

HIGHER-RANK LOCAL MIXING FOR CUSPED POSITIVE REPRESENTATIONS

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ABSTRACT. Let $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ be a torsion-free, non-elementary, finitely generated Fuchsian group with parabolic elements, and let $\rho : \Gamma \rightarrow G$ be a type-preserving, Zariski dense, positive representation into a connected split semisimple real algebraic group. We prove quantitative local mixing for Bowen–Margulis–Sullivan measures on $\rho(\Gamma) \backslash G/M$, where M is the centralizer of a maximal split torus in a compatible maximal compact subgroup.

For every u in the interior of the limit cone of $\rho(\Gamma)$, correlations of smooth compactly supported functions along $\exp(tu)$, normalized by $t^{(\mathrm{rank}_{\mathbb{R}} G - 1)/2}$, admit a complete asymptotic expansion in integer powers of t^{-1} . We also obtain a uniform Gaussian asymptotic for transverse displacements, with an integrable remainder. We deduce Haar local mixing and orbital counting asymptotics in Riemannian balls.

More generally, these conclusions hold for Zariski dense, relatively Borel Anosov representations of Γ satisfying a peripheral root-balance condition. This includes relatively Borel Anosov representations into split simple groups of rank at most three and self-joinings of type-preserving geometrically finite real hyperbolic representations.

To prove these results, we develop an intrinsic cusp geometry on the flag limit set and use it to construct a countable Markov coding. We establish exponential moment bounds and uniform distortion estimates for the unbounded vector-valued return cocycle. Using peripheral Cartan growth and Zariski density, we establish the quantitative non-integrability needed for the spectral analysis.

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1. INTRODUCTION

We establish local mixing with quantitative error terms for higher-rank diagonal flows associated with cusped positive representations and two other classes of representations of finitely generated Fuchsian groups. Our main theorem gives a complete asymptotic expansion of correlations along each direction in the interior of the limit cone, together with a Gaussian asymptotic for transverse displacements and a uniform, integrable remainder. It also yields Haar local mixing and orbital counting in Riemannian balls.

In rank one, the Bowen–Margulis–Sullivan (BMS) measure of a non-elementary geometrically finite locally symmetric manifold is finite [74]. Babillot [1] proved mixing of the geodesic flow for this measure, and Winter [77] established mixing of the frame flow for Zariski dense groups. These results underlie orbital counting and equidistribution theorems; see [51, 67, 60, 59]. For finite-volume locally symmetric manifolds, mixing of regular diagonal flows with respect to Haar measure follows from the Howe–Moore theorem [34].

The higher-rank, infinite-volume setting is different. For Zariski dense discrete subgroups of simple real algebraic groups of higher rank, there is an infinite family of non-proportional BMS measures [2, 64], all of which are infinite [30]. Local mixing describes the decay of correlations after a suitable normalization: for a flow $\{a_t\}$ on X preserving a Radon measure m , one seeks $\mathcal{N}(t) > 0$ and nonzero Radon measures m_1, m_2 on X such that, as $t \rightarrow +\infty$,

$$\mathcal{N}(t) \int_X f_1(xa_t)f_2(x) dm(x) \rightarrow m_1(f_1)m_2(f_2) \quad (f_1, f_2 \in \mathcal{C}_c(X)),$$

where $m_i(f) := \int_X f dm_i$ for $i = 1, 2$. Oh–Pan [57] introduced this terminology and proved local mixing for geodesic flows on $\mathbb{Z}^d$ -covers of convex cocompact rank-one locally symmetric manifolds, with normalization proportional to $t^{d/2}$.

The main new ingredient in this paper is a geometric construction of a countable coding that controls the full vector-valued return cocycle in the presence of arbitrarily long cusp excursions. We develop an intrinsic cusp geometry on the flag limit set and derive quantitative non-integrability in every Cartan direction from peripheral Cartan growth and Zariski density. The required root-balance condition is automatic in the three classes below. These estimates yield complete correlation expansions and uniform transverse asymptotics for every representation in these classes.

