Mathematics III Chapter

of the

Mathematics Framework

for California Public Schools: Kindergarten Through Grade Twelve

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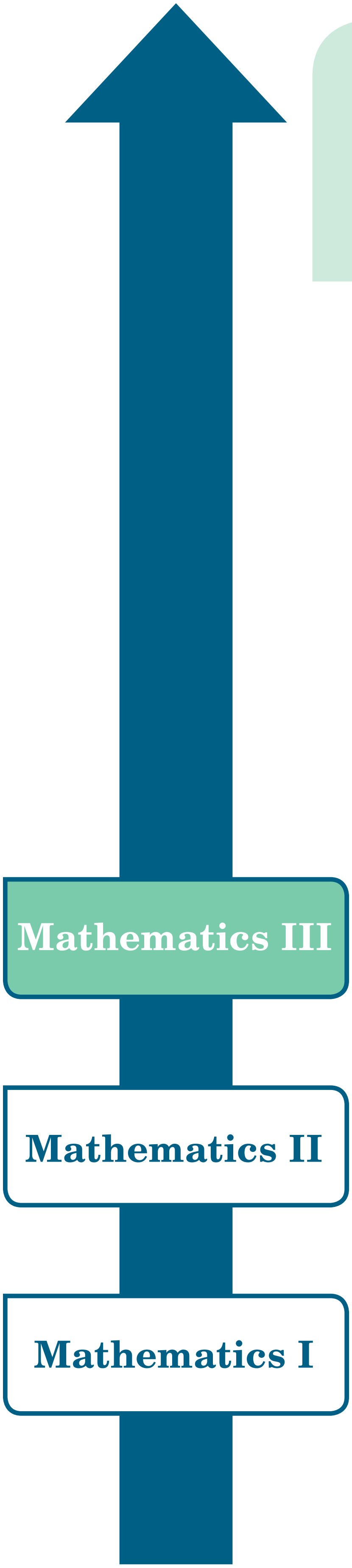
Mathematics III



In the Mathematics III course, students expand their repertoire of functions to include polynomial, rational, and radical functions. They also expand their study of right-triangle trigonometry to include general triangles. And, finally, students bring together all of their experience with functions and geometry to create models and solve contextual problems. The courses in the Integrated Pathway follow the structure introduced in the K–8 grade levels of the California Common Core State Standards for Mathematics (CA CCSSM); they present mathematics as a coherent subject and blend standards from different conceptual categories.

The standards in the integrated Mathematics III course come from the following conceptual categories: Modeling, Functions, Number and Quantity, Algebra, Geometry, and Statistics and Probability. The course content is explained below according to these conceptual categories, but teachers and administrators alike should note that the standards are not listed here in the order in which they should be taught. Moreover, the standards are not topics to be checked off after being covered in isolated units of instruction; rather, they provide content to be developed throughout the school year through rich instructional experiences.

What Students Learn in Mathematics III

In Mathematics III, students understand the structural similarities between the system of polynomials and the system of integers. Students draw on analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property. They connect multiplication of polynomials with multiplication of multi-digit integers and division of polynomials with long division of integers. Students identify zeros of polynomials and make connections between zeros of polynomials and solutions of polynomial equations. Their work on polynomial expressions culminates with the Fundamental Theorem of Algebra. Rational numbers extend the arithmetic of integers by allowing division by all numbers except 0. Similarly, rational expressions extend the arithmetic of polynomials by allowing division by all polynomials except the zero polynomial. A central theme of working with rational expressions is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers.

Students synthesize and generalize what they have learned about a variety of function families. They extend their work with exponential functions to include solving exponential equations with logarithms. They explore the effects of transformations on graphs of diverse functions, including functions arising in an application, in order to abstract the general principle that transformations on a graph always have the same effect, regardless of the type of the underlying functions.

Students develop the Laws of Sines and Cosines in order to find missing measures of general (not necessarily right) triangles. They are able to distinguish whether three given measures (angles or sides) define 0, 1, 2, or infinitely many triangles. This discussion of general triangles opens up the idea of trigonometry applied beyond the right triangle—that is, at least to obtuse angles. Students build on this idea to develop the notion of radian measure for angles and extend the domain of the trigonometric functions to all real numbers. They apply this knowledge to model simple periodic phenomena.

Students see how the visual displays and summary statistics they learned in previous grade levels or courses relate to different types of data and to probability distributions. They identify different ways of collecting data—including sample surveys, experiments, and simulations—and recognize the role that randomness and careful design play in the conclusions that may be drawn.

Finally, students in Mathematics III extend their understanding of modeling: they identify appropriate types of functions to model a situation, adjust parameters to improve the model, and compare models by analyzing appropriateness of fit and by making judgments about the domain over which a model is a good fit. The description of modeling as "the process of choosing and using mathematics and statistics to analyze empirical situations, to understand them better, and to make decisions" (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA/CCSSO] 2010e) is one of the main themes of this course. The discussion about modeling and the diagram of the modeling cycle that appear in this chapter should be considered when students apply knowledge of functions, statistics, and geometry in a modeling context.

Examples of Key Advances from Mathematics II

- Students begin to see polynomials as a system analogous to the integers that they can add, subtract, multiply, and so forth. Subsequently, polynomials can be extended to rational expressions, which are analogous to rational numbers.
- Students extend their knowledge of linear, exponential, and quadratic functions to include a much broader range of classes of functions.
- Students begin to examine the role of randomization in statistical design.

Connecting Mathematical Practices and Content

The Standards for Mathematical Practice (**MP**) apply throughout each course and, together with the Standards for Mathematical Content, prescribe that students experience mathematics as a coherent, relevant, and meaningful subject. The Standards for Mathematical Practice represent a picture of what it looks like for students to *do mathematics* and, to the extent possible, content instruction should include attention to appropriate practice standards. The Mathematics III course offers ample opportunities for students to engage with each MP standard; table M3-1 offers some examples.

