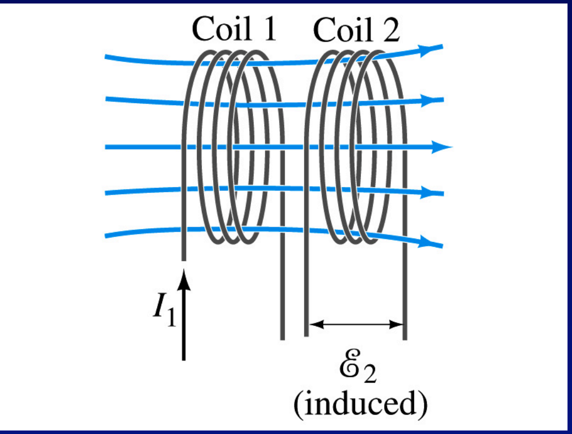
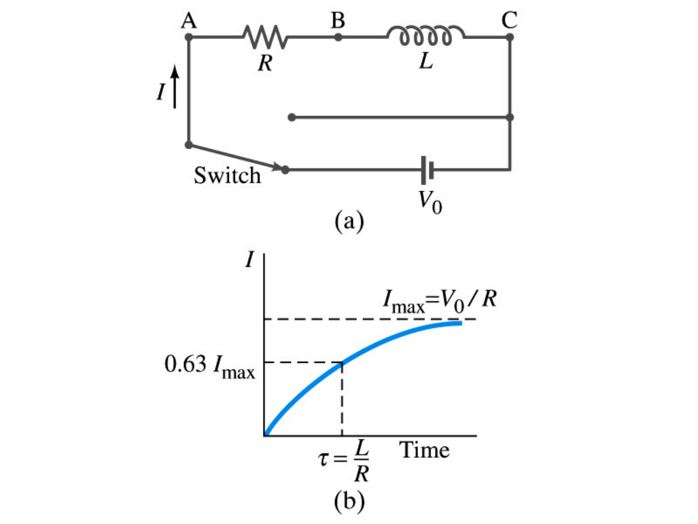
Chapter 30
Inductance

- Mutual Inductance
- Self-Inductance
- Energy stored in a magnetic field
- LR Circuits
- LC Circuits
- LRC Circuits

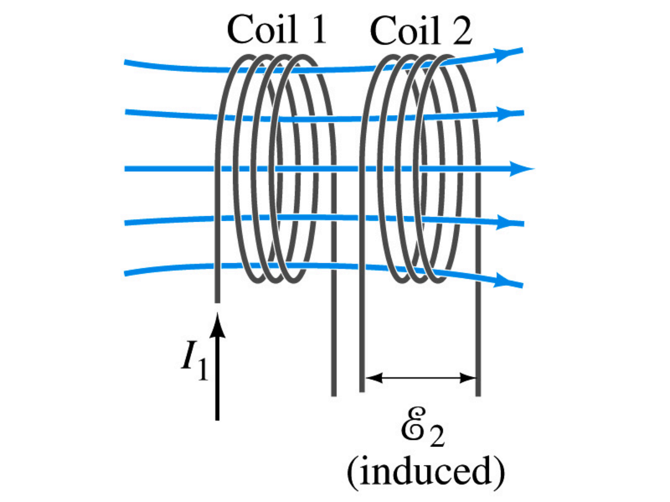




Mutual Inductance

New Topic

Mutual Inductance



The magnetic flux in coil 2 created by coil 1 is proportional to I_1. Define

$$M_{21} = \frac{N_2 \Phi_{21}}{I_1}$$

M_{21} is called the mutual inductance. It depends only on the geometric factors, NOT on the currents.

Faraday's law:

