Web Appendix for "*Stochastic Search Equilibrium*"

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1 Detailed derivation of Equation (14)

All equation numbers in this appendix refer to equations in the main body of the paper.

Equations (12) and (13) are

$$\Pi_t\left(\overline{V}\right) = \sup_{w_t, W_{t+1} \geq U_{t+1}} \left\langle \left(\omega_t p - w_t\right) L_t + \beta \mathbf{E}_t\left[\Pi_{t+1}\left(W_{t+1}\right)\right] \right\rangle \tag{12}$$

$$\overline{V} = w_t + \beta \mathbf{E}_t \Bigg[\delta_{t+1} U_{t+1} + \left(1-\delta_{t+1}\right)\left(1 - s\lambda_{t+1} \overline{F}_{t+1}\left(W_{t+1}\right)\right) W_{t+1} + \left(1-\delta_{t+1}\right) s\lambda_{t+1} \int_{W_{t+1}}^{+\infty} v dF_{t+1}\left(v\right) \Bigg]. \tag{13}$$

Now solve for the wage from (13)

$$w_t = \overline{V} - \beta \mathbf{E}_t \Bigg[\delta_{t+1} U_{t+1} + \left(1-\delta_{t+1}\right)\left(1 - s\lambda_{t+1} \overline{F}_{t+1}\left(W_{t+1}\right)\right) W_{t+1} + \left(1-\delta_{t+1}\right) s\lambda_{t+1} \int_{W_{t+1}}^{+\infty} v dF_{t+1}\left(v\right) \Bigg].$$

This equation pins down w_t once the future promised values W_{t+1} are chosen for each possible date-$(t+1)$ state. Replace this expression into (12), where w_t is no longer a choice

$$\Pi_t\left(\overline{V}\right) = \sup_{W_{t+1} \geq U_{t+1}} \Bigg\langle \omega_t p L_t - \overline{V} L_t + \beta \mathbf{E}_t\left[\Pi_{t+1}\left(W_{t+1}\right)\right] + L_t \beta \mathbf{E}_t \Bigg[\delta_{t+1} U_{t+1} + \left(1-\delta_{t+1}\right)\left(1 - s\lambda_{t+1} \overline{F}_{t+1}\left(W_{t+1}\right)\right) W_{t+1} + \left(1-\delta_{t+1}\right) s\lambda_{t+1} \int_{W_{t+1}}^{+\infty} v dF_{t+1}\left(v\right) \Bigg\rangle$$

Separate the part that the firm chooses from terms that are not under the control of the firm.

$$\Pi_t\left(\overline{V}\right) = \omega_t p L_t - \overline{V} L_t + \beta \mathbf{E}_t\left[\delta_{t+1} U_{t+1} L_t\right] + \sup_{W_{t+1} \geq U_{t+1}} \beta \mathbf{E}_t \Bigg[\Pi_{t+1}\left(W_{t+1}\right) + L_t\left(1-\delta_{t+1}\right)\left(1 - s\lambda_{t+1}\overline{F}_{t+1}\left(W_{t+1}\right)\right) W_{t+1} + L_t\left(1-\delta_{t+1}\right) s\lambda_{t+1} \int_{W_{t+1}}^{+\infty} v dF_{t+1}(v)\Bigg].$$

Moving $\overline{V}L_t$, which does not depend on the choice variables, from the RHS to the LHS, and using the definition

$$S_t = \Pi_t\left(\overline{V}\right) + \overline{V} L_t$$

this last equation becomes

$$S_t = \omega_t p L_t + \beta \mathbf{E}_t\left[\delta_{t+1} U_{t+1} L_t\right] + \sup_{W_{t+1} \geq U_{t+1}} \beta \mathbf{E}_t \Bigg[S_{t+1} - L_{t+1} W_{t+1} + L_t\left(1-\delta_{t+1}\right)\left(1 - s\lambda_{t+1}\overline{F}_{t+1}\left(W_{t+1}\right)\right) W_{t+1} + L_t\left(1-\delta_{t+1}\right) s\lambda_{t+1} \int_{W_{t+1}}^{+\infty} v dF_{t+1}(v)\Bigg].$$

Now consider the maximand on the RHS: the terms

$$\begin{aligned} & - L_{t+1} W_{t+1} + L_t\left(1-\delta_{t+1}\right)\left(1 - s\lambda_{t+1}\overline{F}_{t+1}\left(W_{t+1}\right)\right) W_{t+1} \\ = & - W_{t+1}\left[L_{t+1} - L_t\left(1-\delta_{t+1}\right)\left(1 - s\lambda_{t+1}\overline{F}_{t+1}\left(W_{t+1}\right)\right)\right] \end{aligned}$$

simplify to (using the law of motion of employment to express new hires from both unemployment and employment in square brackets)

$$= -W_{t+1}\left\{\lambda_{t+1}\left[1 - N_t\left(\overline{p}\right)\right] + s\lambda_{t+1}\left(1-\delta_{t+1}\right) N_t\left(\overline{p}\right) G_t\left(W_{t+1}\right)\right\}.$$

