Mechanism Design with Limited Information: The Case of Nonlinear Pricing

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Limited Information and Communication

- "revelation principle": any incentive compatible in any mechanism can be truthfully implemented in the direct revelation mechanism, where every agent reports his private information, his type, truthfully
- yet when the private information of the agents is large, then the direct revelation mechanism requires:
  the agents to have abundant capacity to communicate with the principal, and the principal to have
  abundant capacity to process information
- the objective of this paper is to study the performance of optimal mechanisms, when the agents can communicate only limited information, or equivalently, when the principal can process only limited information

The Case of Nonlinear Pricing

- analysis in the context of a representative, but tractable, mechanism design environment, namely the canonical problem of nonlinear pricing
- the principal, the seller, is offering a variety of choices to the agent, the buyer, who has private information about his willingness-to-pay for the product
- the distinct point of view, relative to the seminal analysis by Mussa and Rosen (1978) and Maskin and Riley (1984) is that the information conveyed by the agents, and subsequently the menu of possible choices offered by the seller, is finite, rather than uncountable

The Limits of Information

- the limits to information may arise for various reasons
- demand side:
  too difficult or too complex for the buyer to communicate his exact preferences and resulting willingness to pay to the seller
- supply side:
  too time-consuming for the seller to identify the consumer's preferences across many goods with close attributes and only subtle differences

The Link to Information Theory

- we adopt a linear-quadratic specification, Mussa & Rosen (1978) where the consumer's gross utility is the product of his willingness-to-pay, his type θ, and of the consumed quantity q of the product
- the socially efficient quantity q for a customer should be equated to his valuation θ if a continuum of choices were available
- when only a finite number of choices are accessible q can take on only finitely many values

The Link to Information Theory

- if we view θ as source signal and q as representation level, then the social welfare can be written as the mean square error between the source signal and the representation signal.
- now the welfare maximization problem can be characterized by the Lloyd-Max optimality conditions, a well-established result in the theory of quantization

From Efficiency to Revenue Maximizing

- we extend the analysis to the revenue maximization problem by reinterpreting the source signal as the virtual utility
- we estimate the welfare and revenue loss resulting from the use of a finite n-class contract (relative to the continuum contract)
- we characterize the rate of convergence for the welfare and revenue loss as a function of n
- we establish that the maximum welfare loss shrinks towards zero at a rate proportional to $1/n^2$.

Priority Classes and Coarse Matching

- Wilson (1989) considers the impact of a finite number of priority classes on the efficient rationing of services (electricity pricing)

  less concerned with optimal priority ranking for finite class but rather with approximation property of finite priority classes
- McAfee (2002) rephrases priority rationing problem as two-sided matching problem (between consumer and services): a binary priority contract ("coarse matching") can achieve at least half of the social welfare of a continuum of priorities
- Madarasz and Prat (2010) suggest "profit-participation" mechanism to establish approximation rather than finite optimality results in nonlinear pricing environment

Many Agent Mechanisms

- beyond single agent mechanisms, specifically single-item auctions among many bidders
- Blumrosen, Nisan and Signal (2007) consider restricted communication in auctions with either two agents or binary messages for every agent
- Kos (2010) allows for a finite number of messages and agent based on optimal auction by Bergemann and Pesendorfer (2007).

Model

- a monopolist facing a continuum of heterogeneous consumers
- each consumer has quasi-linear utility function:

$$u(\theta, q, t) = \theta q - t$$

- $q \in \mathbb{R}_+$: quality, quantity of consumption
- $\theta \in [0, 1]$: willingness-to-pay for the good, private information
- prior distribution of θ is given by $F \in \Delta [0, 1]$
- profit of monopoly seller:

$$\pi(q, t) = t - c(q)$$

  $t \in \mathbb{R}_+$: transfer paid by buyer
- cost of production

$$c(q) = \frac{1}{2} q^2$$

- consider the social welfare maximization problem in the absence of private information by the agent
- In the absence of communication constraints, the social surplus, denoted by SW_∞ is:

$$SW_\infty \triangleq \max_{q(\theta)} \mathbb{E}\left[\theta q(\theta) - \frac{1}{2}q^2(\theta)\right].$$

