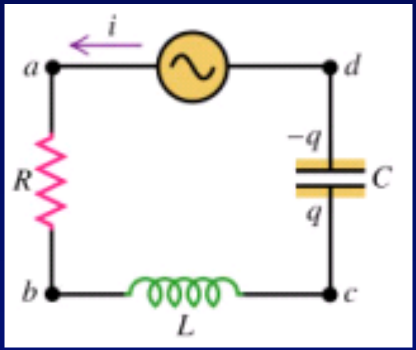
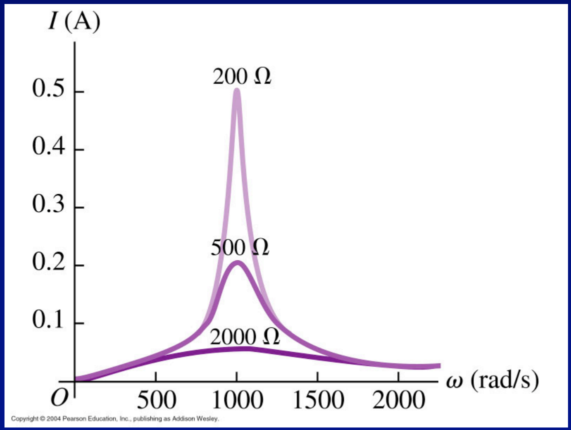
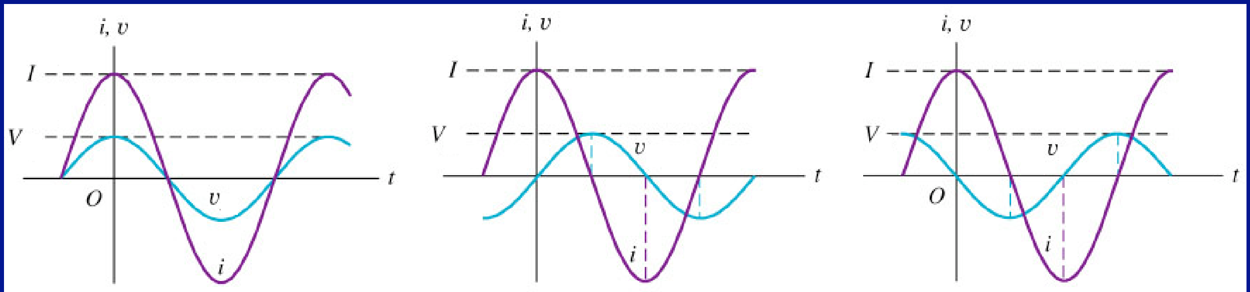
Chapter 31
Alternating Current (AC) Circuits



- AC Circuits
  - AC Circuit with only R
  - AC circuit with only L
  - AC circuit with only C
  - AC circuit with LRC
  - phasors
- Resonance
- Transformers

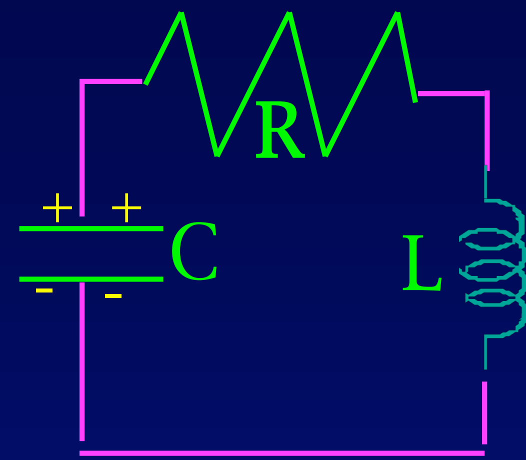




AC Circuits

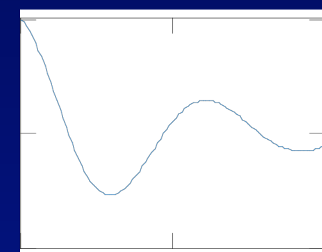
New Topic

Physics Motivation



- **Last time we discovered that a LC circuit was a natural oscillator.**

- **However, any real attempt to construct a LC circuit must account for the resistance of the inductor. This resistance will cause oscillations to damp out.**



- **Question: Is there any way to modify the circuit above to sustain the oscillations without damping?**

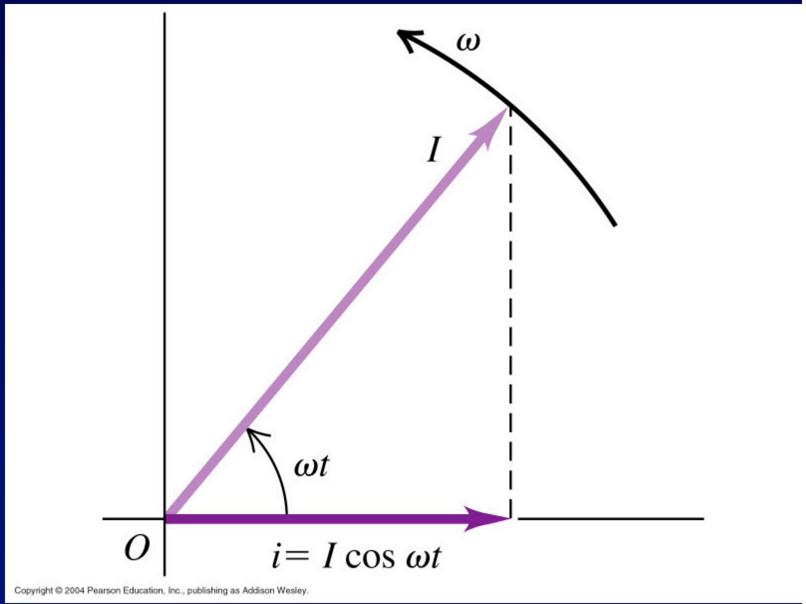
- Answer: Yes, if we can supply energy at the rate the resistor dissipates it! How? A sinusoidally varying emf (AC generator) will sustain sinusoidal current oscillations! $i(t)=I\cos(\omega t)$ or $v(t)=V\cos(\omega t)$

- But first, let us look at R, L, C separately before putting them together.

Phasor Diagram

How to graphically represent an AC source?

$$i = I\cos\omega t$$



- A phasor is a vector in the x-y plane, rotating counter-clockwise with angular velocity ω.
  - Magnitude is the peak value
  - Projection on the x-axis is the value at a given time

Same for emf:

$$v = V\cos\omega t$$

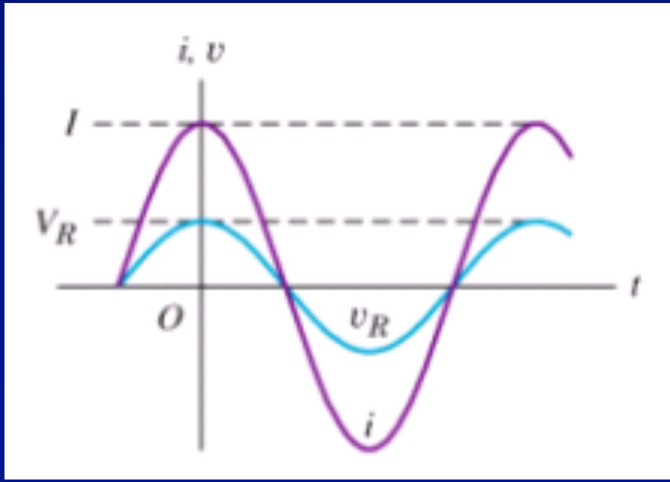
RMS current and voltage

$$i = I\cos\omega t$$

root-mean-square: $$I_{rms} = \sqrt{\overline{i^2}}$$

$$\overline{i^2} = \frac{1}{T}\int_0^T i^2 dt = \frac{I^2}{2T}\int_0^T (1+\cos 2\omega t)dt = \frac{I^2}{2}$$

