of C is p. A similar but simpler argument establishes the existence of a maximal proper periodic continuum that intersects A and has period one. □

4.2. Unruly end layers

Theorem 4.3. *Suppose f is a map of a hereditarily decomposable chainable continuum X onto itself. If each maximal proper periodic continuum of f lies in an end layer of X, and there is a positive integer n such that f^n maps each end layer of X onto X, then f has maximal proper periodic continua of all periods.*

Proof. Denote by A and B the end layers of X. Since $f^{n+1}[A] = X$ and $f^{n+1}[B] = X$, it may be assumed without loss of generality that n is odd. By Theorem 4.2, f^n has a periodic point x of period 3 such that the orbit of x under f^n intersects each of A and B, and lies in $A \cup B$. Since $3n$ is odd and $f^{3n}(x) = x$, one of A and B, say A, contains two consecutive terms of the orbit of x under f. But there is also a term of the orbit of x under f that lies in B, so there is a positive integer p such that $f^{p-1}(x)$ and $f^p(x)$ both belong to A and $f^{p+1}(x)$ belongs to B. Then $f[A]$, which contains both $f^p(x)$ and $f^{p+1}(x)$, is a subcontinuum of X that intersects both A and B and, therefore, is equal to X. Also note that $f[B]$ intersects A.

Let a and b be terms of the orbit of x under f such that $a \in A$, $f(a) \in B$, $b \in B$, and $f(b) \in A$; and let g be a Kuratowski map for X such that $g^{-1}(0) = A$ and $g^{-1}(1) = B$. Since f is continuous at a and $f(a) \in B$, there is $c \in g^{-1}(0, 1]$ such that $f(c) \in g^{-1}(0, 1]$. Denote the subcontinuum of X that is irreducible between b and c by $\overline{bc}$. Then A and $\overline{bc}$ are disjoint. Notice that $f[\overline{bc}]$ intersects both $A = g^{-1}(0)$ and $X - A = g^{-1}(0, 1]$. Since g satisfies (3) of Theorem 3.4, $A \subset f[\overline{bc}]$. Consequently, $A \subset f[\overline{bc}]$, and $A \cup \overline{bc} \subset f[A]$. Every continuum that intersects both A and $X - A$ contains A, and $f[A] = X$, so every proper periodic continuum that intersects A is contained by A. Consequently the collection of proper periodic continua that intersect A is bounded away from $\overline{bc}$. Then, by Theorem 4.2, f has maximal proper periodic continua of all periods. □

4.3. The Sarkovskii order for maximal proper periodic continua

Lemma 4.4. *Suppose f is a map of a hereditarily decomposable chainable continuum X onto itself, and suppose n is a positive integer. The union of all maximal proper periodic continua of f whose periods do not exceed n is closed.*

Proof. First notice that the lemma is trivial if f has only two maximal proper periodic continua. Suppose there are at least three maximal proper periodic continua. Then, by Lemma 3.2, the collection of all maximal proper periodic continua of f is pairwise disjoint. Denote by $\mathcal{K}$ the collection of all maximal proper periodic continua of f whose period is not larger than n, and suppose x is a limit point of $\mathcal{K}^*$. Then there is a (possibly constant) sequence $K_1, K_2, K_3, \ldots$ of terms of $\mathcal{K}$ and a sequence $x_1, x_2, x_3, \ldots$, with $x_i \in K_i$ for each i, that converges to x; furthermore, the sequences $K_1, K_2, K_3, \ldots$ and $x_1, x_2, x_3, \ldots$ may be chosen so that $\text{period}(K_i) = \text{period}(K_j)$ for each pair of positive integers i and j. Denote this period by p, and notice that $p \leqslant n$. There is a subsequence of $K_1, K_2, K_3, \ldots$ that converges to some continuum K (Corollary 4.18 of [9]). It follows that $x \in K$. Since f is continuous, $f^p[K] = K$.