Replacing back into the Bellman equation we obtain the desired equation

$$S_t = \omega_t p L_t + \beta \mathbf{E}_t\left[\delta_{t+1} U_{t+1} L_t\right] + \sup_{W_{t+1} \geq U_{t+1}} \beta \mathbf{E}_t \Bigg[S_{t+1} + L_t\left(1-\delta_{t+1}\right) s\lambda_{t+1} \int_{W_{t+1}}^{+\infty} v dF_{t+1}(v) - W_{t+1}\left\{\lambda_{t+1}\left[1 - N_t\left(\overline{p}\right)\right] + s\lambda_{t+1}\left(1-\delta_{t+1}\right) N_t\left(\overline{p}\right) G_t\left(W_{t+1}\right)\right\}\Bigg]. \quad (14)$$

2 Detailed formal comparison of the steady-state allocation with that of the BM model

In this appendix, we establish formally the equivalence between the steady-state predictions of our model and the solution to the Burdett and Mortensen (1998, henceforth BM) model, or, more specifically, the Bontemps et al. (2000) version of that model.

Consider the steady-state version of our first-order condition (22):

$$V'(p) = [\mu(p) - V(p)]\, 2\frac{d\log H(p)}{dp} \tag{1}$$

In steady state the flow of hires into firm p is

$$H(p) = \lambda u + s\lambda(1-\delta) N^{\star}(p)$$

so that

$$\frac{d\log H(p)}{dp} = \frac{s\lambda(1-\delta) L(p)\gamma(p)}{\lambda u + s\lambda(1-\delta) N^{\star}(p)} = \frac{s\lambda(1-\delta) L(p)\gamma(p)}{H(p)} \tag{2}$$

The flow of hires also equals separations in steady-state: $H(p) = L(p)\left[\delta + (1-\delta) s\lambda\overline{F}(V(p))\right]$. Moreover, our steady-state equilibrium being rank-preserving, we have the additional property that $\overline{F}(V(p)) = \overline{\Gamma}(p)$. Substituting this expression of $H(p)$ into (2):

$$\frac{d\log H(p)}{dp} = \frac{s\lambda(1-\delta)\gamma(p)}{\delta + (1-\delta) s\lambda\overline{\Gamma}(p)}.$$

Further replacing into (1)

$$V'(p) = [\mu(p) - V(p)]\frac{2s\lambda(1-\delta)\gamma(p)}{\delta + (1-\delta) s\lambda\overline{\Gamma}(p)} \tag{3}$$

Next, consider the steady-state RPE version of the Bellman equation defining the worker's value function:

$$V(p) = w(p) + \beta\left[\delta U + (1-\delta)\left(1 - s\lambda\overline{\Gamma}(p)\right)V(p) + (1-\delta) s\lambda\int_p^{+\infty} V(x)\, d\Gamma(x)\right]. \tag{4}$$

This defines the steady-state RPE wage $w(p)$ as a function of p, which inherits the differentiability of $V(p)$. Differentiating and collating terms, we obtain:

$$V'(p) = \frac{w'(p)}{1 - \beta(1-\delta)\left(1 - s\lambda\overline{\Gamma}(p)\right)}. \tag{5}$$

Next, consider the steady-state RPE version of our Euler equation (20):

$$\mu(p) = \omega p + \beta\left[\delta U + (1-\delta) s\lambda\int_p^{+\infty} V(x)\, d\Gamma(x) + (1-\delta)\left(1 - s\lambda\overline{\Gamma}(p)\right)\mu(p)\right]. \tag{6}$$

Subtracting (4) from (6), and solving for $\mu(p) - V(p)$ yields:

$$\mu(p) - V(p) = \frac{\omega p - w(p)}{1 - \beta(1-\delta)\left(1 - s\lambda\overline{\Gamma}(p)\right)}. \tag{7}$$

Finally, substituting (5) and (7) into the first-order condition (3), we obtain:

$$w'(p) = [\omega p - w(p)]\frac{2s\lambda(1-\delta)\gamma(p)}{\delta + (1-\delta) s\lambda\overline{\Gamma}(p)}, \tag{8}$$

which is exactly the equation that characterizes equilibrium wages in Bontemps et al. (2000) (their equation (14) p316). The only formal difference is the presence of $1-\delta$ as a multiplier of $s\lambda$ in our expression, which does not appear explicitly in their equation (14). This is just a minor consequence of the specific timing assumption we made: workers are first separated exogenously and then draw a chance to search on the job. Thus, our model's counterpart of Bontemps et al.'s offer arrival rate to employed job seekers, λ_1, is $(1-\delta)\, s\lambda$. Defining $\lambda_1 := (1-\delta)\, s\lambda$, our expression becomes identical to Bontemps et al.'s. This establishes that the steady-state RPE wages in our model coincide with equilibrium wages in the Bontemps et al. version of the BM model. Finally, simple inspection of the law of motion of employment, taken in steady-state, directly establishes that the allocation of workers to firm types is the same in the steady-state of our model as in BM.

