

The Alternating Series Test

So far we have considered mostly series all of whose terms are positive. If the signs of the terms alternate, then testing convergence is a much simpler matter. On the other hand, the limit to which some of these series converge depends delicately on the order in which the terms are added. How can this be? Let's see.

Suppose all a_n are positive. An *alternating series* is any series of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - \cdots \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + \cdots \quad (1)$$

Leibnitz found conditions under which alternating series converge. We sketch a proof at the end of this section.

Alternating Series Test An alternating series (1) converges if

1. for all n , $a_n \geq a_{n+1}$, and
2. as $n \rightarrow \infty$, $a_n \rightarrow 0$

As we have seen, convergence is not determined by any finite collection of a_n , so the first condition of the Alternating Series Test need hold only for all $n \geq M$ for some M .

Example 1 Does the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n} + 1}{\sqrt{n+2}}$$

converge or diverge? This is certainly an alternating series. To check that the terms are eventually decreasing, compute the derivative of $f(x) = (\sqrt{x} + 1)/(\sqrt{x+2})$, obtaining

$$f'(x) = \frac{2 - \sqrt{x}}{2\sqrt{x}(x+2)^{3/2}}$$

negative for $x > 4$. Thus the terms of the alternating series are decreasing for $n > 4$. However, condition 2 of the Alternating Series Theorem is not satisfied, because

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n} + 1}{\sqrt{n+2}} = 1$$

Alone, not satisfying the hypotheses of the Alternating Series Test is not sufficient reason to conclude that the series does not converge. The theorem says that if its conditions are satisfied, then the series converges. It DOES NOT say that if the conditions are not satisfied, then the series does not converge. However, for this example we can conclude that the series diverges, because the way in which the second condition is not satisfied is a violation of the n^{th} term test.

For another example, the alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{n+1}/n$ converges, because it satisfies the conditions of the Alternating Series Test. That the harmonic series diverges, but the alternating harmonic series converges, motivates these definitions.

The series $\sum_{n=1}^{\infty} a_n$ *converges absolutely* if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

A series *converges conditionally* if it converges but does not converge absolutely.

Example Does the series $\sum_{n=1}^{\infty} (-1)^n/\sqrt{n+1}$ converge absolutely, converge conditionally, or diverge?

Certainly, this is an alternating series, with $1/\sqrt{n+1}$ a decreasing function of n satisfying $\lim_{n \rightarrow \infty} 1/\sqrt{n+1} \rightarrow 0$. So we know the series converges by the Alternating Series Test. Does it converge absolutely? The absolute values of the terms are $1/\sqrt{n+1}$, looking much like $1/\sqrt{n}$. This suggests the Limit Comparison Test:

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n+1}}{\sqrt{n}} = 1$$

Now $\sum_{n=1}^{\infty} 1/\sqrt{n}$ is a p -series with $p = 1/2$, and consequently diverges. The series of this example converges but does not converge absolutely, so must converge conditionally.

One relation between conditional and absolute convergence is more-or-less what we expect.

Absolute Convergence Theorem If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then the series converges.

At the end of this section we sketch a proof of the Absolute Convergence Theorem. On the other hand, the alternating harmonic series shows that a converging series need not converge absolutely.

Absolutely convergent series have a property, of great use to us, that appears to be just an extension of the commutative law of addition to infinite series. We shall see the result is much more subtle than this. For example, this theorem was proved by Riemann in 1867.

Rearrangement Theorem If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $\sum_{n=1}^{\infty} b_n$ is ANY rearrangement of $\sum_{n=1}^{\infty} a_n$ – that is, there is a 1 – 1 correspondence between $\{a_i\}$ and $\{b_j\}$ – then $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$.

For finite series, indeed this is just the commutative law of addition. But in general, this law does not extend to infinite series. In fact, every infinite series that converges conditionally but not absolutely can be rearranged to converge (conditionally) to any number at all.

The alternating harmonic series illustrates this point. The terms with odd denominators all are positive, and

$$\sum_{i=1}^{\infty} \frac{1}{2n-1}$$

diverges to $+\infty$. Fig. 1 shows this by a modification of the Integral Test. The shaded rectangles have areas that sum to $+\infty$, because they cover the area under the graph of $1/x$, which we know diverges to $+\infty$. Now the sum of the areas of these rectangles is

$$2 \cdot 1 + 2 \cdot \frac{1}{3} + 2 \cdot \frac{1}{5} + 2 \cdot \frac{1}{7} + \cdots$$

If $\sum_{n=1}^{\infty} 1/(2n-1)$ converged, then so would $2 \sum_{n=1}^{\infty} 1/(2n-1) = \sum_{n=1}^{\infty} 2/(2n-1)$, but Fig. 1 shows this is not so.