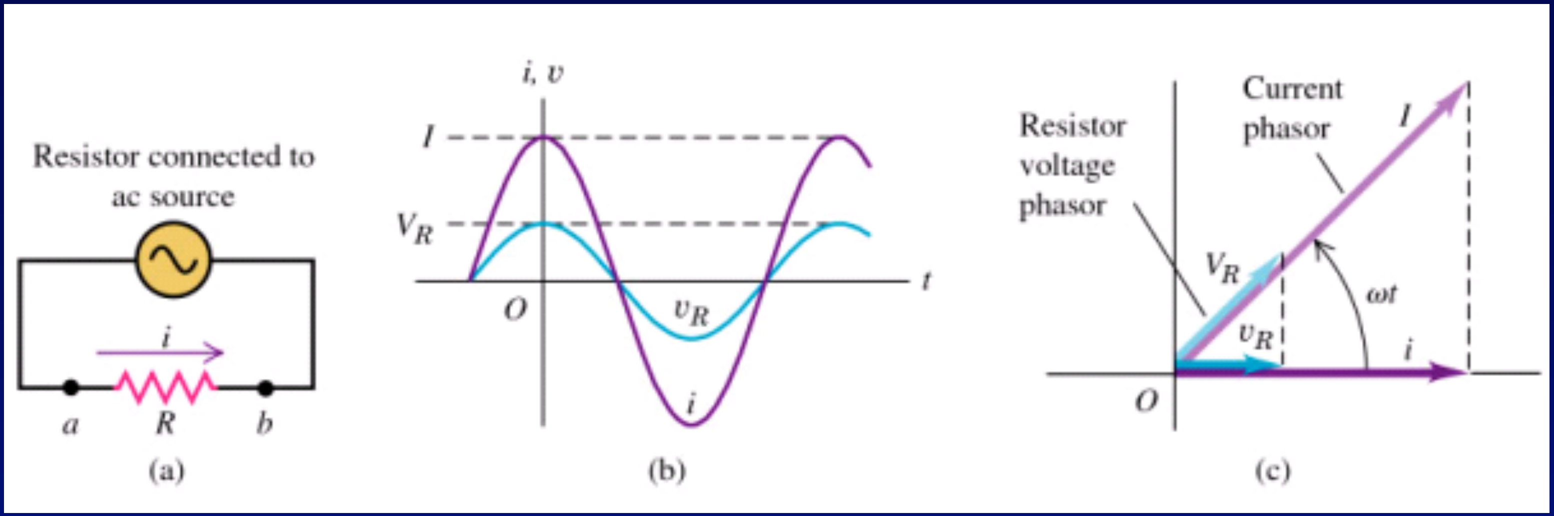
$$I_{rms} = \frac{I}{\sqrt{2}}$$ $$V_{rms} = \frac{V}{\sqrt{2}}$$



In the US, the 120-V AC voltage supplied to household refers to the rms value. What is the peak voltage?

170 V

AC Circuit Containing only Resistance



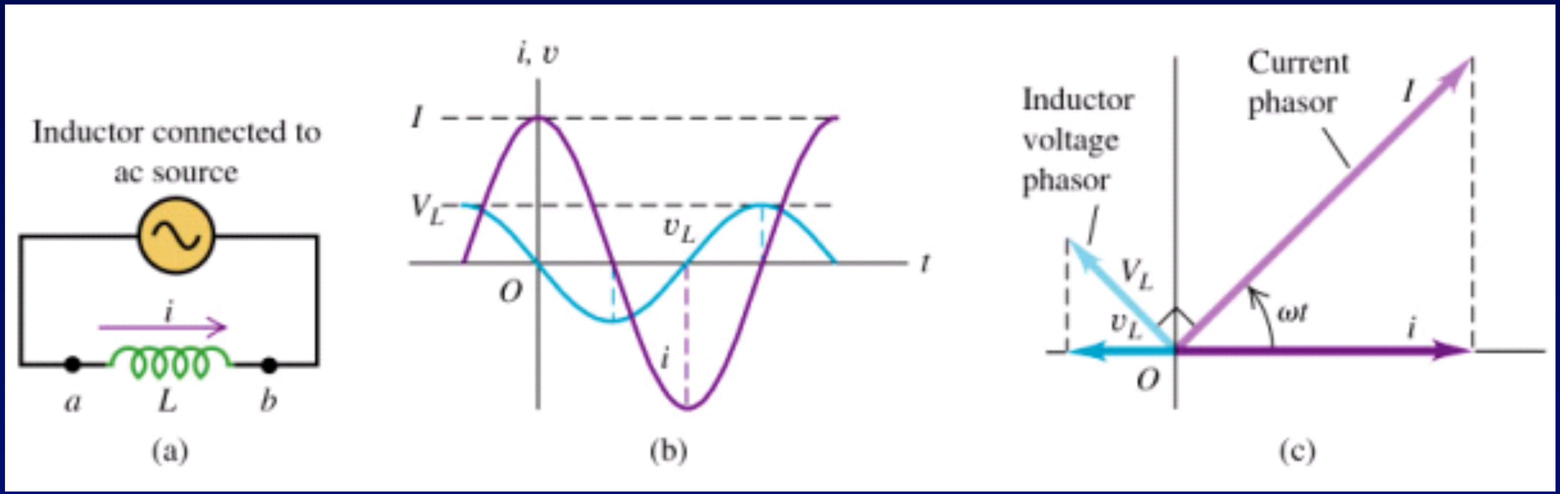
The AC source is given as $i = I\cos\omega t$

So the voltage across the resistor is $v_R = iR = IR\cos\omega t = V_R\cos\omega t$

The resistor voltage is in phase with the current.

The power dissipated in the resistor is p=iv, or at an average rate $\overline{P} = I_{rms}^2 R = \dfrac{V_{rms}^2}{R}$

AC Circuit Containing only Inductor



The AC source is given $i = I\cos\omega t$

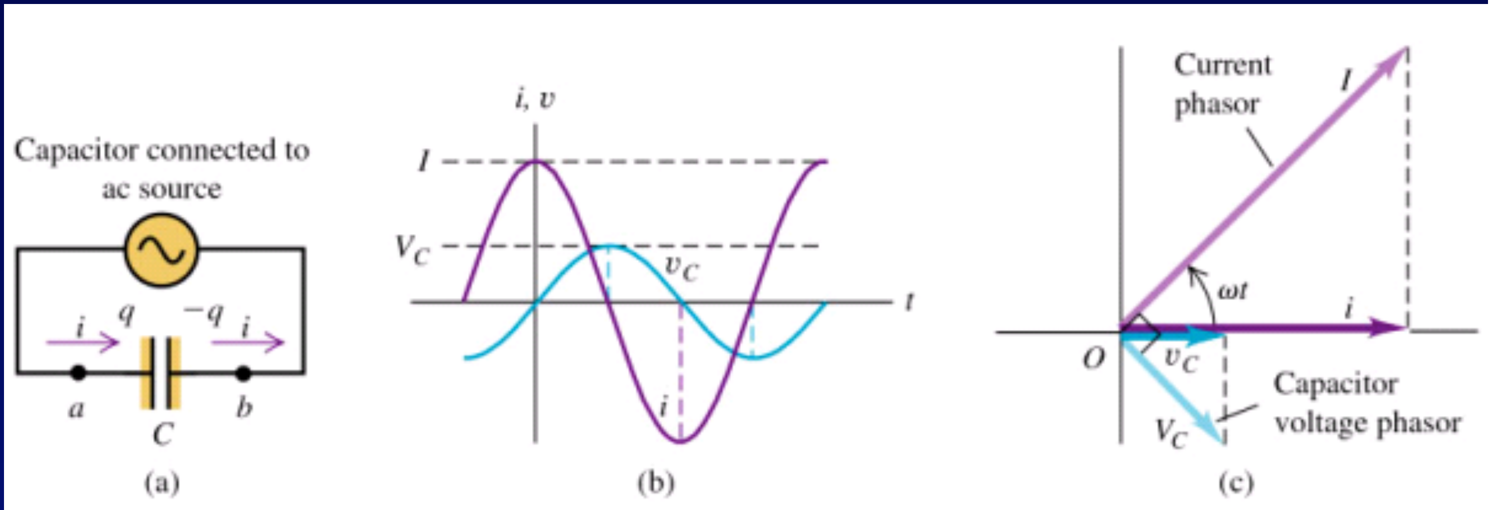
So the voltage across the inductor is $$v_L = L\frac{di}{dt} = -IL\omega\sin\omega t = V_L\cos(\omega t + 90^0)$$

The inductor voltage leads the current by 90 degrees.

Define reactance of inductor $X_L=\omega L$, then $V_L = I X_L$.
Its unit is Ohm. For example, a 0.3-H inductor connected to a 60-Hz ac-source has a reactance of $X_L=2\pi x60x0.3=113\ \Omega$.