To see that x belongs to a maximal proper periodic continuum, two cases will be considered. Either each of $K_1, K_2, K_3, \ldots$ is contained in an end layer of X, or some one of them fails to lie in an end layer. In the former case, it follows that K is contained in an end layer of X that contains all but finitely many terms of $K_1, K_2, K_3, \ldots$. Consequently, by (2) of Theorem 3.5, there is a maximal proper periodic continuum $\widetilde{K}$ containing K. If one of $K_1, K_2, K_3, \ldots$ fails to lie in an end layer of X, then, by (1) of Theorem 3.5, there is a maximal proper periodic continuum $\widetilde{K}$ containing K. In either case, it follows that $\widetilde{K}$ contains x.

Since $\widetilde{K}$ and $f^p[\widetilde{K}]$ are both maximal proper periodic continua that contain K, and distinct maximal proper periodic continua are disjoint, it follows that $f^p[\widetilde{K}] = \widetilde{K}$. Hence $\widetilde{K} \in \mathcal{K}$, from which it follows that $x \in \mathcal{K}^*$. □

The following version of Theorem 4.5 together with its proof were suggested by the referee.

Lemma 4.5. *Suppose X is a hereditarily decomposable chainable continuum and K is a closed subset of X. Let Y denote the decomposition space of the partition of X into components of K and single points in $X - K$, and let m denote the natural map of X onto Y. Then Y is a hereditarily decomposable chainable continuum and m is a monotone map.*

Furthermore, if f is a map from X into itself such that $f[K] \subset K$, then there is a map $g : Y \to Y$ such that $m \circ f = g \circ m$.

Proof. As the natural map onto the decomposition space, m is continuous. It is monotone as the elements of the decomposition are connected. Since the decomposition of X into components of K and single points in $X - K$ is upper semi-continuous, Y is a continuum [9, Theorem 3.10]. Suppose Z is an arbitrary nondegenerate subcontinuum of Y. Since m is monotone, $E = m^{-1}[Z]$ is a continuum. Since E is hereditarily decomposable and $m[E] = Z$, Z is decomposable [5, Theorem 5]. Hence Y is hereditarily decomposable. It is chainable as a monotone image of a chainable continuum [9, Theorem 12.14].

If f is a map of X into itself such that $f[K] \subset K$, then $m \circ f[m^{-1}(y)]$ is a single point for each $y \in Y$. Define $g(y) = m \circ f[m^{-1}(y)]$ for each $y \in Y$ and observe that g is a map such that $m \circ f = g \circ m$. □

Lemma 4.6. *Suppose f, g, and m are functions such that $m \circ f = g \circ m$. If x is a point of the domain of g such that $m^{-1}(x)$ is periodic under f, then x is periodic under g, and* $\text{period}(x) = \text{period}(m^{-1}(x))$.

Proof. It follows from the hypothesis that $m \circ f^n = g^n \circ m$ for each positive integer n. Denote the period of $m^{-1}(x)$ by p. Then $g^p(x) = g^p \circ m[m^{-1}(x)] = m \circ f^p[m^{-1}(x)] = m[m^{-1}(x)] = x$. Consequently, x is periodic under g. Denote the period of x by q, and note that the previous computation implies that $q \leqslant p$.

To complete the proof of the lemma it remains only to show that $p \leqslant q$. First note that $m \circ f^q[m^{-1}(x)] = g^q \circ m[m^{-1}(x)] = g^q(x) = x$. Since each point of $f^q[m^{-1}(x)]$ is mapped to x by m, it follows that $f^q[m^{-1}(x)] \subset m^{-1}(x)$. But $m^{-1}(x)$ is periodic, so $f^q[m^{-1}(x)] = m^{-1}(x)$. Consequently, $p \leqslant q$. □

Lemma 4.7. *Suppose f is a map of a hereditarily decomposable chainable continuum X onto itself with disjoint maximal proper periodic continua, and suppose $n \in \mathbb{N}$. Then there are a map g of a hereditarily decomposable continuum Y onto itself and a monotone map m from X onto Y such that each of the following holds.*

(1) *$m \circ f = g \circ m$.*
(2) *Two points are identified by m if and only if they belong to a single maximal proper periodic continuum whose period does not exceed $2n$.*
(3) *If K is a maximal proper periodic continuum of f with period $p \leqslant n$, then $m[K]$ is a periodic point of g with period p.*
(4) *If x is a periodic point of g with period $p \leqslant n$, then $m^{-1}(x)$ is either a maximal proper periodic continuum of f with period p or a periodic point of f with period p that fails to lie in a maximal proper periodic continuum.*

