Assignment 5

Yale Math 230
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Due: Friday October 5, 2018 at 9:25am

Archimedes (c. 287 – c. 212 BCE) was one of the best mathematicians of all time and he was *by far* the best European mathematician of antiquity. His work with geometry and infinitesimals was two thousand years ahead of his time, and his insights into what are now termed analysis and integral calculus were unparalleled. His works were marked by an astounding rigor uncharacteristic of his time, and which was not matched until the formalizations of limits by Cauchy and Weierstrass in 1821. Archimedes was a native of Syracuse, which was besieged by Rome from 214 to 212 BCE as part of the Second Punic War. By then an old man, Archimedes is said to have developed a series of truly *incredible* weapons to defend his city, and there are accounts that the Roman soldiers trembled in fear whenever they saw the wizened figure of Archimedes standing atop the city walls. According to Plutarch, when the city fell, a Roman soldier burst into Archimedes's home to find him completely preoccupied with a geometric problem. The soldier demanded the old man follow him to the Roman general, but Archimedes refused saying he first must finish the mathematics problem that he was working on. In anger, the soldier struck him down, and—according to Valerius Maximus—the dying man reached to the mathematics he was working on "protecting the dust with his hands, [saying] 'I beg of you, do not disturb this.'"

Sophie Germain (1776–1831) read this account of Archimedes when she was a girl. She was fascinated by his story, and she decided to devote herself to the study of mathematics. She became extremely passionate, reading every book in her family's library and even teaching herself foreign languages just so she could read mathematical texts written in Latin and Greek.

But Sophie's parents were deeply upset that their daughter wanted to learn mathematics. It was not at all fashionable for a woman to study such things, and it would not improve her eligibility as an unmarried girl in Parisian society. In her book *Women in Mathematics*, Lynn Osen writes of Germain:

> *Her family firmly and stubbornly opposed her decision, but her determination was only strengthened by the vehemence of their opposition. . . . They denied her light and heat for her bedroom and confiscated her clothing after she retired at night in order to force her to sleep. . . . but after her parents were in bed, she would wrap herself in quilts, take out a store of hidden candles, and work at her books all night. After finding her asleep at her desk in the morning, the ink frozen in the ink horn and her slate covered with calculations, her parents finally [allowed] Sophie to study and use her genius as she wished. . . . and Sophie, still without a tutor, spent the years of the Reign of Terror studying differential calculus.*

Today, Archimedes is honored as the image on the front of the Fields Medal with his famous proof of the volume of a sphere on the reverse. As a man, he was able to freely pursue mathematics, but as a woman, Sophie Germain was forbidden even to read about it. And though her parents eventually relented, she was still never allowed to receive any formal training.

Archimedes is celebrated as one of the greatest mathematicians ever, but it is difficult to separate his work from the privilege he had. Sophie Germain's love of math was firmly opposed by society—and even by her own family—for her entire life, and we can never know if she would have surpassed even Archimedes himself had she been given the opportunity. Yet despite all of this opposition... *More of Sophie Germain's story next time*

You are encouraged to learn LaTeX if you feel like it, but you can still just write it all by hand and scan it in. You may collaborate, but each student must submit their own work. Mark each problem indicating anybody with whom you worked.

If you collaborated with nobody, **please note this** on your pset.

Archimedes (c. 287 – c. 212 BCE)

Sophie Germain (1776–1831)

Left: image of the Fields Medal, which is the highest award in math research

1. (Topology) For this problem, suppose S and T are non-empty subsets of $\mathbb{R}^n$. The unions in each of the following parts are allowed to be infinite unions.
   (a) Consider the set $U = \{(x, y) \in \mathbb{R}^2 : 2 < x < 4 \text{ and } 3 < y < 6\} \subseteq \mathbb{R}^2$. Sketch U and prove that it is open.
   (b) If $\emptyset \neq S$ is open, prove we can write S as some union of open balls.
   (c) For *any* set $\emptyset \neq T$, prove we can write T as some union of closed sets.