Table M3-1. Standards for Mathematical Practice—Explanation and Examples for Mathematics III

Standards for Mathematical Practice	Explanation and Examples
MP.1 Make sense of problems and persevere in solving them.	Students apply their understanding of various functions to real-world problems. They approach complex mathematics problems and break them down into smaller problems, synthesizing the results when presenting solutions.
MP.2 Reason abstractly and quantitatively.	Students deepen their understanding of variables—for example, by understanding that changing the values of the parameters in the expression $A\sin(Bx+C)+D$ has consequences for the graph of the function. They interpret these parameters in a real-world context.
MP.3 Construct viable arguments and critique the reasoning of others. **Students build proofs by induction and proofs by contradiction. CA 3.1** (for higher mathematics only).	Students continue to reason through the solution of an equation and justify their reasoning to their peers. Students defend their choice of a function when modeling a real-world situation.
MP.4 Model with mathematics.	Students apply their new mathematical understanding to real-world problems, making use of their expanding repertoire of functions in modeling. Students also discover mathematics through experimentation and by examining patterns in data from real-world contexts.
MP.5 Use appropriate tools strategically.	Students continue to use graphing technology to deepen their understanding of the behavior of polynomial, rational, square root, and trigonometric functions.
MP.6 Attend to precision.	Students make note of the precise definition of *complex number,* understanding that real numbers are a subset of complex numbers. They pay attention to units in real-world problems and use unit analysis as a method for verifying their answers.
MP.7 Look for and make use of structure.	Students understand polynomials and rational numbers as sets of mathematical objects that have particular operations and properties. They understand the periodicity of sine and cosine and use these functions to model periodic phenomena.
MP.8 Look for and express regularity in repeated reasoning.	Students observe patterns in geometric sums—for example, that the first several sums of the form $\sum_{k=0}^{n} 2^k$ can be written as follows: $1 = 2^1 - 1$ $1+2 = 2^2 - 1$ $1+2+4 = 2^3 - 1$ $1+2+4+8 = 2^4 - 1$ Students use this observation to make a conjecture about any such sum.

Standard **MP.4** holds a special place throughout the higher mathematics curriculum, as Modeling is considered its own conceptual category. Although the Modeling category does not include specific standards, the idea of using mathematics to model the world pervades all higher mathematics courses and should hold a significant place in instruction. Some standards are marked with a star (★) symbol to indicate that they are *modeling standards*—that is, they may be applied to real-world modeling situations more so than other standards.

Examples of places where specific Mathematical Practice standards can be implemented in the Mathematics III standards are noted in parentheses, with the standard(s) also listed.

Mathematics III Content Standards, by Conceptual Category

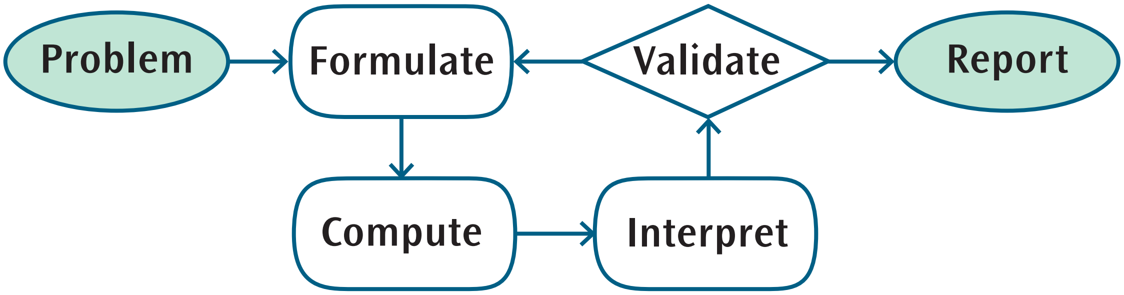
The Mathematics III course is organized by conceptual category, domains, clusters, and then standards. The overall purpose and progression of the standards included in Mathematics III are described below, according to each conceptual category. Standards that are considered new for secondary-grades teachers are discussed more thoroughly than other standards.

Conceptual Category: Modeling

Throughout the CA CCSSM, specific standards for higher mathematics are marked with a ★ symbol to indicate that they are modeling standards. Modeling at the higher mathematics level goes beyond the simple application of previously constructed mathematics and includes real-world problems. True modeling begins with students asking a question about the world around them, and mathematics is then constructed in the process of attempting to answer the question. When students are presented with a real-world situation and challenged to ask a question, all sorts of new issues arise (e.g., Which of the quantities present in this situation are known, and which are unknown?). Students need to decide on a solution path that may need to be revised. They make use of tools such as calculators, dynamic geometry software, or spreadsheets. They try to use previously derived models (e.g., linear functions), but may find that a new formula or function will apply. Students may see when trying to answer their question that solving an equation arises as a necessity and that the equation often involves the specific instance of knowing the output value of a function at an unknown input value.

Modeling problems have an element of being genuine problems, in the sense that students care about answering the question under consideration. In modeling, mathematics is used as a tool to answer questions that students really want answered. Students examine a problem and formulate a *mathematical model* (an equation, table, graph, or the like), compute an answer or rewrite their expression to reveal new information, interpret and validate the results, and report out; see figure M3-1. This is a new approach for many teachers and may be challenging to implement, but the effort should show students that mathematics is relevant to their lives. From a pedagogical perspective, modeling gives a concrete basis from which to abstract the mathematics and often serves to motivate students to become independent learners.

Figure M3-1. The Modeling Cycle



The examples in this chapter are framed as much as possible to illustrate the concept of mathematical modeling. The important ideas surrounding rational functions, graphing, solving equations, and rates of change are explored through this lens. Readers are encouraged to consult appendix B (Mathematical Modeling) for further discussion of the modeling cycle and how it is integrated into the higher mathematics curriculum.

Conceptual Category: Functions

The standards in the Functions conceptual category can serve as motivation for the study of standards in the other Mathematics III conceptual categories. Students have already worked with equations in which they have to "solve for x" as a search for the input of a function f that gives a specified output; solving the equation amounts to undoing the work of the function. The types of functions that students encounter in Mathematics III have new properties. Students previously learned that quadratic functions exhibit different behavior from linear and exponential functions; now they investigate polynomial, rational, and trigonometric functions in greater generality. As in the Mathematics II course, students must discover new techniques for solving the equations they encounter. Students see how rational functions can model real-world phenomena, in particular in instances of inverse variation ($x \cdot y = k$, k a constant), and how trigonometric functions can model periodic phenomena. In general, functions describe how two quantities are related in a precise way and can be used to make predictions and generalizations, keeping true to the emphasis on modeling that occurs in higher mathematics. As stated in the University of Arizona (UA) Progressions Documents for the Common Core Math Standards, "students should develop ways of thinking that are general and allow them to approach any function, work with it, and understand how it behaves, rather than see each function as a completely different animal in the bestiary" (UA Progressions Documents 2013c, 7).

Interpreting Functions F-IF

Interpret functions that arise in applications in terms of the context. [Include rational, square root and cube root; emphasize selection of appropriate models.]

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. ★

6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

Analyze functions using different representations. [Include rational and radical; focus on using key features to guide selection of appropriate type of model function.]

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. ★
   c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. ★
   e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★

8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

As in Mathematics II, students work with functions that model data and choose an appropriate model function by considering the context that produced the data. Students' ability to recognize rates of change, growth and decay, end behavior, roots, and other characteristics of functions becomes more sophisticated; they use this expanding repertoire of families of functions to inform their choices of models. This group of standards focuses on applications and how key features relate to characteristics of a situation, making selection of a particular type of function model appropriate (**F-IF.4–9**). The following example illustrates some of these standards. (Note that only sine, cosine, and tangent are treated in Mathematics III.)