Proof. Denote by $\mathcal{K}$ the collection of all maximal proper periodic continua of f whose periods do not exceed $2n$. By Lemma 4.4, $\mathcal{K}^*$ is closed. Thus, by Lemma 4.5, there are a hereditarily decomposable continuum Y, a map g of Y onto itself, and a monotone map m of X onto Y such that $m \circ f = g \circ m$ and such that m identifies two points of X if and only if they belong to the same member of $\mathcal{K}$. Hence (1) and (2) both hold.

Consider (3). If K is a maximal proper periodic continuum of f with period $p \leqslant n$, then $K \in \mathcal{K}$ and, hence, $m[K]$ is a point. Furthermore, $m^{-1}(m[K]) = K$. Consequently, by Lemma 4.6, $m[K]$ is a periodic point of g with period p.

To see that (4) holds, suppose x is a periodic point of g with period $p \leqslant n$. Notice that $m^{-1}(x)$ is either a term of $\mathcal{K}$ or a point of $X - \mathcal{K}^*$. In the former case, $m^{-1}(x)$ is clearly a maximal proper periodic continuum. Its period is equal to that of x by Lemma 4.6, so the conclusion of (4) follows.

Suppose $m^{-1}(x)$ is a point of $X - \mathcal{K}^*$. Then, since $m \circ f^p(m^{-1}(x)) = g^p \circ m(m^{-1}(x)) = g^p(x) = x$, it follows that $f^p(m^{-1}(x)) = m^{-1}(x)$. Thus $m^{-1}(x)$ is periodic, and, by Lemma 4.6, has period p. For the purpose of establishing contradiction, suppose $m^{-1}(x)$ belongs to a maximal proper periodic continuum. Denote by K the maximal proper periodic continuum that contains $m^{-1}(x)$. It follows from Theorem 2.8 that the period of K divides $2p$. Consequently, the period of K is no greater than $2n$. Hence, $K \in \mathcal{K}$, and $m^{-1}(x) = K$. Consequently, $m^{-1}(x) \in \mathcal{K}^*$. But this is impossible because $m^{-1}(x)$ is also in $X - \mathcal{K}^*$. Therefore $m^{-1}(x)$ fails to belong to a maximal proper periodic continuum, and the conclusion of (4) follows. □

Notation. For each of the following two theorems, $p \prec q$ means p precedes q according to the Sarkovskii ordering of the positive integers:

$$3 \prec 5 \prec 7 \prec 9 \prec 11 \prec \cdots \prec 3 \cdot 2 \prec 5 \cdot 2 \prec \cdots \prec 3 \cdot 2^2 \prec 5 \cdot 2^2 \prec \cdots \prec 2^3 \prec 2^2 \prec 2 \prec 1.$$

Theorem 4.8. *(See Minc and Transue [8].) Let Y be a hereditarily decomposable chainable continuum and let X be a subcontinuum of Y. If $p \prec q$, f is a continuous map of X into Y, and f has a periodic point of period p, then f has a point of period q.*

Theorem 4.9. *Suppose f is a map of a hereditarily decomposable chainable continuum X onto itself with disjoint maximal proper periodic continua. If $p \prec q$ and f has a maximal proper periodic continuum of period p, then f has a maximal proper periodic continuum of period q.*

Proof. Let n be the maximum of p and q. By Lemma 4.7, there are a hereditarily decomposable chainable continuum Y, a map g from Y onto Y, and a monotone map m from X onto Y such that each of the following holds.

(a) Two points are identified by m if and only if they belong to a single maximal proper periodic continuum whose period does not exceed $2n$.
(b) If K is a maximal proper periodic continuum of f with period $r \leqslant n$, then $m[K]$ is a degenerate maximal proper periodic continuum of g with period r.
(c) If x is a periodic point of g with period $r \leqslant n$, then $m^{-1}(x)$ is either a maximal proper periodic continuum of f with period r or a periodic point of f with period r that fails to lie in a maximal proper periodic continuum.

