# Chromatic numbers of random subgraphs (SUMRY 2018 project description)

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**High-level description:** This project explores a problem due to Bukh [4], which is firmly at the intersection of probability and graph theory. Participants will learn (and use) fancy probabilistic techniques to study how the chromatic number of a graph is related to those of its random subgraphs. This will have a heavy probabilistic flavor. The first week will get everyone up to speed, and all interested applicants should apply.

## Background and motivation

Beginning in 1959, Paul Erdős published a series of *ludicrously influential* papers introducing random graph theory [7, 6, 8, 9]. Since then, the subject has exploded becoming one of the main thrusts within discrete mathematics. The central idea is that by introducing randomness to classical combinatorial settings, things often (magically!) become easier, and it enables us to study more nuanced aspects of what's going on (i.e., whereas combinatorial results are usually statements about *fringe* objects, probabilistic results are about *most* objects).

One of the key notions in probabilistic combinatorics is a random subgraph. For a fixed graph $G$ and a fixed constant $p \in [0, 1]$, we let $G_p$ denote the random subgraph of $G$ where each edge is independently included in $G_p$ with probability $p$. For example, $G_1$ is deterministically equal to $G$, $G_0$ has no edges, and $G_{1/2}$ is uniformly distributed over all subgraphs of $G$.

Everything that people care about in classical graph theory is immediately interesting (and often well-studied) in $G_p$ (see [5] as an example, [10] as a survey, and [2] as a reference text), and the case when $G$ is the complete graph has gotten particular attention. For this project, the independence number[1] and the chromatic number[2] ($\alpha$ and $\chi$ resp.) will be the main focus. In particular, we will be concerned with the following

> **Main focus:** Let $G$ be a fixed graph. How is $\chi(G_p)$ related to $\chi(G)$?

From a purely mathematical view, this question is extremely natural, and it offers the chance to understand the chromatic number—a fundamental graph parameter—through the lens of "typical" substructures. But this problem also has applications in statistical mechanics, where physicists use

[1] An *independent set* is a collection of vertices containing no edges, and the *independence number* of a graph—denoted $\alpha(G)$—is the size of the largest independent set.

[2] The *chromatic number* of a graph—denoted $\chi(G)$—is the size of the smallest partition of the vertices into independent sets.

random subgraphs[3] to model molecular interactions, and properties of the resulting graph colorings are predictive of various macroscopic features.

## Brief overview of what's known

The case that $G = K_n$ (the complete graph) is referred to as the Erdős-Rényi random graph, and it has been extensively studied. Surprisingly, the behavior of the independence number is remarkably well-understood, and for large $n$, the random variable $\alpha((K_n)_p)$ is virtually a deterministic constant[4] approximately equal to $c\log(n)$.

Because the chromatic number involves partitioning into independent sets, every graph satisfies $\chi(G) \geq |V(G)|/\alpha(G)$, and it turns out that for random subgraphs of $K_n$, this lower bound is asymptotically tight (see [2]).

In general, it is always true that $\chi(G_p) \leq \chi(G)$ [because $G_p$ is a subgraph], and there are in fact graphs for which this is (almost always) tight. As a lower bound, Alon, Krivelevich, and Sudakov [1] proved there is a universal constant for which $\mathbb{E}[\chi(G_p)] \geq C\chi(G)/\log(|V(G)|)$, and Bukh [4] conjectured this could be improved to $\mathbb{E}[\chi(G_p)] \geq C\chi(G)/\log(\chi(G))$.

Mild progress towards this conjecture was made in [3, 11], mainly via proofs of large deviation results. They also note $\mathbb{E}[\chi(G_{1/2})] \geq \sqrt{\chi(G)}$ (by a simple coupling and $\chi(H \cup K) \leq \chi(H)\chi(K)$), and they prove Bukh's conjecture in the case $\chi(G) \leq 1000|V(G)|/\alpha(G)$ [a rather weak hypothesis]. But for general graphs, Bukh's conjecture is still very much open.