3 Endogenous contact rates: Proofs

3.1 Proof of Proposition 6

This proof is adapted from the proof of Proposition 2 presented in detail in Appendix A in the paper, and follows the same general strategy. We first introduce the following notation:

$$\varphi\left(p, L, \omega, N\right) := \omega p L + \beta \int_{\Omega} \delta^{\omega'} U\left(\omega', \mathscr{N}\left(\omega', N\right)\right) L Q\left(d\omega' \mid \omega\right),$$

$$\begin{aligned}\Phi\left(L, W\left(\omega'\right), \omega', N, a\left(\omega'\right)\right) := -c\left(a\left(\omega'\right)\right) + L\left(1-\delta^{\omega'}\right) s\lambda^{\omega'} \int_{W(\omega')}^{+\infty} v dF\left(v \mid \omega', N\right) \\ - W\left(\omega'\right) \eta^{\omega'} a\left(\omega'\right) \mathscr{P}_A\left(W\left(\omega'\right) \mid \omega'\right).\end{aligned}$$

where

$$\mathscr{P}_A\left(W \mid \omega'\right) := \frac{\lambda^{\omega'}\left(1-N\left(\overline{p}\right)\right) + s\lambda^{\omega'}\left(1-\delta^{\omega'}\right) N\left(\overline{p}\right) G\left(W \mid \omega'\right)}{\lambda^{\omega'}\left(1-N\left(\overline{p}\right)\right) + s\lambda^{\omega'}\left(1-\delta^{\omega'}\right) N\left(\overline{p}\right)}$$

is the acceptance probability of a job offer of W in state ω'. Finally, we let:

$$\mathscr{L}\left(L, W, a, \omega', N\right) := L\left(1-\delta^{\omega'}\right)\left(1 - s\lambda^{\omega'} \overline{F}\left(W \mid \omega', N\right)\right) + \eta^{\omega'} a \mathscr{P}_A\left(W \mid \omega'\right),$$

and:

$$\mathscr{N}\left(\omega', N\right) := \int_{\underline{p}}^{p} \mathscr{L}\left[L\left(x\right), V\left(x, L\left(x\right), \omega', N\right), a\left(x, L\left(x\right), \omega', N\right), \omega', N\right] d\Gamma\left(x\right).$$

Fix N to be some given c.d.f. over $\left[\underline{p}, \overline{p}\right]$. Then, for any function $\mathscr{S}\left(p, L, \omega\right)$, we define the following operator $\mathbf{M^N}$:

$$\begin{aligned}\mathbf{M^N}\mathscr{S}\left(p, L, \omega\right) := \varphi\left(p, L, \omega, N\right) + \beta \int_{\Omega} \max_{W(\omega'), a(\omega')} \Big\langle \mathscr{S}\left[p, \mathscr{L}\left(L, W\left(\omega'\right), \omega', N, a\left(\omega'\right)\right), \omega'\right] \\ + \Phi\left(L, W\left(\omega'\right), \omega', N, a\left(\omega'\right)\right)\Big\rangle Q\left(d\omega' \mid \omega\right). \quad (9)\end{aligned}$$

Lemma 1 *Let $\mathscr{S}(p, L, \omega)$ be bounded, continuous in p and L, increasing and convex in L and with increasing differences in (p, L) over $(\underline{p}, \overline{p}) \times (0,1)$. Then:*

1. *$\mathbf{M^N}\mathscr{S}$ is bounded and continuous in p and L;*
2. *There exists a measurable selection $(V(p, L, \omega, N), a(p, L, \omega, N))$ from the maximizing correspondence associated with $\mathbf{M^N}\mathscr{S}$;*
3. *Any such measurable selection is such that V and a are both increasing in p and L;*
4. *$\mathbf{M^N}\mathscr{S}$ is increasing and convex in L and with increasing differences in (p, L) over $(\underline{p}, \overline{p}) \times (0,1)$.*

Proof. In this proof, wherever possible without causing confusion, we will make the dependence of all functions on aggregate state variables ω and N implicit to streamline the notation.

Points 1 and 2 of this lemma follow immediately from the same arguments as in the proof of Lemma 1 in Appendix A. To prove point 3, we establish that the maximand in (9) has increasing differences in (p, W), (p, a), (L, W), (L, a), and (W, a). Monotonicity of V and a in p and L then follows from standard monotone comparative statics arguments. Proving that the maximand in (9) has increasing differences in (p, W) and (p, a) is immediate as Φ is independent of p: letting $\tau > 0$, differences in p of the maximand equal $\mathscr{S}(p + \tau, \mathscr{L}(L, W, a)) - \mathscr{S}(p, \mathscr{L}(L, W, a))$ which is increasing in W and a because $\mathscr{L}$ is increasing in W and a by construction and $\mathscr{S}$ has increasing differences in (p, L) by assumption.

We thus now fix p and establish that the maximand in (9) has increasing differences in (L, W) and (L, a). Since $\mathscr{S}$ is continuous and convex in L, it has left and right derivatives everywhere (and those two can at most differ at countably many points). Now take L and $\varepsilon > 0$ and define the difference in L of the maximand in (9):

$$\mathscr{D}_L(W, a) := \mathscr{S}(p, \mathscr{L}(L + \varepsilon, W, a)) - \mathscr{S}(p, \mathscr{L}(L, W, a)) \\ + \varepsilon(1 - \delta^{\omega'}) s \lambda^{\omega'} \int_W^{+\infty} v dF\left(v \mid \omega'\right).$$

(The dependence of $\mathscr{D}$ on p is kept implicit.) The fact that $\mathscr{D}_L(W, a)$ is increasing in W is established in exactly the same way as in the proof of Lemma 1 in Appendix A. The fact that $\mathscr{D}_L(W, a)$ is increasing in a is a simple consequence of the convexity of $\mathscr{S}(p, L)$ w.r.t. L and the fact that $\mathscr{L}(L, W, a)$ is increasing in a.

