Pg 8 Oct 12 2017

Proof of Bolzano-Weierstrass

Pictures!

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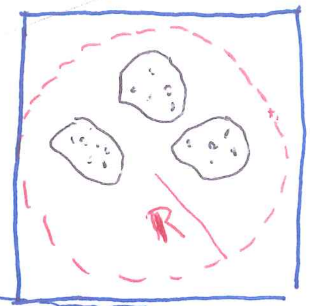
Given S closed, bounded subset of $\mathbb{R}^n$ and $x_1, x_2, \ldots \in S$.

Goal: Find a subseq of $x_1, \ldots$ that converges (to something in S)

Step 0

Here we use S is bounded

- Draw a big box containing S (and hence $x_1, x_2, \ldots$). Call box R_0



ok because S is bounded

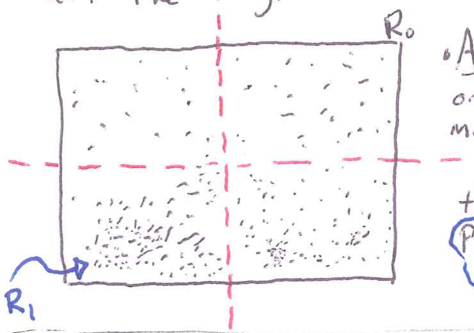
$S \subseteq B_R(0) \subseteq \{\vec{x} : |x_i| < R \ \forall i\}$

Ball fits into a box

∞'ly many pts in R_0

Step 1.a (Divide R_0)

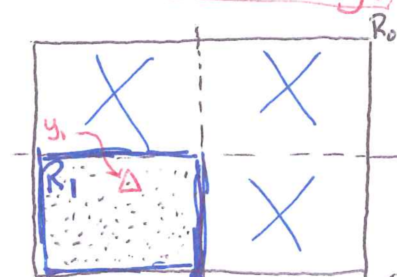
- Cut R_0 into boxes of half the length



- At least one box must have ∞'ly many elements of the sequence. Pick one, call it R_1

Step 1.b (Pick y_1 and update the sequence)

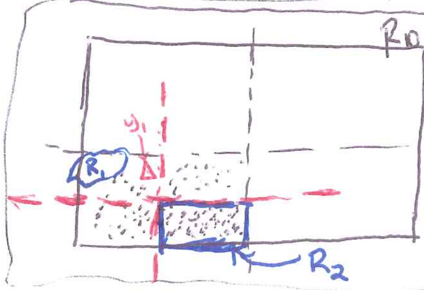
- Pick any point of sequence that's in R_1. Call it y_1



- Throw away all terms of the sequence that precede y_1 and everything that's not in R_1.

Still ∞'ly many pts in R_1

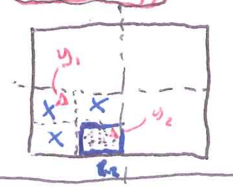
Step 2.a (Divide R_1)



- Pick a box within R_1 with ∞'ly many points. Call it R_2

Step 2.b (Pick y_2 and update the sequence)

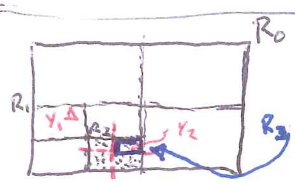
- Pick any pt in seq. in R_2. Call it y_2



- Throw away all terms in seq. that precede y_2 and everything that's not in R_2

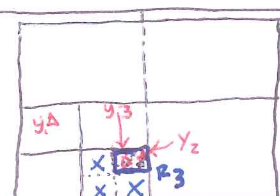
still have ∞'ly many pts in R_2

Step 3.a (Divide R_2)



- Pick a box within R_2 with ∞'ly many points. Call it R_3.

Step 3.b (Pick y_3 and update the sequence)



- Pick any point in seq. in R_3. Call it y_3
- Throw away all terms in seq. that precede y_3 and everything that's not in R_3

