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Robust Implementation in Direct Mechanisms

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A social choice function is robustly implementable if there is a mechanism under which the process of iteratively eliminating strictly dominated messages lead to outcomes that agree with the social choice function for all beliefs at every type profile. In an interdependent-value environment with single-crossing preferences, we identify a contraction property on the preferences which together with strict *ex post* incentive compatibility is sufficient to guarantee robust implementation in the direct mechanism. Strict *ex post* incentive compatibility and the contraction property are also necessary for robust implementation in *any* mechanism, including indirect ones. The contraction property requires that the interdependence is not too high. In a linear signal model, the contraction property is equivalent to an interdependence matrix having all eigenvalues smaller than one.

1. INTRODUCTION

The mechanism design literature provides a powerful characterization of which social choice functions can be achieved when the designer has incomplete information about agents' types. If we assume a commonly known common prior over the possible types of agents, the revelation principle establishes that if the social choice function can arise as an equilibrium in some mechanism, then it will arise in a truthtelling equilibrium of the direct mechanism (where each agent truthfully reports his type and the designer chooses an outcome assuming that they are telling the truth). Thus, the Bayesian incentive compatibility constraints characterize whether a social choice function is implementable in this sense.

But, even if a truthtelling equilibrium of the direct mechanism exists, there is no guarantee that there do not exist non-truthtelling equilibria that deliver unacceptable outcomes. For this reason, the literature on *full* implementation has sought to show the existence of a mechanism all of whose equilibria deliver the social choice function. A classic literature on Bayesian implementation–Postlewaite and Schmeidler (1986), Palfrey and Srivastava (1989*b*) and Jackson (1991)–characterized when this is possible: a *Bayesian monotonicity* [1] condition is necessary for full implementation, in addition to the Bayesian incentive compatibility

1. The Bayesian monotonicity condition is an incomplete information analogue of the classic "Maskin monotonicity" condition shown to be necessary and almost sufficient for complete information implementation by Maskin (1999).

conditions. Bayesian monotonicity and Bayesian incentive compatibility are also "almost" sufficient for full implementation.

This important literature has had a limited impact on the more applied mechanism design literature, despite the fact that the problem of multiple equilibria is real. An important difficulty is that, in general, positive results rely on complicated indirect, or "augmented", mechanisms in which agents report more than their types. Such mechanisms appear impractical to many researchers. We believe that the difficulty arises because the standard formulation of the Bayesian implementation problem–assuming common knowledge of a common prior on agents' types and using equilibrium as solution concept–endows the planner with more information than would be available in practice. The implementing mechanism and equilibrium then rely on that information in an implausible way.

In this paper, we characterize when a social choice function can be *robustly* implemented. We fix a social choice environment including a description of the set of possible payoff types for each agent. We ask when does there exist a mechanism with the property that every outcome consistent with common knowledge of rationality agrees with the social choice function, making no assumptions about agents' beliefs and higher-order beliefs about other agents' payoff types. This requirement gives rise to an iterative deletion procedure: fix a mechanism and iteratively delete messages for each payoff type that are strictly dominated by another message for each payoff type profile and message profile that has survived the procedure. Consequently, our notion of robust implementation requires that truthtelling is the unique rationalizable outcome in the incomplete information game defined using the mechanism. This notion of robust implementation is equivalent to requiring that every equilibrium on every type space corresponding to the social choice environment delivers the right outcome. An operational advantage of the iterative definition is that it is defined relative to the payoff type space rather than the much larger universal type space or union of all possible type spaces.

This paper identifies a class of environments where there are tight and easily understood characterizations of when robust implementation is possible. As always, there will be an incentive compatibility condition that is necessary: strict *ex post* incentive compatibility is necessary for robust implementation. We show that if, in addition, a *contraction* property–which we explain shortly–is satisfied, robust implementation is possible in the *direct mechanism*, where each agent reports only his payoff type. If strict *ex post* incentive compatibility or the contraction property fails, then robust implementation is not possible in *any* mechanism. Thus the *augmented mechanisms* used in the earlier complete information and Bayesian implementation literatures do not perform better than the simpler direct mechanisms. An intuition for this result is that the strong common knowledge assumptions used in the complete information and the classic Bayesian implementation literatures can be exploited via complex augmented mechanisms. Thus an attractive feature of our approach is that the robustness requirement reduces the usefulness of complexity in mechanism design (without any *ad hoc* restrictions on complexity).

In the case of private values, strict *ex post* incentive compatibility is equivalent to strict dominant-strategy incentive compatibility. Thus full implementation is obtained for free. It follows that the contraction property must have bite only if there are interdependent values. In fact, the contraction property requires exactly that there is *not too much* interdependence in agents' types. The contraction property can be nicely illustrated in a class of interdependent preferences in which the private types of the agents can be linearly aggregated. If θ_j is the type of agent j, then agent i's utility depends on $\theta_i + \gamma \sum_{j \neq i} \theta_j$. Thus if $\gamma \neq 0$, there are interdependent values–agent j's type will enter agent i's utility assessment–but each agent i cares differently about his own type than about other agents' types. In this example, the contraction property reduces to the requirement that $|\gamma| < 1/(I-1)$, where I is the number of agents. We provide characterizations of the contraction property–equivalent to the intuition that there is no too much interdependence–in more general environments.

An important paper of Chung and Ely (2001) analysed auctions with interdependent valuations under iterated elimination of weakly dominated strategies. In a linear and symmetric setting, they reported sufficient conditions for direct implementation that coincide with the ones derived here. We show that in the environment with linear aggregation, under strict incentive compatibility, the basic insight extends from the single-unit auction model to general allocations models, with elimination of strictly dominated actions only (thus Chung and Ely (2001) require deletion of weakly dominated strategies only because incentive constraints are weak). We also prove a converse result: if there is too much interdependence, then neither the direct nor any augmented mechanism can robustly implement the social choice function.[2]

The main results of this paper apply to environments where each agent's type profile can be aggregated into a one-dimensional sufficient statistic for each player, where preferences are single crossing with respect to that statistic. These restrictions incorporate many economic models with interdependence in the literature. In particular, these restrictions immediately hold in some well-known settings. They automatically hold in single- or multi-unit auctions when each bidder demands at most one unit of the good. In this case, the aggregation function is the utility function itself. The restrictions also encompass a widely-used statistical model with interdependent values. Since the seminal contributions of Wilson (1977) and Milgrom (1981), the canonical model of common values is one in which each agent receives a conditionally independent and identical signal about a one-dimensional common value. In this case, the aggregator function is naturally given directly using Bayes' rule. Subsequently, many allocation models with interdependent–but not necessarily common–values use the same conditionally independent and identically distributed information structure. Reny and Perry (2006) develop strategic foundations for the rational expectations equilibrium within a large double auction. Here the value of the object for each agent is determined by a (perhaps non-linear) function of a private and common value. A private signal jointly informs the agent about the private value and the common value. In consequence, the value of the object is again given by a Bayesian estimate as the natural aggregator function. We shall illustrate informational aggregation via Bayes' law and the relationship with the contraction property in Section 4.

We focus in this paper on economically important environments and well-behaved mechanisms where we get clean and tight characterizations of the robust implementation problem with direct or augmented mechanisms. The *ex post* incentive constraints necessary for robust implementation are already strong (even without the contraction property): Jehiel, Moldovanu, Meyer-Ter-Vehn, and Zame (2006) have shown that in an environment with multi-dimensional signals, the *ex post* incentive constraints are "generically" impossible to satisfy. If *ex post* incentive compatibility fails, our positive results are moot. While this provides a natural limit for our analysis, there are many interesting applications for which *ex post* incentive compatibility holds and the contraction property binds.

First, there is the large and important literature on one-dimensional interdependent type models, including papers on auction environments (Dasgupta and Maskin, 2000; Perry and Reny, 2002; Bergemann and Välimäki, 2002), the bilateral trading model (Gresik, 1991; Fieseler, Kittsteiner, and Moldovanu, 2003), and public and team decision problems without transferable utilities (Gruener and Kiel, 2004). We illustrate our results with a public good example with transfers, with a linear aggregator as described above; we also apply our results to

2. Bergemann and Morris (2009) described how to derive a strong converse to the original Chung and Ely (2001) result for iterated deletion of weakly dominated strategies.

the classic problem of allocating a single private good with quasilinear utility (*i.e.* a single-unit auction with interdependent utility).

Second, even with multi-dimensional signals, there are many environments where economically natural "special" assumptions lead to a failure of the generic conditions as described by Jehiel, Moldovanu, Meyer-Ter-Vehn, and Zame (2006). For example, in any environment with common interests and private information (e.g. Piketty (1999)) there will be *ex post* incentive compatibility (whatever the dimension of private information), yet our contraction property will imply the impossibility of robust implementation, as we discuss in section 6 for the one-dimensional linear aggregator case. In section 8.1, we use a multi-dimensional version of our public good example to illustrate an extension of our results to multi-dimensional signals without the aggregation property. In this case, symmetry allows the existence of *ex post* incentive compatible transfers. Further examples in the literature show that *ex post* incentive compatibility (EPIC) is satisfied in multi-dimensional signal models without allocative externalities (Bikhchandani (2006)) or with a separable structure (Eso and Maskin (2002)).

While we prove the necessity of the contraction property within single-crossing aggregator environments, the necessity argument extends to general environments. In Bergemann and Morris (2005*a*), we show that a robust monotonicity condition is necessary and almost sufficient for robust implementation in general environments with general mechanisms. The robust monotonicity condition is equivalent to assuming the Bayesian monotonicity necessary condition (from Postlewaite and Schmeidler (1986), Palfrey and Srivastava (1989*b*) and Jackson (1991)) on all possible type spaces. The contraction property is the simpler expression of this robust monotonicity condition in the single-crossing aggregator environments of this paper. The definition of the contraction property depends on the aggregation of the preferences. But in section 8.1, we discuss a generalized contraction property–again capturing the idea of moderate interdependence in preferences–that does not depend on an aggregation property or one-dimensional signals.

The remainder of the paper is organized as follows. Section 2 describes the formal environment and solution concepts. Section 3 considers a public good example that illustrates the main ideas and results of the paper. Section 4 establishes necessary conditions for robust implementation in the direct mechanism. Section 5 reports sufficient conditions for robust implementation. Section 6 considers the preference environment with a linear aggregation of types and obtains sharp implementation results. Section 7 considers a single-unit auction with interdependent values as a second example of robust implementation. Section 8 concludes.

2. SETUP

Payoff environment. We consider a finite set of agents, $1, 2, \ldots, I$. Agent i's *payoff type* is $\theta_i \in \Theta_i$, where Θ_i is a compact subset of the real line. We write $\theta = (\theta_1, \ldots, \theta_1) \in \Theta_1 \times \cdots \times \Theta_I = \Theta$. Let X be a compact set of deterministic outcomes and let $Y = \Delta(X)$ be the lottery space generated by the deterministic outcome space X. Each agent has a von Neumann–Morgenstern expected utility function $u_i : Y \times \Theta \to \mathbb{R}$. Let agent i's utility if outcome y is chosen and agents' type profile is θ be $u_i(y, \theta)$. We emphasize that the utility function of agent i is allowed to depend on the type profile θ_{-i} of the other agents. A social choice function is a mapping $f : \Theta \to Y$.

Mechanisms. A planner must choose a *game form* or *mechanism* for the agents to play in order to determine the social outcome. Let M_i be a compact set of messages available to agent i. Let $g(m)$ be the outcome chosen if action profile m is chosen. Thus a mechanism is a collection

$$\mathcal{M} = (M_1, ..., M_I, g(\cdot)),$$

where $g : M \to Y$. The *direct mechanism* has the property that $M_i = \Theta_i$ for all i and $g(\theta) = f(\theta)$.