- optimal solution for every type θ can be obtained pointwise

$$q^*(\theta) = \theta.$$

- the socially optimal menu offers a continuum of choices:

$$M_\infty^* = \{q^*(\theta) = \theta\}$$

- the welfare maximizing problem is equivalent to minimizing the mean square error (MSE):

$$\mathbb{E}_\theta[(\theta - q)^2]$$

- discretized contract is $n-$class contract or $n-$class menu:

$$M_n = \{q_k\}_{k=1}^{n},$$

- $M_n \in \mathcal{M}_n$ is the set of contracts which offer at most a finite number n of quantity choices:
- the social welfare problem for a given number n of choices:

$$SW_n = \max_{q(\theta)} \mathbb{E}_\theta \left[\theta q(\theta) - \frac{1}{2} (q(\theta))^2 \right],$$

subject to $\{q(\theta)\}_{\theta=0}^{1} \in \mathcal{M}_n$.

Single Crossing and Monotone Partition

- the valuation of the buyer is supermodular:

$$\partial^2 u(\theta, q) / \partial\theta\partial q > 0$$

- the optimal assignment of types to quantities has a monotone partitional structure:

$$\{A_k = [\theta_{k-1}, \theta_k)\}_{k=1}^n$$

represent a partition of the set of consumer types where

$$0 = \theta_0 < \cdots < \theta_{k-1} < \theta_k < \cdots < \theta_n = 1.$$

- all consumers with type $\theta \in A_k$ will be assigned $q^*(\theta) = q_k^*$
- the socially optimal menu $M_n^* = \{q_k^*\}_{k=0}^n$ is increasing in k:

$$q_1^* < q_2^* < \cdots < q_k^*.$$

- if we view θ as the source signal and q_k as the representation points of θ on the quantization intervals

$$A_k = [\theta_{k-1}, \theta_k),$$

  the solution to the social welfare maximizing contract is the n-level quantization problem in information theory
- find the quantization intervals A_k and the corresponding representation points q_k to minimize the mean square error (MSE):

$$MSE_n \equiv \min_{q(\theta)} \mathbb{E}_\theta\left[(\theta - q)^2\right], \quad \text{subject to } \{q(\theta)\}_{\theta=0}^1 \in \mathcal{M}_n,$$

Theorem (Necessary Conditions)

The optimal menu M_n^* of the social welfare problem satisfies:

$$\theta_k^* = \frac{1}{2}\left(q_k^* + q_{k+1}^*\right), \quad q_k^* = \mathbb{E}_\theta\left[\theta | \theta \in \left[\theta_{k-1}^*, \theta_k^*\right)\right], \quad k = 0, \ldots, n.$$

- q_k^*, the production level for the interval $A_k^* = \left[\theta_{k-1}^*, \theta_k^*\right)$, must be the conditional mean for θ given that $\theta \in A_k^*$
- θ_k^*, which separates two neighboring intervals A_k^* and A_{k+1}^*, must be the arithmetic average of q_k^* and q_{k+1}^*
- Lloyd (1982) - Max (1960) optimality conditions

- we are interested in the relative performance of finite contracts and evaluate the difference between SW_{∞}^{*} and SW_{n}^{*}

Definition

Given any $F \in \Delta$, the welfare loss of an n-class contract compared with the optimal continuous contract is defined by
$L(F; n) \equiv SW_{\infty}^{*} - SW_{n}^{*}$

- provide an upper bound over all distributions, i.e., the worst-case scenario from point of view of social welfare.

Definition

The maximum welfare loss of an n-class contract over all $F \in \Delta$ is given by $L(n) \equiv \sup_{F \in \Delta} L(F; n)$.

- for certain family of distributions we obtain closed-form solutions from the Lloyd-Max optimality conditions
- suppose that θ is uniformly distributed over $[0,1]$
- earlier optimality conditions have unique solution:

$$\theta_k^* = \frac{k}{n}, \qquad q_k^* = \frac{k-1/2}{n}, \ \ k = 0, 1, \ldots, n.$$

