Math 250 - Section C2
Fall 2013
Midterm 2
November 13, 2013
Time Limit: 80 Minutes

Name (Print): ______________________________

Instructor: Pat Devlin

This exam has 10 pages (including this cover page) and 9 problems. Check to see if any pages are missing. Put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the place without a clear ordering is a pain in the butt to read.
- **Mysterious or unsupported answers rarely receive credit**. Some answers require no work. However, the majority require at least some work or explanation or both.
- **When in doubt, show work and explanations**. But no question requires all that much explanation—don't waste your time writing a book!
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	25	
3	25	
4	25	
5	20	
6	20	
7	20	
8	25	
9	0	
Total:	180	

1. (20 points) Consider the following matrices and vectors:

$$A = \begin{pmatrix} 1 & 4 & 0 \\ 2 & 1 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & 2 & 1 & 1 \\ 7 & 7 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 6 & 6 & 9 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 3 & 2 \\ 0 & -1 \\ 2 & 1 \end{pmatrix},$$

$$\vec{u} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \qquad \vec{v} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \qquad \vec{w} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}.$$

Compute each expression given or state that it does not make sense.

(a) $A\vec{v} + \vec{u}$

(b) $\det(C)$

(c) $B^T D - 3A^T$

(d) $\det(\vec{w}) - \det(\vec{u})$

(e) BD^T

2. (25 points) Consider the following matrices

$$M = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} -6 & -3 & 1 \\ 5 & 2 & -1 \\ 2 & 3 & -5 \end{pmatrix}.$$

(a) What is the characteristic polynomial of M? What are the eigenvalues of M? What is the algebraic multiplicity of each of its eigenvalues?

(b) The matrix A has characteristic polynomial $p_A(t) = -(t+1)(t+4)^2$. Find a basis for each eigenspace of A, and find the geometric multiplicity of each of its eigenvalues.

3. (25 points) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation satisfying

$$T\left(\begin{bmatrix} 3 \\ -5 \end{bmatrix}\right) = \begin{bmatrix} 8 \\ -16 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad T\left(\begin{bmatrix} 8 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 8 \\ -16 \\ 0 \end{bmatrix}.$$

Find a basis for $\text{Im}(T)$ and for $\text{null}(T)$. Be sure to either show your work or (if you didn't do that much work) explain how you know your answer is correct.

4. (25 points) Let W be the set of vectors in $\mathbb{R}^4$ of the form $W = \left\{ \begin{bmatrix} r \\ s \\ t \\ r \end{bmatrix} : r, s, t \in \mathbb{R} \right\}$ (that is, W is the set of all vectors in $\mathbb{R}^4$ whose first and fourth entries are equal). Then W is a subspace.

(a) Find a basis for W. What is the dimension of W?

(b) Find a set of vectors in W that are linearly independent but that do not span W.

(c) Find a set of vectors in W that span W but that are not linearly independent.

(d) Let $V = \text{span}\left(\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \right)$. Is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 8 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \\ 9 \end{bmatrix} \right\}$ a basis for V? Explain.

5. (20 points) Consider the following matrix and vectors

$$A = \begin{pmatrix} -7 & -3 & -6 \\ 0 & -4 & 0 \\ 3 & 3 & 2 \end{pmatrix}, \quad \vec{w} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \vec{z} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

(a) Are any of the vectors $\vec{w}$, $\vec{x}$, $\vec{y}$, or $\vec{z}$ eigenvectors of A? For those that are, what are their corresponding eigenvalues?

(b) For each eigenvector in part (a), compute what happens when you multiply that vector by $A^{8675309}$. [You (of course!) don't need to simplify any numbers that show up.]

6. (20 points) Let $T : \mathbb{R}^4 \to \mathbb{R}^5$ be a linear transformation.

   (a) What is the largest possible value for $\dim(\mathrm{Im}(T))$? What is the smallest possible value for $\dim(\mathrm{Im}(T))$?

   (b) For each of the possible values of $\dim(\mathrm{Im}(T))$ found in (a), what would be $\dim(\mathrm{null}(T))$? In particular, what is the largest possible value for $\dim(\mathrm{null}(T))$, and what is the smallest possible value for $\dim(\mathrm{null}(T))$?

   (c) Is it possible for T to be one-to-one[1]? Explain.

   (d) Is it possible for T to be onto[2]? Explain.

[1] Recall that one-to-one means that no matter what $\vec{b}$ is, the equation $T(\vec{x}) = \vec{b}$ always has **at most** one solution.

[2] Recall that onto means that no matter what $\vec{b}$ is, the equation $T(\vec{x}) = \vec{b}$ always has **at least** one solution.

7. (20 points) Consider

$$A = \begin{bmatrix} 1 & 3 & -1 & -1 & -1 \\ 1 & 2 & 0 & 1 & -1 \\ 2 & 5 & -1 & 0 & -2 \\ 2 & 3 & 1 & 4 & -1 \end{bmatrix}, \qquad \text{which has rref} \qquad \text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 5 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) What is the rank and nullity of A?

(b) Are the columns of A linearly independent? Do the columns of A span $\mathbb{R}^4$? Explain.

(c) Find a basis for the row space of A. What's the dimension of row(A)?

(d) Find a basis for the column space of A. What's the dimension of col(A)?

(e) Find a basis for the null space of A. What's the dimension of null(A)?

8. (a) (7 points) What does it mean to say that $\vec{x}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = 5$?

(b) (7 points) What does it mean to say that a space is 13-dimensional? [**Note:** You may *not* reference $\mathbb{R}^{13}$ in your answer.]

(c) (11 points) What properties must a set of vectors have to be called a *subspace*? Give an example of set of vectors in $\mathbb{R}^2$ that is *not* a subspace (your example can be a drawing or a description if you prefer).

9. **Extra credit:**

   (a) Suppose M is an $n \times n$ matrix such that 0 is *not* an eigenvalue of M. Then what can you say about the column space of M? [Hint: what can you say about the rank of M?]

   (b) Let A and B be any appropriately-sized matrices such that AB makes sense. Show that null(B) is a subspace of null(AB) and use this to show rank(B) $\geq$ rank(AB).

   (c) An $n \times n$ matrix, C, is called *nilpotent* iff $C^n = 0$ (the zero-matrix). Show that if C is nilpotent, then all its eigenvalues must be equal to 0. [Hint: you just need to show that if $C\vec{x} = \lambda\vec{x}$ and $\vec{x} \neq 0$, then $\lambda = 0$. What would $C^n\vec{x}$ be?]

   (d) On the other hand, use the Cayley-Hamilton theorem[3] to show that if all the eigenvalues of a matrix are equal to 0, then that matrix is nilpotent.

[3]Recall the Cayley-Hamilton theorem is that if you "plug in" any matrix into its characteristic polynomial, then you get the zero-matrix.