First price auctions with general information structures: Implications for bidding and revenue

Dirk Bergemann
Yale

Benjamin Brooks
BFI/UChicago

Stephen Morris
Princeton

New York University Abu Dhabi

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Premises

1. Classical auction theory makes stylized assumptions about information
2. Assumptions about information are hard to test
3. Equilibrium behavior can depend a lot on how we specify information

Promises

- Goal: a theory of bidding that is robust to specification of information
- First attempt: First price auction
- Hold fixed underlying *value distribution*,
- Consider *all* specifications of *information* and *equilibrium*
- We deliver:
  - A tight lower bound on the winning bid distribution
  - A tight lower bound on revenue
  - A tight upper bound on bidder surplus
- Other results on max revenue, min bidder surplus, min efficiency

A (toy) model of a first price auction

- Two bidders
- Pure common value $v \sim U[0,1]$
- Submit bids $b_i \in \mathbb{R}_+$
- High bidder gets the good and pays bid
  $\implies$ winner's surplus is $v - b_i$
- Allocation of good is always efficient, total surplus $1/2$
- Seller's expected revenue is $R = \mathbb{E}[\max\{b_1, b_2\}]$
- Bidder surplus $U = 1/2 - R$
- What predictions can we make about U and R in equilibrium?

Filling in beliefs

- What do bidders know about the value?
- What do they know about what others know?
- Assume beliefs are consistent with a common prior
- Still, many possible ways to "fill in" information:
  - Bidders observe nothing;
    Unique equilibrium: $b_1 = b_2 = R = 1/2$
  - Bidders observe everything;
    $b_1 = b_2 = v$, $R = 1/2$
- True information structure is likely somewhere in between:
  - Bidders have some information about v, but not perfect
  - But exactly how much information do they have?

Lower revenue?

- Engelbrecht-Wiggans, Milgrom, Weber (1983, EMW):
- Bidder 1 observes v, bidder 2 observes nothing
  - $b_1 = v/2$, $b_2 \sim U[0, 1/2]$ and independent of v
- Bidder 2 is indifferent:
  With a bid of $b_2 \in [0, 1/2]$, will win whenever $v \leq 2b_2$
  Expected value is exactly b_2!
- Bidder 1 wins with a bid of b_1 with probability $2b_1$
  Surplus is $(v - b_1)2b_1$
  $\implies$ optimal to bid $b_1 = v/2$!
- $U_1 = \int_{v=0}^{1} v(v - v/2)dv = 1/6$, $U_2 = 0$, $R = 1/3$

How we model beliefs matters

- Welfare outcomes are sensitive to modelling of information
- Why? Optimal bid depends on distribution of others' bids, and on correlation between others' bids and values
- Problem: hard to say which specification is "correct"
- What welfare predictions do not depend on how we model information?

Uniform example continued

- Can we characterize minimum revenue?
- Must be greater than zero!
- But seems likely to be lower than EMW
- At min R, winning bids have been pushed down "as far as they can go"
- Force pushing back must be incentive to deviate to higher bids
- In EMW, informed bidder strictly prefers equilibrium bid

Towards a Bound: Winning Bid

- Consider symmetric equilibria in which *winning bid* is an increasing and deterministic function $\beta(v)$ of true value v
- Which β could be incentive compatible in equilibrium?
- Consider the following *uniform upward deviation* to b: Whenever equilibrium bid, winning or not, is $b' < b$, bid b instead!
- Now let bids b', b be winning bids for some values x, v respectively:

$$b' = \beta(x) < \beta(v) = b$$

Towards a Bound: Uniform Upward Deviation

- Now let bids b', b be winning bids for some values x, v respectively:

$$b' = \beta(x) < \beta(v) = b$$

- Bid b' could have been a loosing or a winnig bid
- Uniform upward deviation to $b = \beta(v)$ is not attractive if

$$\underbrace{\frac{1}{2}\int_{x=0}^{v}(\beta(v) - \beta(x))dx}_{\text{loss when would have won}} \geq \underbrace{\frac{1}{2}\int_{x=0}^{v}(x - \beta(v))dx}_{\text{gain when would have lost}}$$

- Using symmetry $(1/2)$ and deterministic winning bid $\beta(v)$

Restrictions on β

- Uniform upward deviation to $b = \beta(v)$

$$\underbrace{\frac{1}{2}\int_{x=0}^{v}(\beta(v)-\beta(x))dx}_{\text{loss when would have won}} \geq \underbrace{\frac{1}{2}\int_{x=0}^{v}(x-\beta(v))dx}_{\text{gain when would have lost}}$$

  rearranges to

$$\beta(v) \geq \frac{1}{2v}\int_{x=0}^{v}(x+\beta(x))dx \qquad \text{(IC)}$$

- What is the smallest β subject to (IC) and $\beta \geq 0$?
- Must solve (IC) with equality for all v