Robust implementation. In a fixed mechanism $\mathcal{M}$, we call a correspondence $S = (S_1, ..., S_I)$, with each $S_i : \Theta_i \to 2^{M_i} / \varnothing$, a *message profile* of the agents. We will refer to a message profile in the direct mechanism where truthtelling is always possible as a *report profile*. Thus a report profile $\beta = (\beta_1, ..., \beta_I)$ is described by

$$\beta_i : \Theta_i \to 2^{\Theta_i} / \varnothing, \text{ for all } i,$$

and $\theta_i \in \beta_i(\theta_i)$ for all i and θ_i. Let β^* be the truthful report, with $\beta_i^*(\theta_i) = \{\theta_i\}$ for all i and θ_i.

Next we define the process of iterative elimination of never-best responses. We denote the belief of agent i over message and payoff type profiles of the remaining agents by a Borel measure λ_i:

$$\lambda_i \in \Delta(M_{-i} \times \Theta_{-i}).$$

Let $S_i^0(\theta_i) = M_i$ for all i and θ_i and define inductively:

$$S_i^{k+1}(\theta_i) \triangleq \left\{ m_i \in M_i \left| \exists \lambda_i \text{ s.th.:} \begin{array}{cc} (1) & \lambda_i \left[\left\{ (m_{x-i}, \theta_{-i}) \middle| m_j \in S_j^k(\theta_j), \ \forall j \neq i \right\} \right] = 1 \\ (2) & \begin{array}{c} \int u_i(g(m_i, m_{-i}), (\theta_i, \theta_{-i}))\, d\lambda_i \ \geq \\ \int u_i\left(g\left(m_i', m_{-i}\right), (\theta_i, \theta_{-i})\right) d\lambda_i, \ \forall m_i' \in M_i. \end{array} \end{array} \right. \right\}.$$

We observe that $S_i^k(\theta_i)$ is non-increasing in k in the set-inclusion order for each θ_i. We denote the limit set $S_i^{\mathcal{M}}(\theta_i)$ by:

$$S_i^{\mathcal{M}}(\theta_i) \triangleq \bigcap_{k \geq 0} S_i^k(\theta_i), \text{ for all } \theta_i \in \Theta_i \text{ and all } i.$$

We refer to the messages $m_i \in S_i^{\mathcal{M}}(\theta_i)$ as the *rationalizable messages* of type θ_i of agent i in mechanism $\mathcal{M}$. We call a social choice function f *robustly implementable* if there exists a mechanism $\mathcal{M}$ under which the social choice can be recovered through a process of iterative elimination of never-best responses.

Definition 1. **(Robust implementation)** Social choice function f is robustly implemented using mechanism $\mathcal{M}$ if $m \in S^{\mathcal{M}}(\theta) \Rightarrow g(m) = f(\theta)$.

As we mentioned in the introduction, the above notion of robust implementation is equivalent to requiring that every equilibrium on every type space corresponding to the social choice environment delivers the right outcome. In other words, the set of rationalizable messages for mechanism $\mathcal{M}$ is equal to the set of messages that could be played in a

Bayesian equilibrium of the game generated using the mechanism $\mathcal{M}$ and some type space. The basic logic of this equivalence result follows the well-known argument of Brandenburger and Dekel (1987) who established, in complete information games, the equivalence of correlated rationalizable actions and the set of actions that could be played in a subjective correlated equilibrium. Battigalli and Siniscalchi (2003) described a general incomplete information extension of this observation. We report a formal version of the equivalence result for our environment in Proposition 1 in Bergemann and Morris (2005*a*). As all subsequent results work directly with the above iterative notion, we refer for the formal statements about the equivalence to Bergemann and Morris (2005*a*).

Monotone aggregator. We now describe the structural assumptions that will be maintained throughout the rest of the paper. We assume the existence of a monotonic aggregator $h_i(\theta)$ for every i that allows us to rewrite the utility function of every agent i as:

$$u_i(y,\theta) \triangleq v_i(y, h_i(\theta)),$$

where $h_i : \Theta \to \mathbb{R}$ is assumed to be continuous, strictly increasing in θ_i and $v_i : Y \times \mathbb{R} \to \mathbb{R}$ is continuous in the aggregator (continuity with respect to lotteries follows from the vNM assumption). The content of the aggregation assumption comes from the continuity requirement and the following single-crossing condition.

Definition 2. **(Strict single crossing)** The utility function $v_i(y,\phi)$ satisfies strict single crossing (SSC) if for all $\phi < \phi' < \phi''$:

$$v_i(y,\phi) > v_i\left(y',\phi\right) \text{ and } v_i\left(y,\phi'\right) = v_i\left(y',\phi'\right) \quad \Rightarrow \quad v_i\left(y,\phi''\right) < v_i\left(y',\phi''\right).$$

The single-crossing property is defined relative to the aggregation $\phi = h_i(\theta)$ of all agents' types. The combination of a monotonic aggregator representation of preferences and the SSC condition will drive our results.

3. A PUBLIC GOOD EXAMPLE

We precede the formal results with an example illustrating the main insights of the paper and reviewing some key ideas from the implementation literature. The example involves the provision of a public good with quasilinear utility. The utility of each agent is given by:

$$u_i(x,\theta) = \left(\theta_i + \gamma \sum_{j \neq i} \theta_j\right) x_0 + x_i,$$

where x_0 is the level of public good provided and x_i is the monetary transfer to agent i. The utility of agent i depends on his own type $\theta_i \in [0,1]$ and the type profile of other agents, $\theta_{-i} \in [0,1]^{I-1}$. The weight $\gamma \geq 0$ represents the strength of the interdependence in the preferences of agent i. The utility function of agent i has the aggregation property with

$$h_i(\theta) = \theta_i + \gamma \sum_{j \neq i} \theta_j.$$

We notice that the aggregator function $h_i(\theta)$ depends on the identity of agent i. In particular, a given type profile θ leads to a different aggregation result for i and j, provided that $\theta_i \neq \theta_j$.

The cost of establishing the public good is given by $c(x_0) = \frac{1}{2}x_0^2$. The planner must choose $(x_0, x_1, ..., x_I) \in \mathbb{R}_+ \times \mathbb{R}^I$ to maximize social welfare, i.e. the sum of gross utilities minus the cost of the public good:

$$\left((1+\gamma(I-1))\sum_{i=1}^{I}\theta_i\right)x_0 - \frac{1}{2}x_0^2.$$

The socially optimal level of the public good is therefore equal to

$$f_0(\theta) = (1+\gamma(I-1))\sum_{i=1}^{I}\theta_i.$$

The generalized Vickrey–Clarke–Groves (VCG) transfers, unique up to a constant, that give rise to *ex post* incentive compatibility are:

$$f_i(\theta) = -(1+\gamma(I-1))\left(\gamma\theta_i\sum_{j\neq i}\theta_j + \frac{1}{2}\theta_i^2\right). \quad (1)$$

It is useful to observe that the generalized VCG transfers given by equation (1) guarantee *ex post* incentive compatibility for any $\gamma \in \mathbb{R}_+$. Hence, *ex post* incentive compatibility does not impose *any* constraint on the interdependence parameter γ. In contrast, the dominant-strategy property of the VCG mechanism only holds with private values, or $\gamma = 0$, and fails for all $\gamma > 0$.

Now we shall argue that if $\gamma < 1/(I-1)$, then the social choice function f is robustly implementable in the *direct mechanism* where each agent reports his payoff type θ_i and the planner chooses outcomes according to f on the assumption that agents are telling the truth. Consider an iterative deletion procedure. Let $\beta^0(\theta_i) = [0, 1]$ and, for each $k = 1, 2, ...$, let $\beta^k(\theta_i)$ be the set of reports that agent i might send, for some conjecture over his opponents' types and reports, with the only restriction on his conjecture being that each type θ_j of agent j sends a message in $\beta^{k-1}(\theta_j)$.

Suppose that agent i has payoff type θ_i, but reports himself to be type θ_i' and has a point conjecture that other agents have type profile θ_{-i} and report their types to be θ_{-i}'. Then his expected payoff is a constant $(1+\gamma(I-1))$ times:

$$\left(\theta_i + \gamma\sum_{j\neq i}\theta_j\right)\left(\theta_i' + \sum_{j\neq i}\theta_j'\right) - \left(\gamma\theta_i'\sum_{j\neq i}\theta_j' + \frac{1}{2}\left(\theta_i'\right)^2\right).$$

The first order condition with respect to the report θ_i' is then:

$$\theta_i + \gamma\left(\sum_{j\neq i}\theta_j\right) - \gamma\left(\sum_{j\neq i}\theta_j'\right) - \theta_i' = 0,$$

so he would wish to set

$$\theta_i' = \theta_i + \gamma\sum_{j\neq i}\left(\theta_j - \theta_j'\right).$$

In other words, his best response to a misreport θ_{-i}' by the other agents is to report a type so that the aggregate type from his point of view is exactly identical to the aggregate type generated by the true type profile θ. Note that the above calculation also verifies the strict *ex post* incentive compatibility of f, as setting $\theta'_i = \theta_i$ is the unique best response if $\theta'_j = \theta_j$ for all $j \neq i$. The quadratic payoff/linear best response nature of this problem means that we can characterize $\beta^k(\theta_i)$ restricting attention to such point conjectures. In particular, we have

$$\beta^k(\theta_i) = \left[\underline{\beta}^k(\theta_i), \overline{\beta}^k(\theta_i)\right],$$

where

$$\begin{aligned}\overline{\beta}^k(\theta_i) &= \min\left\{1, \theta_i + \gamma \max_{\left\{\left(\theta'_{-i}, \theta_{-i}\right) \middle| \theta'_j \in \beta^{k-1}(\theta_j) \text{ for all } j \neq i\right\}} \sum_{j \neq i} \left(\theta_j - \theta'_j\right)\right\} \\ &= \min\left\{1, \theta_i + \ \gamma \max_{\theta_{-i}} \sum_{j \neq i}\left(\theta_j - \underline{\beta}^{k-1}(\theta_j)\right)\right\}.\end{aligned}$$

Analogously,

$$\underline{\beta}^k(\theta_i) = \max\left\{0, \theta_i - \ \gamma \max_{\theta_{-i}} \sum_{j \neq i}\left(\overline{\beta}^{k-1}(\theta_j) - \theta_j\right)\right\}.$$

Thus

$$\overline{\beta}^k(\theta_i) = \min\left\{1, \theta_i + (\gamma(I-1))^k\right\},$$

and

$$\underline{\beta}^k(\theta_i) = \max\left\{0, \theta_i - (\gamma(I-1))^k\right\}.$$

Thus $\theta'_i \neq \theta_i \Rightarrow \theta'_i \notin \beta^k(\theta_i)$ for sufficiently large k, provided that $\gamma < \frac{1}{I-1}$.

Now consider what happens when this condition fails, *i.e.* $\gamma \geq \frac{1}{I-1}$. In this case, it is possible to exploit the large amount of interdependence to construct beliefs over the opponents' types such that all types are indistinguishable. Suppose that every type $\theta_i \in [0, 1]$ has a degenerate belief over the types of his opponents. In particular, type θ_i is convinced that each of his opponents is of type $\theta_j[\theta_i]$ given by:

$$\theta_j[\theta_i] \triangleq \frac{1}{2} + \frac{1}{\gamma(I-1)}\left(\frac{1}{2} - \theta_i\right),$$

where the belief of i about j evidently depends on his type θ_i. In this case the aggregation of the types leads to:

$$\theta_i + \gamma \sum_{j \neq i} \theta_j[\theta_i] = \frac{1}{2}(1 + \gamma(I-1)),$$

independent of θ_i. Thus in *any* mechanism, for each type, we can construct beliefs so that there are no differences across types of agent i in terms of the actions which get deleted in the iterative process.

4. ROBUST IMPLEMENTATION

In our earlier work on robust mechanism design (Bergemann and Morris, 2005*b*), we showed that *ex post* incentive compatibility is necessary and sufficient for partial robust implementation (*i.e.* ensuring that there exists *an* equilibrium consistent with the social choice function).

Definition 3. (***ex post*** **incentive compatibility**) Social choice function f satisfies EPIC if for all i, θ and θ_i':

$$u_i\left(f\left(\theta_i,\theta_{-i}\right),\left(\theta_i,\theta_{-i}\right)\right)\geq u_i\left(f\left(\theta_i',\theta_{-i}\right),\left(\theta_i,\theta_{-i}\right)\right).$$

In the subsequent analysis we use the strict version of the incentive constraints as we require full implementation.