Example: The Juice Can **F-IF.4–9**

Students are asked to find the minimal surface area of a cylindrical can of a fixed volume. The surface area is represented in units of square centimeters (cm^2), the radius in units of centimeters (cm), and the volume is fixed at 355 milliliters (ml), or 355 cm^3. Students can find the surface area of this can as a function of the radius:

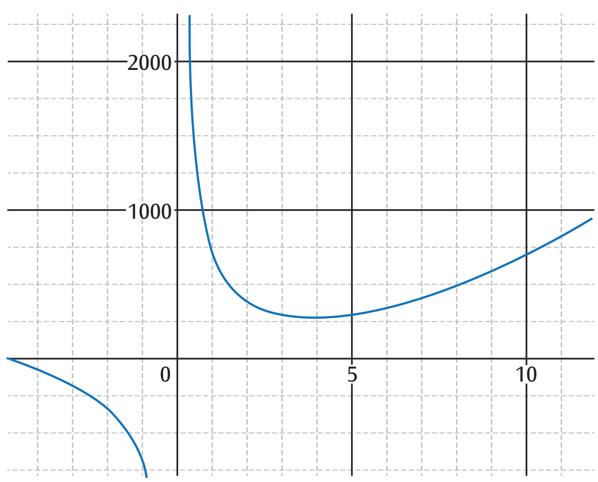
$$S(r)=\frac{2(355)}{r}+2\pi r^2$$

(See The Juice-Can Equation example that appears in the Algebra conceptual category of this chapter.)

This representation allows students to examine several things. First, a table of values will provide a hint about what the minimal surface area is. The table below lists several values for S based on r:

r(cm)	S(cm^2)
0.5	1421.6
1.0	716.3
1.5	487.5
2.0	380.1
2.5	323.3
3.0	293.2
3.5	279.8
4.0	278.0
4.5	284.9
5.0	299.0
5.5	319.1
6.0	344.4
6.5	374.6
7.0	409.1
7.5	447.9
8.0	490.7

The data suggest that the minimal surface area occurs when the radius of the base of the juice can is between 3.5 and 4.5 centimeters. Successive approximation using values of r between these values will yield a better estimate. But how can students be sure that the minimum is truly located here? A graph of $S(r)$ provides a clue:



Furthermore, students can deduce that as r gets smaller, the term $\frac{2(355)}{r}$ gets larger and larger, while the term $2\pi r$ gets smaller and smaller, and that the reverse is true as r grows larger, so that there is truly a minimum somewhere in the interval [3.5, 4.5].

Graphs help students reason about rates of change of functions (**F-IF.6**). In grade eight, students learned that the *rate of change* of a linear function is equal to the slope of the graph of that function. And because the slope of a line is constant, the phrase "rate of change" is clear for linear functions. For non-linear functions, however, rates of change are not constant, and thus average rates of change over an interval are used. For example, for the function g defined for all real numbers by $g(x) = x^2$, the average rate of change from $x = 2$ to $x = 5$ is

$$\frac{g(5) - g(2)}{5 - 2} = \frac{25 - 4}{5 - 2} = \frac{21}{3} = 7.$$

This is the slope of the line containing the points (2,4) and (5,25) on the graph of g. If g is interpreted as returning the area of a square of side length x, then this calculation means that over this interval the area changes, on average, by 7 square units for each unit increase in the side length of the square (UA Progressions Documents 2013c, 9). Students could investigate similar rates of change over intervals for the Juice-Can problem shown previously.

Building Functions — F-BF

Build a function that models a relationship between two quantities. [Include all types of functions studied.]

1. Write a function that describes a relationship between two quantities. ★
   b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ★

Build new functions from existing functions. [Include simple, radical, rational, and exponential functions; emphasize common effect of each transformation across function types.]

3. Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*
4. Find inverse functions.
   a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) = (x+1)/(x-1)$ *for* $x \neq 1$.

Students in Mathematics III develop models for more complex or sophisticated situations than in previous courses, due to the expansion of the types of functions available to them (**F-BF.1**). Modeling contexts provide a natural place for students to start building functions with simpler functions as components. Situations in which cooling or heating are considered involve functions that approach a limiting value according to a decaying exponential function. Thus, if the ambient room temperature is 70 degrees Fahrenheit and a cup of tea is made with boiling water at a temperature of 212 degrees Fahrenheit, a student can express the function describing the temperature as a function of time by using the constant function $f(t)=70$ to represent the ambient room temperature and the exponentially decaying function $g(t)=142e^{-kt}$ to represent the decaying difference between the temperature of the tea and the temperature of the room, which leads to a function of this form:

$$T(t)=70+142e^{-kt}$$

Students might determine the constant k experimentally (**MP.4, MP.5**).

With standard **F-BF.4a,** students learn that some functions have the property that an input can be recovered from a given output; for example, the equation $f(x)=c$ can be solved for x, given that c lies in the range of f. Students understand that this is an attempt to "undo" the function, or to "go backwards." Tables and graphs should be used to support student understanding here. This standard dovetails nicely with standard **F-LE.4** described below and should be taught in progression with it. Students will work more formally with inverse functions in advanced mathematics courses, and so standard **F-LE.4** should be treated carefully to prepare students for deeper understanding of functions and their inverses.

Linear, Quadratic, and Exponential Models — F-LE

Construct and compare linear, quadratic, and exponential models and solve problems.

4. For exponential models, express as a logarithm the solution to $ab^{ct}=d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology. ★ [Logarithms as solutions for exponentials]

4.1. Prove simple laws of logarithms. CA ★

4.2 Use the definition of logarithms to translate between logarithms in any base. CA ★

4.3 Understand and use the properties of logarithms to simplify logarithmic numeric expressions and to identify their approximate values. CA ★

Students worked with exponential models in Mathematics II. Based on the graph of the exponential function $f(x)=b^x$, students in Mathematics III can deduce that this function has an inverse—which is called *the logarithm to the base* b and denoted by $g(x)=\log_b x$. The logarithm has the property that $\log_b x=y$ if and only if $b^y=x$. Students find logarithms with base b equal to 2, 10, or e, by hand and with the assistance of technology (**MP.5**). Students may be encouraged to explore the properties of logarithms (such as $\log_b xy=\log_b x+\log_b y$) and to connect these properties to those of exponents. For example, the property just mentioned comes from the fact that the logarithm is essentially an exponent and that $b^{n+m}=b^n \bullet b^m$. Students solve problems involving exponential functions and logarithms and express their answers by using logarithm notation (**F-LE.4**). In general, students understand logarithms as functions that *undo* their corresponding exponential functions; instruction should emphasize this relationship.

