The Role of the Common Prior in Robust Implementation

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Epistemic Foundations

- how to elicit private and decentralized information to solve social choice problem
- wide range of applications from bilateral trading, auctions to constitutional design
- first question: is truthtelling in direct mechanism Bayesian incentive compatible

1. truthtelling in direct mechanism (= partial implementation), but there might be other equilibria which do not lead to the realization of the social choice function
2. Bayesian incentive compatible for specific prior/posterior, but wat if agents have richer beliefs and higher-order beliefs

Robust Implementation

- we address these issues by:

1. requiring that every equilibrium is consistent with social choice function
   $\Rightarrow$ implementation rather than partial implementation
2. allowing for all possible beliefs and higher order beliefs of the agents
   $\Rightarrow$ robust implementation

- robust implementation: every equilibrium in “every type space” is consistent with social choice function
- intermediate notion of robustness:
  all possible common prior type spaces versus all possible type spaces

Common Prior

- importance of common prior assumption for the possibility of robust implementation
- develop necessary and sufficient conditions for robust implementation depending
  - for all types spaces
  - for all type spaces with a common prior

Epistemic Foundations

- analysis of robust implementation with and without a common prior relies on epistemic results for incomplete information games
- for complete information games, Aumann (1987) and Brandenburger and Dekel (1987) show that correlated equilibrium and rationalizability characterize, the consequences of common knowledge of rationality with and without common prior
- in “Belief Free Incomplete Information Games” (2007) we report incomplete information and belief free generalizations of these solution concepts
  - incomplete information correlated equilibrium
  - incomplete information rationalizability

Epistermic Analysis in Direct Mechanism

- apply these results to the specific game given by the direct mechanism
  - action is reported type
- a message of a payoff type is incomplete information rationalizable if and only if there is a hierarchical type space and a BNE s.th. message is an equilibrium action for a type with a given payoff type
- a message of a payoff type is an element of incomplete information correlated equilibrium if and only if there is a common prior type space and a BNE s.th. message is an equilibrium action for a type with a given payoff type

Informational Externalities

- develop the arguments in the context of a public good model with interdependent values
- allow for positive as well as negative informational externalities
- in the direct revelation mechanism, the reporting strategies by the agents are
  - strategic complements with negative informational externalities
  - strategic substitutes with positive informational externalities.

Robust Implementation

- rephrase the conditions for robust implementation with and without common prior with epistemic results in the background
- find conditions for a unique solution under
  - incomplete information correlated equilibrium
  - incomplete information rationalizability
- with strategic complements necessary and sufficient conditions do not depend on the existence of a common prior
- with strategic substitutes the common prior assumption changes the implementation conditions

Set Up

- $i \in \{1, 2, ..., I\}$ agents
- i has payoff type $\theta_i \in \Theta_i = [\underline{\theta}_i, \overline{\theta}_i] \subset \mathbb{R}$
- i gets utility from social choice $x \in X$ and transfers $t_i \in \mathbb{R}$;

$$u_i(x, \theta) - t_i$$

- direct mechanism specifies social choice function $f : \Theta \to X$, and transfer rule $t_i : \Theta \to \mathbb{R}$.
- direct mechanism $(f, (t_i)_i)$ is ex post incentive compatible if $\forall i, \forall \theta, \forall m_i$:

$$u_i(f(\theta_i, \theta_{-i}), \theta) - t_i(\theta_i, \theta_{-i}) \geq u_i(f(m_i, \theta_{-i}), \theta) - t_i(m_i, \theta_{-i}).$$

Public Good Example

- provision of a public good $x \in \mathbb{R}_+$
- utility of i for public good

$$u_i(x, \theta) = \left(\theta_i + \gamma \sum_{j \neq i} \theta_j \right) \cdot x$$

and for each i, aggregator $h_i(\theta)$:

$$h_i(\theta) = \theta_i + \gamma \sum_{j \neq i} \theta_j$$

- weight γ represents preference interdependence
  - $\gamma < 0$ represents negative informational externalities
  - $\gamma = 0$ represents private value model
  - $\gamma > 0$ represents positive informational externalities