Definition 4. (**Strict** ***ex post*** **incentive compatibility**) Social choice function f satisfies strict *ex post* incentive compatibility (strict EPIC) if for all i, $\theta_i'\neq\theta_i$ and θ_{-i}:

$$u_i\left(f\left(\theta_i,\theta_{-i}\right),\left(\theta_i,\theta_{-i}\right)\right)>u_i\left(f\left(\theta_i',\theta_{-i}\right),\left(\theta_i,\theta_{-i}\right)\right).$$

The key property for our analysis is the following contraction property.

Definition 5. (**Contraction property**) The aggregator functions $h=(h_i)_{i=1}^I$ satisfy the contraction property if, for all $\beta\neq\beta^*$, there exists i, θ_i and $\theta_i'\in\beta_i\left(\theta_i\right)$ with $\theta_i'\neq\theta_i$, such that

$$\operatorname{sign}\left(\theta_i-\theta_i'\right)=\operatorname{sign}\left(h_i\left(\theta_i,\theta_{-i}\right)-h_i\left(\theta_i',\theta_{-i}'\right)\right),\tag{2}$$

for all θ_{-i} and $\theta_{-i}'\in\beta_{-i}\left(\theta_{-i}\right)$.

The contraction property essentially says that for some agent i the direct impact of his private signal θ_i on the aggregator $h_i\left(\theta\right)$ is always sufficiently strong such that the difference in the aggregated value between the true type profile and the reported type profile always has the same sign as the difference between the true and reported type of agent i by itself.

How strong is the aggregator restriction on the environment? It requires that the payoff types of the players can be aggregated into a variable that changes preferences in a monotonic way. To get some sense of the strength of this restriction, we next consider two examples. The first example involves a binary outcome space which automatically guarantees the aggregation property; the second example uses an informational foundation by means of Bayes' law to obtain the aggregation property.

Binary allocation model. Let each agent's utility from outcomes depend on the payoff type profile only via a binary partition of the deterministic outcome space X. Thus, for each i, there exists $f_i:X\rightarrow\{0,1\}$, $w_i^1:\{0,1\}\times\Theta\rightarrow\mathbb{R}$ and $w_i^2:X\rightarrow\mathbb{R}$ such that

$$u_i\left(x,\theta\right)=w_i^1\left(f_i\left(x\right),\theta\right)+w_i^2\left(x\right).$$

In this case, we can write the agent's utility over lotteries $y\in Y=\Delta\left(X\right)$ as

$$u_i\left(y,\theta\right)=\int_{\{x|f_i(x)=1\}}w_i^1\left(1,\theta\right)dy+\int_{\{x|f_i(x)=0\}}w_i^1\left(0,\theta\right)dy+\int_{x\in X}w_i^2\left(x\right)dy.$$

An equivalent representation of this agent's preferences is

$$\int_{\{x|f_i(x)=1\}} \left[w_i^1(1,\theta) - w_i^1(0,\theta)\right]dy + \int_{x\in X} w_i^2(x)dy \ = v_i(y, h_i(\theta)),$$

with

$$h_i(\theta) \triangleq w_i^1(1,\theta) - w_i^1(0,\theta),$$

and

$$v_i(y, h_i(\theta)) \triangleq \int_{\{x:f_i(x)=1\}} dy \cdot h_i(\theta) + \int_{x\in X} w_i(x)dy.$$

Thus in such a binary allocation model, the aggregation property is satisfied automatically.

A natural example of a binary allocation model is an auction of many identical units of a good to agents with unit demand for the good and quasilinear preferences. In this case, an allocation is a pair $(Z, z) \in X = 2^{\{1,\ldots,I\}} \times \mathbb{R}^I$ where Z is the set of agents who are allocated a unit of the good and z_i is the payment of agent i. Now if agent i's utility from being allocated the good if the payoff type profile is θ is given by $v_i(\theta)$, then this fits the above framework as

$$u_i((Z,z),\theta) = \begin{cases} v_i(\theta) - z_i, & \text{if } i \in Z, \\ -z_i, & \text{otherwise.} \end{cases}$$

Information aggregation. A natural source of interdependence in preferences is informational, when an agent's payoff type corresponds to a signal which ends up being correlated with all agents' expected values of a state. In particular, suppose that each agent i's utility depends on the expected value of an additive random variable $\omega_0 + \omega_i$, where ω_0 is a common value component and ω_i is the private value component. We describe the additive model with two agents i and j (the generalization to many agents is immediate). The random variables $\omega_0, \omega_1, \omega_2$ are assumed to be independently and normally distributed with zero mean and variance σ_0^2, σ_1^2 and σ_2^2 respectively. Let each agent i observe one signal $\theta_i = \omega_0 + \omega_i + \varepsilon_i$, where each ε_i is independently normally distributed with mean 0 and variance τ_i^2. We are thus assuming that each agent observes only a one-dimensional signal, θ_i, of both the common and the idiosyncratic component. Thus agent i, with his noisy signal θ_i, is unable to distinguish between the common and the private value component. But naturally his own signal is more informative about his valuation than the others' signals because it contains his own idiosyncratic shock.

Now standard properties of the normal distribution (see DeGroot, 1970) imply that agent i's expected value of $\omega_0 + \omega_i$, given the vector of signals (θ_i, θ_j) is a constant

$$\frac{\sigma_0^2\tau_i^2 + \sigma_0^2\tau_j^2 + \sigma_0^2\sigma_i^2 + \sigma_0^2\sigma_j^2 + \tau_i^2\tau_j^2 + \tau_i^2\sigma_j^2 + \tau_j^2\sigma_i^2 + \sigma_i^2\sigma_j^2}{\sigma_0^2\tau_i^2 + \sigma_0^2\sigma_i^2 + \sigma_0^2\sigma_j^2 + \tau_j^2\sigma_i^2 + \sigma_i^2\sigma_j^2}$$

times

$$h_i(\theta) = \theta_i + \frac{\sigma_0^2\tau_i^2}{\sigma_0^2\tau_j^2 + \sigma_0^2\sigma_i^2 + \sigma_0^2\sigma_j^2 + \tau_j^2\sigma_i^2 + \sigma_i^2\sigma_j^2}\theta_j, \tag{3}$$

where we write j for the other agent $3 - i$. The calculations are reported in the Appendix. Now if we assume that each agent i's preferences conditional on $h_i(\theta)$ satisfy SSC with respect to

$h_i(\theta)$, then we have an informational microfoundation for the SSC environment of the paper. Moreover, in this example the aggregator takes the linear form $h_i(\theta) = \theta_i + \gamma_{ij}\theta_j$, with

$$\gamma_{ij} = \frac{\sigma_0^2 \tau_i^2}{\sigma_0^2 \tau_j^2 + \sigma_0^2 \sigma_i^2 + \sigma_0^2 \sigma_j^2 + \tau_j^2 \sigma_i^2 + \sigma_i^2 \sigma_j^2}.$$

This conclusion is quite intuitive. If the variance of the common component (σ_0^2) is small or if the noise in one's own signal (τ_i^2) is small, then the interdependence goes away. But a reduction in variance of one's own idiosyncratic component (σ_i^2), in one's opponent's idiosyncratic component (σ_j^2) or in one's opponent's noise (τ_j^2) all tend to increase the interdependence.[3] With this interpretation the single-crossing property with respect to the aggregator reduces to assuming that there is a one-dimensional parameter whose expected value affects the preferences and that there is a sufficient statistic for the vector of signals that agents observe.

We now state our first positive result.

Theorem 1. **(Robust implementation)** *If a social choice function f satisfies strict EPIC and the aggregator functions satisfy the contraction property, then f can be robustly implemented in the direct mechanism.*

Proof. We argue by contradiction. Let $\beta = S^{\mathcal{M}}$ and suppose that $\beta \neq \beta^*$. Continuity of each u_i with respect to θ implies that each $\beta_i(\theta_i)$ will be a compact set. By the contraction property, there exists i and $\theta_i' \in \beta_i(\theta_i)$ with $\theta_i' \neq \theta_i$ such that

$$\text{sign}\left(\theta_i - \theta_i'\right) = \text{sign}\left(h_i(\theta_i, \theta_{-i}) - h_i\left(\theta_i', \theta_{-i}'\right)\right),$$

for all θ_{-i} and $\theta_{-i}' \in \beta_{-i}(\theta_{-i})$. Thus by compactness

$$\delta \triangleq \min_{\theta_{-i} \in \Theta_{-i} \text{ and } \theta_{-i}' \in \beta_{-i}(\theta_{-i})} \left| h_i(\theta_i, \theta_{-i}) - h_i\left(\theta_i', \theta_{-i}'\right)\right|,$$

is well defined and strictly positive. Suppose (without loss of generality) that $\theta_i > \theta_i'$. Let

$$\xi(\varepsilon) \triangleq \max_{\theta_{-i}'} \left\{h_i\left(\theta_i' + \varepsilon, \theta_{-i}'\right) - h_i\left(\theta_i', \theta_{-i}'\right)\right\}.$$

As $h_i(\cdot)$ is strictly increasing in θ_i, we know that $\xi(\varepsilon)$ is increasing in ε and by continuity of h_i in θ_i, $\xi(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Thus we have

$$h_i(\theta_i, \theta_{-i}) - h_i\left(\theta_i', \theta_{-i}'\right) \geq \delta, \tag{4}$$

for all θ_{-i} and $\theta_{-i}' \in \beta_i(\theta_{-i})$; and

$$h_i\left(\theta_i', \theta_{-i}'\right) \geq h_i\left(\theta_i' + \varepsilon, \theta_{-i}'\right) - \xi(\varepsilon), \tag{5}$$

3. The additive model with a private and a common component appears as described by Hong and Shum (2003). Interestingly, they prove the existence and uniqueness of an increasing bidding strategy by appealing to a dominant diagonal condition, which is implied by the contraction property. The example of a normal distribution fails the compact type space assumption of our model, but we use the normal distribution here merely for its transparent updating properties.

for all θ'_{-i}. By strict EPIC,

$$v_i\left(f\left(\theta'_i,\theta'_{-i}\right),h_i\left(\theta'_i,\theta'_{-i}\right)\right)>v_i\left(f\left(\theta'_i+\varepsilon,\theta'_{-i}\right),h_i\left(\theta'_i,\theta'_{-i}\right)\right),$$

for all $\varepsilon>0$ and

$$v_i\left(f\left(\theta'_i+\varepsilon,\theta'_{-i}\right),h_i\left(\theta'_i+\varepsilon,\theta'_{-i}\right)\right)>v_i\left(f\left(\theta'_i,\theta'_{-i}\right),h_i\left(\theta'_i+\varepsilon,\theta'_{-i}\right)\right),$$

for all $\varepsilon>0$. Now continuity of u_i with respect to θ implies that for each $\varepsilon>0$ and θ'_{-i}, there exists $\phi^*\left(\varepsilon,\theta'_{-i}\right)\in\mathbb{R}$ such that:

$$h_i\left(\theta'_i,\theta'_{-i}\right)<\phi^*\left(\varepsilon,\theta'_{-i}\right)<h_i\left(\theta'_i+\varepsilon,\theta'_{-i}\right), \tag{6}$$

and

$$v_i\left(f\left(\theta'_i,\theta'_{-i}\right),\phi^*\left(\varepsilon,\theta'_{-i}\right)\right)=v_i\left(f\left(\theta'_i+\varepsilon,\theta'_{-i}\right),\phi^*\left(\varepsilon,\theta'_{-i}\right)\right);$$

and SSC implies that:

$$v_i\left(f\left(\theta'_i,\theta'_{-i}\right),\phi\right)<v_i\left(f\left(\theta'_i+\varepsilon,\theta'_{-i}\right),\phi\right), \tag{7}$$

for all $\phi>\phi^*\left(\varepsilon,\theta'_{-i}\right)$. Now fix any ε with

$$\xi(\varepsilon)<\delta. \tag{8}$$

Now for all $\theta'_{-i}\in\beta_{-i}(\theta_{-i})$,

$$\begin{aligned}h_i(\theta_i,\theta_{-i}) &\geq h_i\left(\theta'_i,\theta'_{-i}\right)+\delta, \text{ using equation (4)}\\ &\geq h_i\left(\theta'_i+\varepsilon,\theta'_{-i}\right)-\xi(\varepsilon)+\delta, \text{ using equation (5)}\\ &> h_i\left(\theta'_i+\varepsilon,\theta'_{-i}\right), \text{ using equation (8)}\\ &\geq \phi^*\left(\varepsilon,\theta'_{-i}\right), \text{ using equation (6).}\end{aligned}$$

So

$$v_i\left(f\left(\theta'_i+\varepsilon,\theta'_{-i}\right),h_i(\theta_i,\theta_{-i})\right)>v_i\left(f\left(\theta'_i,\theta'_{-i}\right),h_i(\theta_i,\theta_{-i})\right),$$

for every θ_{-i} and $\theta'_{-i}\in\beta_{-i}(\theta_{-i})$ using equation (7). This contradicts our assumption that $\beta=S^{\mathcal{M}}$. ‖

A surprising element in this result is that we do not need to impose any conditions on how the social choice function varies with the type profile. It does not have to respond to the reported profile θ in a manner similar to the response of any of the aggregators h_i. The SSC condition is sufficient to make full use of the contraction property.