Trigonometric Functions — F-TF

Extend the domain of trigonometric functions using the unit circle.

1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

2.1 Graph all 6 basic trigonometric functions. CA

Model periodic phenomena with trigonometric functions.

5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline. ★

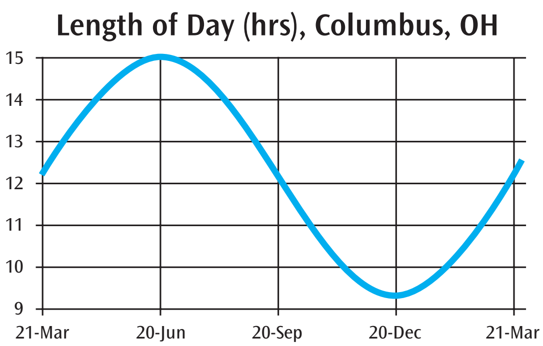
This set of standards calls for students to expand their understanding of trigonometric functions, which was first developed in Mathematics II. At first, the trigonometric functions apply only to angles in right triangles; for example, $\sin\theta$, $\cos\theta$, and $\tan\theta$ make sense only for $0<\theta<\frac{\pi}{2}$. By representing right triangles with hypotenuse 1 in the first quadrant of the plane, students see that $(\cos\theta, \sin\theta)$ represents a point on the unit circle. This leads to a natural way to extend these functions to any value of θ that remains consistent with the values for acute angles: interpreting θ as the radian measure of an angle traversed from the point (1,0) counterclockwise around the unit circle, $\cos\theta$ is taken to be the x-coordinate of the point corresponding to this rotation and $\sin\theta$ is the y-coordinate of this point. This interpretation of sine and cosine immediately yields the Pythagorean Identity: that $\cos^2\theta+\sin^2\theta=1$. This basic identity yields other identities through algebraic manipulation and allows students to find values of other trigonometric functions for a given θ if one value is known (**F-TF.1–2**).

Students should explore the graphs of trigonometric functions, with attention to the connection between the unit-circle representation of the trigonometric functions and their properties—for example, to illustrate the periodicity of the functions, the relationship between the maximums and minimums of the sine and cosine graphs, zeros, and so on. Standard **F-TF.5** calls for students to use trigonometric functions to model periodic phenomena. This is connected to standard **F-BF.3** (families of functions), and students begin to understand the relationship between the parameters appearing in the general cosine function $f(x)=A\bullet\cos(Bx-C)+D$ (and sine function), as well as the graph and behavior of the function (e.g., amplitude, frequency, line of symmetry). Additionally, students use their understanding of inverse functions to explore the inverse sine, cosine, and tangent functions at a basic level. It is important for students to understand that a function is well defined only when its domain is specified. For example, the general sine function $\sin x$, defined for all real numbers x, does not have an inverse, whereas the function $s(x)=\sin x$, defined only for values of x such that $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$, *does* have an inverse function.

Example: Modeling Daylight Hours	F-TF.5

By looking at data for length of days in Columbus, Ohio, students see that the number of daylight hours is approximately sinusoidal, varying from about 9 hours, 20 minutes on December 21 to about 15 hours on June 21. The average of the maximum and minimum gives the value for the midline, and the amplitude is half the difference of the maximum and minimum. Approximations of these values are set as $A = 12.17$ and $B = 2.83$. With some support, students determine that for the period to be 365 days (per cycle), or for the frequency to be $\frac{1}{365}$ cycles per day, $C = \frac{2\pi}{365}$, and if day 0 corresponds to March 21, no phase shift would be needed, so $D = 0$.

Thus, $f(t) = 12.17 + 2.83\sin\left(\frac{2\pi t}{365}\right)$ is a function that gives the approximate length of day for t, the day of the year from March 21. Considering questions such as when to plant a garden (i.e., when there are at least 7 hours of midday sunlight), students might estimate that a 14-hour day is optimal. Students solve $f(t) = 14$ and find that May 1 and August 10 mark this interval of time.



Students can investigate many other trigonometric modeling situations, such as simple predator–prey models, sound waves, and noise-cancellation models.

Source: UA Progressions Documents 2013c, 19.

Conceptual Category: Number and Quantity

Students continue to expand their understanding of the number system by finding complex-number roots when solving quadratic equations. Complex numbers have a practical application, and many phenomena involving real numbers become simpler when real numbers are viewed as a subsytem of complex numbers. As an example, complex solutions of differential equations can present a clear picture of the behavior of real solutions. Students are introduced to this when they study complex solutions of quadratic equations—and when complex numbers are involved, each quadratic polynomial can be expressed as a product of linear factors.

The Complex Number System	N-CN

Use complex numbers in polynomial identities and equations. [Polynomials with real coefficients; apply N-CN.9 to higher degree polynomials.]

8. (+) Extend polynomial identities to the complex numbers.

9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Standards **N-CN.8–9** call for students to continue working with complex numbers as solutions to polynomial equations. This builds on student work with quadratics that started in Mathematics II. For example, students can draw upon the Mathematics III algebra standards (e.g., **A-APR.2**) and find roots of equations such as $x^3+5x^2+8x+6=0$. They experiment by using the remainder theorem and find that $x+3$ is a root, since the polynomial expression evaluated at $x=-3$ is 0. Using polynomial long division or other factoring techniques, students find that

$$x^3+5x^2+8x+6=(x+3)(x^2+2x+2).$$

They use the quadratic formula to find the roots of the quadratic, $\{-1+i, -1-i\}$, and they write

$$x^3+5x^2+8x+6=(x+3)(x^2+2x+2)$$
$$x^3+5x^2+8x+6=(x+3)(x-(-1+i))(x-(-1-i)).$$

Experimentation with examples of such polynomials and an understanding that the quadratic formula always yields solutions to a quadratic equation help students understand the Fundamental Theorem of Algebra (**N-CN.9**).

Conceptual Category: Algebra

Along with the Number and Quantity standards in Mathematics III, the standards from the Algebra domain of the Mathematics III course develop the structural similarities between the system of polynomials and the system of integers. Students draw on analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property. Students connect multiplication of polynomials with multiplication of multi-digit integers and connect division of polynomials with long division of integers. Similar to the way that rational numbers extend the arithmetic of integers by allowing division by all numbers except 0, rational expressions extend the arithmetic of polynomials by allowing division by all polynomials except the zero polynomial. A central theme that arises is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers.

Seeing Structure in Expressions — A-SSE

Interpret the structure of expressions. [Polynomial and rational]

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. ★
2. Use the structure of an expression to identify ways to rewrite it.