$$\varepsilon_2 = -N_2 \frac{d\Phi_{21}}{dt}$$

$$\varepsilon_2 = -M_{21} \frac{dI_1}{dt}$$

The reverse situation is

$$\varepsilon_1 = -M_{12} \frac{dI_2}{dt}$$

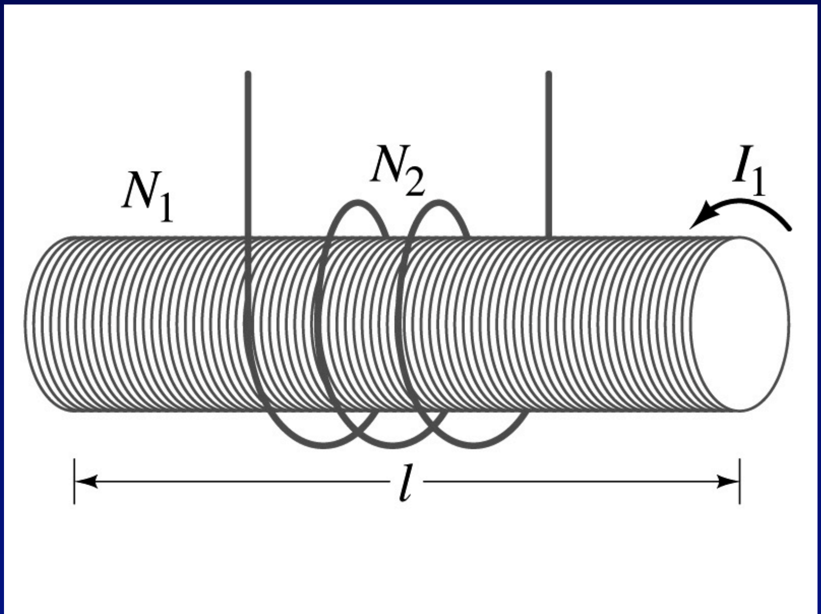
$$M_{12} = M_{21} = M$$

The SI unit for M is henry (H). 1 H = 1 V.s/A = 1 Ω.s

Example: Solenoid and coil

The magnetic field inside the solenoid is

$$B = \mu_0 \frac{N_1}{l} I_1$$



The magnetic flux through the coil is

$$\Phi_{21} = BA = \mu_0 \frac{N_1}{l} I_1 A$$

Hence the mutual inductance is

$$M = \frac{N_2 \Phi_{21}}{I_1}$$

$$M = \mu_0 \frac{N_1 N_2 A}{l}$$

Examples of Mutual Inductance

- Transformer
- Heart pacemaker
  - Power in an external coil is transmitted via mutual inductance to a 2^{nd} coil inside the body.
  - No surgery is needed to replace a battery like in battery-operated pacemakers.

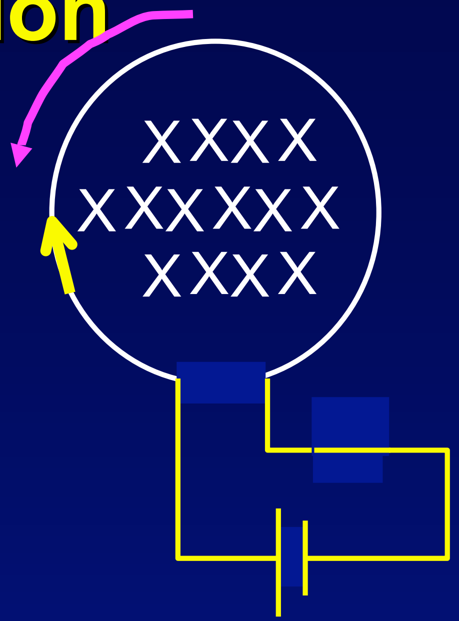
- It can be a problem: interference
  - Any changing current can induce an emf in another part of the same circuit or in a different circuit

Self Inductance

New Topic

The concept of self-induction

- Consider the loop at the right.

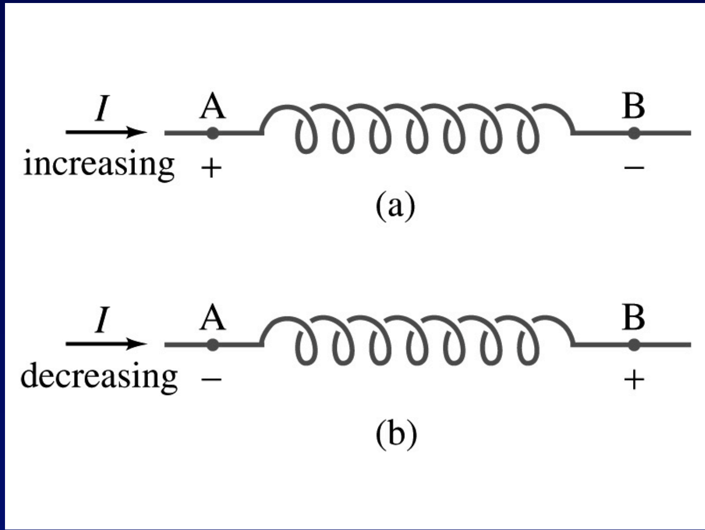


  - switch closed ⇒ current starts to flow in the loop.
  - magnetic field is produced in the area enclosed by the loop.
  - flux through loop changes
  - **emf induced in loop opposing initial emf**

- **Self-Induction: the act of a changing current through a loop inducing an opposing current in that same loop.**

Self-Inductance

The total magnetic flux in the coil is proportional to the current I. Define



$$L = \frac{N\Phi_B}{I}$$

L is called self-inductance. It depends only on the geometric factors, NOT on the current. Such coil is called an inductor.

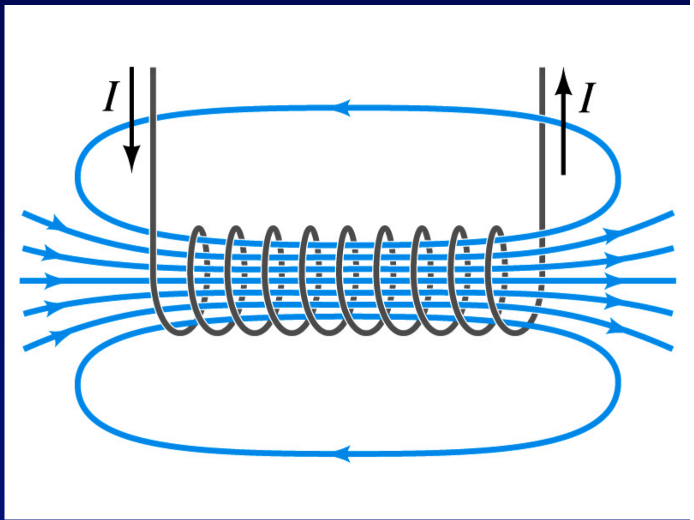
Use Faraday's law: $\varepsilon = -N\frac{d\Phi_B}{dt}$ $\quad \varepsilon = -L\frac{dI}{dt}$

The SI unit for L is also henry (H). 1 H =1 Ω.s.