still ∞'ly many pts in R_3

Step 4.a

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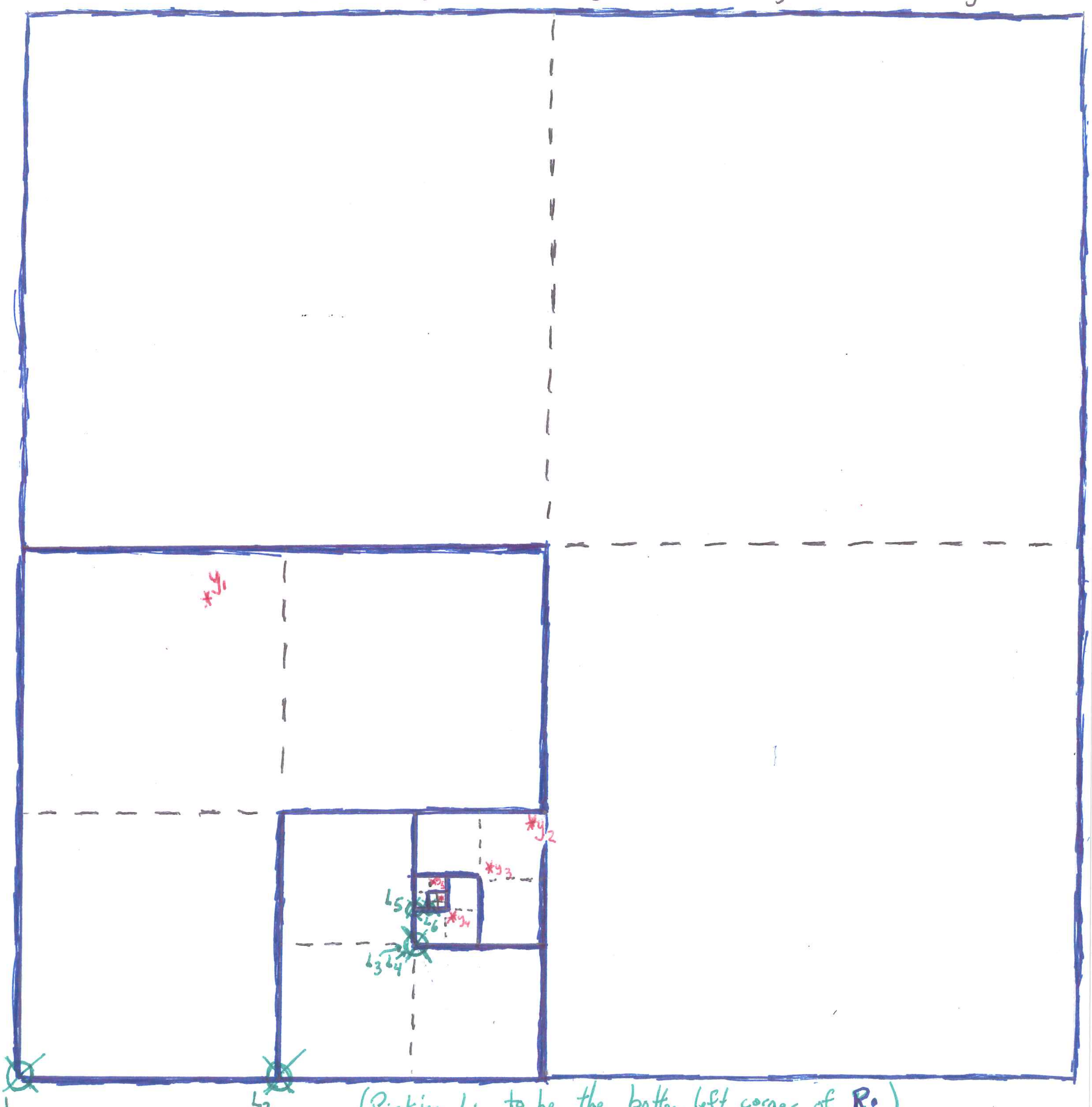
We now have a sequence $y_1, y_2, \ldots$, which is a subsequence of $x_1, x_2, \ldots$. To prove $y_1, y_2, \ldots$ converges to <u>something</u>, consider the sequence

$\boxed{L_i \text{ is the "bottom left" corner of } R_i.}$

Each coordinate of the sequence L_i is increasing, and each is bounded by the "top right" corner of R_0. So L_i converges to <u>something</u> since each coordinate converges to <u>something</u>.



(Picking L_i to be the bottom left corner of R_i)

So the sequence L_i converges to something,

and $d(L_i, y_i) = \|L_i - y_i\|_2$.

By construction, R_i is a box with side lengths

$$\underbrace{(2R)}_{\text{(side lengths of } R_0)} 2^{-i}.$$

Therefore since $L_i, y_i \in R_i$, we must have

$$\|L_i - y_i\|_\infty \le \text{side length}(R_i) \underset{(= \text{ actually})}{\le} 2R2^{-i}$$

So $\|L_i - y_i\|_\infty \longrightarrow 0$, which implies [by a homework problem]

$d(L_i, y_i) = \|L_i - y_i\|_2 \longrightarrow 0$ (Hint: $\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\,\|x\|_\infty$)

So $L_i - y_i$ converges to 0 vector.

So $y_i = L_i - (L_i - y_i)$ is the sum of two convergent sequences, $\boxed{\text{So } y_i \text{ converges to something}}$.

Finally, since y_i converges to something, its limit must be in S since S is closed. (You prove this last step on homework.)

$\square$