Minimal Winning Bid $\underline{\beta}$

- uniform upward deviation solves

$$\beta(v) = \frac{1}{2v} \int_{x=0}^{v} (x + \beta(x)) dx \qquad \text{(IC)}$$

- $\underline{\beta}$ is conditional expectation of (average of) value and $\underline{\beta}$:

$$\underline{\beta}(v) = \frac{1}{\sqrt{v}} \int_{x=0}^{v} x \frac{1}{2\sqrt{x}} dx = \frac{v}{3}$$

- Conditional Expectation with respect to $F(v)^{1/2} = v^{1/2}$.
- Compare to the bid $b(v) = v/2$, not even winning bid in EMW.

A lower bound on revenue

- Induced distribution of winning bids is $U[0, 1/3]$
- Revenue is $1/6$
- In fact, symmetry/deterministic winning bid are not needed
- Distribution of winning bid has to FOSD $U[0, 1/3]$ in *all* equilibria under *any* information
- $1/6$ is a *global* lower bound on equilibrium revenue

Bound is tight

- Can construct information/equilibrium that hits bound
- Bidders get i.i.d. signals $s_i \sim F(x) = \sqrt{x}$ on $[0,1]$
- Value is highest signal
- Distribution of highest signal is $U[0,1]$
- Equilibrium bid: $\sigma_i(s_i) = s_i/3 \ \left(= \underline{\beta}(s_i)\right)$
- Defer proof until general results

Beyond the example

- Argument generalizes to:
- Any common value distribution!
  - Any number of bidders!
  - Arbitrarily correlated values!!!
- Assume symmetry of value distribution for some results
- Minimum bidding is characterized by a *deterministic winning bid* given the true values
- In general model, only depends on a one-dimensional statistic of the value profile
- Bound is characterized by binding *uniform upward incentive constraints*

The plan

- Detailed exposition of minimum bidding
- Maximum revenue/minimum bidder surplus
- Restrictions on information
- Other directions in welfare space (e.g., efficiency)

General model

- N bidders
- Distribution of values: $P(dv_1, \ldots, dv_N)$
- Support of marginals $V = [\underline{v}, \overline{v}] \subseteq \mathbb{R}_+$
- An *information structure* $\mathcal{S}$ consists of
  - A measurable space S_i of signals for each player i, $S = \times_{i=1}^{N} S_i$
  - A conditional probability measure

$$\pi : V^N \to \Delta(S)$$

Equilibrium

- Bidders' strategies map signals to distributions over bids in $[0, \overline{v}]$

$$\sigma_i : S_i \to \Delta(B)$$

- Assume "*weakly undominated strategies*": bidder i never bids strictly above the support of first-order beliefs about v_i
- Bidder i's payoff given strategy profile $\sigma = (\sigma_1, \dots, \sigma_N)$:

$$U_i(\sigma, \mathcal{S}) = \int_{v \in V} \int_{s \in S} \int_{b \in B^N} (v_i - b_i) \frac{\mathbb{I}_{\{b_i \geq b_j, \ \forall j\}}}{|\arg\max_j b_j|} \sigma(db|s) \pi(ds|v) P(dv)$$

- σ is a Bayes Nash *equilibrium* if

$$U_i(\sigma, \mathcal{S}) \geq U_i(\sigma_i', \sigma_{-i}, \mathcal{S}) \ \forall i, \sigma_i'$$

Other welfare outcomes

$$\text{Bidder surplus: } U(\sigma,\mathcal{S}) = \sum_{i=1}^{N} U_i(\sigma,\mathcal{S})$$

$$\text{Revenue: } R(\sigma,\mathcal{S}) = \int_{v\in V^N}\int_{s\in S}\int_{b\in B^N} \max_i b_i \sigma(b|s)\pi(ds|v)P(dv$$

$$\text{Total surplus: } T(\sigma,\mathcal{S}) = R(\sigma,\mathcal{S}) + U(\sigma,\mathcal{S})$$

$$\text{Efficient surplus: } \overline{T} = \int_{v\in V} \max_i v_i P(dv)$$

General common values

- As we generalize, minimum bidding continues to be characterized by a *deterministic winning bid* given values: $\underline{\beta}(v_1, \ldots, v_N)$
- $\underline{\beta}$ has an explicit formula
- Consider pure common values with $v \sim P \in \Delta([\underline{v}, \overline{v}])$
- Minimum winning bid generalizes to

$$\underline{\beta}(v) = \frac{1}{\sqrt{P(v)}} \int_{x=\underline{v}}^{v} x \frac{P(dx)}{2\sqrt{P(x)}}$$