Example: Solenoid inductance

The magnetic field inside the solenoid is

$$B = \mu_0 \frac{N}{l} I$$



The total magnetic flux through the coil is

$$\Phi_B = BA = \mu_0 \frac{N}{l} IA$$

Hence the self-inductance is $$L = \frac{N\Phi_B}{I} \text{ or } \quad L = \mu_0 \frac{N^2 A}{l}$$

For N=100, l=5 cm, A=0.3 cm^2, L=4π x10^{-7}x100^{2}x0.3x10^{-4}/0.05=7.5 μH.

If filled with an iron core (μ=4000 μ_0), L= 30 mH.

Voltage across an inductor

- Inductor does not oppose current that flows through it. It opposes the change in the current.
  - It's a current stabilizer in the circuit.

$$V_{ab} = iR$$

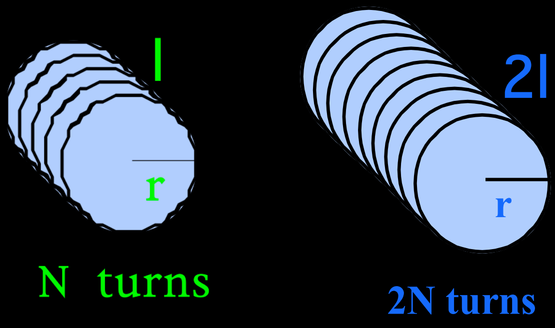
(a) Resistor with current i flowing from a to b: potential drops from a to b

$$V_{ab} = L\frac{di}{dt}$$

(b) Inductor with current i flowing from a to b:
- If $di/dt > 0$: potential drops from a to b
- If $di/dt < 0$: potential increases from a to b
- If i is constant ($di/dt = 0$): no potential difference

ConcepTest 30.1 Inductance

- Consider the two inductors shown:
  - Inductor 1 has length l, N total turns and has inductance L_1.
  - Inductor 2 has length 2l, 2N total turns and has inductance L_2.
- What is the relation between L_1 and L_2?

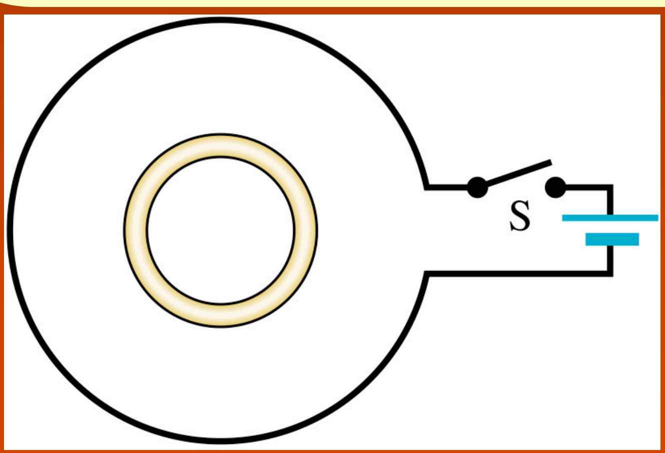


(1) $L_2 < L_1$ (2) $L_2 = L_1$ (3) $L_2 > L_1$

ConcepTest 30.2 Inductance

- A small, circular ring of wire is inside a larger loop that is connected to a battery and a switch S. The small ring and the larger loop both lie in the same plane. When the switch S is closed,

1) a clockwise current flows in the ring, caused by self-inductance
2) a counterclockwise current flows in the ring, caused by self-inductance
3) a clockwise current flows in the ring, caused by mutual inductance
4) a counterclockwise current flows in the ring, caused by mutual inductance
5) nothing happens

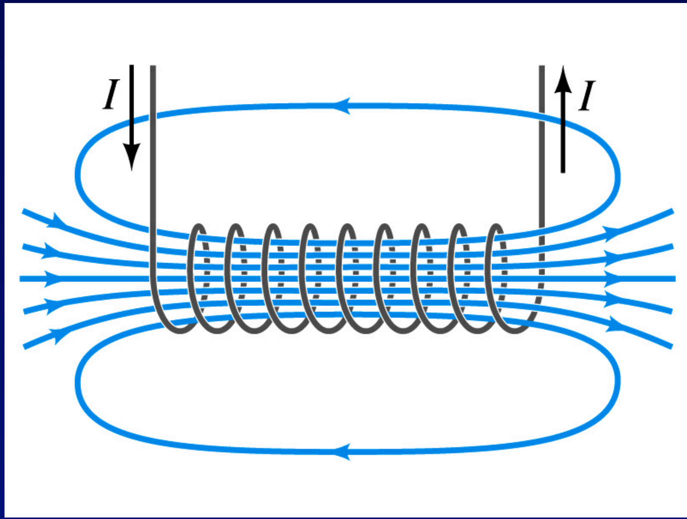


Energy Stored in a Magnetic Field

New Topic

Energy Stored in a Magnetic Field

When an inductor is carrying a current which is changing at a rate dI/dt, the energy is being supplied to the inductor at a rate