Setting. Let $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ be a torsion-free, non-elementary, finitely generated Fuchsian group with parabolic elements, and let G be a connected semisimple real linear algebraic group. Let $\rho : \Gamma \rightarrow G$ be a Zariski dense representation. Assume that one of the following holds:

- G is $\mathbb{R}$ -split, and ρ is type-preserving and positive in the sense of Fock–Goncharov [29]; see Definitions 3.2 and 4.4.
- G is $\mathbb{R}$ -split and simple with $\mathrm{rank}_{\mathbb{R}} G \leq 3$, and ρ is relatively Borel Anosov.

- $G = \prod_{i=1}^k \mathrm{SO}(n_i, 1)^\circ$, with $n_i \geq 2$, and ρ is type-preserving and each projection ρ_i is geometrically finite.

In all three cases, ρ is relatively Borel Anosov, and hence discrete and faithful; see Theorem 4.5 and Proposition 4.7.

Positive representations admit explicit Fock–Goncharov coordinates associated with ideal triangulations [29, Section 6.1 and Theorem 1.6]. When Γ is a lattice, the connected component containing a principal Fuchsian representation, with its peripheral conjugacy classes fixed, consists entirely of type-preserving positive representations [28, Theorem G].

More generally, our results hold for Zariski dense, relatively Borel Anosov representations of Γ satisfying the peripheral root-balance condition in Definition 4.2; see Theorem 4.1.

Fix a maximal $\mathbb{R}$ -split torus $A < G$, a compatible maximal compact subgroup $K < G$, and a closed positive Weyl chamber $A^+ \subset A$, so that the Cartan decomposition $G = KA^+K$ holds. Set

$$\mathfrak{a} := \mathrm{Lie} A, \quad \mathfrak{a}^+ := \log A^+, \quad r := \mathrm{rank}_{\mathbb{R}} G = \dim \mathfrak{a}, \quad M := Z_K(A),$$

where $Z_K(A)$ denotes the centralizer of A in K .

Write $\Gamma_\rho := \rho(\Gamma) < G$ and let $\mathcal{L}_\rho \subset \mathfrak{a}^+$ and ψ_ρ denote the limit cone and growth indicator of Γ_ρ , respectively. The limit cone has nonempty interior [2]. For each $u \in \mathrm{int} \mathcal{L}_\rho$, there is a unique linear form $\psi_u \in \mathfrak{a}^*$ tangent to ψ_ρ at u , meaning that $\psi_u \geq \psi_\rho$ and $\psi_u(u) = \psi_\rho(u)$ [13, 14]. Set $V_u := \ker \psi_u$.

Set $X_\rho := \Gamma_\rho \backslash G/M$, and let m_u^{BMS} be the BMS measure on X_ρ associated with ψ_u , as defined in (3.9). Write $a_v := \exp(v)$ for $v \in \mathfrak{a}$.

BMS local mixing. The following is our main result. In this higher-rank cusped setting, the leading local mixing asymptotic is itself new. We also obtain a complete correlation expansion and uniform transverse estimates with an integrable remainder.

Theorem 1.1. *Let $u \in \mathrm{int} \mathcal{L}_\rho$.*

- (1) *There exist $c_u > 0$ and continuous bilinear forms*

$$\mathfrak{c}_{j,u} : \mathcal{C}_c^\infty(X_\rho) \times \mathcal{C}_c^\infty(X_\rho) \rightarrow \mathbb{R}, \quad j \geq 0$$

such that, for every integer $N \geq 0$ and all $f_1, f_2 \in \mathcal{C}_c^\infty(X_\rho)$,

$$t^{\frac{r-1}{2}} \int_{X_\rho} f_1(x a_{tu}) f_2(x) dm_u^{\mathrm{BMS}}(x) = \sum_{j=0}^N \mathfrak{c}_{j,u}(f_1, f_2) t^{-j} + O_{u,f_1,f_2,N}(t^{-N-1})$$

as $t \rightarrow +\infty$, where $\mathfrak{c}_{0,u}(f_1, f_2) = c_u m_u^{\mathrm{BMS}}(f_1) m_u^{\mathrm{BMS}}(f_2)$.

