Econ 121b: Intermediate Microeconomics

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Week of 1/29 - 2/4

1 Lecture 7: Expenditure Minimization

Instead of maximizing utility subject to a given income we can also minimize expenditure subject to achieving a given level of utility $\bar{u}$. In this case, the consumer wants to spend as little money as possible to enjoy a certain utility. Formally, we write

$$\min_x \; p_1 x_1 + p_2 x_2 \quad \text{s.t. } u(x) \geq \bar{u}. \tag{1}$$

We can set up the Lagrange expression for this problem as the following:

$$\mathcal{L}(x_1, x_2, \lambda) = p_1 x_1 + p_2 x_2 + \lambda[\bar{u} - u(x_1, x_2)]$$

The F.O.C.s are now:

$$\frac{\partial \mathcal{L}}{\partial x_1} = p_1 - \lambda \frac{\partial u}{\partial x_1} = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = p_2 - \lambda \frac{\partial u}{\partial x_2} = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = \bar{u} - u(x_1, x_2) = 0$$

Comparing the first two equations we get,

$$\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \frac{p_1}{p_2}$$

This is the exact relation we got in the utility maximization program. Therefore these two programs are equivalent exercises. In the language of mathematics it is called the duality. But the values of x_1 and x_2 that minimizes the expenditure is a function of the utility level $\bar{u}$ instead of income as in the case of utility maximization. The result of this optimization problem is a demand function again, but in general it is different from $x^*(p_1, p_2, I)$. We call the demand function derived from problem (1) *compensated demand* or Hicksian demand.[1] We denote it by,

$$h_1(p_1, p_2, \bar{u}) \text{ and } h_2(p_1, p_2, \bar{u})$$

Note that compensated demand is a function of prices and the utility level whereas uncompensated demand is a function of prices and income. Plugging compensated demand into the objective function ($p_1x_1 + p_2x_2$) yields the *expenditure function* as function of prices and $\bar{u}$

$$E(p_1, p_2, \bar{u}) = p_1 h_1(p_1, p_2, \bar{u}) + p_2 h_2(p_1, p_2, \bar{u}).$$

Hence, the expenditure function measures the minimal amount of money required to buy a bundle that yields a utility of $\bar{u}$.

Uncompensated and compensated demand functions usually differ from each other, which is immediately clear from the fact that they have different arguments. There is a special case where they are identical. First, note that indirect utility and expenditure function are related by the following relationships

$$\begin{aligned} V(p_1, p_2, E(p_1, p_2, \bar{u})) &= \bar{u} \\ E(p_1, p_2, V(p_1, p_2, I)) &= I. \end{aligned}$$

That is, if income is exactly equal to the expenditure necessary to achieve utility level $\bar{u}$, then the resulting indirect utility is equal to $\bar{u}$. Similarly, if the required utility level is set equal to the indirect function when income is I, then minimized expenditure will be equal to I. Using these relationships, we have that uncompensated and compensated demand are equal in the following two cases:

$$\begin{aligned} x_i^*(p_1, p_2, I) &= h_i^*(p_1, p_2, V(p_1, p_2, I)) \\ x_i^*(p_1, p_2, E(p_1, p_2, \bar{u})) &= h_i^*(p_1, p_2, \bar{u}) \text{ for } i = 1, 2. \end{aligned} \tag{2}$$

Now we can express income and substitution effects analytically. Start with one component of equation (2):

$$h_i^*(p_1, p_2, \bar{u}) = x_i^*(p_1, p_2, E(p_1, p_2, \bar{u}))$$

and take the derivative with respect to p_j using the chain rule

$$\frac{\partial h_i^*}{\partial p_j} = \frac{\partial x_i^*}{\partial p_j} + \frac{\partial x_i^*}{\partial I}\frac{\partial E}{\partial p_j}. \tag{3}$$

[1] After the British economist Sir John Hicks, co-recipient of the 1972 Nobel Prize in Economic Sciences.

