Polynomials*

Putnam Seminar

Week 9

Polynomials: polynomials look like $p(x) = \sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ (if $a_n \neq 0$, then $p(x)$ is said to have *degree* n).

Single variable: If $\deg(p(x)) \leq n$ and $p(x)$ has at least $n+1$ distinct roots, then $p(x) = 0$ [i.e., $a_i = 0$ for all i]. (This works over any field.)

Useful thing about modular arithmetic: Given integers a and b and a polynomial $f(x)$ with integer coefficients, $a - b$ must divide $f(a) - f(b)$.

Rational roots test: Say $P(x) = \sum_{i=0}^{n} a_i x^i$ has integer coefficients, $a_n \neq 0$, and r and s are integers. If $P(r/s) = 0$, then r divides a_0 and s divides a_n.

Two variables: Suppose $P(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} x^i y^j$. And suppose there are sets A, B where $|A| = |B| = n + 1$ and that $P(a, b) = 0$ for all $(a, b) \in A \times B$. Then $P(x, y) = 0$ [i.e., all $a_{i,j} = 0$]

Multiplicity: If r is a root with multiplicity m, then $p^{(j)}(r) = 0$ for $1 \leq j \leq m-1$ and $p^{(m)}(r) \neq 0$. (where $p^{(j)}(x)$ is the j^{th} derivative)

Viète's relations: Suppose $a_n \neq 0$ and $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ factors as $P(x) = a_n(x - r_1)(x - r_2) \cdots (x - r_n)$. Then we have

$$\begin{aligned}
-a_{n-1}/a_n &= r_1 + r_2 + \cdots + r_n \\
a_{n-2}/a_n &= r_1 r_2 + r_1 r_3 + \cdots + r_{n-1} r_n \\
&\vdots \\
(-1)^k a_{n-k}/a_n &= \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} r_{i_1} r_{i_2} \cdots r_{i_k} \\
&\vdots \\
(-1)^n a_0/a_n &= r_1 r_2 r_3 \cdots r_{n-1} r_n.
\end{aligned}$$

- Let $f(x) = 7x^5 - x^2 + 2x - 8$. Find all the rational solutions to $f(x) = 0$.
- Say $P(x)$ is a degree 3 polynomial and $P(2) = P(5) = P(4) = 0$. Also suppose $P(1) = 4$. Find $P(0)$.

1. Find all polynomials $P(x)$ such that $2P(x^2) = P(x) + 6$.
2. Find all polynomials $f(x)$ such that $xf(x-1) = (x+1)f(x)$ for all x.
3. Let $f(x)$ be a polynomial with integer coefficients each between 1 and 1000. Is there an integer N where $f(x)$ can be completely determined from $f(N)$?

*Some material taken from *Putnam and Beyond* by Razvan Gelca and Titu Andreescu

4. Suppose $p_1(x) \neq p_2(x)$ are polynomials with degree at most 7. What's the maximum number of times that the graphs of p_1 and p_2 can intersect?

5. (Putnam 1971-A2) Find all polynomials $p(z)$ where $p(0) = 0$ and $p(z^2+1) = p(z)^2+1$.

6. (USAMO 1974) Let a, b, c be distinct integers. Show there is no polynomial $P(x)$ with integer coefficients such that $P(a) = b$, $P(b) = c$, and $P(c) = a$.

7. (Putnam 2007-A1) Let f be a nonconstant polynomial with positive integer coefficients. Prove that for any positive n, $f(n)$ divides $f(f(n)+1)$ iff $n = 1$.

8. (Putnam 2003-B1) Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

hold identically?

9. (IMO 1976) Let $P_1(x) = x^2 - 2$ and for $j \geq 2$ define $P_j(x) = P_1(P_{j-1}(x))$. Show that for each integer $n > 0$, the roots of $P_n(x) = x$ are real and distinct.

10. (Putnam 1999-A2) Let $p(x)$ be a polynomial that is nonnegative for all real x. Prove that for some k, there are polynomials $f_1(x), \dots, f_k(x)$ such that

$$p(x) = \sum_{j=1}^{k} (f_j(x))^2.$$

11. Suppose $f(x) = (x-a_1)(x-a_2)\cdots(x-a_n)-2$ for distinct integers $a_1, \dots, a_n$. Also suppose that $f(x) = g(x)h(x)$ where g and h are both non-constant polynomials with integer coefficients. Prove that $n \leq 3$.

12. Suppose P is a polynomial with integer coefficients and a, b, c, d are distinct integers such that $P(a) = P(b) = P(c) = P(d) = 7$. Prove $P(m) = 14$ has no integer solutions.