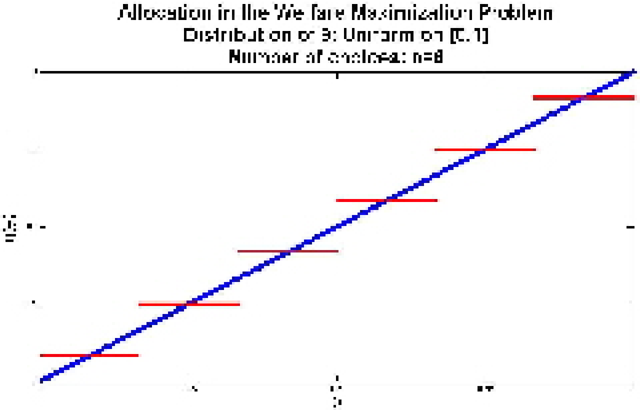
- expected social welfare is

$$SW_n^* = \frac{1}{6} - \frac{1}{24n^2}$$

and welfare loss is

$$SW_\infty^* - SW_n^* = \frac{1}{24n^2}$$

- the cutoff points are uniformly distributed as the underlying distribution of θ is uniform


Allocation in the Welfare Maximization Problem
Distribution of θ: Uniform on [0,1]
Number of choices: n=6

Approximation

- a direct approach would require the explicit form of the optimal quantizer of Lloyd-Max conditions
- in the absence of an explicit characterization, we pursue an indirect approach to obtain a bound through a series of suboptimal quantizers
- for any given $F \in \Delta$, we have:

$$SW_n = \mathbb{E}_\theta \left[\theta q - \frac{1}{2} q^2 \right] = \frac{1}{2} \mathbb{E} \left[\theta^2 \right] - \frac{1}{2} MSE_n$$

- with $SW_\infty = \frac{1}{2} \mathbb{E} \left[\theta^2 \right]$, we obtain:

$$SW_\infty - SW_n = \frac{1}{2} MSE_n$$

Menus

- for a given distribution $F \in \Delta$, the menu

$$M_n = \{q_k\}_{k=1}^{n}$$

can be generated by a finite partition A_k through

$$q_k = \mathbb{E}\left[\theta | \theta \in A_k\right], \; k = 1, \ldots, n,$$

- for any $M_n \in \mathcal{M}_n^*$:

$$MSE_n = \mathbb{E}_\theta\left[(q-\theta)^2\right] = \sum_{k=1}^{n} \left(F\left(\theta_k\right) - F\left(\theta_{k-1}\right)\right) var\left(\theta | \theta \in A_k\right)$$

- we estimate the variance of θ conditional on the interval A_k to provide an upper bound on $L\left(n\right)$.

Theorem

For $F \in \Delta$, and any $n \geq 1$, $L(F; n) \leq \frac{1}{8n^2}$.

Theorem

For any $n \geq 1$, $\frac{1}{24n^2} \leq L(n) \leq \frac{1}{8n^2}$, i.e. $L(n) = \Theta\left(\frac{1}{n^2}\right)$.

- Wilson (1989) establishes that finite priority ranking of order n induces a welfare loss of order $1/n^2$
- his method of proof is different from ours, using numerical approximation arguments, but for the limit results implicitly uses uniform quantization of the relevant distribution

- revenue maximizing mechanism has two sets of constraints:
- participation constraint:

$$\theta q(\theta) - t(\theta) \geq 0, \quad \text{for all } \theta \in [0,1],$$

  and incentive constraints:

$$\theta q(\theta) - t(\theta) \geq \theta q(\theta') - t(\theta'),$$

- the revenue maximization problem, finding optimal solution for $\{q(\theta), t(\theta)\}$ is distinct from welfare maximization problem involving $q(\theta)$ only
- use incentive constraints to eliminate the transfers and rewrite the problem in terms of the allocation alone
- replace the *true valuation* θ of the buyer with the corresponding *virtual valuation*:

$$\hat{\theta} \equiv \psi(\theta) = \theta - \frac{1 - F(\theta)}{f(\theta)}.$$

- the expected profit of the seller (without information constraints) is:

$$\Pi_\infty^* = \max_{q(\theta)} \mathbb{E}_\theta \left[q(\theta)\,\psi(\theta) - \frac{1}{2} q^2(\theta) \right].$$

- with limited information, the seller can only offer a finite menu

$$\{(q_k, t_k), k = 1, \ldots, n\}$$

rewrite the problem in terms of a choice over a finite set of allocations $\mathcal{M}_n$:

$$\Pi_n^* = \max_{q(\theta) \in \mathcal{M}_n} \mathbb{E}_\theta \left[q\psi(\theta) - \frac{1}{2} q^2 \right].$$