AC Circuit Containing only Capacitor

$$i = I\cos\omega t$$



The voltage across the capacitor is $v = q/C$

$$q = \int_0^t i\,dt = \frac{I}{\omega}\sin\omega t = \frac{I}{\omega}\cos(\omega t - 90^0)$$

$$v_C = \frac{I}{\omega C}\cos(\omega t - 90^0)$$

The capacitor voltage lags the current by 90 degrees.

Define reactance of capacitor $X_C=1/\omega C$, then $V_C = I\,X_C$. Its unit is Ohm. For example, a 1-μF capacitor connected to a 60-Hz ac-source has a reactance of $X_C=1/(2\pi x60x10^{-6})=2.7$ kΩ.

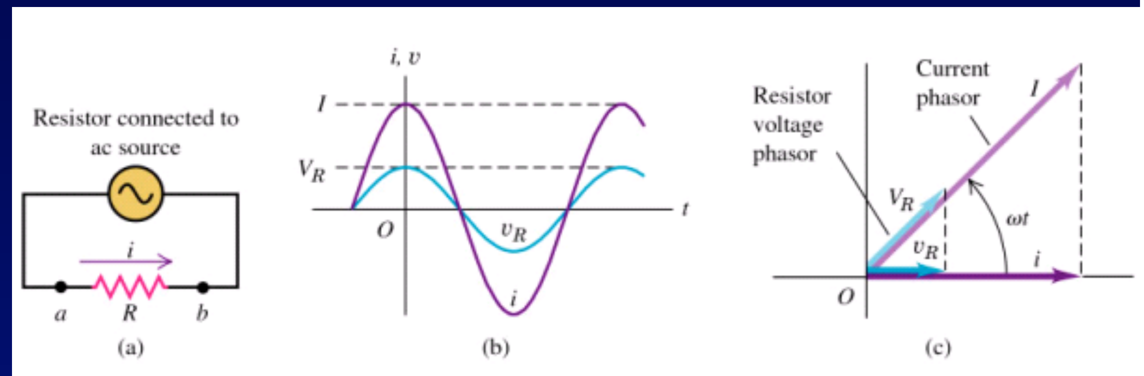
Summary: AC source

$$i = I\cos\omega t$$

Define $V = IX$

$$v_R = IR\cos\omega t$$

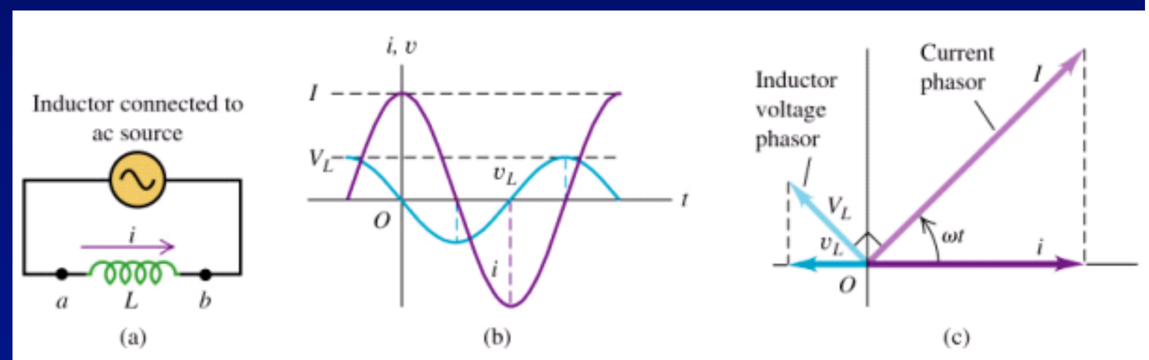
resistance $X_R = R$.



$$v_L = IL\omega\cos(\omega t + 90^0)$$

inductive reactance

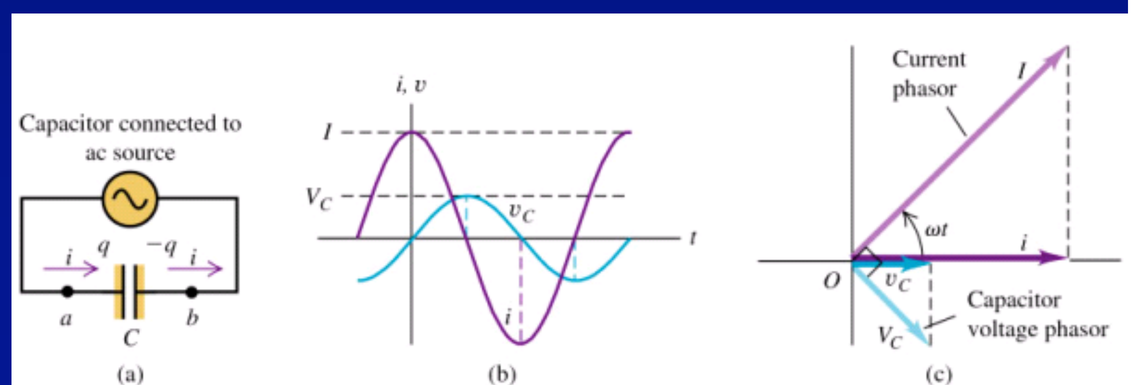
$X_L = L\omega$.



$$v_C = \frac{I}{\omega C}\cos(\omega t - 90^0)$$

capacitive reactance

$X_C = 1/(\omega C)$.

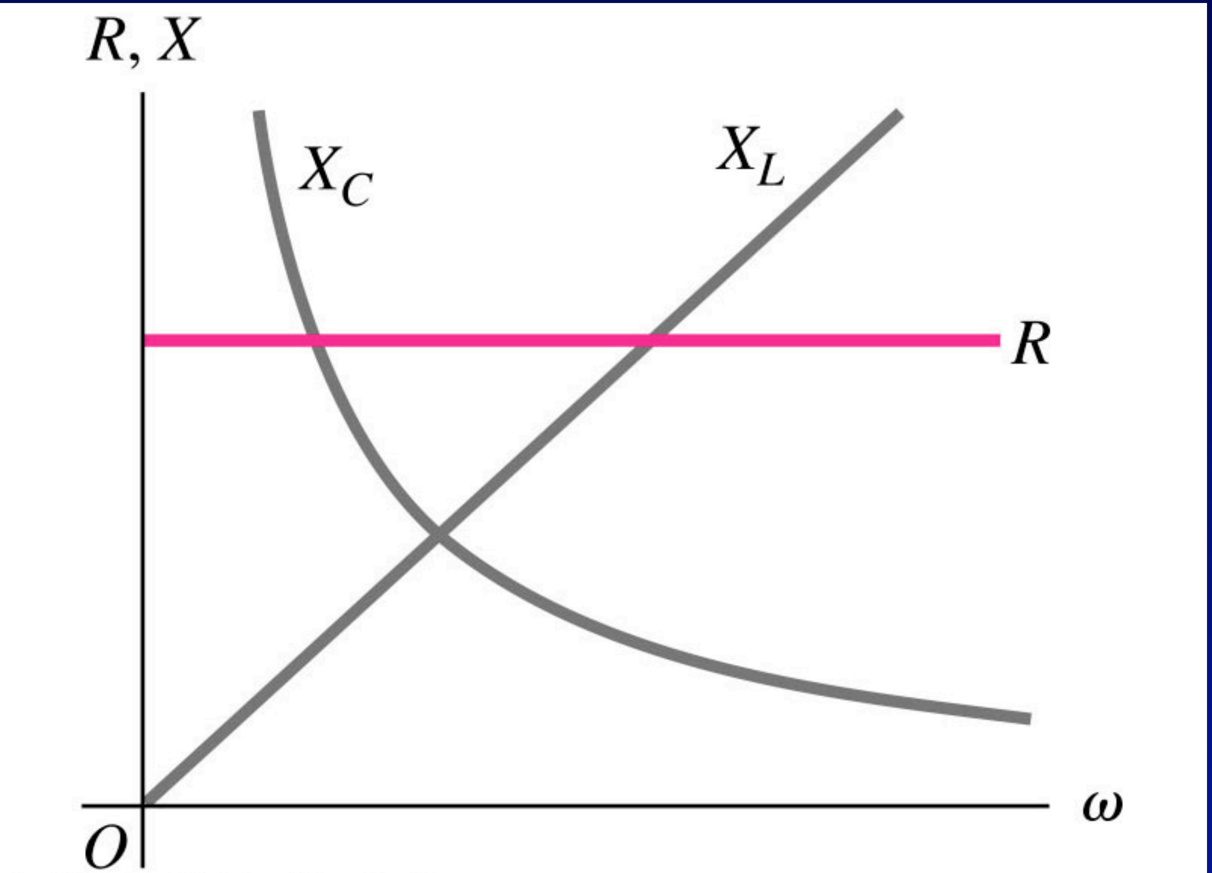


Frequency Dependence

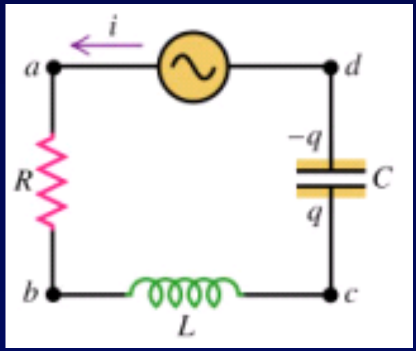
resistance $X_R=R$.

inductive reactance
$X_L=\omega L$.

capacitive reactance
$X_C=1/(\omega C)$.