- $P(v)^{1/2}$ generalizes to $P(v)^{(N-1)/N}$ with N bidders
- Minimum revenue:

$$\underline{R} = \int_{v=\underline{v}}^{\overline{v}} \underline{\beta}(v) P(dv)$$

$N = 2$ and general value distributions

- Write $P(dv_1, dv_2)$ for value distribution
- Similarly, lots of binding uniform upward IC
- Incentive to deviate up depends on value when you *lose*
- On the whole, efficient allocation reduces gains from deviating up
- Suggests minimizing equilibrium is efficient, winning bid is constrained by *loser's (i.e., lowest) value*

General bounds for $N = 2$

- Similar $\underline{\beta}$, but now depends on *lowest* value
- $Q(dm)$ is distribution of $m = \min\{v_1, v_2\}$ (assume non-atomic)
- Minimum winning bid is

$$\underline{\beta}(m) = \frac{1}{\sqrt{Q(m)}} \int_{x=\underline{v}}^{v} x \frac{Q(dx)}{2\sqrt{Q(x)}}$$

- Minimum revenue:

$$\underline{R} = \int_{m=\underline{v}}^{\overline{v}} \underline{\beta}(m) Q(dm)$$

Losing values when $N > 2$

- With $N > 2$, bid minimizing equilibrium should still be efficient
- Intuition: coarse information about losers' values lowers revenue
- Consider complete information, all values are common knowledge
- High value bidder wins and pays second highest value

Average losing values I

- Simple variation: Bidders only observe
  (i) High value bidder's identity
  (ii) *Distribution* of values
- Winner is still high value bidder, but losing bidders don't know who has which value
- If prior is symmetric, believe they are equally likely to be at any point in the distribution *except* the highest
- In equilibrium, winner pays *average of $N-1$ lowest values*:

$$\mu(v_1, \ldots, v_N) = \frac{1}{N-1}\left(\sum_{i=1}^{N} v_i - \max_i v_i\right)$$

General bounds

- $Q(dm)$ is distribution of $m = \mu(v)$ (assume non-atomic)
  - Minimum winning bid and revenue:

$$\underline{\beta}(m) = \frac{1}{Q^{\frac{N-1}{N}}(v)} \int_{x=\underline{v}}^{v} x \frac{N-1}{N} \frac{Q(dx)}{Q^{\frac{1}{N}}(x)}$$
$$= \frac{1}{Q^{\frac{N-1}{N}}(v)} \int_{x=\underline{v}}^{v} x \, Q^{\frac{N-1}{N}}(dx)$$

  - Minimum revenue:

$$\underline{R} = \int_{m=\underline{v}}^{\overline{v}} \underline{\beta}(m) \, Q(dm)$$

  - Let $\underline{H}(b) = Q(\underline{\beta}^{-1}(b))$

Main result

Theorem (Minimum Winning Bids)

1. *In any equilibrium under any information structure in which the marginal distribution of values is P, the distribution of winning bids must first-order stochastically dominate $\underline{H}$.*
2. *Moreover, there exists an information structure and an efficient equilibrium in which the distribution of winning bids is exactly $\underline{H}$.*

Implications

Corollary (Minimum revenue)

Minimum revenue over all information structures and equilibria is $\underline{R}$.

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Corollary (Maximum bidder surplus)

Maximum total bidder surplus over all information structures and equilibria is $\overline{T} - \underline{R}$.

Proof methodology

1. Obtain a bound via relaxed program
2. Construct information and equilibrium that attain the bounds

(start with #2)

Minimizing equilibrium and information

- Bidders receive independent signals $s_i \sim Q^{1/N}(s_i)$
  $\implies$ distribution of highest signal is $Q(s)$
- Signals are correlated with values s.t.
  - Highest signal is true average lowest value, i.e.,

$$\mu(v_1, \dots, v_n) = \max\{s_1, \dots, s_n\}$$

  - Bidder with highest signal is also bidder with highest value, i.e.,

$$\arg\max_i s_i \subseteq \arg\max_i v_i$$

- All bidders use the monotonic pure-strategy $\underline{\beta}(s_i)$

Proof of equilibrium

- $\underline{\beta}$ is the equilibrium strategy for an "as-if" IPV model, in which $v_i = s_i$
- IC for IPV model with independent draws from $Q^{1/N}$:

$$(s_i - \sigma(s_i))Q^{\frac{N-1}{N}}(s_i)$$

- Local IC:

$$(s_i - \sigma(s_i))Q^{\frac{N-1}{N}}(ds_i) - \sigma'(s_i)Q^{\frac{N-1}{N}}(s_i) = 0$$

- Solution is precisely

$$\sigma(s_i) = \frac{1}{Q^{\frac{N-1}{N}}(s_i)} \int_{x=\underline{v}}^{s_i} x\, Q^{\frac{N-1}{N}}(dx) = \underline{\beta}(s_i)$$