To prove point 3, we finally need to establish that the maximand in (9) has increasing differences in (W, a). To this end, fix p, L and a and define, for $\varepsilon > 0$:

$$\begin{aligned}\mathscr{D}_a(W) := \mathscr{S}(p, \mathscr{L}(L, W, a+\varepsilon)) - \mathscr{S}(p, \mathscr{L}(L, W, a)) \\ - [c(a+\varepsilon) - c(a)] - W\varepsilon\eta^{\omega'}\mathscr{P}_A(W \mid \omega').\end{aligned}$$

We want to establish that $\mathscr{D}_a(W)$ is an increasing function of W. For $x > 0$:

$$\begin{aligned}\mathscr{D}_a(W+x) - \mathscr{D}_a(W) &= \mathscr{S}(p, \mathscr{L}(L, W+x, a+\varepsilon)) - \mathscr{S}(p, \mathscr{L}(L, W, a+\varepsilon)) \\ &+ [\mathscr{S}(p, \mathscr{L}(L, W+x, a)) - \mathscr{S}(p, \mathscr{L}(L, W, a))] \\ &- \left[(W+x)\varepsilon\eta^{\omega'}\mathscr{P}_A(W+x \mid \omega') - W\varepsilon\eta^{\omega'}\mathscr{P}_A(W \mid \omega')\right].\end{aligned}$$

Dividing through by x and taking the limit superior as $x \to 0+$ (using the definition of $\mathscr{L}$, continuity of F and G which implies continuity of $\mathscr{P}_A$, and some basic properties of Dini derivatives), we obtain:

$$\begin{aligned}D^+\mathscr{D}_a(W) &= [\mathscr{S}_{L,r}(p, \mathscr{L}(L, W, a+\varepsilon)) - \mathscr{S}_{L,r}(p, \mathscr{L}(L, W, a))] \\ &\times \left[L(1-\delta^{\omega'})s\lambda^{\omega'}D^+F(W) + \eta^{\omega'}aD^+\mathscr{P}_A(W)\right] \\ &+ [\mathscr{S}_{L,r}(p, \mathscr{L}(L, W, a+\varepsilon)) - W] \times \varepsilon\eta^{\omega'}\mathscr{P}_A(W \mid \omega').\end{aligned}$$

The first term in the above sum is positive by convexity of $\mathscr{S}$ and the fact that both F and G (and hence $\mathscr{P}_A$) are increasing. The second term is also positive if W is an optimal selection, as was shown in the proof of Lemma 1 in Appendix A (see the optimality condition (38) which still applies here). This proves that $D^+\mathscr{D}_a(W) \geq 0$. Finally, by inspection, $\mathscr{D}$ is a continuous function of W. Those two properties are sufficient to ensure that $\mathscr{D}$ is strictly increasing (see, e.g., Proposition 2 p99 in Royden, 1988). Point 3 of the lemma is thus proven.

Now on to point 4. Again the proof of this point is a simple extension of its Lemma-1, Appendix-A analogue. Take $(p_0, L_0) \in (\underline{p}, \overline{p}) \times (0,1)$ and $\varepsilon > 0$ such that $(p_0 + \varepsilon, L_0 + \varepsilon)$ are still in $(\underline{p}, \overline{p}) \times (0,1)$. We first consider right-differentiability of $\mathbf{M^N}\mathscr{S}$ w.r.t. L at L_0. Fixing an arbitrary selection (V, a) from the optimal policy correspondence, and using the same steps as in Appendix A equations (39) and (40), one can establish:

$$\begin{aligned}\mathbf{M^N}\mathscr{S}(p_0, L_0+\varepsilon) - \mathbf{M^N}\mathscr{S}(p_0, L_0^+) &\geq \left(\omega p_0 + \beta\int_\Omega \delta^{\omega'}U(\omega')Q(d\omega'|\omega)\right)\cdot\varepsilon \\ + \beta\int_\Omega\Big\langle &\mathscr{S}\left[p_0, \mathscr{L}\left(L_0+\varepsilon, V(p_0, L_0^+, \omega'), a(p_0, L_0^+, \omega')\right)\right] \\ &- \mathscr{S}\left[p_0, \mathscr{L}\left(L_0, V(p_0, L_0^+, \omega'), a(p_0, L_0^+, \omega')\right)\right] \\ &+ \varepsilon\cdot(1-\delta^{\omega'})s\lambda^{\omega'}\int_{V(p_0, L_0^+, \omega')}^{+\infty} v\,dF(v \mid \omega')\Big\rangle Q(d\omega' \mid \omega),\end{aligned}$$

and:

$$\mathbf{M^N}\mathscr{S}\left(p_0, L_0+\varepsilon\right) - \mathbf{M^N}\mathscr{S}\left(p_0, L_0^+\right) \leq \left(\omega p_0 + \beta \int_\Omega \delta^{\omega'} U\left(\omega'\right) Q\left(d\omega'|\omega\right)\right) \cdot \varepsilon$$
$$+ \beta \int_\Omega \Big\langle \mathscr{S}\left[p_0, \mathscr{L}\left(L_0+\varepsilon, V\left(p_0, L_0+\varepsilon, \omega'\right), a\left(p_0, L_0+\varepsilon, \omega'\right)\right)\right]$$
$$- \mathscr{S}\left[p_0, \mathscr{L}\left(L_0, V\left(p_0, L_0+\varepsilon, \omega'\right), a\left(p_0, L_0+\varepsilon, \omega'\right)\right)\right]$$
$$+ \varepsilon \cdot (1-\delta^{\omega'}) s\lambda^{\omega'} \int_{V(p_0, L_0+\varepsilon, \omega')}^{+\infty} v dF\left(v \mid \omega'\right) \Big\rangle Q\left(d\omega' \mid \omega\right).$$