By Theorem 3.6, at least one of the following is true.

(1) Every periodic point of f belongs to a maximal proper periodic continuum of f.
(2) There is a positive integer j such that f^j maps each end layer of X onto X, and each maximal proper periodic continuum of f is contained in an end layer of X.
(3) All maximal proper periodic continua lie in a single end layer of X.

If (2) is true, then f has maximal proper periodic continua of all periods by Theorem 4.3, and the conclusion of the present theorem holds trivially.

Suppose (1) is true, and suppose K is a maximal proper periodic continuum of f with period p. By (b), $m[K]$ is a periodic point of g with period p. Then g has a periodic point of period q by Theorem 4.8. Finally, it follows from (c) and (1) that f has a maximal proper periodic continuum of period q.

The proof in case (3) holds is similar, but more delicate. Suppose (3) is true, and suppose K is a maximal proper periodic continuum of f with period p. Denote by A the end layer of X that contains all of the maximal proper periodic continua of f. By (2) of Theorem 3.5, every periodic point of f in A belongs to a maximal proper periodic continuum of f.

By (b), $m[K]$ is a periodic point of g with period p. Since A contains every maximal proper periodic continuum of f, $f^k[K]$ lies in A for each k. Hence $g^k[m[K]] = m[f^k[K]] \subset m[A]$ for each k, and $m[K]$ is a periodic point of $g|m[A]$ with period p. Then, by Theorem 4.8, $g|m[A]$ has a periodic point x of period q.

Since m identifies two points if and only if they belong to a single maximal proper periodic continuum of f whose period does not exceed $2n$ and every such continuum is a subset of A, it follows that $m^{-1}[m[A]] = A$. Consequently, $m^{-1}(x) \subset A$. By (c), $m^{-1}(x)$ is either a maximal proper periodic continuum of f with period q, or $m^{-1}(x)$ is a periodic point of f that fails to lie in a maximal proper periodic continuum. But $m^{-1}(x) \subset A$, and every periodic point of f in A belongs to a maximal proper periodic continuum by (2) of Theorem 3.5, so $m^{-1}(x)$ is a maximal proper periodic continuum of f with period q. □

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References

[1] M. Barge, J. Martin, Chaos, periodicity, and snakelike continua, Trans. Amer. Math. Soc. 289 (1985) 355–365.

[2] M. Barge, J. Martin, The construction of global attractors, Proc. Amer. Math. Soc. 110 (1990) 523–525.
[3] R.H. Bing, Snake-like continua, Duke Math. J. 18 (1951) 653–663.
[4] W.T. Ingram, Concerning periodic points in mappings of continua, Proc. Amer. Math. Soc. 104 (2) (1988) 643–649.
[5] J. Krasinkiewicz, P. Minc, Mappings onto indecomposable continua, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 25 (1977) 675–680.
[6] K. Kuratowski, Topology, vol. II, Academic Press/Polish Scientific Publishers, New York/Warsaw, 1968.
[7] W. Lewis, Periodic homeomorphisms of chainable continua, Fund. Math. 117 (1983) 81–84.
[8] P. Minc, W.R.R. Transue, Sarkovskii's theorem for hereditarily decomposable chainable continua, Trans. Amer. Math. Soc. 315 (1989) 173–188.
[9] S.B. Nadler Jr., Continuum Theory, Marcel Dekker, Inc., 1992.
[10] D.J. Ryden, Irreducibility of inverse limits on intervals, Fund. Math. 165 (2000) 29–53.
[11] D.J. Ryden, Composant structure in inverse limits of Markov maps, preprint.
[12] A.N. Sharkovskii, Coexistence of cycles of a continuous map of the line into itself, Ukrain. Mat. Zh. 16 (1964) 61–71 (in Russian, English summary).
[13] A.N. Sharkovskii, Coexistence of cycles of a contiguous map of the line into itself, Internat. J. Bifurcation Chaos 5 (5) (1995) 1263–1273 (English translation).
[14] R.H. Sorgenfrey, Concerning triodic continua, Amer. J. of Math. (1944) 439–460.