$$P = I\varepsilon = LI\frac{dI}{dt}$$

The work needed to increase the current from 0 to I is

$$W = \int P dt = \int_0^I LI dI = \frac{1}{2}LI^2$$

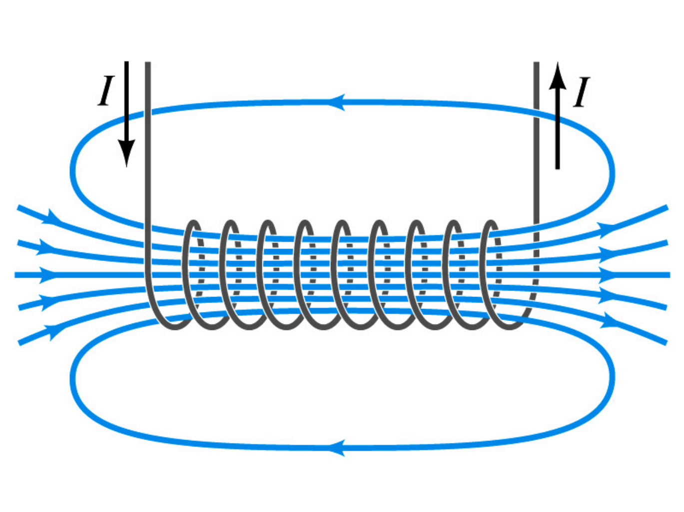
By energy conservation, the energy stored in the inductor is

$$U = \frac{1}{2}LI^2$$

Energy Stored in a Magnetic Field

Question: Where exactly does the energy reside?

$$U = \frac{1}{2} L I^2$$



Answer: It resides in the magnetic field.

Using $L = \mu_0 \frac{N^2 A}{l}$ and $B = \mu_0 \frac{N}{l} I$

$$U = \frac{1}{2}\left(\mu_0 \frac{N^2 A}{l}\right)\left(\frac{Bl}{\mu_0 N}\right)^2 = \frac{1}{2}\frac{B^2}{\mu_0} A l$$

Or energy density

$$u = \frac{1}{2}\frac{B^2}{\mu_0}$$

The conclusion is valid for any region of space where a magnetic field exists.

Compare with the electric case: $U = \frac{1}{2} C V^2$ $u = \frac{1}{2} \varepsilon_0 E^2$

ConcepTest 30.3 Inductors

If you want to double the energy stored in an inductor, the current in the inductor must

(1) stay the same
(2) double
(3) triple
(4) quadruple
(5) increase by a factor of 1.4

LR Circuit

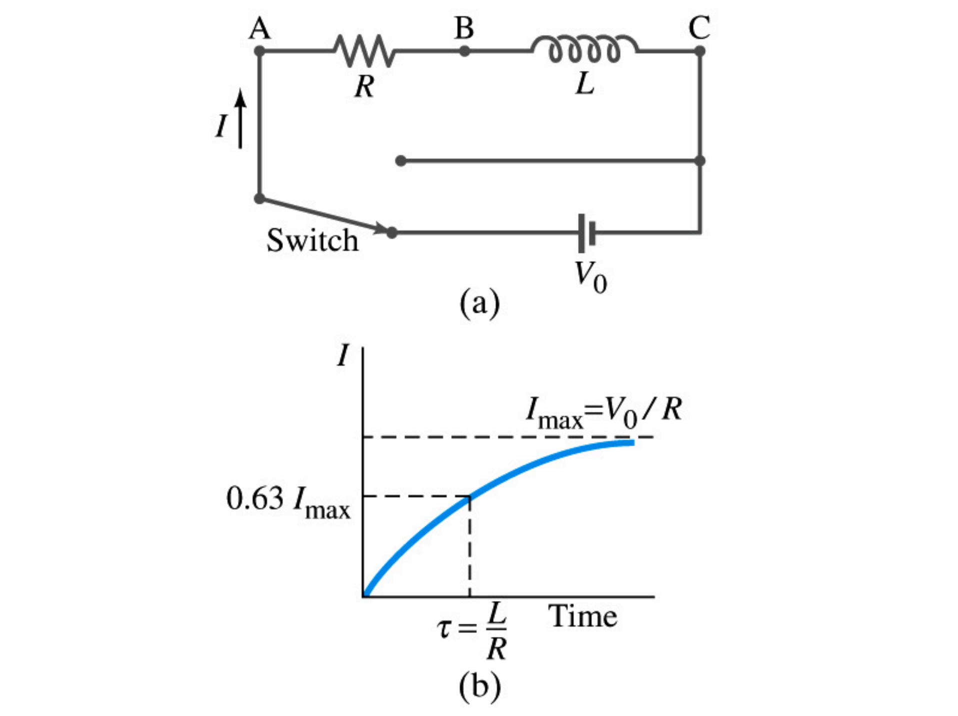
New Topic

LR Circuits

Question: What happens when the switch is thrown to connect with the battery?

Answer: Current starts to flow, eventually reaching the steady value of V_0/R.

$$\text{loop rule}: V_0 - IR - L\frac{dI}{dt} = 0$$



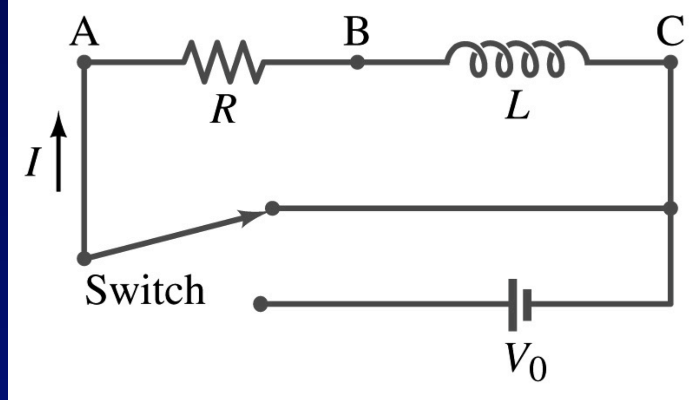
Solve differential equation

$$I(t) = \frac{V_0}{R}\left(1 - e^{-t/\tau}\right)$$

Where $\tau = L/R$ is called the time constant.