Now dividing through by ε in both inequalities above, and invoking continuity w.r.t. V of $\mathscr{L}_L(L,V,a) = (1-\delta^{\omega'})\left(1 - s\lambda^{\omega'}\overline{F}(V)\right)$ (by continuity of F), everywhere right-differentiability of $\mathscr{S}$ w.r.t. L (by convexity of $\mathscr{S}$), and existence of a right limit of V and a at any L_0 (by monotonicity of V and a established in point 1 of this lemma), we see that the lower and upper bounds of $\frac{1}{\varepsilon}\left[\mathbf{M^N}\mathscr{S}\left(p_0, L_0+\varepsilon\right) - \mathbf{M^N}\mathscr{S}\left(p_0, L_0^+\right)\right]$ exhibited above both converge to the same limit as $\varepsilon \to 0^+$, which, together with continuity of $\mathbf{M^N}\mathscr{S}$ in L at L_0 which implies $\mathbf{M^N}\mathscr{S}\left(p_0, L_0^+\right) = \mathbf{M^N}\mathscr{S}\left(p_0, L_0\right)$, establishes right-differentiability of $\mathbf{M^N}\mathscr{S}$ w.r.t L with the following expression for $\left[\mathbf{M^N}\mathscr{S}\right]_{L,r}(p,L)$

$$\left[\mathbf{M^N}\mathscr{S}\right]_{L,r}(p,L) = \omega p + \beta \int_\Omega \delta^{\omega'} U\left(\omega'\right) Q\left(d\omega'|\omega\right)$$
$$+ \beta \int_\Omega \Big\langle \mathscr{S}_{L,r}\left[p, \mathscr{L}\left(L, V\left(p, L^+, \omega'\right), a\left(p, L^+, \omega'\right)\right)\right]$$
$$\times \mathscr{L}_L\left(L, V\left(p, L^+, \omega'\right), a\left(p, L^+, \omega'\right)\right)$$
$$+ (1-\delta^{\omega'}) s\lambda^{\omega'} \int_{V(p, L^+, \omega')}^{+\infty} v dF\left(v \mid \omega'\right) \Big\rangle Q\left(d\omega' \mid \omega\right).$$

Straightforward inspection shows that $\left[\mathbf{M^N}\mathscr{S}\right]_{L,r}(p,L) > 0$, so that $\mathbf{M^N}\mathscr{S}$ is increasing in L. We finally show that $\left[\mathbf{M^N}\mathscr{S}\right]_{L,r}(p,L)$ is increasing in L and p, for which it is sufficient to show that the term under the $\int$ is increasing in L and p for all $\omega' \in \Omega$. Let $L_1 < L_2 \in [0,1]^2$. To lighten the notation, let $V_k = V\left(p, L_k^+, \omega'\right)$ and $a_k = a\left(p, L_k^+, \omega'\right)$

for $k=1,2$. Because V and a are increasing in L, $V_2 \geq V_1$ and $a_2 \geq a_1$. Then:

$$
\begin{aligned}
&\mathscr{S}_{L,r}\left[p,\mathscr{L}\left(L_2,V_2,a_2\right)\right]\cdot\mathscr{L}_L\left(L_2,V_2,a_2\right) \\
&\qquad -\mathscr{S}_{L,r}\left[p,\mathscr{L}\left(L_1,V_1,a_1\right)\right]\cdot\mathscr{L}_L\left(L_1,V_1,a_1\right)-(1-\delta^{\omega'})s\lambda^{\omega'}\int_{V_1}^{V_2} vdF\left(v\mid\omega'\right) \\
&=\mathscr{L}_L\left(L_1,V_1,a_1\right)\cdot\left(\mathscr{S}_{L,r}\left[p,\mathscr{L}\left(L_2,V_2,a_2\right)\right]-\mathscr{S}_{L,r}\left[p,\mathscr{L}\left(L_1,V_1,a_1\right)\right]\right) \\
&\qquad\qquad +\mathscr{S}_{L,r}\left[p,\mathscr{L}\left(L_2,V_2,a_2\right)\right]\cdot\left(\mathscr{L}_L\left(L_2,V_2,a_2\right)-\mathscr{L}_L\left(L_1,V_1,a_1\right)\right) \\
&\qquad\qquad\qquad -\left(1-\delta^{\omega'}\right)\lambda_1^{\omega'}\int_{V_1}^{V_2} vdF\left(v\mid\omega'\right) \\
&=\mathscr{L}_L\left(L_1,V_1,a_1\right)\cdot\left(\mathscr{S}_{L,r}\left[p,\mathscr{L}\left(L_2,V_2,a_2\right)\right]-\mathscr{S}_{L,r}\left[p,\mathscr{L}\left(L_1,V_1,a_1\right)\right]\right) \\
&\qquad\qquad +(1-\delta^{\omega'})s\lambda^{\omega'}\int_{V_1}^{V_2}\left(\mathscr{S}_{L,r}\left[p,\mathscr{L}\left(L_2,V_2,a_2\right)\right]-v\right)dF\left(v\mid\omega'\right),
\end{aligned}
$$