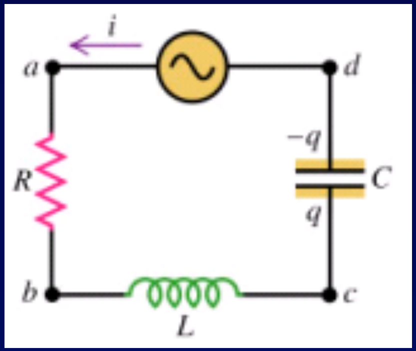


AC-driven LRC Circuit in Series



- Given ac-source $i=I\cos(\omega t)$
  - Let v_R, v_L, v_C represent the voltages at a given time.
- Physics questions:
  - What is the impedance Z of the entire circuit as defined by the peak values $V=IZ$?
  - Does the source voltage lead or lag the current? By how much?
- In other words: If one can write $v= v_R+v_L+v_C = V\cos(\omega t+\varphi)$, then what is V and φ ?
- Or: How to add the voltages $v= v_R+v_L+v_C$, noting that they are varying with time and in general out of phase with each other?

AC-driven LRC Circuit in Series
(Graphical Method)

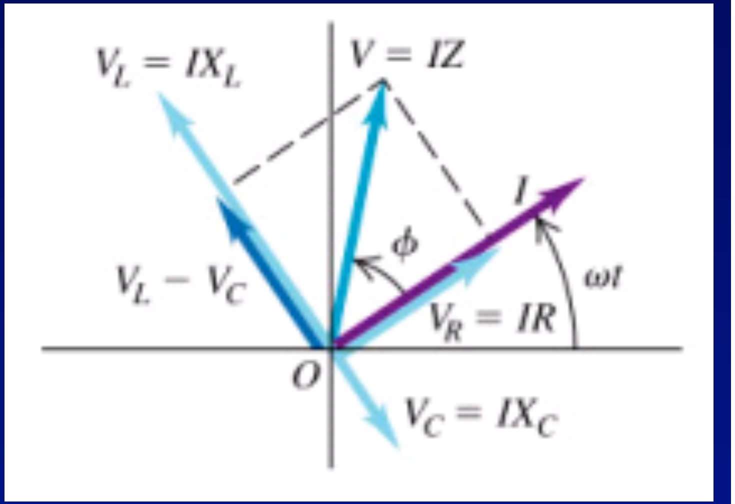


The goal is to cast the total voltage at any time into the form

$$i = I\cos\omega t$$

$$v = v_R + v_L + v_C = V\cos(\omega t + \phi)$$

where ϕ is the phase angle with which v leads i.



From the triangle:

$$V = \sqrt{V_R^2 + (V_L - V_C)^2}$$
$$= \sqrt{(IR)^2 + (IX_L - IX_C)^2}$$
$$= I\sqrt{R^2 + (X_L - X_C)^2}$$

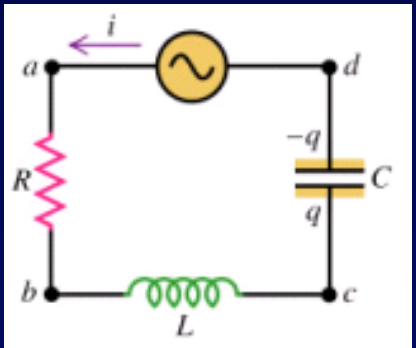
The impedance

$$V = IZ \text{ with } Z = \sqrt{R^2 + (X_L - X_C)^2}$$

The phase difference

$$\tan\varphi = \frac{X_L - X_C}{R} \quad \text{or } \cos\varphi = \frac{R}{Z}$$

AC-driven LRC Circuit in Series (Trigonometry Method)



Given: $i = I\cos\omega t$ and

$$v_R = iR = IR\cos\omega t$$

$$v_L = IX_L\cos(\omega t + \pi/2) \text{ with } X_L = L\omega$$

$$v_C = IX_C\cos(\omega t - \pi/2) \text{ with } X_C = \frac{1}{\omega C}$$

Find the sum:

$$\begin{aligned} v &= v_R + v_L + v_C \\ &= IR\cos(\omega t) - I(X_L - X_C)\sin(\omega t) \\ &= V\cos(\omega t + \varphi) \text{ (desired form)} \\ &= V\cos\varphi\cos(\omega t) - V\sin\varphi\sin(\omega t) \end{aligned}$$

Used: $\cos(A+B) = \cos A\cos B - \sin A\sin B$

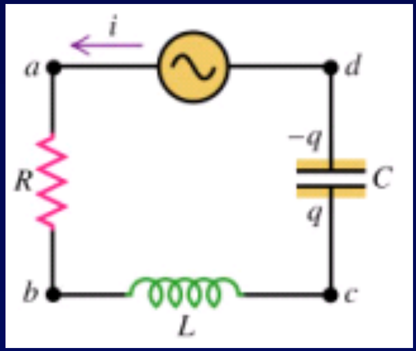
Solution:

$$V\cos\varphi = IR$$
$$V\sin\varphi = I(X_L - X_C)$$

$$V = IZ \text{ with } Z = \sqrt{R^2 + (X_L - X_C)^2}$$

$$\tan\varphi = \frac{X_L - X_C}{R} \text{ or } \cos\varphi = \frac{R}{Z}$$

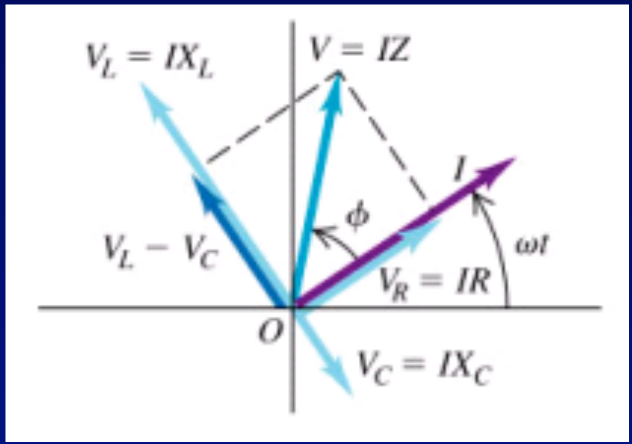
The sign of the phase difference ϕ



The total voltage at time t is $i = I\cos\omega t$

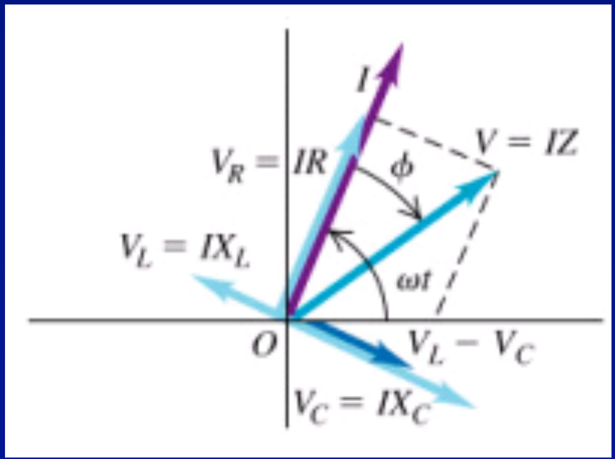
$$v = V\cos(\omega t + \phi)$$

where ϕ is the phase angle by which voltage leads the current.