Write expressions in equivalent forms to solve problems.

4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.* ★

In Mathematics III, students continue to pay attention to the meaning of expressions in context and interpret the parts of an expression by "chunking"—that is, by viewing parts of an expression as a single entity (**A-SSE.1–2**). For example, their facility with using special cases of polynomial factoring allows them to fully factor more complicated polynomials:

$$x^4-y^4=\left(x^2\right)^2-\left(y^2\right)^2=\left(x^2+y^2\right)\left(x^2-y^2\right)=\left(x^2+y^2\right)(x+y)(x-y)$$

Additionally, with their new understanding of complex numbers, students can factor this further into $x^4-y^4=(x+iy)(x-iy)(x+y)(x-y)$. In a physics course, students may encounter an expression such as $L_0\sqrt{1-\frac{v^2}{c^2}}$, which arises in the theory of special relativity. They can see this expression as the product of a constant L_0 and a term that is equal to 1 when $v=0$ and equal to 0 when $v=c$. Furthermore, students might be expected to see this mentally, without having to go through a laborious process of evaluation. This involves combining large-scale structure of the expression—a product of L_0 and another term—with the meaning of internal components such as $\frac{v^2}{c^2}$ (UA Progressions Documents 2013b, 4).

By examining the sum of a finite geometric series, students can look for a pattern to justify why the equation for the sum holds: $\sum_{k=0}^{n} ar^k=\frac{a\left(1-r^{n+1}\right)}{(1-r)}$. They may derive the formula, either with Proof by Mathematical Induction or by other means (**A-SSE.4**).

Example: Sum of a Geometric Series **A-SSE.4**

Students should investigate several concrete examples of finite geometric series (with $r \neq 1$) and use spreadsheet software to investigate growth in the sums and patterns that arise (**MP.5, MP.8**).

Geometric series have applications in several areas, including calculating mortgage payments, calculating totals for annual investments such as retirement accounts, finding total payout amounts for lottery winners, and more (**MP.4**). In general, a finite geometric series has this form:

$$\sum_{k=0}^{n} ar^k = a\left(1 + r + r^2 + \ldots r^{n-1} + r^n\right)$$

If the sum of this series is denoted by S, then some algebraic manipulation shows that

$$S - rS = a - ar^{n+1}.$$

Applying the distributive property to the common factors and solving for S shows that

$$S(1-r) = a\left(1 - r^{n+1}\right),$$

so that

$$S = \frac{a\left(1 - r^{n+1}\right)}{1-r}.$$

Students develop the ability to flexibly see expressions such as $A_n = A_0\left(1 + \frac{.15}{12}\right)^n$ as describing the total value of an investment at 15% interest, compounded monthly, for a number of compoundings, n. Moreover, they can interpret the following equation as a type of geometric series that would calculate the total value in an investment account at the end of one year if \$100 is deposited at the beginning of each month (**MP.2, MP.4, MP.7**):

$$A_1 + A_2 + \ldots + A_{12} = 100\left(1 + \frac{.15}{12}\right)^1 + 100\left(1 + \frac{.15}{12}\right)^2 + \ldots + 100\left(1 + \frac{.15}{12}\right)^{12}$$

They apply the formula for geometric series to find this sum.

Arithmetic with Polynomials and Rational Expressions — A-APR

Perform arithmetic operations on polynomials. [Beyond quadratic]

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Understand the relationship between zeros and factors of polynomials.

2. Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x-a$ is $p(a)$, so $p(a)=0$ if and only if $(x-a)$ is a factor of $p(x)$.
3. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Use polynomial identities to solve problems.

4. Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2+y^2)^2=(x^2-y^2)^2+(2xy)^2$ *can be used to generate Pythagorean triples.*
5. (+) Know and apply the Binomial Theorem for the expansion of $(x+y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[1]

Rewrite rational expressions. [Linear and quadratic denominators]

6. Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x)+r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.
7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a non-zero rational expression; add, subtract, multiply, and divide rational expressions.

Students in Mathematics III continue to develop their understanding of the set of polynomials as a system analogous to the set of integers that exhibits certain properties, and they explore the relationship between the factorization of polynomials and the roots of a polynomial (**A-APR.1–3**). When a polynomial $p(x)$ is divided by $(x-a)$, $p(x)$ is written as $p(x)=q(x)\bullet(x-a)+r$, where r is a constant. This can be done by inspection or by polynomial long division (**A-APR.7**). It follows that $p(a)=q(a)\bullet(a-a)+r=q(a)\bullet 0+r=r$, so that $(x-a)$ is a factor of $p(x)$ if and only if $p(a)=0$. This result is generally known as the Remainder Theorem (**A-APR.2**) and provides an easy check to see if a polynomial has a given linear factor. This topic should not be reduced to "synthetic division," which limits the theorem to a method of carrying numbers between registers—something easily done by a computer—while obscuring the reasoning that makes the result evident. It is important to regard the Remainder Theorem as a theorem, not a technique (**MP.3**) [UA Progressions Documents 2013b, 7].

1. The Binomial Theorem may be proven by mathematical induction or by a combinatorial argument.

Students use the zeros of a polynomial to create a rough sketch of its graph and connect the results to their understanding of polynomials as functions (**A-APR.3**). The notion that polynomials can be used to approximate other functions is important in advanced mathematics courses such as Calculus, and standard **A-APR.3** is the first step in a progression that can lead students, as an extension topic, to constructing polynomial functions whose graphs pass through any specified set of points in the plane (UA Progressions Documents 2013b, 7).

Additionally, polynomials form a rich ground for mathematical explorations that reveal relationships in the system of integers (**A-APR.4**). For example, students can explore the sequence of squares 1, 4, 9, 16, 25, 36, … and notice the differences between them—3, 5, 7, 9 , 11—are consecutive odd integers. This mystery is explained by the polynomial identity $(n+1)^2 - n^2 = 2n+1$, which can be justified by using pictures (UA Progressions Documents 2013b, 6).

In Mathematics **III**, students explore rational functions as a system analogous to the rational numbers. They see rational functions as useful for describing many real-world situations—for instance, when rearranging the equation $d = rt$ to express the rate as a function of the time for a fixed distance d_0 and obtaining $r = \frac{d_o}{t}$. Now students see that any two polynomials can be divided in much the same way that numbers are (provided the divisor is not 0). Students first understand rational expressions as similar to other expressions in algebra, except that rational expressions have the special form $\frac{a(x)}{b(x)}$ for both $a(x)$ and $b(x)$ polynomials in x. Students should evaluate various rational expressions for many values of x, perhaps discovering that when the degree of $b(x)$ is larger than the degree of $a(x)$, the value of the expression gets smaller in absolute value as $|x|$ gets larger. Developing an understanding of the behavior of rational expressions in this way helps students to see these expressions as functions and sets the stage for working with simple rational functions in the Functions domain.

