Week 4: Invariants*

Putnam Seminar – Yale Math 199

Tuesday 25-Sept-2018

Sometimes a problem can be much better understood by looking for some piece of information that changes in a predictable way (e.g., something that always stays the same, always increases by 1, tends to a limit, et cetera). Parameters like this are useful to show two configurations are not equivalent, certain processes must terminate, et cetera. Finding these invariants is always the tricky part.

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1. We start with an 5×8 rectangular chocolate bar, which is internally subdivided into an array of 40 squares. We start by picking up the chocolate bar and breaking it along one of its internal lines. We then pick up a piece and repeat this process until there's nothing left to break apart. Show that regardless of how we go about this process, it always takes the same number of steps.

2. For which n can an $n \times n$ chessboard be tiled by non-overlapping 4×1 pieces?

3. A bug starts at the number 1. Each turn, if she is at x, she can either jump to x^2, to $10 + x$, or to $3x - 4$. Is it possible, after some series of hops, for her to ever land on 2018?

4. For which n can an $n \times n$ chessboard be tiled with non-overlapping "T" shaped tetris pieces?

5. We have a 6×6 array of light switches, all of which are initially off. We pick any row (or column) that we like and flip all of the switches in that row (or column). After repeatedly doing this, is it possible to get to an arrangement where exactly 1 light is on?

6. A real number is written in each square of an $n \times n$ array. We can pick any row we like and multiply all the entries in that row by -1. We can do the same with any column. Prove that after performing these operations a finite number of times, we can make it so that all the row sums and all the column sums are positive.

7. An infection spreads among the squares of an $n \times n$ checkerboard in the following manner: If a square has two or more infected neighbors, then it becomes infected itself. Neighbors are orthogonal only, so each square has at most four neighbors. Prove that you cannot infect the whole board if you begin with fewer than n infected squares.

*Some material taken from *Putnam and Beyond* by Razvan Gelca and Titu Andreescu and also *Mathematical Puzzles: A Connoisseur's Collection* by Peter Winkler.

8. There is a heap of 1001 stones on a table. You are allowed to perform the following operation: you choose one of the heaps containing more than one stone, throw away a stone from the heap, then divide it into two smaller (not necessarily equal) heaps. Is it possible to reach a situation in which all the heaps on the table contain exactly 3 stones?

9. Four bugs begin on the XY-plane on the corners of a square. At any time you may cause one bug to jump over a second, placing the first on the opposite side of the second, but at the same distance as before. The jumped bug reains in place. Can you maneuver the bugs to the corners of a larger square?

10. Let S be the smallest set with the following properties: (i) $1 \in S$; (ii) if $x \in S$, then $x^2 \in S$; and (iii) if $x, y \in S$ then $2x - 4y \in S$. Is $2018 \in S$?

11. We start with four congruent right triangles. We then repeatedly pick one of the triangles and cut it into two by dropping an altitude. Prove that after any number of cuts, we can always find at least two triangles that are congruent.

12. Start with three stones located at coordinates $(0, 0), (1, 0)$, and $(0, 1)$. At each step, you must increase the number of stones in the following way. Remove one of the stones (with coordinates (x, y)) and instead put two stones, one at $(x + 1, y)$ and another at $(x, y + 1)$, assuming there are no stones at these coordinates. Is it possible to devise a sequence of steps so that after a finite number of steps, all three of the initially occupied coordinates will be empty?

13. Start with a finite sequence $a_1, a_2, \ldots, a_n$ of positive integers. If possible, choose two indices $j < k$ such that a_j does not divide a_k, and replace a_j and a_k by $\gcd(a_j, a_k)$ and $\mathrm{lcm}(a_j, a_k)$, respectively. Prove that if this process is repeated, it must eventually stop and the final sequence does not depend on the choices made. (Note: gcd means greatest common divisor and lcm means least common multiple.) (Putnam 2008)

14. Given an ordered triple, we are allowed to change two of the entries from a and b to $(a + b)/\sqrt{2}$ and $(a - b)/\sqrt{2}$. Is it possible to obtain $(1, \sqrt{2}, 1 + \sqrt{2})$ from the triple $(2, \sqrt{2}, 1/\sqrt{2})$?

15. Are there two powers of 2 with the same number of digits such that the digits of one can be rearranged to form the digits of the other?

16. Each point on $\{1, 2, 3, \ldots, n\}^2$ in the XY plane is occupied by a peg. At any time a peg can jump horizontally or vertically over another peg. The peg which was jumped over is then removed. Prove that if n is divisible by 3, then it is impossible to finish with only 1 peg left on the board.

17. For an integer $n \geq 4$, consider an n-gon inscribed in a circle. Dissect the n-gon into $n-2$ triangles by non-intersecting diagonals. Prove the sum of the radii of the incircles of these $n-2$ triangles doesn't depend on the dissection.

18. (Conway's solidiers) Each grid point on the XY plane on or below the X-axis is occupied by a peg. At any time, a peg can be made to jump over a neighbor peg (horizontally or vertically adjacent) and onto the next grid point in line, provided that the new location was unoccupied. The jumped peg is then removed. Can you get a peg arbitrarily far above the X-axis?