Integration by substitution

This integration technique is based on the chain rule for derivatives. It allows us to change some complicated functions into pairs of nested functions that are easier to integrate.

1 The idea behind integration by substitution

Let us first recall the chain rule, which we studied in the "Algebra of derivatives" module. The rule says:

$$(f(g))' = f'(g)g'$$

Integrating this, we find that

$$\int f'(g(x))g'(x)dx = f(g(x)) + c$$

The expression is rather involved, and can be hard to understand at first. This is to be expected: the chain rule was the most complicated one of our rules for derivatives.

The best thing to do is illustrate this on a few examples, to get a feeling for how it works.

Example 1: First we will calculate

$$\int e^{2x}dx$$

This integral is not difficult to guess: in fact, we have seen this function several times in previous modules. We know that $(e^{2x})' = 2e^{2x}$, which gives us the answer:

$$\int e^{2x}dx = \frac{e^{2x}}{2} + c$$

.

Secretly, we using the chain rule here, with $g(x) = 2x$. Then $g'(x) = 2$, and we can write

$$\int e^{2x}dx = \int \frac{e^{2x}}{2}2dx = \frac{e^{g(x)}}{2}g'(x)dx = \frac{e^{g(x)}}{2} + c = \frac{e^{2x}}{2} + c$$

We wish to make this process more formal, in order to apply it in more complicated cases. To this end, we introduce a new variable $u = 2x$. Then we perform a change of variables, which gives this technique its name: we substitute u for x. For the function, this means replacing x with $u/2$ everywhere, getting

$$e^{2x} = e^u$$

This is not enough: we also have to switch the integrated variable from x to u. We can differentiate $u = 2x$, getting $du = 2dx$; or $dx = \frac{1}{2}du$. Now we are done:

$$\int e^{2x} dx = \int e^u \cdot \frac{1}{2} du = \frac{1}{2} e^u + c = \frac{e^{2x}}{2} + c$$

In the last step, we put x back into the expression. We need to report the answer in terms of the original variable, not in terms of some temporary variable that we used to evaluate the integral.

There is another reason to switch back to x: if this integral were a definite one, the limits for x are not the same as the limits for u. To illustrate this, suppose that the limits for x were a to b. We know that $u = 2x$, so while x goes from a to b, the new variable u will travel from $2a$ to $2b$:

$$\int_a^b e^{2x} dx = \int_{2a}^{2b} \frac{e^u}{2} du$$

We can check that this is true, since we know how to evaluate both integrals:

$$\int_a^b e^{2x} dx = \left. \frac{e^{2x}}{2} \right|_a^b = \frac{e^{2a} - e^{2b}}{2}$$

$$\int_{2a}^{2b} \frac{e^u}{2} du = \left. \frac{e^u}{2} \right|_{2a}^{2b} = \frac{e^{2a} - e^{2b}}{2}$$

We can always switch the limits as we perform the change from x to u. Usually it is much less trouble to calculate the indefinite integral by switching to u, convert the answer back to x, and evaluate using the original limits.

Example 2: We can make our first example a bit more complicated:

$$\int x e^{x^2} dx$$

It is the e^{x^2} that is giving us trouble, so we should substitute a variable that separates the polynomial and the exponential: $u = x^2$, so $du = 2xdx$. Now we have

$$\int x e^{x^2} dx = \int \frac{e^{x^2}}{2} \cdot 2xdx = \int \frac{e^u}{2} du = \frac{e^u}{2} + c = \frac{e^{x^2}}{2} + c$$

2 Integrating trigonometric functions

Substitution is ideal for many expressions involving trigonometric functions.

Example 1: Let us try to integrate

$$\int \sin(x)\cos(x)dx$$

It is not so easy to see how to do this, but note that $\cos(x)$ is the derivative of $\sin(x)$. Now we're in business: this is a perfect situation for substitution. We will use $u = \sin(x)$, which means that $du = \cos(x)dx$ and our integral becomes

$$\int \sin(x)\cos(x)dx = \int udu = \frac{1}{2}u^2 + c = \frac{1}{2}\cos^2(x) + c$$

We can use this technique for integrating many other expressions involving $\sin(x)$, so long as they are multiplied by a $\cos(x)$ at the end.

Example 1a: Polynomials, using the same substitution:

$$\int \sin^{10}(x)\cos(x)dx = \int u^{10}du = \frac{1}{11}u^{11} + c = \frac{1}{11}(\sin(x))^{11} + c$$

Example 1b: Other functions, same substitution:

$$\int e^{\sin(x)}\cos(x)dx = \int e^u du = e^u + c = e^{\sin(x)} + c$$

The next question is, what do we do when the $\sin(x)$ is alone, with no $\cos(x)$ around. Let us discuss powers of $\sin(x)$, which will let us integrate all polynomials in $\sin(x)$.