Example: The Juice-Can Equation **A-CED.4**

If someone wanted to investigate the shape of a juice can of minimal surface area, the investigation could begin in the following way. If the volume V_0 is fixed, then the expression for the volume of the can is $V_0 = \pi r^2 h$, where h is the height of the can and r is the radius of the circular base. On the other hand, the surface area S is given by the following formula:

$$S = 2\pi rh + 2\pi r^2$$

This is because the two circular bases of the can contribute $2\pi r^2$ units of surface area, and the outside surface of the can contributes an area in the shape of a rectangle with length equal to the circumference of the base, $2\pi r$, and height equal to h. Since the volume is fixed, h can be found in terms of r: $h = \frac{V_0}{\pi r^2}$. Then this can be substituted into the equation for the surface area:

$$S = 2\pi r \bullet \frac{V_0}{\pi r^2} + 2\pi r^2$$

$$= \frac{2V_0}{r} + 2\pi r^2$$

This equation expresses the surface area S as a (rational) function of r, which can then be analyzed. (Also refer to standard **F-IF.8**.)

Additionally, students are able to rewrite rational expressions in the form $a(x) = q(x) \bullet b(x) + r(x)$, where $r(x)$ is a polynomial of degree less than $b(x)$, by inspection or by using polynomial long division. They can flexibly rewrite this expression as $\frac{a(x)}{b(x)} = q(x) + \frac{r(x)}{b(x)}$ as necessary—for example, to highlight the end behavior of the function defined by the expression $\frac{a(x)}{b(x)}$. In order to make working with rational expressions more than just an exercise in the proper manipulation of symbols, instruction should focus on the characteristics of rational functions that can be understood by rewriting them in the ways described above (e.g., rates of growth, approximation, roots, axis intersections, asymptotes, end behavior, and so on).

Creating Equations — A-CED

Create equations that describe numbers or relationships. [Equations using all available types of expressions, including simple root functions]

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* **CA** ★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. ★

Students in Mathematics III work with all available types of functions, including root functions, to create equations (**A-CED.1**). Although functions referenced in standards **A-CED.2–4** will often be linear, exponential, or quadratic, the types of problems should draw from more complex situations than those addressed in Mathematics I and Mathematics II. For example, knowing how to find the equation of a line through a given point perpendicular to another line allows one to find the distance from a point to a line. The Juice-Can Equation example presented previously in this section is connected to standard **A-CED.4.**

Reasoning with Equations and Inequalities — A-REI

Understand solving equations as a process of reasoning and explain the reasoning. [Simple radical and rational]

2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Represent and solve equations and inequalities graphically. [Combine polynomial, rational, radical, absolute value, and exponential functions.]

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. ★

Students extend their equation-solving skills to those involving rational expressions and radical equations, and they make sense of extraneous solutions that may arise (**A-REI.2**). In particular, students understand that when solving equations, the flow of reasoning is generally forward, in the sense that it is assumed a number x is a solution of the equation and then a list of possibilities for x is found. However, not all steps in this process are reversible. For example, although it is true that if $x = 2$, then $x^2 = 4$, it is not true that if $x^2 = 4$, then $x = 2$, as $x = -2$ also satisfies this equation (UA Progressions Documents 2013b, 10). Thus students understand that some steps are reversible and some are not, and they anticipate extraneous solutions. Additionally, students continue to develop their understanding of solving equations as solving for values of x such that $f(x) = g(x)$, now including combinations of linear, polynomial, rational, radical, absolute value, and exponential functions (**A-REI.11**). Students also understand that some equations can be solved only approximately with the tools they possess.

Conceptual Category: Geometry

In Mathematics III, students extend their understanding of the relationship between algebra and geometry as they explore the equations for circles and parabolas. They also expand their understanding of trigonometry to include finding unknown measurements in non-right triangles. The Geometry standards included in the Mathematics III course offer many rich opportunities for students to practice mathematical modeling.

Similarity, Right Triangles, and Trigonometry — G-SRT

Apply trigonometry to general triangles.

9. (+) Derive the formula $A = \frac{1}{2}ab\sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.

11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Students advance their knowledge of right-triangle trigonometry by applying trigonometric ratios in non-right triangles. For instance, students see that by dropping an altitude in a given triangle, they divide the triangle into right triangles to which these relationships can be applied. By seeing that the base of the triangle is a and the height is $b \bullet \sin C$, they derive a general formula for the area of any triangle $A = \frac{1}{2}ab\sin(C)$ (**G-SRT.9**). Additionally, students use reasoning about similarity and trigonometric identities to derive the Laws of Sines and Cosines only in acute triangles, and they use these and other relationships to solve problems (**G-SRT.10–11**). Instructors will need to address the ideas of the sine and cosine of angles larger than or equal to 90 degrees to fully discuss Laws of Sine and Cosine, although full unit-circle trigonometry need not be discussed in this course.

Geometric Measurement and Dimension — G-GMD

Visualize relationships between two-dimensional and three-dimensional objects.

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Modeling with Geometry — G-MG

Apply geometric concepts in modeling situations.

1. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder). ★
2. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot). ★
3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios). ★

These standards are rich with opportunities for students to apply modeling (**MP.4**) with geometric concepts—and although these standards appear later in the sequence of the Mathematics III geometry standards, they should be incorporated throughout the geometry curriculum of the course. Standard **G-MG.1** calls for students to use geometric shapes, their measures, and their properties to describe objects. This standard can involve two- and three-dimensional shapes and is not relegated to simple applications of formulas. Standard **G-MG.3** calls for students to solve design problems by modeling with geometry.

Example: Ice-Cream Cone **G-MG.3**

The owner of a local ice-cream parlor has hired you to assist with his company's new venture: the company will soon sell its ice-cream cones in the freezer section of local grocery stores. The manufacturing process requires that each ice-cream cone be wrapped in a cone-shaped paper wrapper with a flat, circular disc covering the top. The company wants to minimize the amount of paper that is wasted in the process of wrapping the cones. Use a real ice-cream cone or the dimensions of a real ice-cream cone to complete the following tasks:

a. Sketch a wrapper like the one described above, using the actual size of your cone. Ignore any overlap required for assembly.

b. Use your sketch to help develop an equation the owner can use to calculate the surface area of a wrapper (including the lid) for another cone, given that its base had a radius of length r and a slant height s.

c. Using measurements of the radius of the base and slant height of your cone, and your equation from step b, find the surface area of your cone.

d. The company has a large rectangular piece of paper that measures 100 centimeters by 150 centimeters. Estimate the maximum number of complete wrappers sized to fit your cone that could be cut from this single piece of paper, and explain your estimate. (Solutions can be found at https://www.illustrativemathematics.org/ [accessed April 1, 2015].)

