Assignment 2

Yale Math 231
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Due: Friday February 2, 2018 at 11:50pm
—Submit electronically via canvas—

You are encouraged to learn LaTeX if you feel like it, but you can still just write it all by hand and scan it in. You may collaborate, but each student must submit their own work. Mark each problem indicating anybody with whom you worked.

"My methods are really methods of working and thinking; this is why they have crept in everywhere anonymously." —Amalie Emmy Noether (1882-1935). Emmy Noether was a German mathematician who made immense contributions to abstract algebra and theoretical physics despite *profound* systemic barriers and professional exploitation for being a woman in mathematics.

Due to her evident strengths in invariant theory (and intrigued by its possible applications to general relativity), David Hilbert (1862-1943) and Felix Klein (1849-1925) invited Noether to the University of Göttingen in 1915, where they hoped she would be appointed as a professor. Yet despite their push and a letter of praise from Einstein himself, the university flatly refused to offer any position whatsoever arguing: "what will our soldiers think when they return to the university and find that they are required to learn at the feet of a woman?"

For the love of mathematics, Noether worked as a lecturer at Göttingen for years despite being refused *any form of payment* or even recognition from the university. Although the university eventually started paying her, they still would not grant her the title of full professor. All the while, Noether continued to produce groundbreaking work in mathematics and theoretical physics. Nathan Jacobson (1910-1999) [Yale professor from 1947 until retirement] wrote of Noether, "the development of abstract algebra, which is one of the most distinctive innovations of twentieth century mathematics, is largely due to her—in published papers, in lectures, and in personal influence on her contemporaries." Her mathematical work ranged from ring and field theory to the development of algebraic topology. In 1932, she was awarded the Ackermann-Teubner Memorial Prize in Mathematics, and she was invited to deliver the plenary address at the International Congress of Mathematicians. But even still, she was never admitted into the Göttingen academy of sciences nor granted full professorship.

When Hitler came to power in 1933, Noether (a Jewish woman) was forced to flee the country. She was offered a position at Bryn Mawr College, where—for the first time in her life—she was paid a full salary and treated as a proper faculty member. She was happy at Bryn Mawr, but within two years, at the age of 53, she suddenly died after undergoing surgery for an ovarian cyst.

New grading practices: (Recall this from last time)

- Assignments now have two parts (required and optional).
- *Problems marked $\diamond$ are **optional***, and the others are required.
- You are graded separately for the required and optional problems.
- You must do all of the required problems, and that portion of your assignment grade will be based on total points.
- **You must do at least 2 of the optional $\diamond$ problems**, and you get to pick which ones. You can submit more than two, in which case your best two will count towards your grade, and the others will go towards extra credit. All optional problems have equal weight.

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1. (15 pt) An $n \times n$ matrix A is called *nilpotent* iff there is an integer k such that $A^k = 0$. Prove the following are equivalent:

   (i) $A^k = 0$ for some k (i.e., A is nilpotent)

   (ii) Every eigenvalue of A is equal to 0

   (iii) $A^n = 0$ (this is not *the definition* of nilpotent since this requires $k = n$)

2. (10 pt) Suppose A is a real-valued matrix with all its eigenvalues real. Prove/disprove: A is diagonalizable over $\mathbb{C}$ iff it is diagonalizable over $\mathbb{R}$.

3. (10 pt) For square matrices A and B, prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. Generalize this to show $\operatorname{tr}((AB)^k) = \operatorname{tr}((BA)^k)$ for all positive integers k.

   Remark: Do not use the result of 230-pset-12-4d (i.e., that AB and BA have the same characteristic polynomial), since (a) I bet your proof of that fact used a topological argument that doesn't [easily...] extend to arbitrary fields, and (b) I want to use this to get an alternative proof.

4. (15 pt) We say a matrix A has a *square root* iff there is a B such that $A = B^2$.

   (a) Find at least five real square roots of the 2×2 identity matrix.

   (b) Prove that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has no square roots (not even over $\mathbb{C}$).

   (c) Prove that every diagonalizable matrix has a square root (perhaps over $\mathbb{C}$). Moreover, prove that if A is a real diagonalizable matrix with all its eigenvalues non-negative, then A has a real square root.

5.$\diamond$ (Optional) [Putnam] Alan and Barbara play a game in which they take turns filling out entries of an initially empty 2018×2018 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

6.$\diamond$ (Optional) [Recreational math[1]] Turns out there's a positive integer n for which 2^n begins with the digits 8675309... (in fact, $n = 92936642$ works).

[1] Tony and Parker asked me about this problem, and I like it. So here we go.

(a) Prove $\lg(10) \notin \mathbb{Q}$, where $\lg = \log_2$ is the binary log (i.e., $2^{\lg(t)} = t$).

(b) Prove that for every positive integer r, there are positive integers x and y such that $r10^x \leq 2^y < (r+1)10^x$.

7.◇ (Optional)

(a) For which integers k can we replace the number "2" of question 6 with the number k and still have the same conclusion? (Prove your claim)

(b) Find an integer n for which 3^n begins with the digits 8675309…. (I found $2^{92936642}$ essentially by hand, but I suspect finding the *smallest* such numbers would involve some coding.)

8.◇ (Optional) [Putnam] Let S be the set of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ whose entries a, b, c, d (in that order) form an arithmetic progression. Find all matrices M in S for which there is some integer $k > 1$ such that M^k is also in S.

9.◇ (Optional) Turns out that two matrices A and B have the same characteristic polynomial iff $\operatorname{tr}(A^k) = \operatorname{tr}(B^k)$ for all k. Prove this for diagonal matrices.[2]

10.◇ (Optional) Consider the $n \times n$ *Vandermonde matrix*, which is given by

$$V = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & x_3^3 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^{n-1} \end{pmatrix}.$$

(a) Prove that $\det(V) = \prod_{1 \leq i < j \leq n}(x_j - x_i)$

(b) Prove that for any n points $(x_1, y_1), \ldots, (x_n, y_n)$ with distinct x values, there is a unique polynomial of degree at most $n-1$ containing those points. (This is an alternative proof to the one we did over $\mathbb{R}$ via mean value theorem, and it proves the generalization to any field.)

11.◇ (Optional) [Putnam] Let H be an $n \times n$ Hadamard matrix (i.e., its entries are in $\{-1, 1\}$ and $HH^\top = nI_n$). Suppose H has an $a \times b$ submatrix[3] whose entries are all 1. Show that $ab \leq n$.

12.◇ (Optional) A *row stochastic* matrix is a matrix with non-negative real entries such that each row sums to 1.

(a) Prove the set of row stochastic matrices is closed under multiplication.

(b) If A is a row stochastic matrix with real eigenvalue λ, prove $\lambda \leq 1$.

Bonus: Just do more optional problems.

Special bonus: I want to know about that sphere-cube game, and I offer anybody who solves it a free pass on the homework assignment of their choice. Hand this one in to me separately.

[2]Full generalization to come on the next p-set.

[3]A submatrix is a matrix obtained by deleting rows and/or columns.