

Random matrices: A Survey

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Basic models of random matrices

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Related models. Adjacency matrix of a random graphs (Erdős-Rényi $G(n, p)$, random regular graphs, etc).

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Combinatorics. Various combinatorial problems (Komlós 1960s).

There is a wonderful interaction between the theory of random matrices and other areas of mathematics (number theory, additive combinatorics, theoretical computer science etc).

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is it distributed ?

Eigenvectors. How does a typical eigenvector look like ?

The singularity problem: non-symmetric case

Let ξ be Bernoulli (so we consider random ± 1 matrices).

Question. What is p_n , the probability that M_n is singular ?

Conjecture. (folklore/notorious) $p_n = (1/2 + o(1))^n$.

The lower bound is obvious:

$$\mathbf{P}(\text{there are two equal rows/columns}) = (1 + o(1))n^2 2^{-n}.$$

Upper bound: $o(1)$ (Komlós 67).

Short proof of Komlós theorem

$$p_n \leq \sum_{i=1}^{n-1} \mathbf{P}(X_{i+1} \in \text{Span}(X_1, \dots, X_i)).$$

Fact. A subspace V of dim d contains at most 2^d Bernoulli vectors (as any vector in V is determined by a set of d coordinates). So

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It is enough to show that the contribution of the last $k := \log \log n$ terms is $o(1)$. We will show

$$\mathbf{P}(X_n \in \text{Span}(X_1, \dots, X_{n-1})) \leq \frac{1}{\log^{1/3} n}.$$

A $m \times n$ matrix is l -universal if for any set of l indices i_1, \dots, i_l and any set of signs $\epsilon_1, \dots, \epsilon_l$, there is a row X where the i_j th entry of X has sign ϵ_j , for all $1 \leq j \leq l$.

Fact. $l = \log n$. A random $n \times n$ Bernoulli matrix is l -universal with probability at least $1 - \frac{1}{n}$.

Proof. $\mathbf{P}(\text{fails}) \leq \binom{n}{l} 2^l (1 - \frac{1}{2^l})^n \leq \exp(2l \log n - 2^l n) \leq n^{-1}$.

So with probability $1 - \frac{1}{n}$, any vector v orthogonal to X_1, \dots, X_{n-1} should have at least l non-zero coordinate. Then

$$\mathbf{P}(X_n \in \text{Span}(X_1, \dots, X_{n-1})) \leq \mathbf{P}(X_n \cdot v = 0) = O(l^{1/2}) < \frac{1}{\log^{1/3} n}.$$

Lemma (Littlewood-Offord-Erdős, 1940s)

If a_1, \dots, a_l are non zero numbers, then

$$\mathbf{P}(a_1 \xi_1 + \dots + a_l \xi_l = 0) = O(l^{-1/2}).$$

Further developments

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 $(3/4 + o(1))^n$ (Tao-V. 06),
 $(1/\sqrt{2} + o(1))^n$ (Bourgain-V-Wood 09).

Dominating principle (Halász, KKSZ).
Inverse Littlewood-Offord theory (TV)

$$|\cos x| \leq \frac{3}{4} + \frac{1}{4} \cos 2x.$$

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Fractional dimension (BVW)

$$|\cos x|^2 \leq \frac{1}{2} + \frac{1}{2} \cos 2x.$$

General theorem (BVW 09).

Improvements of LOE lemma with extra assumptions on the a_i .
For instance, if the a_i are different, then (Erdős-Moser,
Sárközy-Szemerédi 1960s) showed

$$\mathbf{P}(a_1\xi_1 + \cdots + a_l\xi_l = 0) = O(l^{-3/2}).$$

Stanley showed the extremal set is an arithmetic progression.
Kleitman, Katona, Frankl-Füredi, Halaász, etc.

Tao-V. 05: If the probability in question is large, then $\{a_1, \dots, a_n\}$ can be characterized.

If $P \geq n^{-A}$, then (most of) a_i belong to an AP of length n^B .

The relation between A and B is of importance. A near optimal bound was obtained by Tao-V. (07), that lead to the establishment of the Circular Law Conjecture concerning the eigenvalues of M_n (Budapest 08, Bulletin AMS 09).

Using a different approach, Nguyen-V. (09+) obtained the optimal relation. As a corollary, one obtains all forward LOE results such as Sárkozy-Szemerédi theorem.

One can also obtain an asymptotic, stable, version of Stanley's result (algebra-free).

See: Hoi Nguyen's talk (December).

The singularity problem: symmetric case

Let ξ be Bernoulli (so we consider random ± 1 matrices).

Question. What is p_n^{sym} , the probability that M_n^{sym} is singular ?

Conjecture. (B. Weiss 1980s) $p_n^{sym} = o(1)$.

This is the symmetric version of Komlós 1967 theorem.

Theorem (Costello-Tao-V. 2005)

$$p_n^{sym} = O(n^{-1/4}).$$

Recently, Kevin Costello (2009) improved the bound to $n^{-1/2+\epsilon}$, which seems to be the limit of the method.