Source: Illustrative Mathematics 2013l.

Expressing Geometric Properties with Equations — G-GPE

Translate between the geometric description and the equation for a conic section.

3.1 Given a quadratic equation of the form $ax^2 + by^2 + cx + dy + e = 0$, use the method for completing the square to put the equation into standard form; identify whether the graph of the equation is a circle, ellipse, parabola, or hyperbola and graph the equation. [In Mathematics III, this standard addresses only circles and parabolas.] **CA**

Students further their understanding of the connection between algebra and geometry by applying the definition of circles and parabolas to derive equations and then deciding whether a given quadratic equation of the form $ax^2 + by^2 + cx + dy + e = 0$ represents a circle or a parabola.

Conceptual Category: Statistics and Probability

In Mathematics III, students develop a more formal and precise understanding of statistical inference, which requires a deeper understanding of probability. They explore the conditions that meet *random sampling* of a population and that allow for generalization of results to that population. Students also learn to use significant differences to make inferences about data gathered during the course of experiments.

Interpreting Categorical and Quantitative Data S-ID

Summarize, represent, and interpret data on a single count or measurement variable.

4. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve. ★

Although students in Mathematics III may have heard of the normal distribution, it is unlikely that they will have had prior experience using it to make specific estimates. In Mathematics III, students build on their understanding of data distributions to see how to use the area under the normal distribution to make estimates of frequencies (which can be expressed as probabilities). It is important for students to see that only some data are well described by a normal distribution (**S-ID.4**). Additionally, they can learn through examples the *empirical rule:* that for a normally distributed data set, 68% of the data lie within one standard deviation of the mean and 95% are within two standard deviations of the mean.

Example: The Empirical Rule **S-ID.4**

Suppose that SAT mathematics scores for a particular year are approximately normally distributed, with a mean of 510 and a standard deviation of 100.

a. What is the probability that a randomly selected score is greater than 610?

b. What is the probability that a randomly selected score is greater than 710?

c. What is the probability that a randomly selected score is between 410 and 710?

d. If a student's score is 750, what is the student's percentile score (the proportion of scores below 750)?

Solutions:

a. The score 610 is one standard deviation above the mean, so the tail area above that is about half of 0.32, or 0.16. The calculator gives 0.1586.

b. The score 710 is two standard deviations above the mean, so the tail area above that is about half of 0.05, or 0.025. The calculator gives 0.0227.

c. The area under a normal curve from one standard deviation below the mean to two standard deviations above the mean is about 0.815. The calculator gives 0.8186.

d. Using either the normal distribution given or the standard normal (for which 750 translates to a z-score of 2.4), the calculator gives 0.9918.

Making Inferences and Justifying Conclusions S-IC

Understand and evaluate random processes underlying statistical experiments.

1. Understand statistics as a process for making inferences about population parameters based on a random sample from that population ★
2. Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?* ★

Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

3. Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each. ★
4. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling. ★
5. Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. ★
6. Evaluate reports based on data. ★

Students in Mathematics III move beyond analysis of data to make sound statistical decisions based on probability models. The reasoning process is as follows: develop a statistical question in the form of a hypothesis (supposition) about a population parameter, choose a probability model for collecting data relevant to that parameter, collect data, and compare the results seen in the data with what is expected under the hypothesis. If the observed results are far from what is expected and have a low probability of occurring under the hypothesis, then that hypothesis is called into question. In other words, the evidence against the hypothesis is weighed by probability (**S-IC.1**) [UA Progressions Documents 2012d]. By investigating simple examples of simulations of experiments and observing outcomes of the data, students gain an understanding of what it means for a model to fit a particular data set (**S-IC.2**). This includes comparing theoretical and empirical results to evaluate the effectiveness of a treatment.

In earlier grade levels, students are introduced to different ways of collecting data and use graphical displays and summary statistics to make comparisons. These ideas are revisited with a focus on how the way in which data are collected determines the scope and nature of the conclusions that can be drawn from those data. The concept of *statistical significance* is developed informally through simulation as meaning a result that is unlikely to have occurred solely through random selection in sampling or random assignment in an experiment (NGA/CCSSO 2010a). When covering standards **S-IC.4–5,** instructors should focus on the variability of results from experiments—that is, on statistics as a way of handling, not eliminating, inherent randomness. Because standards **S-IC.1–6** are all modeling standards, students should have ample opportunities to explore statistical experiments and informally arrive at statistical techniques.

Example: Estimating a Population Proportion **S-IC.4**

Suppose a student wishes to investigate whether 50% of homeowners in her neighborhood will support a new tax to fund local schools. If she takes a random sample of 50 homeowners in her neighborhood, and 20 support the new tax, then the *sample proportion* agreeing to pay the tax would be 0.4. But is this an accurate measure of the true proportion of homeowners who favor the tax? How can this be determined?

If this sampling situation (**MP.4**) is simulated with a graphing calculator or spreadsheet software under the assumption that the true proportion is 50%, then the student can arrive at an understanding of the *probability* that her randomly sampled proportion would be 0.4. A simulation of 200 trials might show that 0.4 arose 25 out of 200 times, or with a probability of 0.125. Thus, the chance of obtaining 40% as a sample proportion is not insignificant, meaning that a true proportion of 50% is plausible.

Adapted from UA Progressions Documents 2012d.

Using Probability to Make Decisions S-MD

Use probability to evaluate outcomes of decisions. [Include more complex situations.]

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game). ★

As in Mathematics II, students apply probability models to make and analyze decisions. This skill is extended in Mathematics III to more complex probability models, including situations such as those involving quality control or diagnostic tests that yield both false-positive and false-negative results. See the University of Arizona Progressions document titled "High School Statistics and Probability" for further explanation and examples: http://ime.math.arizona.edu/progressions/ (UA Progressions Documents 2012d [accessed April 6, 2015]).

Mathematics III is the culmination of the Integrated Pathway. Students completing this pathway will be well prepared for advanced mathematics and should be encouraged to continue their study of mathematics with Precalculus or other mathematics electives, such as Statistics and Probability or a course in modeling.

Mathematics III Overview

Number and Quantity

The Complex Number System

- Use complex numbers in polynomial identities and equations.

Algebra

Seeing Structure in Expressions

- Interpret the structure of expressions.
- Write expressions in equivalent forms to solve problems.

Arithmetic with Polynomials and Rational Expressions

- Perform arithmetic operations on polynomials.
- Understand the relationship between zeros and factors of polynomials.
- Use polynomial identities to solve problems.
- Rewrite rational expressions.