The figure shows the case $X_L > X_C$ which means ϕ is positive. Or the voltage leads the current.

If $X_L < X_C$, then ϕ is negative. Or the voltage lags the current. The figure is



It can be seen more easily from $\tan\phi$.

$$\tan\phi = \frac{X_L - X_C}{R}$$

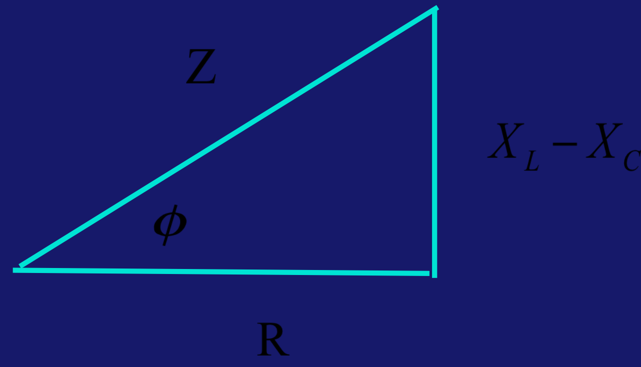
The Impedance Triangle

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

$$X_L = \omega L$$

$$X_C = \frac{1}{\omega C}$$

$$\tan\phi = \frac{X_L - X_C}{R} \quad \text{or} \cos\phi = \frac{R}{Z}$$



General results, valid for any combination of L, R, C in series. For example, if LR circuit, set X_C=0.

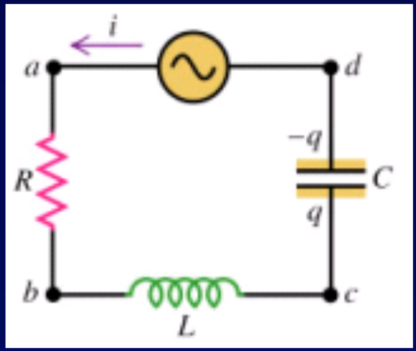
Check special cases:

$$R \text{ only}: \mathrm{X_L} = 0, \mathrm{X_C} = 0, \text{so } \mathrm{Z} = \mathrm{R}, \phi = 0$$

$$L \text{ only}: \mathrm{R} = 0, \mathrm{X_C} = 0, \text{so } \mathrm{Z} = \mathrm{X}_L, \phi = +90^0$$

$$C \text{ only}: \mathrm{R} = 0, \mathrm{X_L} = 0, \text{so } \mathrm{Z} = \mathrm{X}_C, \phi = -90^0$$

Who is driving?



We derived the results with a driving current

$$i = I\cos\omega t$$

$$v = V\cos(\omega t + \phi)$$

ϕ is the angle by which voltage leads the current.

The same results for Z and ϕ apply if we have a driving emf

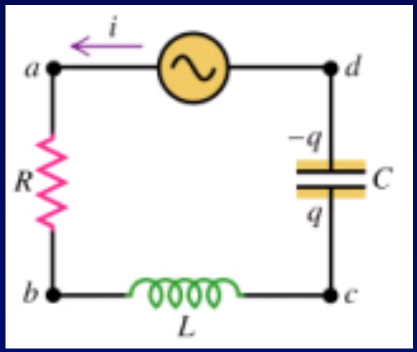
$$v = V\cos\omega t$$

$$i = I\cos(\omega t - \phi)$$

ϕ is the same angle by which current lags the voltage.

It's all relative: the physics is the same.

Frequency Dependence



$$i = I\cos\omega t$$

$$v = V\cos(\omega t + \phi)$$

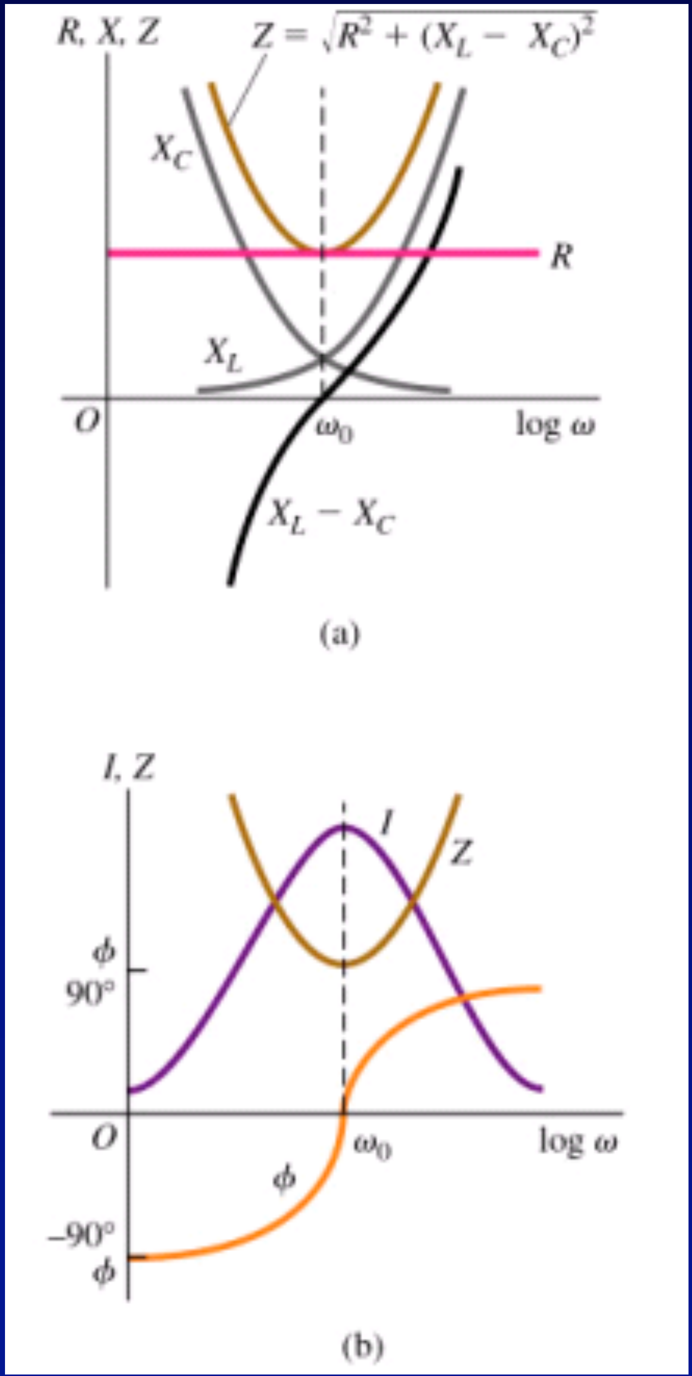
$$V = IZ$$

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

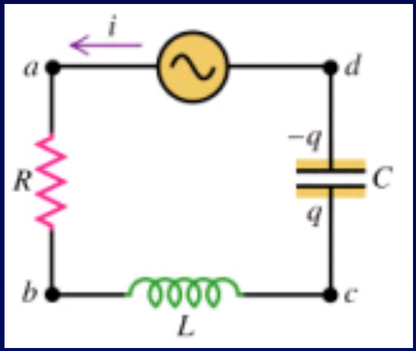
$$X_L = \omega L$$

$$X_C = \frac{1}{\omega C}$$

$$\tan\varphi = \frac{X_L - X_C}{R} \quad \text{or } \cos\varphi = \frac{R}{Z}$$



The Power Factor



The power is only dissipated in the resistor, so the average power is $\overline{P} = I_{rms}^2 R$

But $R = Z\cos\phi$

$$\overline{P} = I_{rms}^2 Z\cos\phi = I_{rms} V_{rms}\cos\phi = \frac{1}{2} IV\cos\phi$$

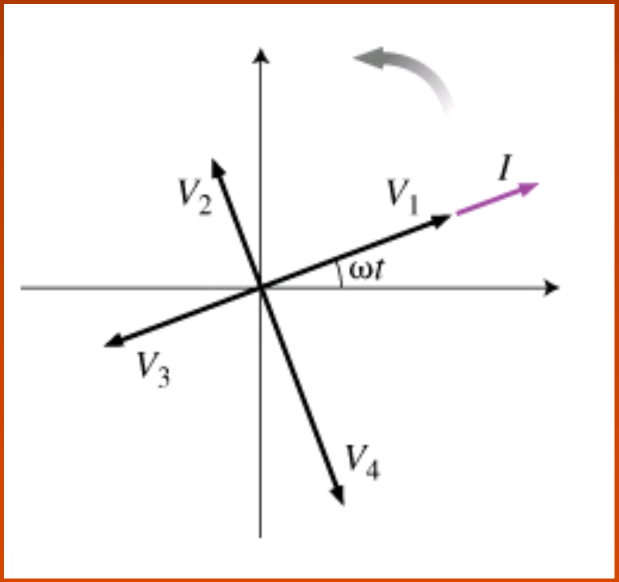
The factor cosϕ is called the power factor.