The argument is based on a true type profile $\theta=(\theta_i,\theta_{-i})$ and a reported profile $\theta'=\left(\theta'_i,\theta'_{-i}\right)$, and without loss of generality $\theta_i>\theta'_i$. We use the contraction property to establish a positive lower bound on the difference $h_i(\theta_i,\theta_{-i})-h_i\left(\theta'_i,\theta'_{-i}\right)$ for all θ_{-i} and $\theta'_{-i}\in\beta_{-i}(\theta_{-i})$. With this positive lower bound, we then show that agent i is made strictly better off by moving his misreport θ'_i marginally upwards in the direction of θ_i, in other words to report $\theta'_i+\varepsilon$. This is achieved by showing that there is an intermediate value ϕ^* for the aggregator, with $h_i\left(\theta'_i,\theta'_{-i}\right)<\phi^*<h_i\left(\theta'_i+\varepsilon,\theta'_{-i}\right)$, such that agent i with the utility profile corresponding to the aggregator value ϕ^* would be indifferent between the social allocations

$f\left(\theta_i', \theta_{-i}'\right)$ and $f\left(\theta_i' + \varepsilon, \theta_{-i}'\right)$. By choosing ε sufficiently small, we know that $h\left(\theta_i, \theta_{-i}\right) > \phi^*$ and SSC then allows us to assert that an agent with a true preference profile $\theta = \left(\theta_i, \theta_{-i}\right)$ would also prefer to obtain $f\left(\theta_i' + \varepsilon, \theta_{-i}'\right)$ rather than $f\left(\theta_i', \theta_{-i}'\right)$. But this yields the contradiction to $\theta_i' \in \beta_i\left(\theta_i\right)$ being part of the fixed point of the iterative elimination. Consequently we show that the misreport θ_i', which established the same sign on the difference between private type profiles and aggregated public profiles can be eliminated as a best response to the set of misreports of the remaining agents.

In the present environment with single crossing and aggregation, the contraction property is equivalent to a notion of "robust monotonicity" by Bergemann and Morris (2005*a*). Social choice function f satisfies *robust monotonicity* if for report profile $\beta \neq \beta^*$, there exist i, θ_i, $\theta_i' \in \beta_i\left(\theta_i\right)$ such that, for all $\theta_{-i}' \in \Theta_{-i}$, there exists y such that

$$u_i\left(y, \left(\theta_i, \theta_{-i}\right)\right) > u_i\left(f\left(\theta_i', \theta_{-i}'\right), \left(\theta_i, \theta_{-i}\right)\right) \tag{9}$$

for all θ_{-i} such that $\theta_{-i}' \in \beta_{-i}\left(\theta_{-i}\right)$; and

$$u_i\left(f\left(\theta_i'', \theta_{-i}'\right), \left(\theta_i'', \theta_{-i}'\right)\right) \geq u_i\left(y, \left(\theta_i'', \theta_{-i}'\right)\right) \tag{10}$$

for all $\theta_i'' \in \Theta_i$.

It is now easy to see that the contraction property guarantees the validity of (9) and (10). Fix θ_i and θ_i' and without loss of generality assume $\theta_i > \theta_i'$. By the contraction property it follows that for every θ_{-i}', we have $h_i\left(\theta_i, \theta_{-i}\right) > h_i\left(\theta_i', \theta_{-i}'\right)$. Hence we can find an $\varepsilon > 0$ such that

$$h_i\left(\theta_i, \theta_{-i}\right) > h_i\left(\theta_i' + \varepsilon, \theta_{-i}'\right) > h_i\left(\theta_i', \theta_{-i}'\right). \tag{11}$$

But now we can choose the allocation y to be $y = f\left(\theta_i' + \varepsilon, \theta_{-i}'\right)$. Now equation (9) follows from equation (11) and single crossing, and equation (10) follows from strict EPIC.

Bergemann and Morris (2005*a*) show that the above robust monotonicity condition is a necessary and almost sufficient condition for robust implementation, by following the classic implementation literature in allowing the use of complicated–perhaps unbounded–augmented mechanisms. In this paper, we show that the contraction property–equivalent to the robust monotonicity condition–is sufficient for implementation in the *direct* mechanism in single-crossing aggregator environments.

5. NECESSITY OF CONTRACTION PROPERTY

We now show that the contraction property is necessary for robust implementation. We impose the following mild restriction on the social choice function for the necessity argument.

Definition 6. (**Responsive social choice function**) Social choice function f is responsive if for all $\theta_i \neq \theta_i'$, there exists θ_{-i} such that

$$f\left(\theta_i, \theta_{-i}\right) \neq f\left(\theta_i', \theta_{-i}\right).$$

Responsiveness requires that a change in agent i's report changes the social allocation for some report of the other agents. The idea behind the necessity argument is to show that the hypothesis of robust implementation leads inevitably to a conflict with a report profile β which fails to satisfy the contraction property.

Theorem 2. (**Necessity**) *If f is robustly implementable and is responsive, then f satisfies strict EPIC and the aggregator functions satisfy the contraction property.*

Proof. Suppose that f is responsive and robustly implemented by mechanism $\mathcal{M}$. The restriction to compact mechanisms ensures that $S^{\mathcal{M}}$ is non-empty. Let $m_i^*(\theta_i)$ be any element of $S_i^{\mathcal{M}}(\theta_i)$. Because mechanism $\mathcal{M}$ robustly implements f, $g(m^*(\theta)) = f(\theta)$, for all $\theta \in \Theta$.

We first establish strict EPIC. Suppose strict EPIC fails, then there exist i, θ and $\theta_i' \neq \theta_i$ such that

$$u_i\left(f\left(\theta_i', \theta_{-i}\right), \theta\right) \geq u_i\left(f(\theta), \theta\right).$$

Now $m^*(\theta) = \left(m_i^*(\theta_i), m_{-i}^*(\theta_{-i})\right) \in S^{\mathcal{M}}(\theta)$ implies that

$$\begin{aligned} \max_{m_i'} \left\{u_i\left(g\left(m_i', m_{-i}^*(\theta_{-i})\right), (\theta_i, \theta_{-i})\right)\right\} &= u_i\left(g\left(m_i^*(\theta_i), m_{-i}^*(\theta_{-i})\right), (\theta_i, \theta_{-i})\right) \\ &= u_i\left(f(\theta), \theta\right). \end{aligned}$$

But

$$u_i\left(g\left(m_i^*\left(\theta_i'\right), m_{-i}^*(\theta_{-i})\right), (\theta_i, \theta_{-i})\right) = u_i\left(f\left(\theta_i', \theta_{-i}\right), \theta\right) \geq u_i\left(f(\theta), \theta\right),$$

and so

$$m_i^*\left(\theta_i'\right) \in \arg\max_{m_i'} \left\{u_i\left(g\left(m_i', m_{-i}^*(\theta_{-i})\right), (\theta_i, \theta_{-i})\right)\right\}$$

which implies that $m_i^*\left(\theta_i'\right) \in S_i^{\mathcal{M}}(\theta_i)$. This in turn implies that

$$f\left(\theta_i', \theta_{-i}\right) = g\left(m_i^*\left(\theta_i'\right), m_{-i}^*(\theta_{-i})\right) = f(\theta_i, \theta_{-i}), \quad \text{for all } \theta_{-i},$$

contradicting our assumption that f is responsive and robustly implemented by mechanism $\mathcal{M}$.

Now we establish the contraction property. First, suppose that $m_i \in M_i$, $\theta_i' \in \Theta_i$, $\theta_{-i}' \in \Theta_{-i}$, $\widehat{m}_{-i} \in S_{-i}^{\mathcal{M}}\left(\theta_{-i}'\right)$ and

$$u_i\left(g(m_i, \widehat{m}_{-i}), \left(\theta_i', \theta_{-i}'\right)\right) > u_i\left(f\left(\theta_i', \theta_{-i}'\right), \left(\theta_i', \theta_{-i}'\right)\right). \tag{12}$$

Then, we have

$$m_i^*\left(\theta_i'\right) \notin \arg\max_{m_i'} \left\{u_i\left(g\left(m_i', \widehat{m}_{-i}\right), \left(\theta_i, \theta_{-i}'\right)\right)\right\},$$

as

$$u_i\left(g\left(m_i^*\left(\theta_i'\right), \widehat{m}_{-i}\right), \left(\theta_i', \theta_{-i}'\right)\right) = u_i\left(f\left(\theta_i', \theta_{-i}'\right), \left(\theta_i', \theta_{-i}'\right)\right) < u_i\left(g(m_i, \widehat{m}_{-i}), \left(\theta_i', \theta_{-i}'\right)\right).$$

Thus $m_i \in M_i$, $\theta_i' \in \Theta_i$, $\theta_{-i}' \in \Theta_{-i}$ and $\widehat{m}_{-i} \in S_{-i}^{\mathcal{M}}\left(\theta_{-i}'\right)$ imply

$$u_i\left(g(m_i, \widehat{m}_{-i}), \left(\theta_i', \theta_{-i}'\right)\right) \leq u_i\left(f\left(\theta_i', \theta_{-i}'\right), \left(\theta_i', \theta_{-i}'\right)\right). \tag{13}$$

Now consider an arbitrary report profile $\beta \neq \beta^*$. Let $\widehat{k}$ be the largest k such that for every i, θ_i and $\theta_i' \in \beta_i(\theta_i)$:

$$S_i^{\mathcal{M}}\left(\theta_i'\right) \subseteq S_i^k(\theta_i).$$

We know that such a $\widehat{k}$ exists because $S_i^0(\theta_i) = M_i$, and, as $\mathcal{M}$ robustly implements f, responsiveness implies $S_i^{\mathcal{M}}(\theta_i) \cap S_i^{\mathcal{M}}(\theta_i') = \varnothing$.