Creating Equations

- Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- Understand solving equations as a process of reasoning and explain the reasoning.
- Represent and solve equations and inequalities graphically.

Functions

Interpreting Functions

- Interpret functions that arise in applications in terms of the context.
- Analyze functions using different representations.

Building Functions

- Build a function that models a relationship between two quantities.
- Build new functions from existing functions.

Linear, Quadratic, and Exponential Models

- Construct and compare linear, quadratic, and exponential models and solve problems.

Trigonometric Functions

- Extend the domain of trigonometric functions using the unit circle.
- Model periodic phenomena with trigonometric functions.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Mathematics III Overview *(continued)*

Geometry

Similarity, Right Triangles, and Trigonometry

- Apply trigonometry to general triangles.

Expressing Geometric Properties with Equations

- Translate between the geometric description and the equation for a conic section.

Geometric Measurement and Dimension

- Visualize relationships between two-dimensional and three-dimensional objects.

Modeling with Geometry

- Apply geometric concepts in modeling situations.

Statistics and Probability

Interpreting Categorical and Quantitative Data

- Summarize, represent, and interpret data on a single count or measurement variable.

Making Inferences and Justifying Conclusions

- Understand and evaluate random processes underlying statistical experiments.
- Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

Using Probability to Make Decisions

- Use probability to evaluate outcomes of decisions.

Number and Quantity

The Complex Number System N-CN

Use complex numbers in polynomial identities and equations. [Polynomials with real coefficients; apply N-CN.9 to higher degree polynomials.]

8. (+) Extend polynomial identities to the complex numbers.[2]
9. (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Algebra

Seeing Structure in Expressions A-SSE

Interpret the structure of expressions. [Polynomial and rational]

1. Interpret expressions that represent a quantity in terms of its context. ★
   a. Interpret parts of an expression, such as terms, factors, and coefficients. ★
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. ★
2. Use the structure of an expression to identify ways to rewrite it.

Write expressions in equivalent forms to solve problems.

4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.* ★

Arithmetic with Polynomials and Rational Expressions A-APR

Perform arithmetic operations on polynomials. [Beyond quadratic]

1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Understand the relationship between zeros and factors of polynomials.

2. Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x-a$ is $p(a)$, so $p(a)=0$ if and only if $(x-a)$ is a factor of $p(x)$.
3. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Use polynomial identities to solve problems.

4. Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity $(x^2+y^2)^2=(x^2-y^2)^2+(2xy)^2$ can be used to generate Pythagorean triples.*
5. (+) Know and apply the Binomial Theorem for the expansion of $(x+y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.[3]

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2. (+) Indicates additional mathematics to prepare students for advanced courses. ★ Indicates a modeling standard linking mathematics to everyday life, work, and decision-making.

3. The Binomial Theorem may be proven by mathematical induction or by a combinatorial argument

Rewrite rational expressions. [Linear and quadratic denominators]

6. Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x)+r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.
7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a non-zero rational expression; add, subtract, multiply, and divide rational expressions.

Creating Equations A-CED

Create equations that describe numbers or relationships. [Equations using all available types of expressions, including simple root functions]

1. Create equations and inequalities in one variable **including ones with absolute value** and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* CA ★
2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. ★
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.★

Reasoning with Equations and Inequalities A-REI

Understand solving equations as a process of reasoning and explain the reasoning. [Simple radical and rational]

2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Represent and solve equations and inequalities graphically. [Combine polynomial, rational, radical, absolute value, and exponential functions.]

11. Explain why the x-coordinates of the points where the graphs of the equations $y=f(x)$ and $y=g(x)$ intersect are the solutions of the equation $f(x)=g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. ★

Functions

Interpreting Functions F-IF

Interpret functions that arise in applications in terms of the context. [Include rational, square root and cube root; emphasize selection of appropriate models.]

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.* ★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.★
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★

Analyze functions using different representations. [Include rational and radical; focus on using key features to guide selection of appropriate type of model function.]

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★
   - b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. ★
   - c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. ★
   - e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. ★
8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

Building Functions F-BF

Build a function that models a relationship between two quantities. [Include all types of functions studied.]

1. Write a function that describes a relationship between two quantities. ★
   - b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* ★

Build new functions from existing functions. [Include simple, radical, rational, and exponential functions; emphasize common effect of each transformation across function types.]

3. Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

4. Find inverse functions.

   a. Solve an equation of the form $f(x)=c$ for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x)=(x+1)/(x-1)$ *for* $x \neq 1$.

Linear, Quadratic, and Exponential Models F-LE

Construct and compare linear, quadratic, and exponential models and solve problems.

4. For exponential models, express as a logarithm the solution to $ab^{ct}=d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology. ★ [Logarithms as solutions for exponentials]

4.1. Prove simple laws of logarithms. CA ★

4.2 Use the definition of logarithms to translate between logarithms in any base. CA ★

4.3 Understand and use the properties of logarithms to simplify logarithmic numeric expressions and to identify their approximate values. CA ★

Trigonometric Functions F-TF

Extend the domain of trigonometric functions using the unit circle.

1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

2.1 Graph all 6 basic trigonometric functions. CA

Model periodic phenomena with trigonometric functions.

5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline. ★

Geometry

Similarity, Right Triangles, and Trigonometry — G-SRT

Apply trigonometry to general triangles.

9. (+) Derive the formula $A = \frac{1}{2}ab\sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.
11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Expressing Geometric Properties with Equations — G-GPE

Translate between the geometric description and the equation for a conic section.

3.1 **Given a quadratic equation of the form $ax^2 + by^2 + cx + dy + e = 0$, use the method for completing the square to put the equation into standard form; identify whether the graph of the equation is a circle, ellipse, parabola, or hyperbola and graph the equation.** [In Mathematics III, this standard addresses only circles and parabolas.] **CA**

Geometric Measurement and Dimension — G-GMD

Visualize relationships between two-dimensional and three-dimensional objects.

4. Identify the shapes of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Modeling with Geometry — G-MG

Apply geometric concepts in modeling situations.

1. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder). ★
2. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot). ★
3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios). ★

Mathematics III

Statistics and Probability

Interpreting Categorical and Quantitative Data — S-ID

Summarize, represent, and interpret data on a single count or measurement variable.

4. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve. ★

Making Inferences and Justifying Conclusions — S-IC

Understand and evaluate random processes underlying statistical experiments.

1. Understand statistics as a process for making inferences about population parameters based on a random sample from that population. ★
2. Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?* ★

Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

3. Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each. ★
4. Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling. ★
5. Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. ★
6. Evaluate reports based on data. ★

Using Probability to Make Decisions — S-MD

Use probability to evaluate outcomes of decisions. [Include more complex situations.]

6. (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator). ★
7. (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game). ★