LR Circuits

Question: What if the switch is thrown to disconnect with the battery?

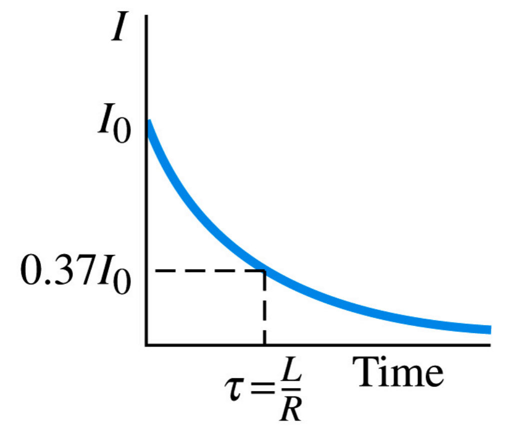


$$\text{loop rule}: -IR - L\frac{dI}{dt} = 0$$

Solve differential equation

$$I(t) = \frac{V_0}{R} e^{-t/\tau}$$

time constant τ=L/R

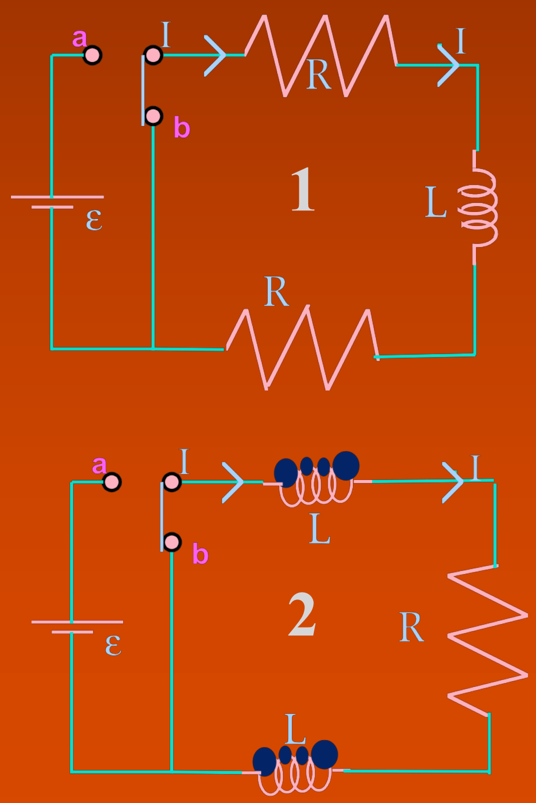


Summary: there is always some reaction time when a LR circuit is turned on or off. The situation is similar to RC circuits, except here the time constant is proportional to 1/R, not R.

ConcepTest 30.4 LR Circuits

Which circuit has the longer time constant after the switch is thrown to position a? (assume R=1 Ω and L=1 H)

(1) Circuit 1
(2) Circuit 2
(3) They are the same
(4) Not enough info

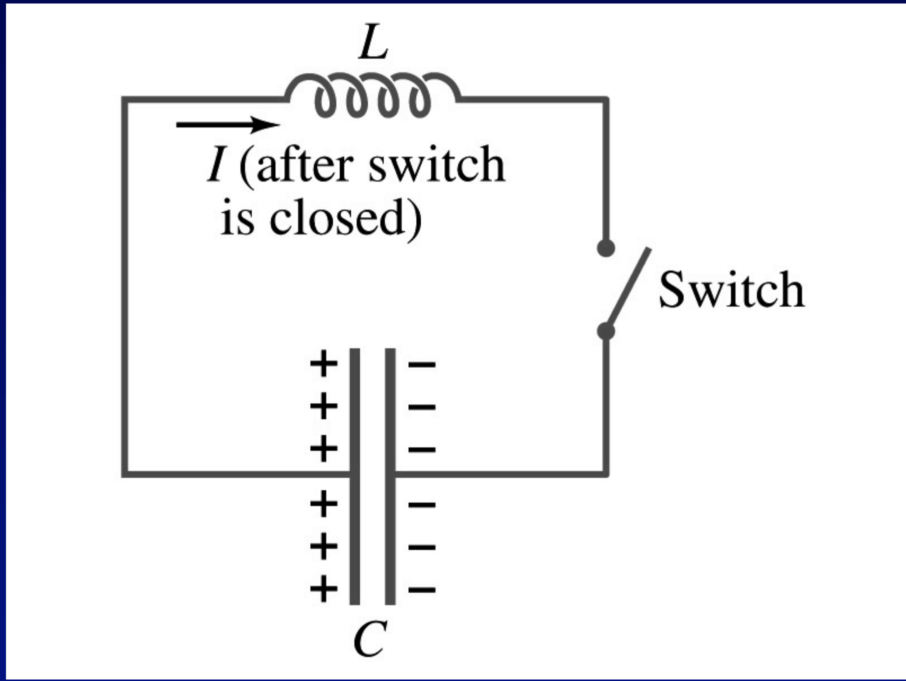


LC Circuit

New Topic

LC Circuits

The capacitor is charged to Q_0.
At t=0, the circuit is closed.
What will happen?