Now we have to find an expression for $\frac{\partial E}{\partial p_j}$. Start with the Lagrangian associated with problem (1) evaluated at the optimal solution $(h^*(p_1, p_2, \bar{u}),\ \lambda^*(p_1, p_2, \bar{u}))$:

$$\mathcal{L}(h^*(p_1, p_2, \bar{u}), \lambda^*(p_1, p_2, \bar{u})) = p_1 h_1^*(p_1, p_2, \bar{u}) + p_2 h_2^*(p_1, p_2, \bar{u}) + \lambda^*(p_1, p_2, \bar{u})[\bar{u} - u(x(p_1, p_2, \bar{u}))].$$

Taking the derivative with respect to any price p_j and noting that $\bar{u} = u(x(p, \bar{u}))$ at the optimum we get

$$\begin{aligned} \frac{\partial \mathcal{L}(h^*(p, \bar{u}), \lambda^*(p, \bar{u}))}{\partial p_j} &= h_j^* + \sum_{i=1}^{I} p_i \frac{\partial h_i^*}{\partial p_j} - \lambda^* \sum_{i=1}^{I} \frac{\partial u}{\partial x_i} \frac{\partial x_i}{\partial p_j} \\ &= h_j^* + \sum_{i=1}^{I} \left(p_i - \lambda^* \frac{\partial u}{\partial x_i} \right) \frac{\partial x_i}{\partial p_j}. \end{aligned}$$

But the first -order conditions for this Lagrangian are

$$p_i - \lambda \frac{\partial u}{\partial x_i} = 0 \text{ for all } i.$$

Hence

$$\frac{\partial E}{\partial p_j} = \frac{\partial \mathcal{L}}{\partial p_j} = h_j^*(p_1, p_2, \bar{u}).$$

This result also follows form the Envelope Theorem. Moreover, from equation (2) it follows that $h_j^* = x_j^*$. Hence, using these two facts and bringing the second term on the RHS to the LHS we can rewrite equation (3) as

$$\frac{\partial x_i^*}{\partial p_j} = \underbrace{\frac{\partial h_i^*}{\partial p_j}}_{SE} - x_j^* \underbrace{\frac{\partial x_i^*}{\partial I}}_{IE}.$$

This equation is known as the *Slutsky Equation*[2] and shows formally that the price effect can be separated into a substitution (SE) and an income effect (IE).

2 Lecture 8: Categories of goods

Definition 1. A **normal good** is a commodity whose Marshallian demand is positively related to income, i.e. as income goes up the uncompensated demand of that good goes up as well. Therefore good i is normal if

$$\frac{\partial x_i^*}{\partial I} > 0$$

[2]After the Russian statistician and economist Eugen Slutsky.

Definition 2. A **inferior good** is a commodity whose Marshallian demand is negatively related to income, i.e. as income goes up the uncompensated demand of that good goes down. Therefore good i is inferior if

$$\frac{\partial x_i^*}{\partial I} < 0$$

Definition 3. Two goods are **gross substitutes** if rise in the price of one good raises the uncompensated demand of the other good. Therefore goods i and j are gross substitutes if

$$\frac{\partial x_i^*}{\partial p_j} > 0$$

Definition 4. Two goods are **net substitutes** if rise in the price of one good raises the compensated or Hicksian demand of the other good. Therefore goods i and j are net substitutes if

$$\frac{\partial h_i^*}{\partial p_j} > 0$$

Definition 5. Two goods are **net complements** if rise in the price of one good reduces the compensated or Hicksian demand of the other good. Therefore goods i and j are net complements if

$$\frac{\partial h_i^*}{\partial p_j} < 0$$

Giffen good

2.1 Shape of Expenditure Function

The expression for expenditure function in a n commodity case is given by,

$$E(p_1, p_2, \ldots, p_n, \bar{u}) \triangleq \sum_{i=1}^{n} p_i h_i^*(p_1, p_2, \ldots, p_n, \bar{u})$$

Now let's look at the effect of changing price p_i on the expenditure. By envelope theorem we get that,

$$\frac{\partial E}{\partial p_i} = h_i^*(p_1, p_2, \ldots, p_n, \bar{u}) > 0$$

Therefore the expenditure function is positively sloped, i.e. when prices go up the minimum expenditure required to meet certain utility level also goes up. Now to find out the curvature of the expenditure function we take the second order derivative:

$$\frac{\partial^2 E}{\partial p_i^2} = \frac{\partial h_i^*}{\partial p_i} < 0$$

This implies that the expenditure function is concave in prices.

Definition 6. A **Giffen good** is one whose Marshallian demand is positively related to its price. Therefore good i is Giffen if,

$$\frac{\partial x_i^*}{\partial p_i} > 0$$

But from the Hicksian demand we know that,

$$\frac{\partial h_i^*}{\partial p_i} < 0$$

Hence from the Slutsky equation,

$$\frac{\partial x_i^*}{\partial p_i} = \frac{\partial h_i^*}{\partial p_i} - x_1^* \frac{\partial x_i^*}{\partial I}$$

we get that for a good to be Giffen we must have,

$$\frac{\partial x_i^*}{\partial I} < 0$$

and x_i^* needs to be large to overcome the effect of substitution effect.