For example:
For a pure resistor (ϕ=0) , cosϕ=1.
For a pure inductor (ϕ=90^0) or capacitor (ϕ=-90^0), cosϕ=0.

ConcepTest 31.1 Phasor Diagram

- Consider this phasor diagram. Let us assume that it describes a series circuit containing a resistor, a capacitor, and an inductor. The current in the circuit has amplitude I, as indicated in the figure. Which of the following choices gives the correct respective labels of the voltages across the resistor, the capacitor, and the inductor ?

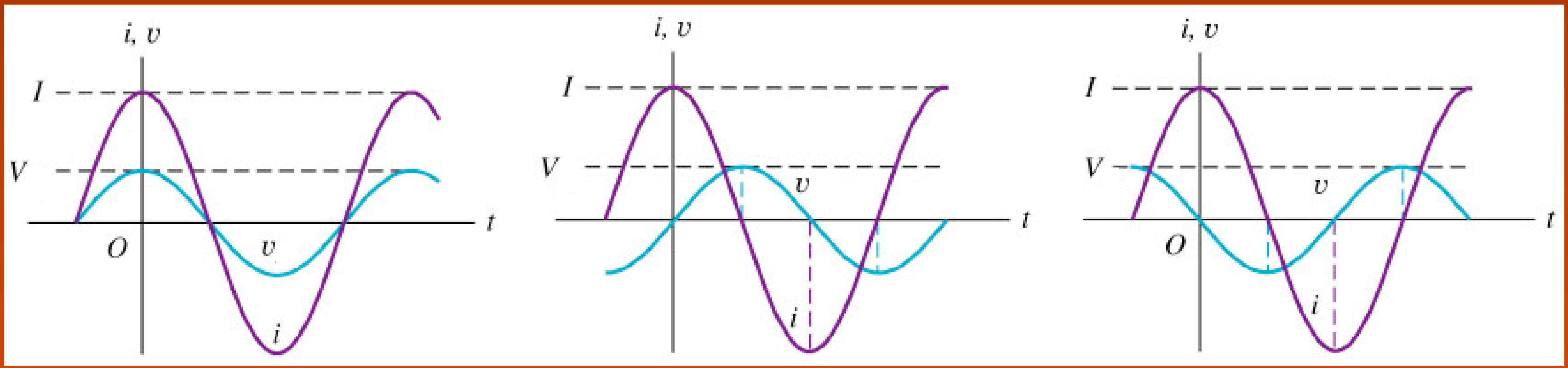
1) V_1; V_2; V_3
2) V_1; V_2; V_4
3) V_1; V_4; V_2
4) V_3; V_2; V_4
5) V_1; V_2; V_3
6) V_3; V_4; V_2



ConcepTest 31.2 AC Circuit

- An AC-current source (purple line) is connected in series to a resistor, an inductor, and a capacitor. The voltage across each element (blue line) is given separately in the pictures shown below. Give the order in which the pictures correspond to the inductor, resistor, and capacitor, respectively.

1) ABC
2) BCA
3) CAB
4) ACB
5) BAC
6) CBA

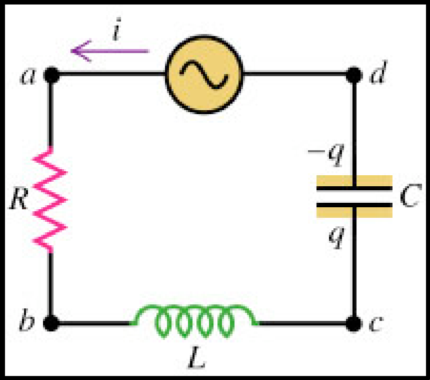


(A) (B) (C)

ConcepTest 31.3 LRC Circuit

In an *L-R-C* series circuit as shown, the current has a very small amplitude if the emf oscillates at a very high frequency. Which circuit element causes this behavior?

1) the resistor *R*
2) the inductor *L*
3) the capacitor *C*
4) misleading question — the current actually has a very *large* amplitude if the frequency is very high



Resonance in AC-circuit

New Topic

Resonance in AC Circuit

Recall:

$$I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + (X_L - X_C)^2}}$$

- **For fixed R,C,L the current I will be a maximum at the resonant frequency ω_0 which makes the impedance Z purely resistive.**

Z is minimum when:

$$\boxed{X_L = X_C}$$

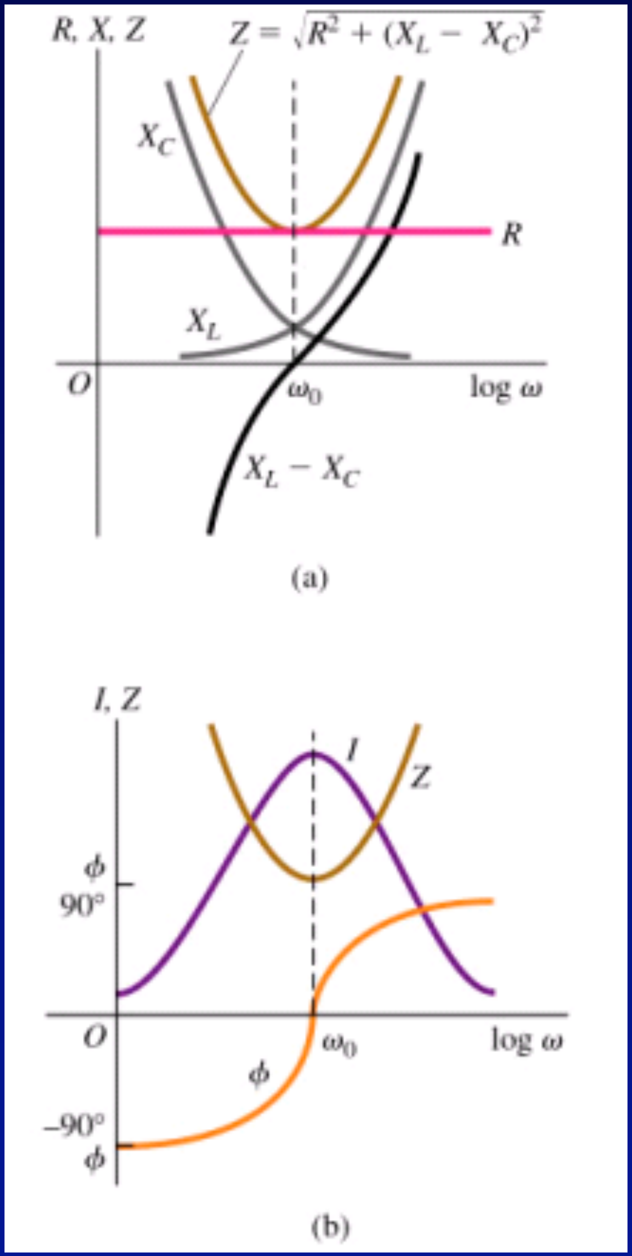
So the frequency at which this condition is satisfied is given from:

$$\omega_o L = \frac{1}{\omega_o C} \quad \Rightarrow \quad \boxed{\omega_o = \frac{1}{\sqrt{LC}}}$$