$$\text{loop rule}: \frac{Q}{C} - L\frac{dI}{dt} = 0$$

Using I=-dQ/dt, one gets

$$\frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0$$

Solve differential equation

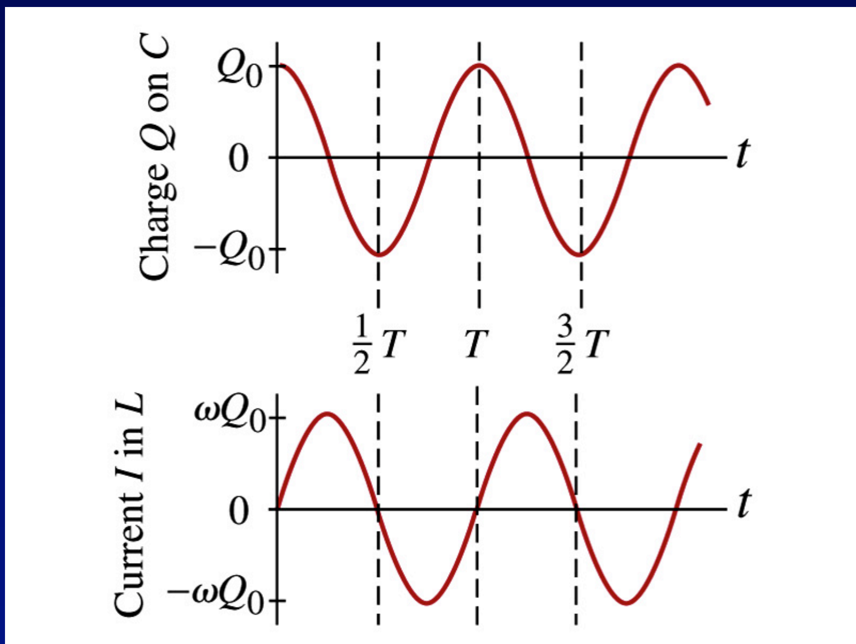
$$Q(t) = Q_0 \cos(\omega t + \phi)$$

where

$$\omega = 1/\sqrt{LC}$$

LC Circuits

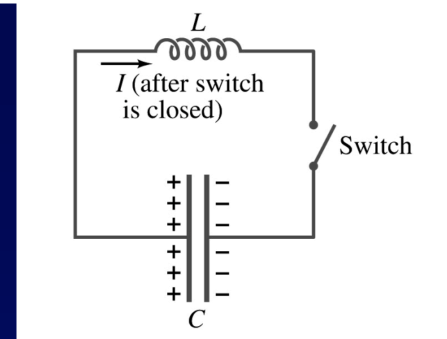
$$Q(t) = Q_0 \cos(\omega t + \phi)$$



$$I(t) = \frac{dQ}{dt} = -Q_0 \omega \sin(\omega t + \phi)$$

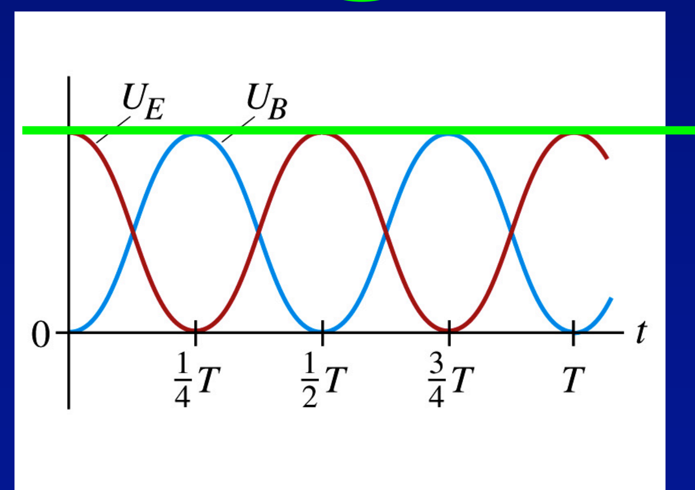
Charge oscillates! So does the current and voltage.

What about energy?



$$U_E = \frac{1}{2}\frac{Q^2}{C} = \frac{Q_0^2}{2C}\cos^2(\omega t + \phi)$$

$$U_B = \frac{1}{2}LI^2 = \frac{Q_0^2}{2C}\sin^2(\omega t + \phi)$$



The total energy is conserved.