Efficient Social Choice

- cost of establishing public good: $c(x) = \frac{1}{2}x^2$
- planner chooses $x = f(\theta)$ to maximize social welfare:

$$f(\theta) = (1 + \gamma(I - 1)) \sum_{i=1}^{I} \theta_i$$

- generalized Vickrey-Clarke-Groves (VCG) transfers

$$t_i(\theta) = (1 + \gamma(I - 1)) (\frac{1}{2}\theta_i^2 + \gamma\theta_i \sum_{j \neq i} \theta_j)$$

- truthtelling is ex post incentive compatible if $\gamma \geq -1/(I - 1)$

Notable Features

- willingness to pay of i is given by an aggregator, namely weighted sum of payoff types of all agents, summarizing the private information of all agents
- cost function of public good is quadratic and transfers are quadratic functions of the reports
  - linear best response property turns the reporting game in the direct mechanism into a potential game
  - analyis of correlated equilibrium by potential game arguments

Ex Post Best Response

- ex post best response:

$$b_i : \Theta \times \Theta_{-i} \to \Theta_i$$

  mapping from true payoff types *and* reported types of all agents but i into report of agent i
- in linear quadratic environment given by:

$$b_i(\theta, m_{-i}) \triangleq \theta_i + \gamma \sum_{j \neq i} (\theta_j - m_j)$$

- verifies strict ex post incentive compatibility of f
- best response by i to (mis)report m_j is to report m_i so that the aggregate type given i's point of view is aggregate under type profile θ:

$$h_i(\theta) = h_i(b_i(\theta, m_{-i}), m_{-i})$$

Strategic Complements and Strategic Substitutes

- strategies of *i* and *j* are *strategic complements* if

$$\partial b_i \left(\theta, m_j, m_{-ij}\right) / \partial m_j > 0$$

and they are *strategic substitutes* if

$$\partial b_i \left(\theta, m_j, m_{-ij}\right) / \partial m_j < 0$$

- here we have

$$b_i \left(\theta, m_{-i}\right) = \theta_i + \gamma \sum_{j \neq i} \left(\theta_j - m_j\right)$$

and the reports of the agents are

  - complements with negative informational externalities $\gamma < 0$
  - substitutes with positive informational externalities $\gamma > 0$

Incomplete Information Rationalizability

- agent i's payoff with type profile θ and reported profile m:

$$u_i^{+}(m,\theta) \triangleq u_i(f(m),\theta) - t_i(m)$$

- iterative elimination of actions which are never best response
- novel elements due to incomplete information
  - elimination payoff type by payoff type
  - elimination for all possible beliefs about type and message profiles
- introduced by Battigalli (1999) and B and Siniscalchi (2003)

Definition (Incomplete Information Rationalizability)

The incomplete information rationalizable actions

$$R_i : \Theta_i \to 2^{\Theta_i} \Big/ \varnothing,$$

are defined recursively with $R_i^0(\theta_i) = \Theta_i$,

$$R_i^{k+1}(\theta_i) = \left\{ m_i \;\middle|\; \begin{array}{l} \exists \mu_i \in \Delta(\Theta_{-i} \times \Theta_{-i}) \text{ s. th.} \\ (1)\ \mu_i \left[\left\{ (m_{-i}, \theta_{-i}) : m_j \in R_j^k(\theta_j)\ \ \forall j \neq i \right\} \right] = 1 \\ (2)\ m_i \in \arg\max\limits_{m_i'} \int\limits_{m_{-i}, \theta_{-i}} u_i^+ \left((m_i', m_{-i}), \theta \right) d\mu_i \end{array} \right\}$$

for each $k = 1, 2, ...$, and

$$R_i(\theta_i) = \bigcap_{k \geq 1} R_i^k(\theta_i).$$

Rationalizability

- consequence of common knowledge of rationality without common prior

Theorem (Rationallizability)

1. *If the interdependence is small and.* $\gamma \in \left(-\frac{1}{I-1}, +\frac{1}{I-1}\right)$, *then for all i and* θ_i, $R_i(\theta_i) = \{\theta_i\}$.
2. *If the interdependence is large and* $\gamma \notin \left(-\frac{1}{I-1}, +\frac{1}{I-1}\right)$, *then for all i and* θ_i, $R_i(\theta_i) = [0, 1]$.