where the second equality use $\mathscr{L}_L\left(L,V,a\right)=\left(1-\delta^{\omega'}\right)\left(1-s\lambda^{\omega'}\overline{F}\left(V\mid\omega'\right)\right)$. Term-by-term inspection of the r.h.s. in the above equation shows that it is positive, using the exact same arguments as in the proof of Lemma 1 in Appendix A. □

The rest of the proof is identical to the proof of Proposition 2 in Appendix A. The proof that $\left[\mathbf{M}^{\mathbf{N}}\mathscr{S}\right]_{L,r}(p,L)$ is increasing in p proceeds along similar lines (details available on request). □

3.2 Further characterization of RPE with endogenous contact rates

We begin with the following partial extension of Proposition 4 to the case of endogenous hiring:

Proposition 1 *The following properties hold in a RPE:*

1. $p\mapsto V\left(p,L^\star\left(p\right),\omega,N^\star\right)$ *is continuous;*
2. *the joint value* $S_t=S\left(p,L,\omega_t,N_t^\star\right)$ *of a firm of type* p *and of its* L *employees is differentiable in* L *at* $L=\mathscr{L}\left(L,V^\star\left(p,L,\omega',N^\star\right),a^\star\left(p,L,\omega',N^\star\right),\omega',N^\star\right)$.

Proof. Point 1 is proven using the exact same improvement argument as in Lemma 5 in Appendix B, as that argument does not depend on hiring effort being exogenous.

The proof of point 2 differs from the exogenous hiring case in that it relies on the optimality of hiring effort. We know from the proof of Proposition 6 (see Lemma 1 in the previous sub-section of this appendix) that S is convex in L, implying existence of increasing right- and left- derivatives everywhere, such that $S_{L,\ell}\leq S_{L,r}$. Moreover, $\mathscr{L}$ is increasing and continuous (indeed, differentiable) in hiring effort a. Consider a selection $\left[V\left(\cdot\right),a\left(\cdot\right)\right]$ from the maximizing correspondence. Letting $V^\star=V\left(p,L^\star\left(p\right),\omega,N^\star\right)$,

$a^\star = a(p, L^\star(p), \omega, N^\star)$ and keeping the direct dependence of S on ω and N implicit to lighten the notation, and assuming that $a^\star$ is interior, optimal choice of hiring effort then requires:

$$\begin{aligned}\left\{S_{L,r}\left[p, \mathscr{L}\left(L, V^\star, a^\star, \omega', N^\star\right)\right] - V^\star\right\} \times \eta^{\omega'} \mathscr{P}_A\left(V^\star \mid \omega'\right) \\ \leq c'\left(a^\star\right) \leq \left\{S_{L,\ell}\left[p, \mathscr{L}\left(L, V^\star, a^\star, \omega', N^\star\right)\right] - V^\star\right\} \times \eta^{\omega'} \mathscr{P}_A\left(V^\star \mid \omega'\right).\end{aligned}$$

This implies $S_{L,\ell}[p, \mathscr{L}(L, V^\star, a^\star, \omega', N^\star)] \geq S_{L,r}[p, \mathscr{L}(L, V^\star, a^\star, \omega', N^\star)]$. But the converse inequality also holds by convexity of S, so that $S_{L,\ell}[p, \mathscr{L}(L, V^\star, a^\star, \omega', N^\star)] = S_{L,r}[p, \mathscr{L}(L, V^\star, a^\star, \omega', N^\star)]$ at optimally chosen $a^\star$ and $V^\star$. □

We now turn to the First-order and Envelope conditions that we report in the main text as the basis for a simulation algorithm. To that end, we simplify the notation again using the same notational shortcuts as in Appendix B:

$$V^\star(p, \omega) := V(p, L^\star(p), \omega, N^\star) \quad \text{and} \quad a^\star(p, \omega) := a(p, L^\star(p), \omega, N^\star).$$

First, point 2 in Proposition 1 implies existence of S_L along the equilibrium path, i.e. at $L = L^\star(p)$. This further implies that the r.h.s. of (32) is differentiable in L at $L = L^\star(p)$, and so we can write the envelope condition (34).

Second, the proof of Proposition 1 readily establishes that the following NFOC must hold along the equilibrium path:

$$c'\left[a^\star\left(p, \omega'\right)\right] = \left\{S_L\left[p, \mathscr{L}\left(L, V^\star\left(p, \omega'\right), a^\star\left(p, \omega'\right), \omega', N^\star\right)\right] - V^\star\left(p, \omega'\right)\right\} \times \eta^{\omega'} \mathscr{P}_A\left[V^\star\left(p, \omega'\right) \mid \omega'\right],$$

which is (35) in the main text.