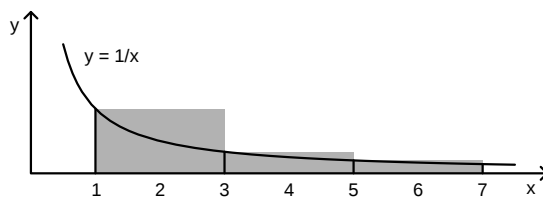


Figure 1: The sum of the odd terms of the alternating harmonic series diverges to ∞ .

A similar argument shows that

$$\sum_{n=1}^{\infty} -\frac{1}{2n}$$

diverges to $-\infty$.

Now pick any number at all, say $\pi/2$. Add the positive terms of the alternating harmonic series until the sum first exceeds $\pi/2$. Because the series of positive terms diverges, we know the sum of positive terms eventually will exceed $\pi/2$.

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} \approx 1.67619 > \frac{\pi}{2}$$

Now add the negative terms until the sum drops below $\pi/2$.

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{2} \approx 1.17619 < \frac{\pi}{2}$$

Now add the remaining positive terms until the sum exceeds $\pi/2$.

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{2} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} + \frac{1}{17} \approx 1.58062 > \frac{\pi}{2}$$

Now add the remaining negative terms until the sum drops below $\pi/2$.

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{2} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} + \frac{1}{17} - \frac{1}{4} \approx 1.33062 < \frac{\pi}{2}$$

Continue in this fashion. At each step, we have used finite collections of positive and of negative terms. Removing the first few (trillion) terms of a diverging series still leaves a diverging series, so always enough positive and negative terms remain to continue the sum to exceed $\pi/2$ and to drop below $\pi/2$. Moreover, as $n \rightarrow \infty$, both $1/(2n-1)$ and $-1/(2n)$ go to 0, so the amount of overshoot and of undershoot of $\pi/2$ goes to 0. That is, this recipe gives a rearrangement of the alternating harmonic series that converges to $\pi/2$.

If the commutative law of addition extended to allow arbitrary rearrangements of infinite series, all rearrangements of every convergent series would have the same sum.

So now maybe it seems strange that all rearrangements of an absolutely convergent series have the same sum, but this is the content of Riemann's Rearrangement Theorem. The proof is a bit involved, so we omit it here.

Sketch of a proof of the Alternating Series Test

Recall S_n denotes the sum of the first n terms of the series. Because this is a finite sum, terms can be associated in any way

$$S_{2m} = (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + \cdots + (a_{2m-1} - a_{2m}) \quad (2)$$

$$= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2m-2} - a_{2m-1}) - a_{2m} \quad (3)$$

Recalling all the a_i are positive, it follows from condition 1 of the Alternating Series Test that all the bracketed terms of Eq. (2) are nonnegative, so the sequence $\{S_{2m}\}$ is nondecreasing. Similarly, each bracketed term $(a_{2i} - a_{2i+1})$ in Eq. (3) is positive, so all S_{2m} are bounded above by a_1 . That is, the even partial sums constitute a nondecreasing sequence bounded above and so by the Monotone Convergence Theorem,

$$\lim_{m \rightarrow \infty} S_{2m} = L \quad (4)$$

Next, because $S_{2m+1} = S_{2m} + a_{2m+1}$ and $a_{2m+1} \rightarrow 0$ as $m \rightarrow \infty$, so

$$\lim_{m \rightarrow \infty} S_{2m+1} = \lim_{m \rightarrow \infty} (S_{2m} + a_{2m+1}) = \lim_{m \rightarrow \infty} S_{2m} = L \quad (5)$$

Combining Eqs. (4) and (5), we see $\lim_{n \rightarrow \infty} S_n = L$. That is, $\sum_{n=1}^{\infty} a_n$ converges.

Sketch of a proof of the Absolute Convergence Theorem

Suppose $\sum_{n=0}^{\infty} a_n$ converges absolutely. We show that this series converges. First, observe that for all n , $-|a_n| \leq a_n \leq |a_n|$, and so

$$0 \leq a_n + |a_n| \leq 2|a_n| \quad (6)$$

Because $\sum_{n=0}^{\infty} a_n$ converges absolutely, $2 \sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{\infty} 2|a_n|$ converges. Then $\sum_{n=0}^{\infty} (a_n + |a_n|)$ converges by the Comparison Theorem applied to Eq. (6). Because the two series on the left below converge, so does their difference:

$$\sum_{n=0}^{\infty} (a_n + |a_n|) - \sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{\infty} (a_n + |a_n| - |a_n|) = \sum_{n=0}^{\infty} a_n$$

That is, the series $\sum_{n=0}^{\infty} a_n$ converges.