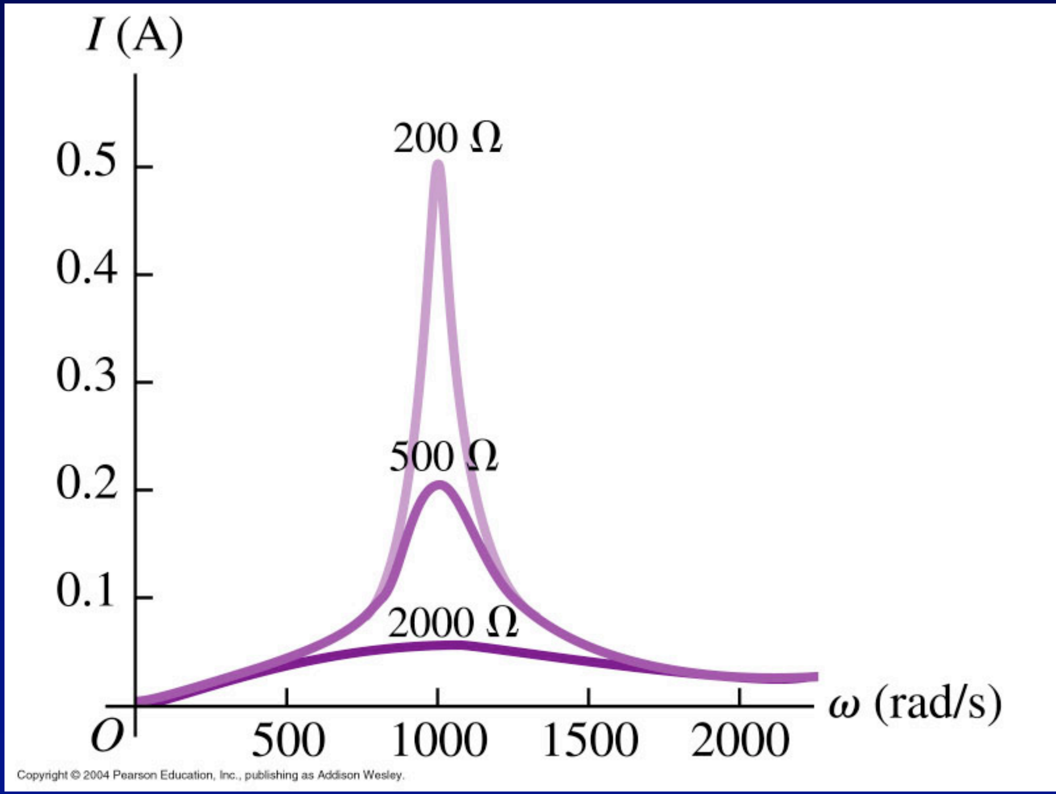
- **Note that this resonant frequency is identical to the natural frequency of the LC circuit by itself!**
- **At this frequency, the current and the driving voltage are in phase!**

$$\cos\phi = \frac{R}{Z} = 1, \text{ so } \phi = 0$$

Resonance in AC Circuit



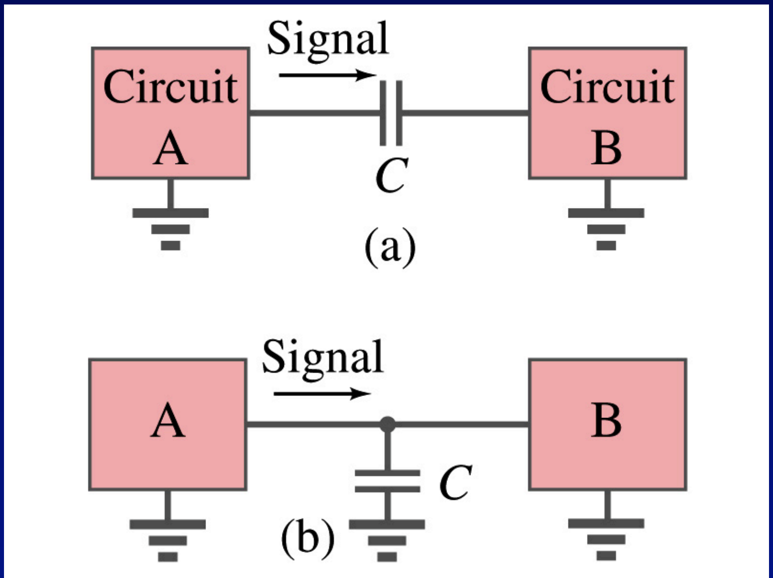
At resonance, I and V are in phase. V, I and P are at their maximum.



(V=100 V, L=2.0 H, C=0.5 μF)

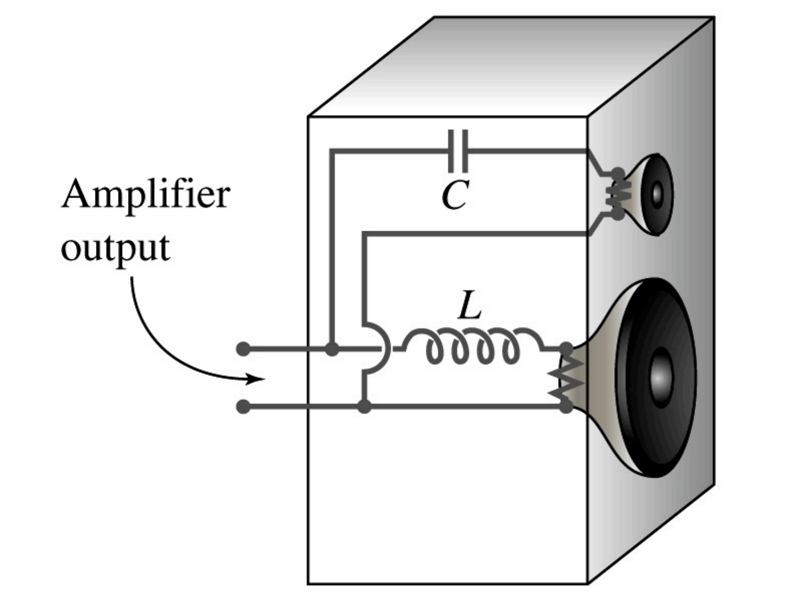
Applications of AC Circuits

Capacitor as filters



$$X_C = \frac{1}{2\pi f C}$$

Tweeter and woofer



$$X_L = 2\pi f L$$

Tuning in a Radio or TV station by resonance.

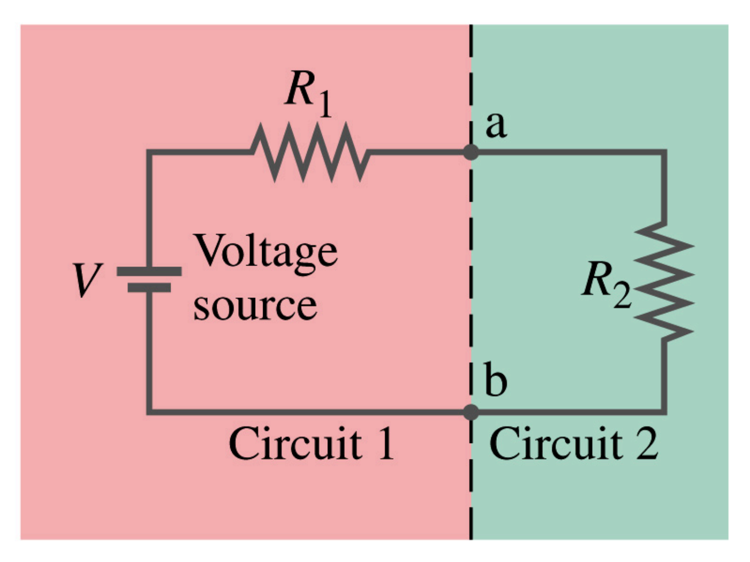
$$f = \frac{1}{2\pi\sqrt{LC}}$$

More about AC circuits

Impedance matching: when

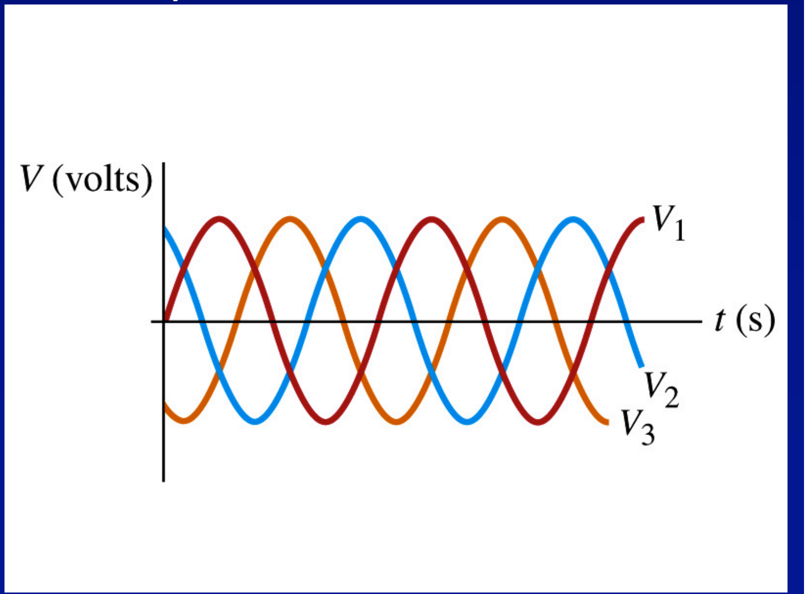
$$R_1 = R_2 \text{ (or } Z_1 = Z_2\text{)}$$

maximum power is delivered from circuit 1 to circuit 2.