Now we know that there exists i and $\theta_i' \in \beta_i(\theta_i)$ such that

$$S_i^{\mathcal{M}}(\theta_i') \subsetneq S_i^{\widehat{k}+1}(\theta_i).$$

Thus there exists $\widehat{m}_i \in M_i$ such that $\widehat{m}_i \in S_i^{\widehat{k}}(\theta_i) \cap S_i^{\mathcal{M}}(\theta_i')$ and $\widehat{m}_i \notin S_i^{\widehat{k}+1}(\theta_i) \cap S_i^{\mathcal{M}}(\theta_i')$. As message $\widehat{m}_i$ gets deleted for θ_i at round $\widehat{k}+1$, we know that for every $\lambda_i \in \Delta(M_{-i} \times \Theta_{-i})$ such that

$$\lambda_i(m_{-i}, \theta_{-i}) > 0 \Rightarrow m_j \in S_j^{\widehat{k}}(\theta_j) \text{ for all } j \neq i,$$

there exists m_i^* such that

$$\sum_{m_{-i},\theta_{-i}} \lambda_i(m_{-i}, \theta_{-i}) u_i\left(g\left(m_i^*, m_{-i}\right), (\theta_i, \theta_{-i})\right) > \sum_{m_{-i},\theta_{-i}} \lambda_i(m_{-i}, \theta_{-i}) u_i\left(g(\widehat{m}_i, m_{-i}), (\theta_i, \theta_{-i})\right).$$

Fix any $\theta'_{-i} \in \Theta_{-i}$ and any $\widehat{m}_j \in S_j^{\mathcal{M}}(\theta_j')$, for each $j \neq i$. Now the above claim remains true if we restrict attention to distributions λ_i putting probability 1 on $\widehat{m}_{-i}$. Thus for every $\psi_i \in \Delta(\Theta_{-i})$ such that

$$\psi_i(\theta_{-i}) > 0 \Rightarrow \widehat{m}_j \in S_j^{\widehat{k}}(\theta_j) \text{ for all } j \neq i,$$

there exists m_i^* such that

$$\sum_{\theta_{-i}} \psi_i(\theta_{-i}) u_i\left(g\left(m_i^*, \widehat{m}_{-i}\right), (\theta_i, \theta_{-i})\right) > \sum_{\theta_{-i}} \psi_i(\theta_{-i}) u_i\left(g(\widehat{m}_i, \widehat{m}_{-i}), (\theta_i, \theta_{-i})\right).$$

Because $\widehat{m}_i$ is never a best response, there must exist a mixed strategy $\mu_i \in \Delta(M_i)$ such that

$$\sum_{m_i} \mu_i(m_i) u_i\left(g(m_i, \widehat{m}_{-i}), (\theta_i, \theta_{-i})\right) > u_i\left(g(\widehat{m}_i, \widehat{m}_{-i}), (\theta_i, \theta_{-i})\right)$$

for all θ_{-i} such that $\widehat{m}_{-i} \in S_{-i}^{\widehat{k}}(\theta_{-i})$ (by the equivalence of "strictly dominated" and "never a best response" (see Lemma 3 by Pearce, 1984).

But $\widehat{m} \in S^{\mathcal{M}}(\theta')$, so (as $\mathcal{M}$ robustly implements f), $g(\widehat{m}_i, \widehat{m}_{-i}) = f(\theta')$. Also observe that if $\theta'_{-i} \in \beta_{-i}(\theta_{-i})$, then $\widehat{m}_{-i} \in S_{-i}^{\widehat{k}}(\theta_{-i})$. Thus

$$\sum_{m_i} \mu_i(m_i) u_i\left(g(m_i, \widehat{m}_{-i}), (\theta_i, \theta_{-i})\right) > u_i\left(f(\theta'), (\theta_i, \theta_{-i})\right) \tag{14}$$

for all θ_{-i} such that $\theta'_{-i} \in \beta_{-i}(\theta_{-i})$. Now let y be the lottery outcome generated by selecting outcome $g(m_i, \widehat{m}_{-i})$ with distribution μ_i on m_i. Now we have established that for any $\beta \neq \beta^*$, there exist i, θ_i and $\theta_i' \in \beta_i(\theta_i)$ with $\theta_i' \neq \theta_i$ such that, for any θ_{-i} and $\theta'_{-i} \in \beta_{-i}(\theta_{-i})$,

$$u_i\left(y, (\theta_i', \theta'_{-i})\right) \leq u_i\left(f(\theta_i', \theta'_{-i}), (\theta_i', \theta'_{-i})\right),$$

using equation (13);

$$u_i\left(y,(\theta_i,\theta_{-i})\right) > u_i\left(f\left(\theta'\right),(\theta_i,\theta_{-i})\right),$$

using equation (14); and

$$u_i\left(y,\left(\theta_i,\theta'_{-i}\right)\right) > u_i\left(f\left(\theta'\right),\left(\theta_i,\theta'_{-i}\right)\right), \tag{15}$$

which also follows from equation (14), as $\theta'_{-i} \in \beta_{-i}\left(\theta'_{-i}\right)$).

Thus using the aggregator representation $u_i(y,\theta) \triangleq v_i(y,h_i(\theta))$, we have

$$v_i\left(y,h_i\left(\theta'_i,\theta'_{-i}\right)\right) \leq v_i\left(f\left(\theta'_i,\theta'_{-i}\right),h_i\left(\theta'_i,\theta'_{-i}\right)\right), \tag{16}$$

and

$$v_i\left(y,h_i(\theta_i,\theta_{-i})\right) > v_i\left(f\left(\theta'\right),h_i(\theta_i,\theta_{-i})\right), \tag{17}$$

and

$$v_i\left(y,h_i\left(\theta_i,\theta'_{-i}\right)\right) > v_i\left(f\left(\theta'\right),h_i\left(\theta_i,\theta'_{-i}\right)\right). \tag{18}$$

Now strict monotonicity of h_i with respect to θ_i implies

$$\operatorname{sign}\left(\theta_i-\theta'_i\right) = \operatorname{sign}\left(h_i\left(\theta_i,\theta'_{-i}\right)-h_i\left(\theta'_i,\theta'_{-i}\right)\right);$$

combining this with the preference rankings in equations (16)–(18) and SSC, we have

$$\operatorname{sign}\left(\theta_i-\theta'_i\right) = \operatorname{sign}\left(h_i(\theta_i,\theta_{-i})-h_i\left(\theta'_i,\theta'_{-i}\right)\right).$$

But now we have just stated the contraction property. ||

The proof of the necessity of the contraction property (Theorem 2), but not of the sufficiency (Theorem 1), uses the fact that the outcome space includes lotteries. We do not know if the contraction property would be necessary for robust implementation with a deterministic domain. However, if the deterministic social choice function is continuous in θ, we can prove the weaker result that the contraction property is necessary for robust implementation in the direct mechanism.

Restricting attention to responsive social choice functions simplifies the statement of the necessity result. The result could be re-stated to allow for non-responsive social choice functions, with appropriate weakenings of the strict EPIC and contraction property conditions. The weakened strict EPIC condition would require only that $f(\theta_i,\theta_{-i}) \neq f\left(\theta'_i,\theta_{-i}\right)$ for some θ_{-i} implies $u_i\left(f(\theta_i,\theta_{-i}),(\theta_i,\theta_{-i})\right) > u_i\left(f\left(\theta'_i,\theta_{-i}\right),(\theta_i,\theta_{-i})\right)$ for all θ_{-i}. The weakened contraction property would require only *unacceptable* report profiles β to satisfy the properties required for all $\beta \neq \beta^*$ in definition 4, where β is unacceptable only if there exists $\theta'_i \in \beta_i(\theta_i)$ with $f(\theta_i,\theta_{-i}) \neq f\left(\theta'_i,\theta_{-i}\right)$ for some θ_{-i}. The weakened strict EPIC and contraction properties are equivalent to the original strict EPIC and contraction properties if f is responsive, and are both automatically satisfied if f is constant. The weakened contraction property is a joint property of the aggregator functions and the social choice function.

We briefly sketch the idea of the proof. We establish the contraction property directly from the robust implementation of the social choice function. We fix an arbitrary report profile $\beta \neq \beta^*$ and consider the iterative process of deleting strictly dominated messages. We identify a step $\hat{k}$ in the process as follows: let $\hat{k}$ be the earliest step at which for some agent i a rationalizable action $\hat{m}_i$ for some type θ_i' fails to be rationalizable *at step* $\hat{k}+1$ for some other type θ_i of agent i given that $\theta_i' \in \beta_i(\theta_i)$. As message $\hat{m}_i$ is deleted for type θ_i, it is never a best response for any message and type profile by the remaining agents. It follows that the message $\hat{m}_i$ is strictly dominated for type θ_i of agent i by a possibly mixed strategy $\mu_i(m_i)$ of agent i. For every given message profile $\hat{m}_{-i}$ of the other agents, the mixed strategy $\mu_i(m_i)$ generates a lottery y over deterministic outcomes. We can now establish the preference ranking of agent i with respect to the allocations y and $f\left(\theta_i', \theta_{-i}'\right)$ for any $\hat{\theta}_{-i}$ such that $\hat{m}_{-i}$ is a rationalizable action for types θ_{-i}' of the remaining agents. In turn, the contraction property follows immediately from these rankings and the single-crossing property.

6. THE LINEAR MODEL

In this section, we consider the special case in which the preference aggregator $h_i(\theta)$ is linear for each i and given by

$$h_i(\theta) = \sum_{j=1}^{I} \gamma_{ij}\theta_j,$$

with $\gamma_{ij} \in \mathbb{R}$ for all i, j and $\gamma_{ii} > 0$ for all i. Without loss of generality, we set $\gamma_{ii} = 1$ for all i:

$$h_i(\theta) = \theta_i + \sum_{j \neq i} \gamma_{ij}\theta_j.$$

The parameters γ_{ij} represent the influence of the signal of agent j on the value of agent i. With the exception of $\gamma_{ii} > 0$ for all i, we do not impose any further *a priori* sign restrictions on γ_{ij}. We denote the square matrix generated by the absolute values of γ_{ij}, namely $\left|\gamma_{ij}\right|$, for all i, j with $i \neq j$ and zero entries on the diagonal by Γ:

$$\Gamma \triangleq \begin{bmatrix} 0 & \left|\gamma_{12}\right| & \cdots & \left|\gamma_{1I}\right| \\ \left|\gamma_{21}\right| & 0 & & \\ \vdots & & \ddots & \\ \left|\gamma_{I1}\right| & & & 0 \end{bmatrix}.$$

We refer to the matrix Γ as the *interdependence matrix*. The specific matrix $\Gamma = \mathbf{0}$ constitutes the case of pure private values. We shall give necessary and sufficient conditions for the matrix Γ to satisfy the contraction property. We then use duality theory to give a characterization of the contraction property in terms of the eigenvalue of the matrix Γ. The proofs of all auxiliary results are in the Appendix.

Lemma 1. (Linear aggregator) The linear aggregator functions $\{h_i(\theta)\}_{i=1}^{I}$ satisfy the contraction property if and only if, for all $\mathbf{c} \in \mathbb{R}_+^I$ with $\mathbf{c} \neq \mathbf{0}$, there exists i such that

$$c_i > \sum_{j \neq i} \left|\gamma_{ij}\right| c_j. \tag{19}$$

The absolute values in the matrix Γ are required to guarantee that the linear inequality (19) implies the contraction property. Condition (19) is required to hold only for a single agent i.

The proof of the contraction property is constructive. We identify for each player i an initial report of the form $\beta_i(\theta_i) = [\theta_i - c_i\varepsilon, \theta_i + c_i\varepsilon]$ for some $\varepsilon > 0$, common across all agents. The size of c_i is therefore proportional to the size of the set of candidate misreports by agent i. It can be thought of as the set of rationalizable strategies at an arbitrary stage k. The inequality of the contraction property then says that for any arbitrary set of reports, characterized by the vector $\mathbf{c}$, there is always an agent i whose set of reports is too large (in the sense of being rationalizable) relative to the set of reports by the remaining agents. It then follows that the set of reports for this agent can be chosen smaller than c_i, allowing us to reduce the set of possible reports for a given agent i with a given type θ_i. Moreover, if the set of reports by i is too large, then there is an "overhang" which can be "nipped and tucked". Now a dual interpretation of the condition (19) leads us from the idea of the overhang directly to the contraction property. With the dual interpretation, we obtain the following simple test of the contraction property.

Proposition 1. **(Contraction property and eigenvalue)** *The matrix Γ has the contraction property if and only if its largest eigenvalue $\lambda < 1$.*

The matrix algebra underlying the above characterization of the contraction property arises in many economic problems which depend on the stability and uniqueness of solutions to a system of linear equations, *e.g.* the uniqueness of equilibrium and rationalizable outcomes in complete information games with linear best responses (see Luenberger, 1978; Gabay and Moulin, 1980; Bernheim, 1984; Weinstein and Yildiz, 2007).