2. An ***interval*** is a set $I \subseteq \mathbb{R}$ with the property that if $x, y \in I$ and $x \leq t \leq y$, then $t \in I$ as well. Suppose $A, B \subseteq \mathbb{R}$ are intervals and $f : A \to B$. Prove the following are equivalent:
   (i) f is a continuous bijection
   (ii) f is a monotone[1] bijection

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for all real numbers x, y, and z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$.

[1]Recall monotone means either f is always increasing or f is always decreasing.
In particular, f is increasing on A iff for all inputs $x, y \in A$, if $x \leq y$ then $f(x) \leq f(y)$.
And f is decreasing on A iff for all inputs $x, y \in A$, if $x \leq y$ then $f(x) \geq f(y)$.

4. Let $S \subseteq \mathbb{R}^n$, and recall the *closure* of S is defined as

$$\bar{S} = \{\vec{x} \in \mathbb{R}^n : \forall \varepsilon > 0,\ \exists \vec{y} \in S,\ d(\vec{y}, \vec{x}) < \varepsilon\}.$$

   (a) For each element $\vec{x} \in \bar{S}$, prove there is a sequence $\vec{s}_1, \vec{s}_2, \ldots$ of elements in S such that $\lim_{i\to\infty} \vec{s}_i = \vec{x}$.

   (b) Suppose $S \subseteq \mathbb{R}^n$ and that $f, g : \bar{S} \to \mathbb{R}^m$ are two functions each mapping from the closure of S to $\mathbb{R}^m$. Also suppose f and g are continuous on $\bar{S}$. If $f(\vec{x}) = g(\vec{x})$ for all $\vec{x} \in S$, prove $f(\vec{x}) = g(\vec{x})$ for all $\vec{x} \in \bar{S}$.

5. (Point-set topology) Let $C \subseteq \mathbb{R}^n$, and prove the following are equivalent.

   (i) C is closed (i.e., its complement is open);

   (ii) for any sequence $\vec{x}_1, \vec{x}_2, \ldots$ of elements of C, if $\lim\limits_{i\to\infty} \vec{x}_i$ converges, then $\lim\limits_{i\to\infty} \vec{x}_i \in C$;

   (iii) $C = \bar{C}$;

   (iv) there is a set $S \subseteq \mathbb{R}^n$ for which $C = \bar{S}$.

   Remark 1: The equivalence of these gives us a very helpful intuitive understanding of closed sets. They are sets that are "closed under limits" and the "closure" of a set S is the smallest closed set containing S.

   Remark 2: Do 4.a before this problem to save yourself time.

   Remark 3: The implication (iii) $\Rightarrow$ (iv) is absolutely immediate, but I included it so that this problem also proves for us that "the closure of any set is closed" [because (iv) $\Rightarrow$ (i)] and also "taking the closure of any set twice is the same as just doing so once [i.e., $\bar{S} = \bar{\bar{S}}$]" [because (iv) $\Rightarrow$ (iii)].

6. Suppose $A, B \subseteq \mathbb{R}$ and $f : A \to B$.

   (a) If f is continuous and A is an interval, prove $f(A)$ is an interval.
   (Recall: $f(A) = \{f(a) : a \in A\}$)

   (b) If $f : A \to B$ is a monotone bijection, then so is $f^{-1} : B \to A$.
   Note: for this part, do not assume that A is an interval.

   (c) Suppose A is an interval and $f : A \to B$ is a bijection. Prove that if f is continuous, then so is $f^{-1} : B \to A$.

   Remark: The conclusion of (c) doesn't extend at all to functions that don't map from $\mathbb{R}$ to $\mathbb{R}$. In fact, it doesn't even extend to the case that $A \subseteq \mathbb{R}$ isn't an interval! (Recall the example from class of a continuous bijection whose inverse isn't continuous [from the day we defined continuity].)

Bonus 1: Prove that if $\emptyset \neq S$ is any open subset of $\mathbb{R}^n$, then S can be written as the union of at most *countably many* open balls.

Bonus 2: (Prove *or* disprove) If $\emptyset \neq S$ is any open subset of $\mathbb{R}$, then S can be written as the union of finitely many open intervals.