## More in-depth discussion of existing proofs

As an example of what participants will learn (and use), we'll show how to estimate the independence number of $(K_n)_p$. **Note:** I do not expect applicants to already know how to do this. A key step is to consider the *number* of independent sets of size $k$—call this random variable $X$—and to estimate the largest $k$ for which this number is at least 1. For each $S \subseteq \{1, 2, \ldots, n\}$ with $|S| = k$, let $X_S$ be the random variable given by

$$X_S = \begin{cases} 1 & \text{if } S \text{ is an independent set in } (K_n)_p, \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly (after thought) $X = \sum_S X_S$, so by linearity of expectation:

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{|S|=k} X_S\right] = \sum_{|S|=k} \mathbb{E}[X_S] = \binom{n}{k}(1-p)^{k(k-1)/2}.$$

Because we wish to estimate the largest $k$ for which $X \geq 1$, a first-order approximation would be to estimate the largest $k$ for which $\mathbb{E}[X] \geq 1$.

[3] Physicists refer to this as *bond percolation*.

[4] With overwhelming probability, it is equal to one of just two values, and for "most" $n$ it's actually concentrated on a single point!

Using the fact[5] that $1+x \le e^x$ for all $x$ and the standard estimate $\binom{n}{k} \le n^k$,

$$\begin{aligned}\mathbb{E}[X] &= \binom{n}{k}(1-p)^{k(k-1)/2} \le n^k \exp[-pk(k-1)/2] \\ &= (\exp[\ln(n) - p(k-1)/2])^k .\end{aligned}$$

The right-hand side is less than 1 precisely when $\ln(n) < p(k-1)/2$, or equivalently when $k > 2\ln(n)/p + 1$. This gives us the heuristic that for fixed $p$, the independence number should be roughly $\log(n)$ (more precision possible by slightly refining the above). As participants will learn, making this rigorous is an easy application of the *second moment method* (see [2]), which simply amounts to estimating the variance $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.

# More details on SUMRY project

## Requirements of applicants

Here are things that will be useful for the project. I *do not* expect applicants to know all of this already [especially not 4], but if you don't care to learn about this stuff (especially random graphs), you likely won't have any fun.

1. Applicants should ideally have some familiarity with graph theory.
2. This project involves a lot of probability. Basic knowledge of probability will be covered quickly (e.g., random variables, linearity of expectation), and the more probability applicants have seen, the better off they'll be. That said, the first week will get everybody up to speed starting from basics and building to fancier stuff.
3. Some programming experience could be useful.
4. (Optional) Knowledge of probabilistic method and random graphs.

## What participants will learn

In addition to the current literature on the subject [1, 3, 11], participants will learn about (and practice) the following

- probability (events, random variables, moment methods)
- basic graph theory
- random graphs (e.g., applications of above to $\alpha$ and $\chi$ via [2])
- fancy probability (coupling, concentration inequalities, martingales)
- (optional) programming (perhaps running simulations)

[5] Fun fact: this is actually a defining property of $e$.

### A few more hints about the project

We will focus on these ideas (among others) following student interest as well as what seems fruitful. Some of these questions would imply others.

- is Bukh's conjecture correct that $10000 \cdot \mathbb{E}[\chi(G_p)] \geq \chi(G)/\log(\chi(G))$
- (from [11]) is there a constant $c_p$ such that $\mathbb{E}[\chi(G_p)] \geq c_p \chi(G)^p$
- can we improve the bound $\mathbb{E}[\chi(G_{1/2})] \geq \chi(G)^{1/2}$ in any way
- (non-probabilistic from [11]) for all graphs $G$, is there a subgraph $G'$ for which $\chi(G') \geq \chi(G)/100$, and $\alpha(G')\chi(G') \leq 100 \cdot |V(G')|$

## References


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