The linear model has the obvious advantage that the local conditions for contraction agree with the global conditions for contraction as the derivatives of the mapping $h_i(\theta)$ are constant and independent of θ. We can naturally extend the idea behind the linear aggregator function to a general non-linear and differentiable aggregator function $h_i(\theta)$, but with a gap between necessary and sufficient conditions. We report the results in the Appendix of a working paper version of this paper (Bergemann and Morris, 2007*b*).

By linking the contraction property to the eigenvalue of the matrix Γ, we can immediately obtain necessary and sufficient conditions for robust implementation for different classes of linear aggregators.

Symmetric preferences. In the symmetric model, the parameters for interdependent values are given by

$$\gamma_{ij} = \begin{cases} 1, & \text{if} \quad j = i, \\ \gamma, & \text{if} \quad j \neq i. \end{cases}$$

The eigenvalue λ of the resulting matrix satisfies: $1 + \lambda = 1 + \gamma(I - 1)$, and hence from Theorem 1, we immediately obtain the necessary and sufficient condition $\gamma < 1/(I - 1)$.

Common interest preferences. An important class of symmetric preferences is the case of common interest preferences with $\gamma = 1$. In this case, for every $I \geq 2$, the contraction property will fail. This result tells us that if there are exact common interests, then robust implementation will be impossible. This observation will hold for more general common interest models, beyond the linear aggregator model of this section. The common interest model is an interesting example in which the *ex post* incentive constraints will automatically be satisfied by any efficient social choice function. Yet the contraction property imposes a constraint leading to the impossibility of robust implementation in common interest environments.

Cyclic preferences. A weaker form of symmetry is incorporated in the following model of cyclic preferences. Here, the interdependence matrix is determined by the distance between i and j (modulo I), or $\gamma_{ij} = \gamma_{(i-j)_{\text{mod}I}}$. In this case, the positive eigenvalue is given by:

$$1 + \lambda = 1 + \sum_{j \neq i} \gamma_{(i-j)},$$

and consequently a necessary and sufficient condition for robust implementation is given by:

$$\sum_{j \neq i} \gamma_{(i-j)} < 1.$$

7. SINGLE UNIT AUCTION

We conclude our analysis with a second economic example, namely a single unit auction with symmetric bidders. The model has I agents and agent i's payoff type is $\theta_i \in [0, 1]$. If the type profile is θ, agent i's valuation of the object is

$$\theta_i + \gamma \sum_{j \neq i} \theta_j,$$

where $0 \leq \gamma \leq 1$.

An allocation rule in this context is a function $y : \Theta \to [0, 1]^I$, where $y_i(\theta)$ is the probability that agent i gets the object and so $\sum_i y_i(\theta) \leq 1$. The symmetric efficient allocation rule is given by:

$$y_i^*(\theta) = \begin{cases} \frac{1}{\#\{j | \theta_j \geq \theta_k \text{ for all } k\}}, & \text{if} \quad \theta_i \geq \theta_k \text{ for all } k, \\ 0, & \text{if} \quad \text{otherwise.} \end{cases}$$

Maskin (1992) and Dasgupta and Maskin (2000) have shown that the efficient allocation can be truthfully implemented in the generalized VCG mechanism, according to which the monetary transfer of the winning agent i is given by

$$x_i(\theta) = \max_{j \neq i} \{\theta_j\} + \gamma \sum_{j \neq i} \theta_j.$$

The winning probability $y_i^*(\theta)$ and the monetary transfer are piecewise constant. The generalized VCG mechanism therefore does not satisfy the strict EPIC conditions which we assumed as part of our analysis. We therefore modify the generalized VCG mechanism to a symmetric ε-efficient allocation rule given by:

$$y_i^{**}(\theta) = \varepsilon \frac{\theta_i}{I} + (1 - \varepsilon) y_i^*(\theta).$$

Under this allocation rule, the object is not allocated with positive probability of order ε.[4] We show that the symmetric ε-efficient allocation rule can be robustly implemented if $\gamma < 1/(I-1)$. Alternatively, we can say that the generalized VCG mechanism itself is robustly virtually implementable if $\gamma < 1/(I-1)$.

4. If the realized payoff type profile is θ, the object will not be allocated to any agent with probability $\varepsilon \sum_i (1 - \theta_i)/I$. At the cost of some additional algebra, we could modify the allocation rule so that it allocates the object with probability 1 by defining $y_i^{**}(\theta) = \varepsilon(\theta_i / \sum_j \theta_j) + (1 - \varepsilon) y_i^*(\theta)$.

It is easy to verify that the resulting generalized VCG transfers satisfy strict EPIC and show that this ε-efficient allocation is robustly implementable. The unique (up to a constant) *ex post* transfer rule is:

$$x_i(\theta) = \frac{\varepsilon}{2I}\theta_i^2 + \frac{\varepsilon\gamma}{I}\left(\sum_{j\neq i}\theta_j\right)\theta_i + (1-\varepsilon)\left(\max_{j\neq i}\left\{\theta_j + \gamma\sum_{j\neq i}\theta_j\right\}\right)y_i^*(\theta).$$

The first two components of the transfers guarantee incentive compatibility with respect to the linear probability assignment and the third component with respect to the efficient allocation rule. The best response of agent i for misreports θ'_{-i} of the remaining agents at a true type profile θ is given as in the public good example by:

$$\theta'_i = \theta_i + \gamma\sum_{j\neq i}\left(\theta_j - \theta'_j\right).$$

We can therefore exactly repeat our earlier argument in the context of the public good and get robust implementation in the direct mechanism if $\gamma < 1/(I-1)$.

8. DISCUSSION

8.1. *Dimensionality and aggregation*

We assumed that agents have one-dimensional payoff types, and each agent's utility depends on the profile of agents' types via an aggregating function. If the agents had multi-dimensional payoff types, and there still existed an aggregator for the multi-dimensional types, then our results would still go through, as the single-crossing condition is defined with respect to the aggregator rather than the types.

More interesting is what happens if the aggregator property fails, with one- or multi-dimensional payoff types. The contraction property formalized the idea of moderate inter-dependence under which the process of iterative elimination of strictly dominated strategies continued until truthtelling remained the only surviving message for each type. Our definition of the contraction property explicitly used the existence of an aggregator. But the notion of moderate interdependence is meaningful in the absence of an aggregator. Informally, the idea of moderate interdependence is that for every possible set of type profiles of all agents, there exists an agent i and a type pair, θ_i and θ'_i, such that the ranking of any pair of alternatives, y and y', is determined by the payoff type of agent i, irrespective of the type profile of the other agents.

More formally, let each Θ_i be a compact subset of $\mathbb{R}^n$ (instead of compact subset of $\mathbb{R}$) and fix a report profile $\beta = \left(\beta_1, ..., \beta_I\right)$ of the agents and consider an agent i with type θ_i and $\theta'_i \in \beta_i(\theta_i)$. We consider the preference ranking of agent i at a type profile θ' with $\theta' \in \beta(\theta)$. A change in the preference profile of agent i from θ'_i in the direction of θ_i is represented by a convex combination $\varepsilon\theta_i + (1-\varepsilon)\theta'_i$. As we change the preference profile of agent i for small $\varepsilon > 0$ in this way, there may be preference reversals such that for some y and y', we observe

$$u_i\left(y', \left(\theta'_i, \theta'_{-i}\right)\right) > u_i\left(y, \left(\theta'_i, \theta'_{-i}\right)\right) \tag{20}$$

and

$$u_i\left(y', \left(\varepsilon\theta_i + (1-\varepsilon)\theta'_i, \theta'_{-i}\right)\right) < u_i\left(y, \left(\varepsilon\theta_i + (1-\varepsilon)\theta'_i, \theta'_{-i}\right)\right). \tag{21}$$

We then say that the preferences of agent i display the contraction property if the direction of the preference reversal at θ_i' is predicated on θ_i in the sense that equations (20) and (21) imply

$$u_i\left(y', (\theta_i, \theta_{-i})\right) < u_i\left(y, (\theta_i, \theta_{-i})\right), \tag{22}$$

for all $\theta'_{-i} \in \beta_{-i}(\theta_{-i})$. This new version of the contraction property is defined independently of an aggregation property or a dimensionality condition on the type of agent i. But it is supposed to hold for all allocations $y, y' \in Y$, even those far away from any realization of the social choice function. This makes it a very strong condition. We therefore propose a weaker, but still sufficient, condition for robust implementation in the direct mechanism by specializing the contraction property to the set of relevant allocations. The two prominent allocations to consider at the type profiles $\left(\theta_i', \theta'_{-i}\right)$ and $\left(\varepsilon\theta_i + (1-\varepsilon)\theta_i', \theta'_{-i}\right)$ are

$$y' \triangleq f\left(\theta_i', \theta'_{-i}\right), \tag{23}$$

and

$$y \triangleq f\left(\varepsilon\theta_i + (1-\varepsilon)\theta_i', \theta'_{-i}\right). \tag{24}$$

Now, if the social choice function f satisfies the *ex post* incentive compatibility, then y and y' as given by equations (23) and (24) satisfy the hypothesis (20) and (21) by force of the *ex post* incentive compatibility condition. A local contraction property can now be defined for the social choice function f.

Definition 7. **(Local contraction property)** Social choice function f satisfies the local contraction property if, for all $\beta \neq \beta^*$, there exists $i, \theta_i, \theta_i' \in \beta_i(\theta_i)$ and $\overline{\varepsilon} > 0$, such that for all $\theta_{-i}, \theta'_{-i} \in \beta_{-i}(\theta_{-i})$ and all $\varepsilon \in (0, \overline{\varepsilon}]$:

$$u_i\left(f\left(\theta_i', \theta'_{-i}\right), (\theta_i, \theta_{-i})\right) < u_i\left(f\left(\varepsilon\theta_i + (1-\varepsilon)\theta_i', \theta'_{-i}\right), (\theta_i, \theta_{-i})\right). \tag{25}$$

By a simple adaptation of the argument in Theorem 1, we can show that the local contraction property of the social choice function is a sufficient condition for robust implementation in the direct mechanism.

Theorem 3. **(Sufficiency)** *If social choice function f satisfies strict EPIC and the local contraction property, then f can be robustly implemented in the direct mechanism.*

It may be useful to illustrate the contraction property of the social choice function with a multi-dimensional generalization of the earlier public good example. We consider an environment with two agents, $i = 1, 2$, where the payoff type of each agent i is a two-dimensional vector $\theta_i = \left(\theta_i^a, \theta_i^b\right) \in [0,1]^2$. Each agent i has a quasilinear utility from a two-dimensional public good $x_0 = \left(x_0^a, x_0^b\right) \in \mathbb{R}_+^2$:

$$u_i(x, \theta) = \left(\alpha_1\theta_i^a + \alpha_2\theta_i^b + \gamma_1\theta_j^a + \gamma_2\theta_j^b\right)x_0^a + \left(\alpha_1\theta_i^b + \alpha_2\theta_i^a + \gamma_1\theta_j^b + \gamma_2\theta_j^a\right)x_0^b + x_i. \tag{26}$$

We assume that $\alpha_1, \alpha_2, \gamma_1, \gamma_2 > 0$. The cost function of providing the public good is given by

$$c(x_0) = \frac{1}{2}\left(x_0^a\right)^2 + \frac{1}{2}\left(x_0^b\right)^2.$$

The (efficient) social choice $f_0(\theta) = \left(f_0^a(\theta), f_0^b(\theta)\right)$ determines the level of the public good along each dimension l:

$$f_0^l(\theta) \triangleq (\alpha_1 + \gamma_1)\left(\theta_1^l + \theta_2^l\right) + (\alpha_2 + \gamma_2)\left(\theta_1^m + \theta_2^m\right), \tag{27}$$

for $l = a, b$ and $l \neq m$. The efficient social choice function $f_0(\theta)$ can be implemented with the generalized VCG transfers:

$$f_i(\theta) = \begin{array}{l} -\left(\theta_i^a\theta_j^a + \theta_i^b\theta_j^b\right)\left(\sum_k (\alpha_k + \gamma_k)\gamma_k\right) - \left(\theta_i^a\theta_j^b + \theta_i^b\theta_j^a\right)\left(\sum_k (\alpha_k + \gamma_k)\gamma_{k+1}\right) \\ -\frac{1}{2}\left(\left(\theta_i^a\right)^2 + \left(\theta_i^b\right)^2\right)\left(\sum_k (\alpha_k + \gamma_k)\alpha_k\right) - \theta_i^a\theta_i^b\left(\sum_k (\alpha_k + \gamma_k)\alpha_{k+1}\right) \end{array}, \tag{28}$$

where the sums over k are modulo 2. The generalized VCG mechanism $(f_0(\theta), f_1(\theta), f_2(\theta))$ satisfies the strict *ex post* incentive constraints in the direct mechanism. The restriction to two agents and a two-dimensional type space is for expositional ease only, and the results generalize to I agents and a K dimensional type and allocation space.