Three-phase AC (4-wire transmission lines)

$$V_1 = V_0 \sin\omega t$$
$$V_2 = V_0 \sin(\omega t + 2\pi/3)$$
$$V_3 = V_0 \sin(\omega t + 4\pi/3)$$



1) More power delivered (3 times)
2) Smoother flow.

Transformers

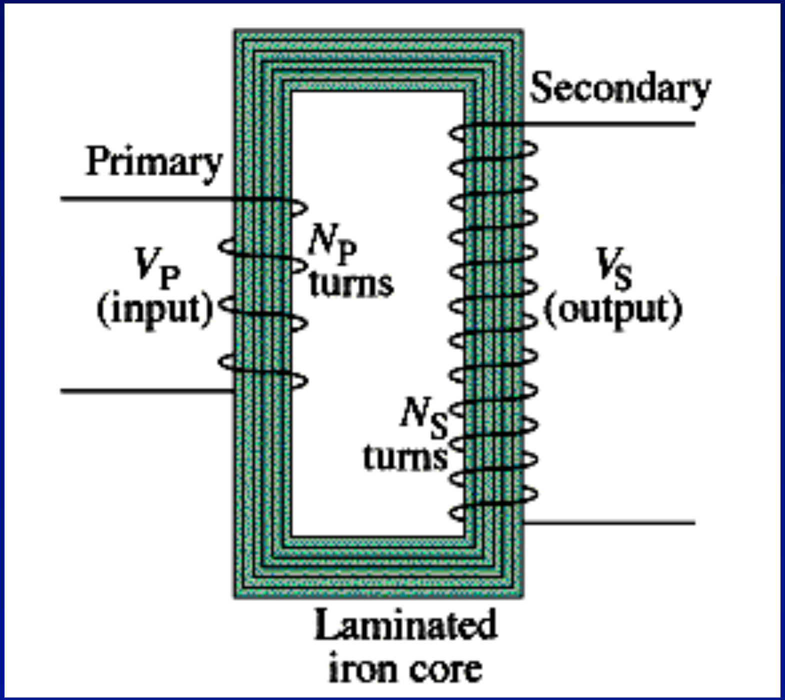
New Topic

Transformers

Transformers change alternating (AC) voltage to a bigger or smaller value

Input AC voltage V_p in the primary produces a flux

$$V_p = N_p \frac{\Delta\Phi_B}{\Delta t}$$



Changing flux in secondary induces emf V_s

$$V_s = N_s \frac{\Delta\Phi_B}{\Delta t}$$

Same $\Delta\Phi / \Delta t$!!

Transformer equation:

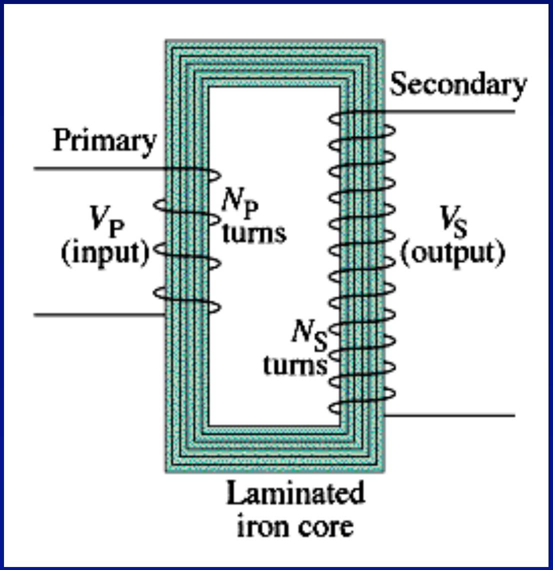
$$V_s = V_p \frac{N_s}{N_p}$$

Transformers

- Nothing comes for free, however!
  - voltage increase comes at the cost of current
  - output power cannot exceed input power
  - power in = power out (assume no heat loss)

$$\boxed{I_p V_p = I_s V_s}$$

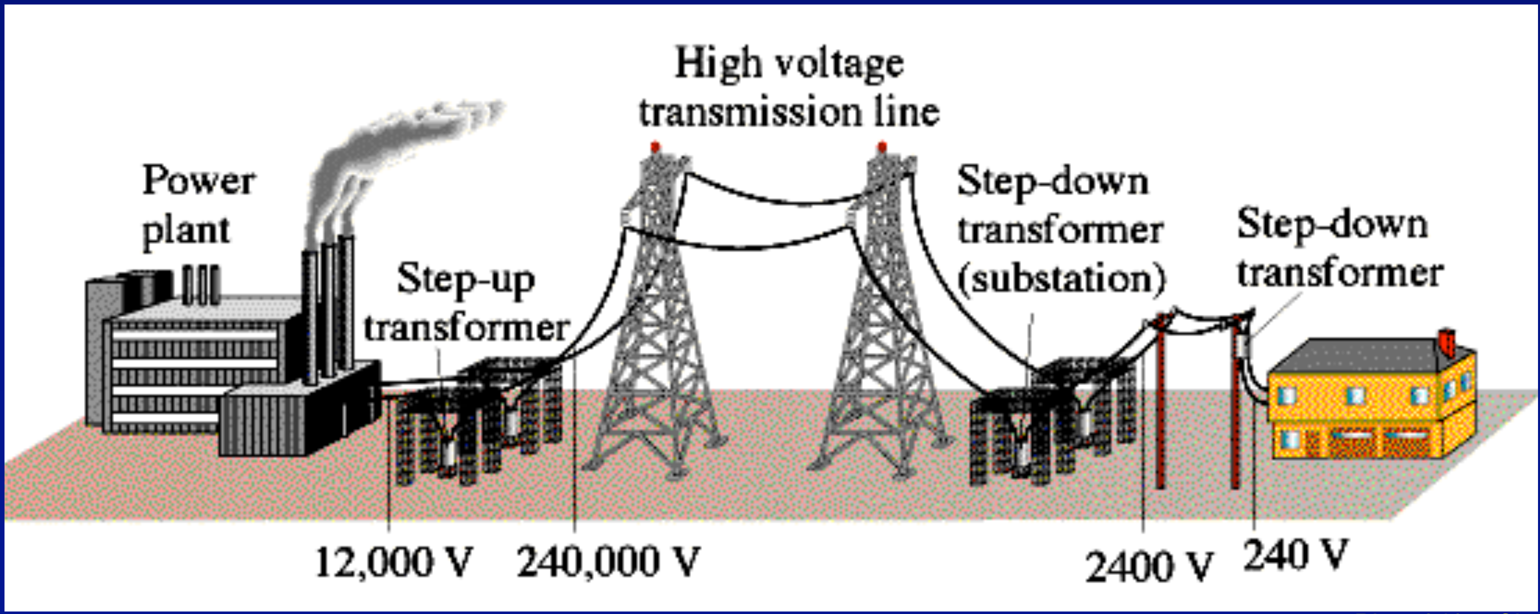
$$\frac{I_s}{I_p} = \frac{V_p}{V_s} = \frac{N_p}{N_s}$$



If voltage increases, then current decreases
If voltage decreases, then current increases

Example: Transmission Lines

- An average of **120 kW** of electric power is sent to a small town from a power plant 10 km away. The transmission lines have a total resistance of **0.4 Ω**. Calculate how much power is saved if the voltage is stepped up from **240 V** to **24,000 V** and then down to **240 V** again, rather than simply transmitting at 240 V.



ConcepTest 31.4 Transformers

- What is the voltage across the light bulb?

(1) 30 V

(2) 60 V

(3) 120 V

(4) 240 V

(5) 480 V