Next, notice that, as in the exogenous contact rate case, offering $W(\omega')$ close to the optimal $V^\star(p, \omega')$ is like choosing q and offering $W(\omega') = V^\star(q, \omega')$, because $V^\star(q, \omega')$ is continuous and increasing in q, so that we can define q to be the inverse of $V^\star(q, \omega') = W(\omega')$. Locally, near $V^\star(p, \omega')$, maximization of the firm-worker surplus with respect to W thus becomes equivalent to:

$$\begin{aligned}\max_q S\left[p, \mathscr{L}\left(L^\star(p), V^\star\left(q, \omega'\right), a^\star\left(p, \omega'\right), \omega', N^\star\right), \omega', \mathscr{N}\left(\omega', N^\star\right)\right] \\ + L\left(1 - \delta^{\omega'}\right) s\lambda^{\omega'} \int_{V^\star(q,\omega')}^{+\infty} v dF\left(v \mid \omega', N^\star\right) - V^\star\left(q, \omega'\right) \eta^{\omega'} a^\star\left(p, \omega'\right) \mathscr{P}_A\left[V^\star\left(q, \omega'\right) \mid \omega'\right]\end{aligned}$$

Next, using the properties of RPE:

$$\overline{F}\left(V^\star\left(q, \omega'\right) \mid \omega', N^\star\right) = \frac{1}{A\left(\omega', N^\star\right)} \int_q^{\overline{p}} a^\star\left(q, \omega'\right) d\Gamma(x)$$

$$\mathscr{P}_A\left[V^\star\left(q, \omega'\right) \mid \omega'\right] = \frac{\lambda^{\omega'}\left(1 - N^\star(\overline{p})\right) + s\lambda^{\omega'}\left(1 - \delta^{\omega'}\right) N^\star(q)}{\lambda^{\omega'}\left(1 - N^\star(\overline{p})\right) + s\lambda^{\omega'}\left(1 - \delta^{\omega'}\right) N^\star(\overline{p})},$$

so that we can write

$$\mathscr{L}\left(L^{\star}(p), V^{\star}\left(q,\omega'\right), a^{\star}\left(p,\omega'\right), \omega', N^{\star}\right) = L\left(1-\delta^{\omega'}\right)\left(1-\frac{s\lambda^{\omega'}}{A\left(\omega',N^{\star}\right)}\int_{q}^{\overline{p}} a^{\star}\left(q,\omega'\right) d\Gamma(x)\right)$$
$$+\eta^{\omega'} a^{\star}\left(p,\omega'\right)\frac{\lambda^{\omega'}\left(1-N^{\star}\left(\overline{p}\right)\right)+s\lambda^{\omega'}\left(1-\delta^{\omega'}\right)N^{\star}(q)}{\lambda^{\omega'}\left(1-N^{\star}\left(\overline{p}\right)\right)+s\lambda^{\omega'}\left(1-\delta^{\omega'}\right)N^{\star}\left(\overline{p}\right)}.$$

Moreover:

$$\int_{V^{\star}(q,\omega')}^{+\infty} v dF\left(v \mid \omega', N^{\star}\right) = \frac{1}{A\left(\omega',N^{\star}\right)}\int_{q}^{\overline{p}} a^{\star}\left(q,\omega'\right) d\Gamma(x).$$

With those identities, existence of S_L established in Proposition 1 and a.e. differentiability of $V^{\star}(q,\omega')$ (which is increasing and continuous) ensures that we can write a NFOC in the surplus maximization problem spelled our above for all q outside of a null set. At all such q:

$$S_L\left[p, \mathscr{L}\left(L^{\star}(p), V^{\star}\left(q,\omega'\right), a^{\star}\left(p,\omega'\right), \omega', N^{\star}\right), \omega', \mathscr{N}\left(\omega',N^{\star}\right)\right]$$
$$\times\frac{d\mathscr{L}\left(L^{\star}(p), V^{\star}(q,\omega'), a^{\star}(p,\omega'), \omega', N^{\star}\right)}{dq}$$
$$= L\left(1-\delta^{\omega'}\right)s\lambda^{\omega'}\frac{V^{\star}(q,\omega')\, a^{\star}(q,\omega')\,\gamma(q)}{A\left(\omega',N^{\star}\right)}$$
$$+\frac{dV^{\star}(q,\omega')}{dq}\eta^{\omega'} a^{\star}\left(p,\omega'\right)\frac{\lambda^{\omega'}\left(1-N^{\star}\left(\overline{p}\right)\right)+s\lambda^{\omega'}\left(1-\delta^{\omega'}\right)N^{\star}(q)}{\lambda^{\omega'}\left(1-N^{\star}\left(\overline{p}\right)\right)+s\lambda^{\omega'}\left(1-\delta^{\omega'}\right)N^{\star}\left(\overline{p}\right)}$$
$$+V^{\star}\left(q,\omega'\right)\eta^{\omega'} a^{\star}\left(p,\omega'\right)\frac{s\lambda^{\omega'}\left(1-\delta^{\omega'}\right)dN^{\star}(q)/dq}{\lambda^{\omega'}\left(1-N^{\star}\left(\overline{p}\right)\right)+s\lambda^{\omega'}\left(1-\delta^{\omega'}\right)N^{\star}\left(\overline{p}\right)}.$$