- (2) *There exist a positive definite quadratic form Q_u on V_u and $a > 0$ such that, for every $w \in V_u$ and all $f_1, f_2 \in \mathcal{C}_c^\infty(X_\rho)$,*

$$\begin{aligned} & t^{\frac{r-1}{2}} \int_{X_\rho} f_1(x a_{tu+\sqrt{t}w}) f_2(x) dm_u^{\mathrm{BMS}}(x) \\ &= c_u e^{-Q_u(w)/2} m_u^{\mathrm{BMS}}(f_1) m_u^{\mathrm{BMS}}(f_2) + O_{u,f_1,f_2} \left(t^{-\frac{1}{2}} e^{-a \min\{\|w\|^2, \|w\|\}} \right) \end{aligned}$$

as $t \rightarrow +\infty$, uniformly for $w \in V_u$.

Here $\|\cdot\|$ is any fixed norm on $\mathfrak{a}$. For functions supported in a fixed compact set, the implied constants are controlled by products of suitable $\mathcal{C}^{k,\beta}$ norms; see Theorem 17.1. The normalization $t^{(r-1)/2}$ reflects the transverse dimension $\dim V_u = r-1$. Part (2) gives control

on the whole transverse space, so the asymptotic can be integrated over V_u with an $O(t^{-1/2})$ error. It also gives decay along rays other than $\mathbb{R}_{>0}u$; see Remark 18.1.

For Anosov subgroups, the leading local mixing asymptotic was established by Sambarino [68, 69] and Chow–Sarkar [17], following work of Thirion in special cases [75]. More recently, Guedes Bonthonneau–Lefeuvre–Weich [31] obtained a complete asymptotic expansion under a Diophantine hypothesis on the representation. Our theorem treats representations with parabolic elements and obtains the quantitative estimates from the peripheral geometry.

The rank-one theory provides important precedents for these estimates. Mohammadi–Oh [55] and Edwards–Oh [26] proved matrix-coefficient asymptotics with exponential errors for large critical exponents. Sarkar–Winter [72] proved exponential mixing of BMS frame flows on convex cocompact real hyperbolic manifolds, and Chow–Sarkar [16] extended this to rank-one locally symmetric spaces. In the presence of cusps, Li–Pan [48] proved exponential mixing of geodesic flows on geometrically finite real hyperbolic manifolds, and Li–Pan–Sarkar [49] established the frame-flow version. Khalil [39] proved exponential mixing of geodesic flows on all geometrically finite rank-one locally symmetric spaces. For infinite invariant measures, Dolgopyat–Nándori–Pène [23] obtained expansions to arbitrary order on $\mathbb{Z}^d$ -covers of compact negatively curved manifolds; Flaminio–Ravotti [27] obtained expansions for horocycle flows on $\mathbb{Z}^d$ -covers of compact hyperbolic surfaces.

Orbital counting and Haar local mixing. The local mixing theorem supplies the input for the counting and equidistribution arguments of Edwards–Lee–Oh [25]. We state two consequences here and give the deductions, together with horospherical equidistribution, in Section 18.

Let d be the Riemannian distance on G/K induced by the Killing form, and let δ_ρ be the critical exponent of $\sum_{\gamma \in \Gamma_\rho} e^{-sd(o, \gamma o)}$, which is independent of $o \in G/K$.

Corollary 1.2. *For every $o \in G/K$, there exists $c(\rho, o) > 0$ such that*

$$\#\{x \in \Gamma_\rho o : d(o, x) < T\} \sim c(\rho, o)e^{\delta_\rho T} \quad (T \rightarrow \infty).$$

We also obtain bisector counting asymptotics (Corollary 18.3). The angular distributions are given by Patterson–Sullivan measures. Canary–Zhang–Zhu–Zimmer [11] established counting asymptotics for relatively Anosov groups with cutoffs defined by linear functionals of the Cartan projection. Linearity reduces counting to equidistribution for a reparametrized flow with finite BMS measure, without requiring higher-rank local mixing. Riemannian balls impose a nonlinear cutoff on the full Cartan projection, so determining the leading constant requires information about the transverse distribution of orbit points. Our uniform transverse local mixing estimates provide this information.