Given the multi-dimensional type space and the multi-dimensional allocation space, it is clear that in general, and specifically in this example, we cannot find an aggregator function to represent the preferences of the agents. Similarly, the preferences will not satisfy a single-crossing condition due to the multi-dimensionality of the allocation space. But because of symmetry across agents and across allocations, the social choice function is strictly *ex post* incentive compatible.[5]

Proposition 1 (Multi-dimensional public goods). *The multi-dimensional social choice function (27) satisfies the local contraction property if*

$$\alpha_1 + \alpha_2 > \gamma_1 + \gamma_2. \tag{29}$$

8.2. *Relation to partial and ex post implementation*

The results in this paper concern full implementation. An earlier paper of ours (Bergemann and Morris, 2005*b*) addresses the analogous questions of robustness to rich type spaces, but looking at the question of *truthtelling* in the *direct* mechanism. In the literature, this is frequently referred to as *partial* implementation. The notion of partial implementation asks whether there exists a mechanism such that *some* equilibrium under that mechanism implements the social choice function. By the revelation principle, it is then sufficient to look at truthtelling in the direct mechanism. In the earlier paper (Bergemann and Morris, 2005*b*), we showed that a social choice function robustly satisfies the interim incentive constraints, *i.e.* satisfies the interim incentive constraints for any type space, if and only if the *ex post* incentive constraints are satisfied.

It is important to note, however, that robust implementation is not equivalent to full *ex post* implementation, *i.e.* the requirement that every *ex post* equilibrium delivers the right

5. The existence of an efficient and *ex post* incentive compatible mechanism in the public good model does not conflict with the generic impossibility of multi-dimensional *ex post* incentive compatible mechanism established by Jehiel, Moldovanu, Meyer-Ter-Vehn, and Zame (2006). The existence of *ex post* incentive compatible transfers here is due to the symmetry across agents and allocations. We emphasize that the objective of the current example is to show how the idea of moderate interdependence extends naturally beyond environments with the aggregation property.

outcome. Often *ex post* implementation will be possible—because there are no undesirable *ex post* equilibria—even though there exist type spaces and interim equilibria that deliver undesirable outcomes. In the earlier paper (Bergemann and Morris, 2008*a*), we identify the *ex post* monotonicity condition that is necessary and sufficient for full *ex post* implementation. It is much weaker than the contraction property (and its equivalent robust monotonicity condition).

8.3. *Robust and virtual implementation in general environments*

The existing Bayesian implementation literature—Postlewaite and Schmeidler (1986), Palfrey and Srivastava (1989*b*) and Jackson (1991)—has shown that on a fixed type space with a common knowledge common prior, Bayesian incentive compatibility and a Bayesian monotonicity condition are necessary and almost sufficient for full implementation. The proof of the sufficiency part of the result relies on complex augmented mechanisms.

In the earlier paper (Bergemann and Morris, 2005*a*), we developed the results in the context of a general approach to robust implementation which allows for complex augmented mechanism. The results reported in this subsection appear in that working paper.

Our robust implementation notion is equivalent to requiring Bayesian implementation on all type spaces. *Ex post incentive compatibility* is equivalent to Bayesian *incentive compatibility* on all type spaces. It is possible to define a notion of robust monotonicity which is equivalent to Bayesian monotonicity on all type spaces. *Ex post* incentive compatibility and robust monotonicity are thus necessary and almost sufficient for full implementation. However, this result relies on allowing complex augmented mechanisms including integer games. If we restrict attention to well-behaved mechanisms—with the compact message space assumption of this paper—then strict EPIC is also necessary.

The contraction property is an implication of robust monotonicity in the environment studied in this paper. The robust monotonicity condition requires the existence of allocations that can be used to reward individuals for reporting deviations from desirable equilibria. In the environment of this paper, we are able to show that we can always use rewards from misreports in the direct mechanism.

In the single good auction example, we used an ε-efficient allocation rule to obtain strict EPIC. An alternative interpretation of the ε-efficient allocation rule is that it virtually implements the efficient social choice function.[6] This naturally leads to the question of how much could be achieved with a robust version of virtual implementation. We pursue this question in Bergemann and Morris (2009) and provide a characterization of robust virtual implementation in general environments. Our general characterization requires complicated augmented mechanisms, building on those by Abreu and Matsushima (1992*a*, 1992*b*). The reliance of positive virtual implementation results on such complex mechanisms has often been criticized (Glazer and Rosenthal, 1992). But in the single-crossing monotonic aggregator environment studied in this paper, we show that *ex post* incentive compatibility and the contraction property are necessary for virtual robust implementation (as well as for exact implementation) and also sufficient for virtual robust implementation in the direct mechanism. Thus, in the current environment, the only implication of going from full to virtual implementation is a relaxation from strict EPIC to EPIC. Thus Bergemann and Morris (2009) show that, by requiring robustness to beliefs and higher-order beliefs, the apparent

6. Abreu and Matsushima (1992*a*) obtain permissive results about virtual implementation under complete information, using iterated deletion of strictly dominated strategies as a solution concept. Abreu and Matsushima (1992*b*) obtain incomplete information analogues of those results.

permissiveness of virtual implementation is greatly reduced relative to exact implementation (the notion studied in the current paper).[7]

8.4. *Social choice correspondences and sets*

We considered necessary and sufficient conditions for the robust implementation of a social choice function. We briefly discuss the relevance of our results for the robust implementation of a social choice *correspondence* or a social choice *set*. A social choice correspondence is a set-valued mapping $F : \Theta \to 2^Y$. A social choice set is a set of social choice functions $\mathcal{F} = \{f \mid f : \Theta \to Y\}$. The concept of a social correspondence, prevalent in the complete information literature (see Maskin, 1999), differs from the concept of a social choice set, prevalent in the incomplete information literature (see Palfrey and Srivastava, 1989*a*; Jackson, 1991).

Our robust implementation results for social choice functions can be applied directly to the notion of a social choice set. In other words a social choice set is robustly implementable if and only if every element of the social choice set is robustly implementable. As the conditions for robust implementation were defined in terms of the utility environment and the social choice function, robust implementation for a social choice set simply requires that every element of the social choice set can be robustly implemented.

In contrast, it is substantially more difficult to obtain necessary or sufficient conditions for the robust implementation of social choice correspondences. In the earlier paper (Bergemann and Morris, 2005*b*), we showed that the difficulties with social choice correspondences already arise with respect to the incentive constraints. In Example 1 of the earlier paper (Bergemann and Morris, 2005*b*) we showed in a setting with two agents and two payoff types for each agent, that a social choice correspondence can satisfy the relevant interim incentive compatibility conditions for all type spaces, yet fail to satisfy the *ex post* incentive compatibility conditions. A comprehensive analysis of the robust implementation of social choice correspondences then first requires additional insights into the nature of incentive constraints in social choice correspondences.

8.5. *The common prior assumption and strategic substitutes/complements*

The definition of robust implementation in this paper is equivalent to requiring that every equilibrium on every type space delivers outcomes consistent with the social choice function. An interesting question is what happens when we look at an intermediate notion of robustness: allowing all possible common prior type spaces. In the earlier paper (Bergemann and Morris, 2008*b*) we pursue this question in the context of our leading example of Section 3.

Consider the case of negative interdependence in valuations, *i.e.* $\gamma < 0$, in the public good example. We recall the *ex post* best response function in that example: if type θ_i is sure that his opponents have type profile θ_{-i} and is sure that they will report themselves to be type

7. Abreu and Matsushima (1992*b*) show that an apparently weak "measurability" condition is the only requirement–beyond incentive compatibility–for virtual implementation under incomplete information. Bergemann and Morris (2009) show that a "robust measurability" condition is the robust analogue of the measurability condition as described by Abreu and Matsushima (1992*b*) when beliefs and higher-order beliefs are not known. This robust measurability condition is shown to be equivalent—in the single-crossing monotonic aggregator environment of this paper—to the contraction property.

profile θ'_{-i}, then his best response is to report himself to be type θ'_i with

$$\theta'_i = \theta_i + \gamma \sum_{j \neq i} \left(\theta_j - \theta'_j\right).$$

We see that there are strategic complements in misreporting strategies (if others misreport upwards, i has an incentive to misreport upwards). This means that when we carry out the iterated deletion procedure, the profile of largest and smallest misreports that survive must constitute an *ex post* equilibrium of the game (Milgrom and Roberts, 1990). Thus a failure of robust implementation also implies that there exists a bad equilibrium on any common prior type space.

In the case with positive interdependence, *i.e.* $\gamma > 0$, there is strategic substitutability in misreports and the above argument does not go through. In fact, Bergemann and Morris (2008*b*) show that if $\gamma \in (0, 1)$, even when the contraction property fails (i.e. $\gamma > \frac{1}{I-1}$), every equilibrium on any common prior type space delivers the right outcome. When we contrast the robust implementation condition for all type spaces with those for all common prior type spaces, it appears that the common prior leads to synchronized beliefs among the agents. This restricts the set of possible best responses and allows for positive implementation results. In the absence of a common prior, a sequential revelation of information may replace the role of the common prior by generating synchronized beliefs among the agents. This theme is developed in the context of an ascending auction as shown by Bergemann and Morris (2007*a*).

8.6. *Informational foundation of interdependence*

In the discussion of the single-crossing condition in Section 4, we present a statistical model of noisy signals which naturally lead to the aggregation property of private signals by means of Bayes' law. There is a possible criticism of using an informational justification for interdependent preferences like this one at the same time as insisting on a stringent robust implementation criterion.[8] This informational microfoundation for the environment depends on the common knowledge of the distribution of signals about the environment—among the agents and the planner. Thus there is common knowledge of a true distribution over the vectors of signals θ. However, we can show that if we allowed that each agent i might receive additional, conditionally independent information–not necessarily consistent with a common prior–about others' signals θ_{-i}, so that the information did not change his expectation of $\omega_0 + \omega_i$, conditional on the vector θ, then our robust implementation results would remain unchanged. Thus there is an admittedly stark story that reconciles the robust implementation environment with an informational justification of the reduced form representation of interdependent preferences.

APPENDIX A

The appendix contains the arguments and proofs missing in the main text.