Using

$$\frac{d\mathscr{L}\left(L^{\star}(p), V^{\star}(q,\omega'), a^{\star}(p,\omega'), \omega', N^{\star}\right)}{dq}$$
$$=\frac{s\lambda^{\omega'} a^{\star}(q,\omega')\,\gamma(q)}{A\left(\omega',N^{\star}\right)}+\eta^{\omega'} a^{\star}\left(p,\omega'\right)\frac{s\lambda^{\omega'}\left(1-\delta^{\omega'}\right)dN^{\star}(q)/dq}{\lambda^{\omega'}\left(1-N^{\star}\left(\overline{p}\right)\right)+s\lambda^{\omega'}\left(1-\delta^{\omega'}\right)N^{\star}\left(\overline{p}\right)}$$

and replacing, we obtain:

$$\frac{dV^{\star}(q,\omega')}{dq}\eta^{\omega'} a^{\star}\left(p,\omega'\right)\frac{\lambda^{\omega'}\left(1-N^{\star}\left(\overline{p}\right)\right)+s\lambda^{\omega'}\left(1-\delta^{\omega'}\right)N^{\star}(q)}{\lambda^{\omega'}\left(1-N^{\star}\left(\overline{p}\right)\right)+s\lambda^{\omega'}\left(1-\delta^{\omega'}\right)N^{\star}\left(\overline{p}\right)}$$
$$=\left\{S_L\left[p, \mathscr{L}\left(L^{\star}(p), V^{\star}\left(q,\omega'\right), a^{\star}\left(p,\omega'\right), \omega', N^{\star}\right), \omega', \mathscr{N}\left(\omega',N^{\star}\right)\right]-V^{\star}\left(q,\omega'\right)\right\}$$
$$\times\frac{d\mathscr{L}\left(L^{\star}(p), V^{\star}(q,\omega'), a^{\star}(p,\omega'), \omega', N^{\star}\right)}{dq}$$

which must hold by optimality at $q = p$ and $L = L^{\star}(p)$. Replacing, we can simplify $a^{\star}(p, \omega')$ and collect terms to obtain:

$$\left\{S_L\left[p, \mathscr{L}\left(L^{\star}(p), V^{\star}\left(p, \omega'\right), a^{\star}\left(p, \omega'\right), \omega', N^{\star}\right), \omega', \mathscr{N}\left(\omega', N^{\star}\right)\right] - V^{\star}\left(p, \omega'\right)\right\}$$
$$\times s\lambda^{\omega'}\left(1 - \delta^{\omega'}\right) L^{\star}(p)\gamma(p)\left\{\frac{1}{\lambda^{\omega'}\left(1 - N^{\star}(\overline{p})\right) + s\lambda^{\omega'}\left(1 - \delta^{\omega'}\right) N^{\star}(\overline{p})} + \frac{1}{\eta^{\omega'} A\left(\omega', N^{\star}\right)}\right\}$$
$$= \frac{dV^{\star}(q, \omega')}{dq} \frac{\lambda^{\omega'}\left(1 - N^{\star}(\overline{p})\right) + s\lambda^{\omega'}\left(1 - \delta^{\omega'}\right) N^{\star}(q)}{\lambda^{\omega'}\left(1 - N^{\star}(\overline{p})\right) + s\lambda^{\omega'}\left(1 - \delta^{\omega'}\right) N^{\star}(\overline{p})}.$$

Further substituting the matching function identity (31), this simplifies into the NFOC (20):

$$\left\{S_L\left[p, \mathscr{L}\left(L^{\star}(p), V^{\star}\left(p, \omega'\right), a^{\star}\left(p, \omega'\right), \omega', N^{\star}\right), \omega', \mathscr{N}\left(\omega', N^{\star}\right)\right] - V^{\star}\left(p, \omega'\right)\right\}$$
$$\times 2s\lambda^{\omega'}\left(1 - \delta^{\omega'}\right) L^{\star}(p)\gamma(p) = \frac{dV^{\star}(q, \omega')}{dq}\left[\lambda^{\omega'}\left(1 - N^{\star}(\overline{p})\right) + s\lambda^{\omega'}\left(1 - \delta^{\omega'}\right) N^{\star}(q)\right].$$

4 Monotone comparative statics

The proof of the standard comparative statics argument referred to in Lemma 1, Appendix A of the paper, goes as follows. Let

$$x^{*}(\alpha) = \arg\max_{x} f(x, \alpha)$$

Simple revealed-preference reasoning leads to the following two inequalities:

$$f\left(x^{*}(\alpha_1), \alpha_1\right) - f\left(x^{*}(\alpha_1), \alpha_0\right) > 0 > f\left(x^{*}(\alpha_0), \alpha_1\right) - f\left(x^{*}(\alpha_0), \alpha_0\right).$$

Rearranging:

$$f\left(x^{*}(\alpha_1), \alpha_1\right) - f\left(x^{*}(\alpha_0), \alpha_1\right) > f\left(x^{*}(\alpha_1), \alpha_0\right) - f\left(x^{*}(\alpha_0), \alpha_0\right)$$

If f has increasing differences, i.e. $f(x_1, \alpha) - f(x_0, \alpha)$ is increasing in α for any $x_1 > x_0$, we have:

$$x^{*}(\alpha_1) > x^{*}(\alpha_0).$$