A natural oscillator in the circuit

Inductor-Capacitor Circuit

Magnetic energy $= \frac{1}{2}Li^2$

Electric energy $= q^2/2C$

$$\tfrac{1}{2}Li^2 + q^2/2C = Q^2/2C$$

$$i = \pm\sqrt{1/LC}\sqrt{Q^2 - q^2}$$

$$i = dq/dt$$

$$\omega = \sqrt{\frac{1}{LC}}$$

$$q = Q\cos(\omega t + \phi)$$

Mass-Spring System

Kinetic energy $= \frac{1}{2}mv_x^2$

Potential energy $= \frac{1}{2}kx^2$

$$\tfrac{1}{2}mv_x^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2$$

$$v_x = \pm\sqrt{k/m}\sqrt{A^2 - x^2}$$

$$v_x = dx/dt$$

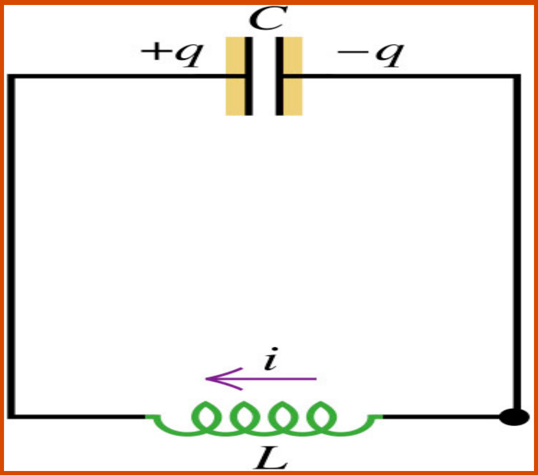
$$\omega = \sqrt{\frac{k}{m}}$$

$$x = A\cos(\omega t + \phi)$$

ConcepTest 30.5 Inductance

- **An inductor (inductance L) and a capacitor (capacitance C) are connected as shown.**
- **If the values of both L and C are doubled, what happens to the time required for the capacitor charge to oscillate through a complete cycle?**

1) it becomes 4 times longer
2) is becomes twice as long
3) it is unchanged
4) it becomes 1/2 as long
5) it becomes 1/4 as long

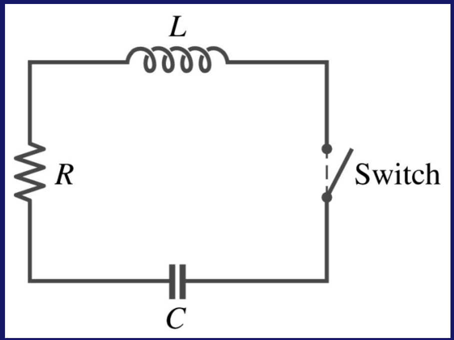


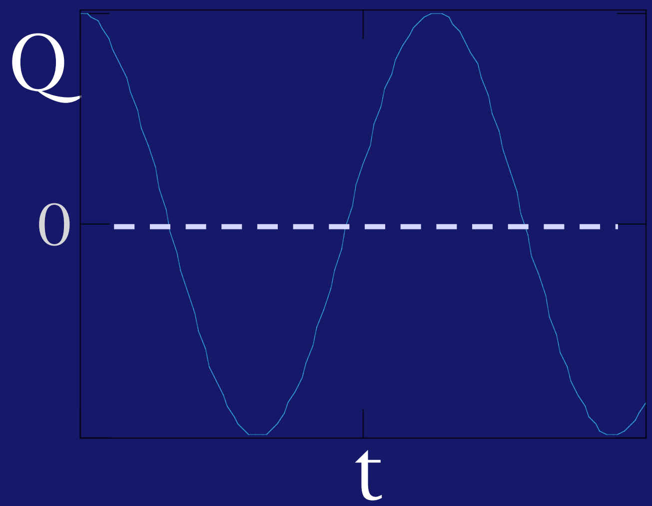
LRC Circuit

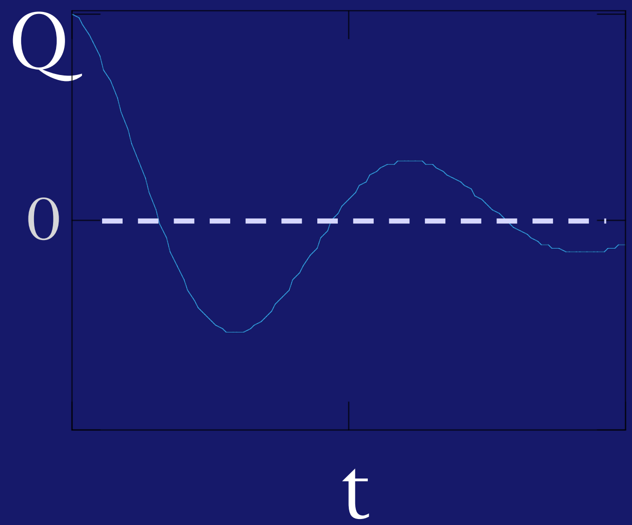
New Topic

LC Oscillations with Resistance

- **If L has finite Resistance, then**
  - **energy will be dissipated in R and**
  - **the oscillations will become damped.**







R = 0

R ≠ 0

Real circuits always have resistance R.

LRC Circuits

Damped oscillations! 3 scenarios.

A) Under-damped if $R^2<4L/C$.

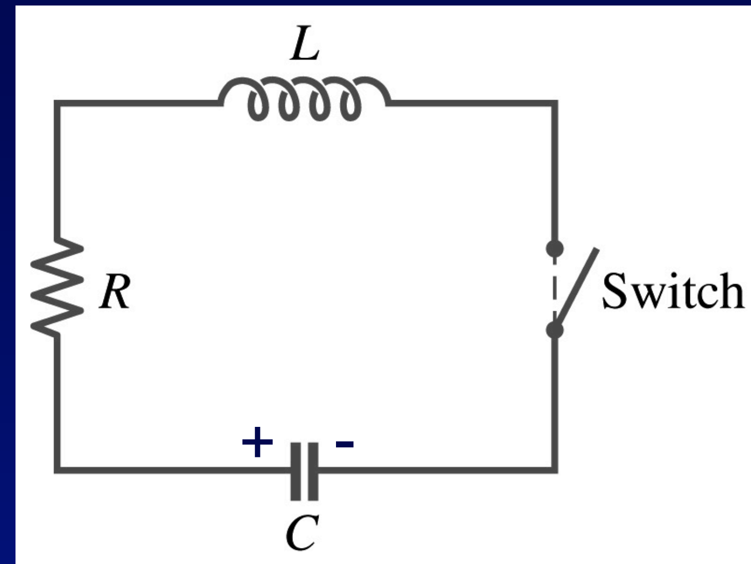
B) Critical damping if $R^2=4L/C$.