We also obtain asymptotics for matrix coefficients of compactly supported M -invariant vectors in $L^2(\Gamma_\rho \backslash G)$. Let $\varrho \in \mathfrak{a}^*$ be the sum of the positive roots, counted with multiplicity, let dx be induced by a fixed Haar measure on G , and let m_u^{BR} and $m_u^{\text{BR}*}$ be the Burger–Roblin measures defined in Section 18.1. Write i for the opposition involution. The transfer argument of Edwards–Lee–Oh [25, Proposition 6.8], extending Roblin’s rank-one result [67], gives the following limit. We also retain a transverse upper bound needed for counting.

Theorem 1.3. *Let $u \in \text{int } \mathcal{L}_\rho$. There exists $\kappa_u > 0$ such that, for every $w \in V_u$ and all $f_1, f_2 \in \mathcal{C}_c(X_\rho)$,*

$$\lim_{t \rightarrow +\infty} t^{\frac{r-1}{2}} e^{(\varrho - \psi_u)(tu + \sqrt{t}w)} \int_{X_\rho} f_1(x a_{tu + \sqrt{t}w}) f_2(x) dx = \kappa_u c_u e^{-Q_u(w)/2} m_{i(u)}^{\text{BR}}(f_1) m_u^{\text{BR}*}(f_2).$$

The convergence is locally uniform in w .

Moreover, there exists $b > 0$ such that, for each compact $K \subset X_\rho$ and all $f_1, f_2 \in \mathcal{C}_c(X_\rho)$ supported in K , the normalized absolute correlation on the left is

$$O_{u,K} \left(\|f_1\|_\infty \|f_2\|_\infty e^{-b \min\{Q_u(w), \sqrt{Q_u(w)}\}} \right)$$

as $t \rightarrow \infty$, uniformly for $w \in V_u$ with $tu + \sqrt{t}w \in \mathfrak{a}^+$.

In rank one, matrix-coefficient asymptotics can also be obtained from a positive L^2 Laplace eigenfunction and a spectral gap [55, 26]. For infinite-volume quotients with no rank-one factors, such an eigenfunction does not exist [24, Theorem 1.1]. The local mixing approach also gives equidistribution of expanding horospheres; the precise statement and measure normalizations appear in Corollary 18.2.

Finally, the Hopf–Tsuji–Sullivan criterion of Burger–Landesberg–Lee–Oh [10] gives the same directional ergodicity threshold as for Anosov subgroups.

Corollary 1.4. *For every $u \in \text{int } \mathcal{L}_\rho$, the flow $\{a_{tu} : t \in \mathbb{R}\}$ on $(X_\rho, m_u^{\text{BMS}})$ is ergodic if and only if $\text{rank}_{\mathbb{R}} G \leq 3$.*

The threshold corresponds to the divergence of $\int_1^\infty t^{-(r-1)/2} dt$, which occurs precisely when $r \leq 3$.

Main ideas of the proof. The support of m_u^{BMS} is a trivial V_u -bundle over a non-compact flow space [45] and the base flow carries a finite invariant measure [42, 6]. For Anosov representations, the corresponding base flow has a compact phase space and admits a finite Markov coding [9]. In our setting, parabolic elements of Γ give rise to arbitrarily long cusp excursions, leading to countably many return branches and an unbounded $\mathfrak{a}$ -valued return cocycle. We construct a countable coding for the right translation by $a_{tu} = \exp(tu)$ on $(X_\rho, m_u^{\text{BMS}})$ and establish uniform distortion estimates and exponential moment bounds for this cocycle.

Our construction builds on the countable coding of Li–Pan [48] for geometrically finite real hyperbolic manifolds. They use Euclidean boundary coordinates, cusp heights in the upper half-space model, and conformality of the boundary action.

On the flag limit set, we develop an intrinsic cusp geometry using a blown-up premetric and a compatible height theory. Separation estimates for parabolic limit points and Patterson–Sullivan mass estimates, together with recurrence of the reparametrized flow, give exponential decay of the residual mass in the coding construction. To control distortion of the full return cocycle, we combine local Busemann estimates on boundary shadows [15] with contraction of the inverse branches.