Given an odd power of $\sin(x)$, we factor it into multiples of $\sin^2(x)$ times a single $\sin(x)$. Then replace each $\sin^2(x)$ by $[1 - cos^2(x)]$ and integrate.

Example 2: Let us try this with $\int \sin^3(x)dx$.

$$\sin^3(x) = \sin^2(x)\sin(x) = [1 - \cos^2(x)]\sin(x) = \sin(x) - \cos^2(x)\sin(x)$$

Now we can evaluate the integral:

$$\int \sin^3(x)dx = \int [\sin(x) - \cos^2(x)\sin(x)]dx = -\cos(x) + \frac{1}{3}\cos^3(x) + c$$

Even powers of $\sin(x)$ are a little more tricky. For this, we need trigonometric identities referred to as "double angle formulas":

$$\begin{aligned}\cos(2x) &= \cos^2(x) - \sin^2(x) \\ \sin(2x) &= 2\sin(x)\cos(x)\end{aligned}$$

The first of these identities can be rewritten to leave only $\sin(x)$ on the right hand side:

$$\cos(2x) = \cos^2(x) - \sin^2(x) = [1 - \sin^2(x)] - \sin^2(x) = 1 - 2\sin^2(x)$$

We can rewrite this to get an expression for $\sin^2(x)$:

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

Now we can integrate $\sin^2(x)$:

$$\int \sin^2(x)dx = \int \frac{1 - \cos(2x)}{2}dx = \frac{x}{2} - \frac{\sin(2x)}{4} + c$$

To integrate $\sin^4(x)$, we first express it in terms of $\cos^2(2x)$. This we can write in terms of $\cos(4x)$, using the first double angle formula, rewritten with cos on the right side.

Everything in this section can be applied to expressions of $\cos(x)$ multiplied by its derivative, $-\sin(x)$. The only change is with even powers of $\cos(x)$ with no $\sin(x)$ present, as indicated above This is left for the exercises.

3 Trig substitution

Trig substitution was created to help with certain sums and differences of squares.

Example 1: The most basic example of trig substitution concerns the integral

$$\int \frac{1}{\sqrt{1 - x^2}}dx$$

This is not an easy integral to do, and certainly not to guess. Trig substitution comes to the rescue: it is designed to eliminate the unfortunate square root.

Here is the trick: we shall use $u = \arcsin(x)$, with the typical convention that u lies in the interval $[-\pi/2 : \pi/2]$.

To use the substitution, we write $\sin(u)$ in place of x everywhere. This turns $1 - x^2$ into $1 - \sin^2(u) = \cos^2(u)$. Here $\cos(u)$ is non-negative, since $u \in [-\pi/2 : \pi/2]$. We can therefore write

$$\sqrt{1 - \sin^2(u)} = \sqrt{\cos^2(u)} = \cos(u)$$

From the substitution expression $x = \sin(u)$, we see that $dx = \cos(u)du$. Altogether, we have:

$$\int \frac{1}{\sqrt{1 - x^2}} dx = \int \frac{1}{\cos(u)} \cos(u) du = \int du = u + c = \arcsin(x) + c$$

There are two other typical forms of trig substitution.

For expressions with $1 + x^2$, we use $x = \tan(u)$. Then

$$1 + x^2 = \frac{\cos^2(u)}{\cos^2(u)} + \frac{\sin^2(u)}{\cos^2(u)} = \frac{1}{\cos^2(u)} = \sec^2(u)$$

This is especially useful when the $1 + x^2$ is on the bottom of a fraction, or under a square root. For change of integration variable, we get

For expressions with $x^2 - 1$, we use $x = \sec(u)$. Then

$$x^2 - 1 = \frac{1}{\cos^2(u)} - 1 = \frac{\sin^2(u) + \cos^2(u)}{\cos^2(u)} - \frac{\cos^2(u)}{\cos^2(u)} = \frac{\sin^2(u)}{\cos^2(u)} = \tan^2(u)$$

All three substitutions are summarized in the table below.

Expression	Substitution	Identity used
$1 - x^2$	$x = \sin(u)$	$1 - \sin^2(u) = \cos^2(u)$
$1 + x^2$	$x = \tan(u)$	$1 + \tan^2(u) = \sec^2(u)$
$x^2 - 1$	$x = \sec(u)$	$\sec^2(u) - 1 = \tan^2(u)$