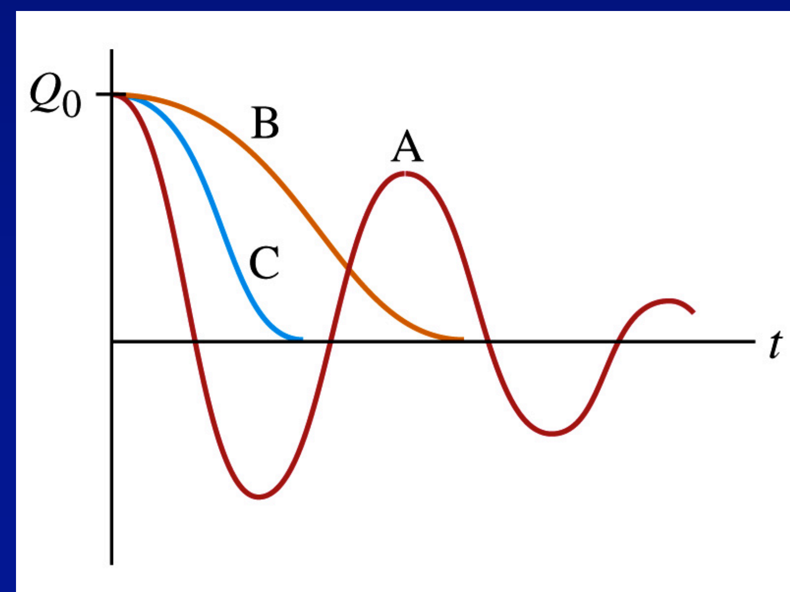
C) Over-damped if $R^2>4L/C$.

For under-damping:

$$Q(t) = Q_0 e^{-\frac{R}{2L}t} \cos(\omega' t + \phi)$$

$$T = \frac{2\pi}{\omega'} = \frac{2\pi\sqrt{LC}}{\sqrt{1-\frac{R^2C}{4L}}}$$





Summary of Various Direct Current Circuits

RC circuit, time constant $\tau = RC$ (transient)

LR circuit, time constant $\tau = L/R$ (transient)

LC circuit, oscillation period $$T = 2\pi\sqrt{LC}$$ (oscillator)

LRC circuit, damped oscillation period $$T = 2\pi\sqrt{LC} / \sqrt{1 - \frac{R^2 C}{4L}}$$

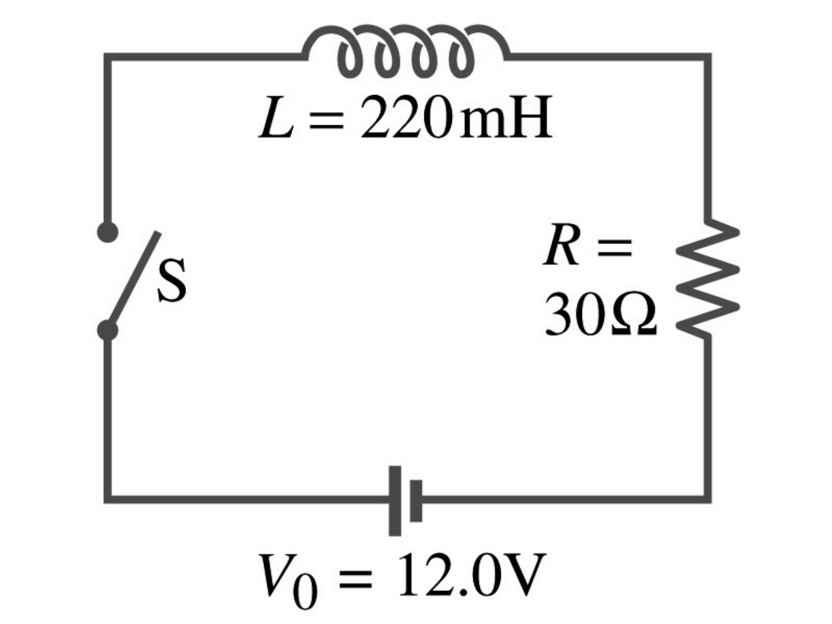
(damped oscillator)

We will discuss what happens if we use a AC-source to replace the DC-source (Chapter 31) in these circuits.

Example: LR circuit

At t=0, the circuit is closed.

a) What is the current at t=0?
b) What is the time constant?
c) What is the maximum current?
d) How long will it take the current to reach half of its maximum value?
e) At this instant in d), at what rate the energy is being delivered by the battery?
f) At what rate is energy is being stored in the inductor?



Example: LC circuit

A 1200-pF capacitor is fully charged by a 500-V dc power supply. It is disconnected from the power supply and is connected, at t=0, to a 75-mH inductor. Determine

a) the initial charge on the capacitor
b) the frequency of oscillation
c) the maximum current
d) the total energy oscillating in the circuit