- as number of agents $I \to \infty$

$$\frac{1}{I-1} \to 0$$

and model converges to private value model

Robust Implementation: General Result

- result is a special case of
  "Robust Implementation: The Case of the Direct Mechanism"
- necessary and sufficient conditions for robust implementation in environments where
  - the payoff types of all agents are aggregated in a one-dimensional variable
- environment there is general:
  - neither the aggregator nor the utility function of *i* has to be linear as in the current example
- robust implementation is possible in any mechanism if and only if it is possible in the direct mechanism;
- robust implementation is possible if and only if aggregator function satisfies a contraction property (= small interdependence)

Correlated Equilibrium

Definition (Incomplete Information Correlated Equilibrium)

A probability distribution $\mu \in \Delta(\Theta \times \Theta)$ is an incomplete information correlated equilibrium (ICE) of the direct mechanism if for each i and each measurable $\phi_i : \Theta_i \times \Theta_i \to \Theta_i$:

$$\int_{m,\theta} u_i^+ \left((m_i, m_{-i}), \theta \right) d\mu \geq \int_{m,\theta} u_i^+ \left((\phi_i(m_i, \theta_i), \theta_{-i}), \theta \right) d\mu.$$

- define $C_i(\theta_i)$ - the set of messages that can be sent by type θ_i in *an* incomplete information correlated equilibrium μ of the direct mechanism (essentially Forges (1993))

Correlated Equilibrium

- consequence of common knowledge of rationality with common prior

Theorem (Incomplete Information Correlated Equilibrium)

The incomplete information correlated equilibrium has $\forall i, \forall \theta_i$, $C_i(\theta_i) = \{\theta_i\}$ *if and only if*

$$\gamma \leq 1.$$

- contrast with rationalizability where

$$\gamma \leq \frac{1}{I-1}$$

Bayesian Potential Game

- belief free incomplete information game

$$\Gamma = \left\{I,\ \{A_i\}_{i=1}^{I}, \{\Theta_i\}_{i=1}^{I}, \{u_i(a,\theta)\}_{i=1}^{I}\right\}$$

  has a *weighted potential* $v : A \times \Theta \to \mathbb{R}$ if there exist $w \in \mathbb{R}^{I}_{++}$ such that

$$u_i(a,\theta) - u_i\left(\left(a_i', a_{-i}\right), \theta\right) = w_i\left[v(a,\theta) - v\left(\left(a_i', a_{-i}\right), \theta\right)\right],$$

  for all i, $a_i, a_i' \in A_i$, $a_{-i} \in A_i$ and $\theta \in \Theta$
- incomplete information generalization of weighted potential in Monderer and Shapley (1996)

Potential Argument

- Neyman (1994) shows in complete information games that if the potential is concave then correlated equilibrium is unique
- in "Belief Free Incomplete Information Games" we show that if belief free game Γ has a strictly concave smooth potential function *and* an ex post equilibrium s^*, then

$$\forall i, \forall \theta_i, \; s_i^* (\theta_i) = C_i (\theta_i)$$

- direct mechanism has truthtelling as an ex post equilibrium
- verify existence of a strictly concave potential of direct mechanism

Robust Implementation

- common prior assumption and robust implementation
- major impact with positive interdependence (strategic substitutes)
- no impact with negative interdependence (strategic complementarities)

Corollary (Robust Implementation)

1. *If the reports are strategic complements, then robust implementation with common prior implies robust implementation without common prior.*
2. *If the reports are strategic substitutes, then robust implementation with common prior fails to imply robust implementation without common prior.*