13. (Putnam 2004-B1) Suppose $p(x) = \sum_{i=0}^{n} c_i x^i$ is a polynomial with integer coefficients, and suppose that r is a rational number such that $p(r) = 0$. Show that the following n numbers are all integers

$$c_n r,\ c_n r^2 + c_{n-1} r,\ c_n r^3 + c_{n-1} r^2 + c_{n-2} r, \dots,\ c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r.$$

14. Suppose $P(x, y, z)$ is a polynomial, and consider the polynomial

$$Q(x,y,z) = P(x,y,z)+P(y,z,x)+P(z,x,y)-P(x,z,y)-P(z,y,x)-P(y,x,z).$$

Show that $(x-y)(y-z)(z-x)$ divides $Q(x, y, z)$.

15. For every integer $n > 1$, prove $P(x) = \sum_{k=0}^{n} x^k/k!$ has no rational roots.

16. Suppose $R_1(x) = p_1(x)/q_1(x)$ and $R_2(x) = p_2(x)/q_2(x)$ are distinct rational functions over $\mathbb{R}$. Show that there exists a real number M such that for all $x \geq M$, $R_1(x) > R_2(x)$ iff $R_1(M) > R_2(M)$. (i.e., eventually either $R_1(x)$ is always bigger than $R_2(x)$ or $R_2(x)$ is always bigger than $R_1(x)$)[1]

[1]This is one way to construct the *hyperreals* (google them because they're awesome!).

17. For every real $n \times n$ matrix A and every $\varepsilon > 0$, show there exists a value $0 < t < \varepsilon$ such that $A + tI_n$ is invertible.

18. (Putnam 2002-A1) Let k be a fixed positive integer. The n^{th} derivative of $(x^k - 1)^{-1}$ has the form $\frac{P_n(x)}{(x^k-1)^{n+1}}$, where $P_n(x)$ is a polynomial. Find $P_n(1)$.

19. (Putnam 2003-B4) Let

$$f(z) = az^4 + bz^3 + cz^2 + dz + e = a(z - r_1)(z - r_2)(z - r_3)(z - r_4),$$

where a, b, c, d, e are integers and $a \neq 0$. Show that if $r_1 + r_2$ is rational and $r_1 + r_2 \neq r_3 + r_4$ then $r_1 r_2$ is a rational number.

20. (Putnam 2016-A1) Find the smallest positive integer j such that for every polynomial $p(x)$ with integer coefficients and for every integer k, the integer

$$p^{(j)}(k) = \left.\frac{d^j}{dx^j} p(x)\right|_{x=k}$$

(the j^{th} derivative of $p(x)$ evaluated at k) is divisible by 2016.

21. Find all polynomials satisfying the equation $(x+1)P(x) = (x-10)P(x+1)$.

22. (IMO 1975) Find all polynomials, P, in two varaibles with the following three properties: [property (i) says "P is homogeneous of degree n"]

    (i) for some integer $n > 0$ and for all $x, y, t \in \mathbb{R}$, $\quad P(xt, yt) = t^n P(x, y)$;

    (ii) for all real a, b, c we have $P(b+c, a) + P(c+a, b) + P(a+b, c) = 0$; and

    (iii) $P(1, 0) = 1$.

23. (USAMO 1984) $P(x)$ is a polynomial of degree $3n$ such that

$$\begin{aligned} P(0) = P(3) = \cdots &= P(3n) = 2, \\ P(1) = P(4) = \cdots &= P(3n-2) = 1, \\ P(2) = P(5) = \cdots &= P(3n-1) = 0, \qquad \text{and} \\ P(3n+1) &= 730. \end{aligned}$$

Determine n.

24. (Russian math olympiad, 2002) Let $P(x)$ be a polynomial with odd degree and real coefficients. Show that the equation $P(P(x)) = 0$ has at least as many distinct real roots as $P(x) = 0$ does [counted without multiplicity].

25. Find all polynomials $P(x)$ with real coefficients for which there is a positive integer n such that for all x, we have $P(x + \frac{1}{n}) + P(x - \frac{1}{n}) = 2P(x)$.

26. (USAMO 1975) Let $P(x)$ be a polynomial of degree n and suppose $P(k) = k/(k+1)$ for all integers $0 \leq k \leq n$. Find $P(n+1)$.

27. (Romanian math olympiad, 1979) Let $P(x)$ be a polynomial with complex coefficients. Prove that $P(x)$ is an even function iff there is some polynomial $Q(x)$ with complex coefficients such that $P(x) = Q(x)Q(-x)$.