Homework 1.

(1) (a) Let $X = \xi_1 + \cdots + \xi_n$, where ξ_i are independent indicator random variables which equal 1 with probability 1/2. What is $\mathbf{P}(X = k)$. Compare $\mathbf{P}(X = k)$ and $\mathbf{P}(X = m)$ for all pairs of numbers (k, m). Which value of k maximizes $\mathbf{P}(X = k)$? Compute the value of $\mathbf{P}(|X - n| \leq 2\sqrt{n})$ approximately (assuming that n is large).

(b) What are the answers if we replace 1/2 by a constant $0 < p < 1$? (again assuming that n is very large and does not depend on p).

Hint: Recall the central limit theorem.

(2) Prove that the function $\binom{y}{n}$ is convex. Read the proof in page 7, Chapter 1.

(3) Let S be a set of n elements (n even). Choose a random subset A of S by selecting each element with probability 1/2. Repeat the process to choose another random subset B. What is the expectation and variance of $|A \cap B|$? What happens if each element of B is chosen with probability 1/3 ? What are the expectation and variance of $|A \cup B|$ in each case ?

(4) Problem 1, page 10.

(5) Problem 2, page 10.

(6) Problem 4, page 11.

Homework 2.

(1) Let σ be a random permutation from S_n. Let X be the number of fixed points of σ.

(a) Compute the expectation and variance of X.

(b) Prove that probability that $X = 0$ tends to $1/e$ as n tends to infinity.

(2) Compute the probability that X contains a cycle of length more than $n/2$ (assuming that n is even).

(3) Problem 7, page 21.

(4) Problem 2, page 21.

(5) Problem 9, page 21.

Homework 3.

(1) Prove, without using the prime number theorem, that $\pi(n) = \Theta(n/\log n)$, where $\pi(n)$ is the number of primes between 1 and n.

(2) Let X be the number of K_4 in $G(n, p)$. Compute the variance of X.

(3) Problem (2), page 58.

(4) Problem (4), page 59.

(5) Problem (5), page 59.