8. We thank Ilya Segal for prompting us to think about this in the context of robust implementation.

A.1. *Informational foundation for interdependence*

The vector $(\omega_0+\omega_1,\theta_1,\theta_2)$ of random variables is normally distributed with mean zero and variance matrix:

$$\begin{pmatrix} \sigma_0^2+\sigma_1^2 & \sigma_0^2+\sigma_1^2 & \sigma_0^2 \\ \sigma_0^2+\sigma_1^2 & \sigma_0^2+\sigma_1^2+\tau_1^2 & \sigma_0^2 \\ \sigma_0^2 & \sigma_0^2 & \sigma^2+\sigma_2^2+\tau_2^2 \end{pmatrix}.$$

By a standard property of the multi-variate normal distribution this implies that the expectation of $\omega_0+\omega_1$ conditional on θ_1 and θ_2 is given by:

$$\begin{pmatrix} \sigma_0^2+\sigma_1^2 & \sigma_0^2 \end{pmatrix}\begin{pmatrix} \sigma_0^2+\tau_1^2+\sigma_1^2 & \sigma_0^2 \\ \sigma_0^2 & \sigma_0^2+\sigma_2^2+\tau_2^2 \end{pmatrix}^{-1}\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix},$$

which equals

$$\frac{\left(\sigma_0^2\tau_2^2+\sigma_0^2\sigma_1^2+\sigma_0^2\sigma_2^2+\tau_2^2\sigma_1^2+\sigma_1^2\sigma_2^2\right)\theta_1+\sigma^2\tau_1^2\theta_2}{\sigma_0^2\tau_1^2+\sigma_0^2\tau_2^2+\sigma_0^2\sigma_1^2+\sigma_0^2\sigma_2^2+\tau_1^2\tau_2^2+\tau_1^2\sigma_2^2+\tau_2^2\sigma_1^2+\sigma_1^2\sigma_2^2}.$$

If we multiply the above expression by the constant

$$\frac{\sigma_0^2\tau_1^2+\sigma_0^2\tau_2^2+\sigma_0^2\sigma_1^2+\sigma_0^2\sigma_2^2+\tau_1^2\tau_2^2+\tau_1^2\sigma_2^2+\tau_2^2\sigma_1^2+\sigma_1^2\sigma_2^2}{\sigma_0^2\tau_2^2+\sigma_0^2\sigma_1^2+\sigma_0^2\sigma_2^2+\tau_2^2\sigma_1^2+\sigma_1^2\sigma_2^2},$$

we obtain, as reported in equation (3):

$$\theta_1+\frac{\sigma^2\tau_1^2}{\sigma_0^2\tau_2^2+\sigma_0^2\sigma_1^2+\sigma_0^2\sigma_2^2+\tau_2^2\sigma_1^2+\sigma_1^2\sigma_2^2}\theta_2.$$

Proof of Lemma 1. We prove the contrapositive. Thus suppose there exists $\mathbf{c}\in\mathbb{R}_+^I$ with $\mathbf{c}\neq\mathbf{0}$, such that for all i:

$$c_i\leq\sum_{j\neq i}\left|\gamma_{ij}\right|c_j.$$

We now show that this implies that the contraction property fails. Choose $\varepsilon>0$ such that $2c_i\varepsilon<\overline{\theta}_i-\underline{\theta}_i$ for all i. Now consider reports of the form:

$$\beta_i\left(\theta_i\right)=\left[\theta_i-\varepsilon c_i,\theta_i+\varepsilon c_i\right]\cap\Theta_i, \tag{A1}$$

for all i. Then for all i and all $j\neq i$, let $\theta_j=\frac{1}{2}\left(\underline{\theta}_j+\overline{\theta}_j\right)$ and let $\theta'_j=\theta_j-\varepsilon c_i$ if $\gamma_{ij}\geq 0$ and $\theta'_j=\theta_j+\varepsilon c_i$ if $\gamma_{ij}<0$. Using equation (A1), we have $\theta'_j\in\beta_j\left(\theta_j\right)$ for each $j\neq i$. Also observe that $\gamma_{ij}\left(\theta_j-\theta'_j\right)=\varepsilon\left|\gamma_{ij}\right|c_j$. Thus

$$\sum_{j\neq i}\gamma_{ij}\left(\theta_j-\theta'_j\right)=\varepsilon\sum_{j\neq i}\left|\gamma_{ij}\right|c_j\geq\varepsilon c_i.$$

Now if $\theta'_i=\theta_i+\varepsilon c_i$, $\theta_i-\theta'_i$ is strictly negative but

$$\theta_i-\theta'_i+\sum_{j\neq i}\gamma_{ij}\left(\theta_j-\theta'_j\right)$$

is non-negative. A symmetric argument works if $\theta_i>\theta'_i$. So the contraction property, which says that for all $\beta\neq\beta^*$, there exists i and $\theta'_i\in\beta_i\left(\theta_i\right)$ with $\theta'_i\neq\theta_i$, such that

$$\begin{aligned}\operatorname{sign}\left(\theta_i-\theta'_i\right) &= \operatorname{sign}\left(h_i\left(\theta_i,\theta_{-i}\right)-h_i\left(\theta'_i,\theta'_{-i}\right)\right)\\ &= \operatorname{sign}\left(\theta_i-\theta'_i+\sum_{j\neq i}\gamma_{ij}\left(\theta_j-\theta'_j\right)\right),\end{aligned} \tag{A2}$$

for all θ_{-i} and $\theta'_{-i} \in \beta_{-i}(\theta_{-i})$ fails. This proves the necessity of condition (19).

($\Leftarrow$) To show sufficiency, suppose that condition (19) of the lemma holds. Fix any report $\beta \neq \beta^*$. For all j, let:

$$c_j = \max_{\theta'_j \in \beta_j(\theta_j)} \left|\theta'_j - \theta_j\right|.$$

By hypothesis, there exists i such that $c_i > \sum_{j \neq i} \left|\gamma_{ij}\right| c_j$. Let $\left|\theta_i - \theta'_i\right| = c_i$, and suppose without loss of generality that $\theta_i > \theta'_i$. Observe that for all θ_{-i} and $\theta'_{-i} \in \beta_{-i}(\theta_{-i})$: $\gamma_{ij}\left(\theta_j - \theta'_j\right) \leq \left|\gamma_{ij}\right| c_j$. Thus

$$\sum_{j \neq i} \gamma_{ij}\left(\theta_j - \theta'_j\right) \leq \sum_{j \neq i} \left|\gamma_{ij}\right| c_j;$$

and so from

$$\begin{aligned}\left(\theta_i - \theta'_i\right) + \sum_{j \neq i} \gamma_{ij}\left(\theta_j - \theta'_j\right) &= c_i + \sum_{j \neq i} \gamma_{ij}\left(\theta_j - \theta'_j\right) \\ &\geq c_i - \sum_{j \neq i} \left|\gamma_{ij}\right| c_j > 0,\end{aligned}$$

it follows that the contraction property, and equivalently (A2), is satisfied. ‖

The following lemma gives a dual representation of the contraction property for the linear case. In turn, it allows us to characterize the contraction property in terms of the largest eigenvalue of the interdependence matrix Γ.

Lemma 2. **(Duality)** *The following two properties of* Γ *are equivalent:*

1. *for all* $c \in \mathbb{R}^I_+$ *with* $c \neq \mathbf{0}$, *there exists* i *such that:*

$$c_i > \sum_{j \neq i} \left|\gamma_{ij}\right| c_j; \tag{A3}$$

2. *there exists* $d \in \mathbb{R}^I_+$ *such that for all* i:

$$d_i > \sum_{j \neq i} \left|\gamma_{ji}\right| d_j. \tag{A4}$$

Proof. Consider the following contrapositive restatement of condition (A3): there does not exist $c \in \mathbb{R}^I_+$ such that

$$\sum_{i=1}^{I} c_i > 0, \tag{a}$$

and for all i :

$$\sum_{j \neq i} \left|\gamma_{ij}\right| c_j - c_i \geq 0. \tag{b$_i$}$$

Writing μ for the multiplier of constraint (a) and d_i for the multiplier of constraint (b$_i$), Farkas' lemma states that such a c does not exist if and only if there exist $d \in \mathbb{R}^I_+$ and $\mu \in \mathbb{R}_+$ such that

$$\mu - d_i + \sum_{j \neq i} \left|\gamma_{ji}\right| d_j = 0 \text{ for all } i, \tag{a'}$$

and

$$\mu > 0. \tag{b'}$$

But this is true if and only if condition (A4) of the lemma holds. ‖

Proof of Proposition 1. If we try to find a solution for the strict inequalities (A4):

$$d_i > \sum_{j \neq i} \left| \gamma_{ji} \right| d_j, \quad \text{for all } i,$$

with the assistance of a contraction constant $\lambda < 1$, or

$$d_i \lambda = \sum_{j \neq i} \left| \gamma_{ji} \right| d_j,$$

then by the Frobenius–Perron Theorem for non-negative matrices (see Minc, 1988, Theorem 1.4.2), there exist positive right and left eigenvectors with the same positive eigenvalue λ. We can use the above dual property to establish that a (λ, d) solution exists for:

$$\lambda d_i = \sum_{j \neq i} \left| \gamma_{ji} \right| d_j,$$

but from the duality relationship (A4), we know that for every $d > 0$,

$$d_i > \sum_{j \neq i} \left| \gamma_{ji} \right| d_j,$$

so it follows that $\lambda < 1$. ||

Proof of Proposition 2. Suppose first the inequality (29) holds. Given a report profile $\beta \neq \beta^*$, we consider an agent i such that

$$\max_{\theta_i, \theta_i' \in \beta_i(\theta_i)} \left| \theta_i^a - \theta_i^{a\prime} + \theta_i^b - \theta_i^{b\prime} \right| \geq \max_{\theta_j, \theta_j' \in \beta_j(\theta_j)} \left| \theta_j^a - \theta_j^{a\prime} + \theta_j^b - \theta_j^{b\prime} \right|. \tag{A5}$$

Consider a pair of types $\theta_i, \theta_i' \in \beta_i(\theta_i)$ which achieves the maximum in equation (A5), and assume without loss of generality that the maximum is positive: $\theta_i^a - \theta_i^{a\prime} + \theta_i^b - \theta_i^{b\prime} > 0$. The indirect utility from the mechanism is given by:

$$\begin{aligned} u_i(f(m), \theta) &= \left(\alpha_1 \theta_i^a + \alpha_2 \theta_i^b + \gamma_1 \theta_j^a + \gamma_2 \theta_j^b\right) f_0^a(m) \\ &\quad + \left(\alpha_1 \theta_i^b + \alpha_2 \theta_i^a + \gamma_1 \theta_j^b + \gamma_2 \theta_j^a\right) f_0^b(m) + f_i(m). \end{aligned}$$

We want to show that the inequality (25) holds for some $\overline{\varepsilon} > 0$. As the indirect utility is a quadratic function in the report profile, it is sufficient to consider the derivative of the indirect utility $u_i\left(f\left((1-\varepsilon)\theta_i' + \varepsilon\theta_i, \theta_{-i}'\right), \theta\right)$ with respect to ε evaluated at $\varepsilon = 0$. After collecting terms, the derivative results in:

$$\begin{aligned} &\left(\left(\theta_i^a - \theta_i^{a\prime} + \theta_i^b - \theta_i^{b\prime}\right)(\alpha_1 + \alpha_2) + \left(\theta_j^a - \theta_j^{a\prime} + \theta_j^b - \theta_j^{b\prime}\right)(\gamma_1 + \gamma_2)\right) \\ &\left(\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2\right) > 0, \end{aligned} \tag{A6}$$

where the strict inequality follows from the hypothesis of $\alpha_1 + \alpha_2 > \gamma_1 + \gamma_2$, the inequality (A5):

$$\left| \theta_i^a - \theta_i^{a\prime} + \theta_i^b - \theta_i^{b\prime} \right| \geq \left| \theta_j^a - \theta_j^{a\prime} + \theta_j^b - \theta_j^{b\prime} \right|,$$

and the positivity of $\theta_i^a - \theta_i^{a\prime} + \theta_i^b - \theta_i^{b\prime} > 0$. (If the later sum were negative, then the sign of the derivative (A6) would change and the same argument would go through.)

We prove the converse by contrapositive and assume that the inequality (29) is reversed to $\alpha_1 + \alpha_2 \leq \gamma_1 + \gamma_2$. We now consider a report profile $\beta \neq \beta^*$ such that

$$\max_{\theta_i, \theta_i' \in \beta_i(\theta_i)} \left| \theta_i^a - \theta_i^{a\prime} + \theta_i^b - \theta_i^{b\prime} \right| = \max_{\theta_j, \theta_j' \in \beta_j(\theta_j)} \left| \theta_j^a - \theta_j^{a\prime} + \theta_j^b - \theta_j^{b\prime} \right|$$

and

$$\theta_i^a - \theta_i^{a\prime} + \theta_i^b - \theta_i^{b\prime} > 0 > \theta_j^a - \theta_j^{a\prime} + \theta_j^b - \theta_j^{b\prime}.$$

It follows that the strict inequality (A6) is reversed and this establishes the failure of the local contraction property of the social choice function. ‖

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