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- IC for IPV model with independent draws from $Q^{1/N}$:

$$(s_i - \sigma(s_i))Q^{\frac{N-1}{N}}(s_i) \geq (s_i - \sigma(m))Q^{\frac{N-1}{N}}(m)$$

- Local IC:

$$(s_i - \sigma(s_i))Q^{\frac{N-1}{N}}(ds_i) - \sigma'(s_i)Q^{\frac{N-1}{N}}(s_i) = 0$$

- Solution is precisely

$$\sigma(s_i) = \frac{1}{Q^{\frac{N-1}{N}}(s_i)} \int_{x=\underline{v}}^{s_i} x\, Q^{\frac{N-1}{N}}(dx) = \underline{\beta}(s_i)$$

Downward deviations

- Expectation of the bidder with the highest signal is $\tilde{v}(s_i) \geq s_i$
- Downward deviator obtains surplus

$$(\tilde{v}(s_i) - \underline{\beta}(m))Q^{\frac{N-1}{N}}(m)$$

and

$$\left(\tilde{v}(s_i) - \underline{\beta}(m)\right) Q^{\frac{N-1}{N}}(dm) - \underline{\beta}'(m)Q^{\frac{N-1}{N}}(m)$$
$$\geq \left(s_i - \underline{\beta}(m)\right) Q^{\frac{N-1}{N}}(dm) - \underline{\beta}'(m)Q^{\frac{N-1}{N}}(m)$$

- Well-known that IPV surplus is single peaked: if $m < s_i$,

$$\Longrightarrow (s_i - \underline{\beta}(m))Q^{\frac{N-1}{N}}(dm) - \underline{\beta}'(m)Q^{\frac{N-1}{N}}(dm) \geq 0$$

Average losing values II

- Winning bids depend on avg of lowest values $=$ average of losing bids (since equilibrium is efficient)
- Suppose winning bid in equilibrium is $\underline{\beta}(m) > \underline{\beta}(s_i)$ $\implies \mu(v) = m$ for true values v
- By symmetry, all permutations of v are in $\mu^{-1}(m)$ and equally likely
- If you only know that

  (i) you lose in equilibrium and

  (ii) $v \in \mu^{-1}(m)$,

  you expect your value to be m!
- By deviating up to win on this event, gain m in surplus

Upward deviations

- Upward deviator's surplus

$$(\tilde{v}(s_i) - \underline{\beta}(m))Q^{\frac{N-1}{N}}(s_i) + \int_{x=s_i}^{m} (x - \underline{\beta}(m))Q^{\frac{N-1}{N}}(dx)$$

- Derivative w.r.t. m:

$$(m - \underline{\beta}(m))Q^{\frac{N-1}{N}}(dm) - \underline{\beta}(m)'Q^{\frac{N-1}{N}}(m) = 0!$$

- In effect, correlation between others bids' and losing values induces adverse selection s.t. losing bidders are indifferent to deviating up

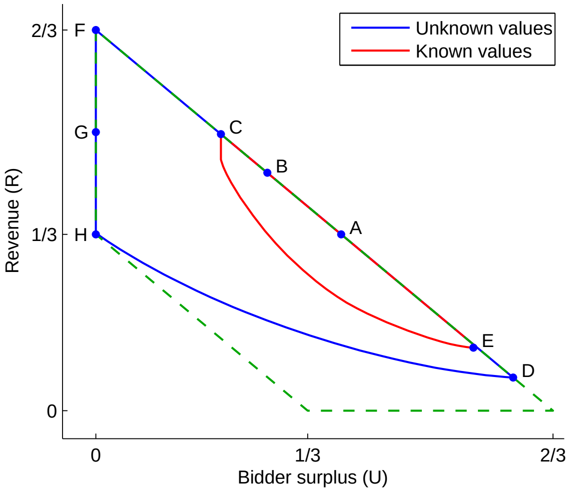
Towards a general bound

- Claim is that construction attains a lower bound
- Show this via relaxed program
- Minimum CDF of winning bids subject to uniform upward IC
- Key WLOG properties of solution (and minimizing equilibrium):
    1. Symmetry
    2. Winning bid depends on average losing value
    3. Efficiency
    4. Monotonicity of winning bids in losing values
    5. All uniform upward IC bind

Other directions

- We talked about max/min revenue, max/min bidder surplus
- What about weighted sums? Minimum efficiency?
- More broadly, what is the *whole set* of possible (U, R) pairs?
- Solved numerically for two bidder i.i.d. $U[0, 1]$ model

Welfare set



- Note: Lower bound on efficiency

What can we do with this?

- Applications/extensions:
  - Many bidder limit
  - Impact of reserve prices/entry fees
  - Identification

  Other directions in welfare space
- Context:
  - Part of a larger agenda on robust predictions and information design