The geometry of the cusp coding. We begin with the unique linear form $\psi \in \mathfrak{a}^*$ tangent to ψ_ρ at u . Rescaling u , we may assume that $\psi(u) = 1$. Set $\bar{\psi} = (\psi + \psi \circ i)/2$. The symmetric form $\bar{\psi}$ defines the cusp geometry, while ψ defines the Patterson–Sullivan measure and the return times. We use a ρ -equivariant continuous Morse embedding $f : \text{hull } \Lambda(\Gamma) \rightarrow G/K$, whose asymptotic Cartan cone equals $\mathcal{L}_\rho$.

Let $\mathcal{F} := G/P$ be the Furstenberg boundary, where P is the minimal parabolic subgroup determined by A^+ . Let $\Lambda_\rho \subset \mathcal{F}$ be the limit set of Γ_ρ , and let ν_ψ be the Patterson–Sullivan measure on Λ_ρ associated with ψ . Recall the visual premetric on Λ_ρ [47, 22]:

$$d_{\bar{\psi}}(\xi, \eta) := e^{-\bar{\psi}(\mathcal{G}(\xi, \eta))} \quad (\xi \neq \eta \in \Lambda_\rho),$$

where $\mathcal{G}$ is the $\mathfrak{a}$ -valued Gromov product defined in (3.2). Fixing a parabolic limit point $p_0 \in \Lambda_\rho$, we introduce the blown-up premetric on $\Lambda_\rho \setminus \{p_0\}$:

$$D_{\bar{\psi}}(\xi, \eta) := \frac{d_{\bar{\psi}}(\xi, \eta)}{d_{\bar{\psi}}(\xi, p_0)d_{\bar{\psi}}(\eta, p_0)} \quad (\xi \neq \eta \in \Lambda_\rho \setminus \{p_0\}).$$

In the upper half-space model of $\mathbb{H}^n$, with $p_0 = \infty$, this is, up to scaling, a positive power of the Euclidean metric on $\partial\mathbb{H}^n \setminus \{\infty\}$ (Remark 5.6). We prove a uniform quasi-triangle inequality for $D_{\bar{\psi}}$ (Corollary 5.10) and establish geometric and measure estimates that allow it to play the role of the Euclidean metric in the cusp geometry of Λ_ρ .

We define heights $h(p)$ of parabolic limit points (Definition 6.1), relate them with Cartan projections of suitably chosen group representatives, (Lemma 6.5), and prove the uniform separation estimate

$$D_{\bar{\psi}}(p, q) \geq C\sqrt{h(p)h(q)} \quad (p \neq q \in \Lambda_p(\Gamma_\rho) \setminus \{p_0\})$$

for a uniform $C > 0$; see Proposition 6.7. Using the global shadow lemma [43] and Lemma 3.5, we prove doubling and reverse doubling for ν_ψ (Proposition 7.8) and obtain estimates for $D_{\bar{\psi}}$ -balls centered at parabolic limit points (Corollary 7.10).

Choose a compact fundamental domain $\Delta_0 \subset \Lambda_\rho \setminus \{p_0\}$ for the stabilizer of p_0 in Γ_ρ . Starting from Δ_0 , we group parabolic limit points into height layers and successively remove the corresponding $D_{\bar{\psi}}$ -balls. The central geometric step combines annular mass estimates with recurrence of the reparametrized flow: there exist $N \geq 1$ and $\varepsilon > 0$ such that, for every sufficiently large n , stages $n+1, \dots, n+N$ remove at least an ε -fraction of the Patterson–Sullivan mass remaining at stage n . Thus the residual mass decays exponentially; see Section 10.

The geometric partition defines an expanding map on a conull invariant subset of the union of the cusp domains. Taking first returns to Δ_0 gives a full-branch expanding map, and we prove that the induced branches retain exponential summability. Finally, we group a fixed number of successive returns so that the resulting scalar roof is uniformly bounded away from zero; see Sections 11 and 12.2.

The following theorem summarizes the construction and the estimates needed for the spectral analysis.