Lemma (Costello 09)

Consider the quadratic form $Q = \sum_{1 \leq i, j \leq n} a_{ij} \xi_i \xi_j$ with $a_{ij} \neq 0$.
Then

$$\mathbf{P}(Q = 0) \leq n^{-1/2+\epsilon}.$$

Question. Higher degree polynomials ?

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Question. Higher degree polynomials ?

Conjecture. (V. 2006) $p^{sym}(n) = (1/2 + o(1))^n$.

Rank of random graphs

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Question. What about non-constant p ?

If $p < (1 - \epsilon) \log n/n$, $A(n, p)$ is almost surely singular, as the graph has isolated vertices.

Theorem (Costello-V. 06)

For $p > (1 + \epsilon) \log n/n$, $A(n, p)$ is a.s. non-singular.

Theorem (Costello-V. 07)

For $p = c \log n/n$ with $0 < c < 1$, the co-rank of $A(n, p)$ (a.s.) comes from small local obstructions.

A set S is a local obstruction if the number of neighbors of S is less than $|S|$. Small means $|S| \leq K(c)$.

For $p = c/n$, Bordenare and Lelarge (09) computed the asymptotics of the rank. I wonder if one can characterize the co-rank.

Conjecture. [V. 2006] $A(n, d)$ (adjacency matrix of a random regular graphs of degree d) is a.s non-singular for all $3 \leq d \leq n/2$.

Conjecture. $A(n, d)$ (adjacency matrix of a random regular graphs of degree d) has second eigenvalue $O(\sqrt{d})$ for all $3 \leq d \leq n/2$.

Known for $d = O(1)$ (Freedman, Kahn-Szemerédi 1989). Recently Freedman showed

$$\lambda = (2 + o(1))\sqrt{d-1}.$$

The KSz argument extends to cover up to $d \leq n^{1/2}$ (Frieze et. al.). For $d = n/2$, the best current bound is $O(\sqrt{n \log n})$.

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Turán (1940s): By linearity of expectation,

$$\mathbf{E}(\det M_n^2) = n! = n^{(1+o(1))n}.$$

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Theorem (Tao-V. 2004)

A.s.

$$|\det M_n| = n^{(1/2+o(1))n}.$$

Determinant is the volume of the parallelepiped spanned by the row vectors.

Use the height times base formula.

Most of the distances are **strongly concentrated**.

Lemma (Tao-V. 04)

*The distance from a random vector in \mathbf{R}^n to a fixed subspace of dimension d is, with high prob, $\sqrt{n-d} +$ **small error**.*

The remaining few distances are not too small !.

Question. We know $\mathbf{P}(\det = 0) = \exp(-\Theta(n))$. What about $\mathbf{P}(\det = z)$, for integer $z \neq 0$?

I think $\mathbf{P}(\det = z) = \exp(-\omega(n))$, perhaps $\leq n^{-cn}$ for some constant $c > 0$.

Question. Limiting distribution of $|\det|$

$$\frac{\log |\det M_n| - \frac{1}{2} \log(n-1)!}{c\sqrt{\log n}} \rightarrow N(0, 1).$$

(Girko (???) 80s).

For the next discussion, consider a slightly more general model:
Let M be a deterministic matrix with entries $0 < c < m_{ij} < C$. Let ξ be a random variable with mean 0 and variance one and M_n be the random matrix with entries $m_{ij}\xi_{ij}$. (So the entries have different variances.)

Question. How strongly is $|\det M_n|$ concentrated ?

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Motivation. Estimating permanent using random determinant.

Given a matrix A , define $m_{ij} := \sqrt{a_{ij}}$, then

$$\text{Per } A := \mathbf{E} |\det M_n|^2.$$

Markov chain: Jerrum-Sinclair, J-S-Vigoda (00). Random determinant: Barvinok (00): $\exp(cn)$ with ξ gaussian, Friedland-Rider-Zeitouni (04) $\exp(\epsilon n)$ with ξ gaussian (under boundedness).

Theorem (Costello-V. 07)

With high probability and ξ gaussian or Bernoulli

$$|\det M_n| / \mathbf{E} |\det M_n| \leq \exp(n^{2/3+o(1)}).$$

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Conjecture. (Kostello-V. 07) With high probability,
 $|\det M_n|/\mathbf{E}|\det M_n| \leq n^{O(1)}.$

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It had been a long standing open conjecture that a.s. $|\text{Per}M_n| > 0$ (the permanent version of Komlós 1967 theorem).

Theorem (Tao-V. 2008)

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Current bounds. $q_n \leq n^{-b}$. The truth may be n^{-bn} .

Determinant-Permanent: Symmetric case

Still by linearity of expectation

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Recently, Tao-V. (2009) confirmed the first conjecture, the second is still open.

Consider M_n^{sym} . Its eigenvectors form an orthonormal system

$$v_1, \dots, v_n, \|v_i\| = 1.$$

Question. How do the v_j look like ?

sub-Question. $\|v_i\|_\infty = ?$ (Linial)

Theorem (Tao-V. 2009)

With high probability

$$\max_i \|v_i\|_\infty = n^{-1/2+o(1)}.$$