- optimality conditions for revenue maximizing contract in the presence of information constraints:

Lemma

The revenue maximizing solution satisfies:

$$\theta_k^* - \frac{1 - F\left(\theta_k^*\right)}{f\left(\theta_k^*\right)} = \frac{1}{2}\left(q_k^* + q_{k+1}^*\right) \qquad k = 0, \ldots, n-1, \qquad (1)$$

and

$$q_k^* = \frac{\theta_{k-1}^*\left(1 - F\left(\theta_{k-1}^*\right)\right) - \theta_k^*\left(1 - F\left(\theta_k^*\right)\right)}{F\left(\theta_k^*\right) - F\left(\theta_{k-1}^*\right)} \qquad k = 1, \ldots, n. \qquad (2)$$

- revenue loss induced by an n-class contract

$$\Lambda\left(F; n\right) \equiv \Pi_\infty^* - \Pi_n^*, \qquad \Lambda\left(n\right) \equiv \sup_{F \in \Delta} \Lambda\left(F; n\right).$$

Example Continued: Uniform Distribution

- θ is uniformly distributed over $[0, 1]$
- the quantization problem has a unique solution:

$$\theta_k^* = \frac{n+k+1}{2n+1}, \qquad q_k^* = \frac{2k}{2n+1}, \qquad k = 0, \ldots, n.$$

- the maximum expected revenue is

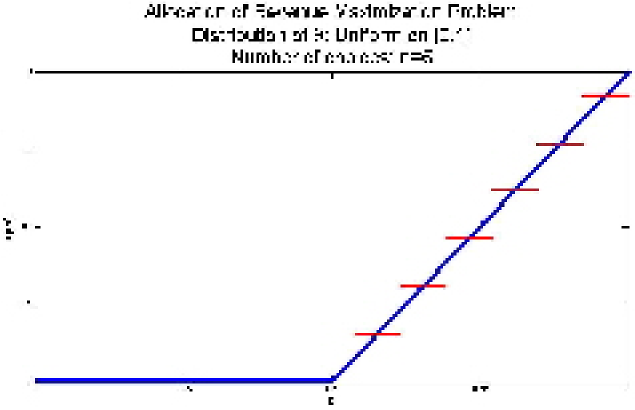
$$\Pi_n^* = \frac{n(n+1)}{3(2n+1)^2}$$

- the revenue loss is

$$\Pi_\infty^* - \Pi_n^* = \frac{1}{12(2n+1)^2}.$$

- the seller tends to serve fewer buyers as compared to the case when a continuous contract is used
- the seller reduces the service coverage to pursue higher marginal revenues

Example Continued: Uniform Distribution



Theorem

For any $n \geq 1$, $1/12\,(2n+1)^2 \leq \Lambda\,(n) \leq 1/8n^2$, *and hence* $\Lambda\,(n) = \Theta(1/n^2)$.

- the approximation result of the revenue maximizing problem is similar to the one of the social welfare program

- by focusing on the linear-quadratic model, we relate the limited information problem directly to the quantization problem in information theory
- the nonlinear pricing environment represents an elementary instance of the general mechanism design environment
- the simplicity is that the principal does not have to solve allocative externalities
- by contrast, in auctions, the allocation is constrained by the allocation to the other agents. For an information-theoretic point of view, the ensuing multi-dimensionality would suggest that the methods of vector quantization are relevant

Competition with Limited Information

- consider competing firms with limited information
- with limited number of items on offer, segmentation across firms
- less competition and positive profits
- with cost of information, fewer items offered with many firms than under single firm

Source Coding versus Channel Coding

- the current analysis focused on limited information: efficient source coding
- from an information-theoretic and economic viewpoint, it is natural to augment the analysis to reliable communication between agent and principal over noisy channels: the problem of channel coding
- larger information rents to the privately informed agents

- A multi-product firm provides d dimensional goods
- each consumer's utility function is:

$$\theta^T \Phi q - t,$$

- $\theta = (\theta_1, ..., \theta_d) \in \Theta = \Theta_1 \times \cdots \times \Theta_d \subset \mathbb{R}_+^d$,
  - θ_i is his preference parameter on feature i
- $q = (q_1, ..., q_d) \in \mathbb{R}_+^d$: vector of quality