Assignment 1

Yale Math 230
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Due: Friday September 8, 2017 at 9:25am

This assignment is to be submitted in class on Friday morning. You are encouraged to learn LaTeX. You may collaborate, but each student must submit their own work. Mark each problem indicating who (if anyone) you worked with.

1. Prove that every odd square is one more than a multiple of 8.
2. Suppose $P_1, P_2, \ldots, P_n$ are points in the plane such that any three of them can be covered by a triangle of area 1.
   (a) Prove the entire set of points can be covered by a rectangle of area 4.
   (b) Prove the entire set of points can be covered by a triangle of area 4.
3. Given a positive integer n, let $f(n)$ be the integer obtained by summing the squares of the digits of n (when written in base 10). For example, $f(234) = 2^2+3^2+4^2 = 4+9+16 = 29$. Prove that for any positive integer, repeatedly applying this map eventually reaches the number 1, or it reaches the number 4 (which then goes in a loop). [For instance, $19 \mapsto 82 \mapsto 68 \mapsto 100 \mapsto 1$.] (Pro-tip: perhaps verify this for 'small' values on a case-by-case basis.)
4. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ be points with distinct x-values. Prove there exists a polynomial $p(x)$ of degree at most 2 passing through these points. State and prove a similar theorem for four points.
5. The *Goldbach conjecture* is one of the biggest unsolved problems in number theory. It is equivalent to the following
   - Every integer $n \geq 2$ can be written as the average of exactly two primes.

   The *Pat-bach conjecture* is the following thing I recently thought about
   - For each $k \geq 2$, every integer $n \geq 2$ can be written as the average of exactly k primes.

   (a) Prove that the Goldbach conjecture implies the Pat-bach conjecture. [So these conjectures are equivalent.]
   (b) Read the wikipedia article on the Goldbach conjecture and (*slightly*) link-dive as necessary. Use with attribution some result you find to prove the Pat-bach conjecture for some fixed value of k.
6. Prove every right triangle with three integer side lengths has an integral area.

Bonus Question:
Suppose we start with an n-tuple of integers $(a_1, a_2, \ldots, a_n)$ and consider the map $(x_1, x_2, \ldots, x_n) \mapsto (|x_1 - x_n|, |x_2 - x_1|, |x_3 - x_2|, \ldots, |x_n - x_{n-1}|)$. For which values of n will repeatedly applying this map necessarily reach $(0, 0, 0, \ldots, 0)$ [eventually, regardless of the initial input]?