Theorem 1.5 (Countable Markov coding). *There exist $\Lambda_+ \subset \Delta_0$ with $\nu_\psi(\Delta_0 \setminus \Lambda_+) = 0$, a countable family $\mathcal{H} \subset \Gamma_\rho$, and a map $T : \Lambda_+ \rightarrow \Lambda_+$ such that*

$$\Lambda_+ = \bigsqcup_{\gamma \in \mathcal{H}} \gamma\Lambda_+, \quad T|_{\gamma\Lambda_+} = \gamma^{-1}.$$

Moreover, the family $\mathcal{H}$ freely generates a Zariski dense subsemigroup, and the coding map $\Psi : \mathcal{H}^{\mathbb{N}} \rightarrow \Lambda_+$ is a Hölder homeomorphism conjugating the one-sided shift to T , for the standard symbolic metric and the premetric $D_{\bar{\psi}}$.

For $x \in \gamma\Lambda_+$, define the $\mathfrak{a}$ -valued return cocycle K and the scalar roof function R by

$$K(x) := \beta_x(e, \gamma) + 2\mathcal{G}(Tx, p_0) - 2\mathcal{G}(x, p_0), \quad R(x) := \psi(K(x)),$$

where β is the $\mathfrak{a}$ -valued Busemann map. Then $K \circ \Psi$ is locally Hölder, $R \geq 1$ on Λ_+ , and there exists $\varepsilon_0 > 0$ such that

$$\sum_{\gamma \in \mathcal{H}} \sup_{x \in \Lambda_+} e^{-(1-\varepsilon_0)R(\gamma x)} < \infty, \quad \int_{\Lambda_+} e^{\varepsilon_0\|K\|} d\nu_\psi < \infty.$$

We also prove quantitative non-integrability of the return cocycle K : in every unit direction in $\mathfrak{a}$, we obtain quantitative lower bounds on the variation of differences of projected cocycle sums along suitable inverse branches (Theorem 14.1). The proof combines non-concentration of Λ_ρ at scales adapted to the contraction rates in the simple-root directions (Proposition 14.3) with Zariski density of the semigroup $\langle \mathcal{H} \rangle_+$ (Corollary 13.7) and the peripheral root-balance condition (4.1).

The symbolic model. Let $\mathfrak{m}_\psi := m_u^{\text{BMS}}$ be the BMS measure on X_ρ associated with ψ . The local Hölder regularity of $R \circ \Psi$ and exponential branch summability (Theorem 1.5) allow us to apply the Ruelle–Perron–Frobenius theory for countable shifts [54, 71], obtaining a unique T -invariant Gibbs probability measure ν_+ on Λ_+ for the potential $-R$. This measure is equivalent to $\nu_\psi|_{\Lambda_+}$, with density bounded above and below by positive constants. Using the Gibbs property and the ergodic theorem, we show that

$$u = \frac{\int_{\Lambda_+} K d\nu_+}{\int_{\Lambda_+} R d\nu_+};$$

see Theorem 15.11. We use the coding to construct a suspension model for the flow $\exp(tu)$ on $(X_\rho, \mathfrak{m}_\psi)$ and establish local mixing in this direction.

Let $\Lambda_- \subset \Lambda_\rho$ be a compact neighborhood of p_0 , disjoint from Δ_0 , chosen so that for every $\gamma \in \mathcal{H}$, γ^{-1} preserves Λ_- and contracts $d_{\bar{\psi}}$ uniformly; see Lemma 16.1. Define $\hat{T}$ on $\Lambda_+ \times \Lambda_-$ by

$$\hat{T}(x, x_-) := (Tx, \gamma^{-1}x_-) \quad (x \in \gamma\Lambda_+, \gamma \in \mathcal{H}),$$

and let $\nu^\diamond$ be its invariant probability measure projecting onto ν_+ on Λ_+ . Set

$$K^\diamond(x) := K(x) - R(x)u \in \ker \psi \quad (x \in \Lambda_+),$$

so that $\int_{\Lambda_+} K^\diamond d\nu_+ = 0$. The suspension has coordinates (x, x_-, s, v) with $0 \leq s \leq R(x)$ and $v \in \ker \psi$, subject to the boundary identification

$$(x, x_-, R(x), v) \sim (\hat{T}(x, x_-), 0, v - K^\diamond(x)).$$

The suspension flow increases s at unit speed. We construct a factor map Φ from this suspension to the $\exp(tu)$ -flow on $(X_\rho, \mathfrak{m}_\psi)$ that intertwines translations in the transverse coordinates; see Section 16. Although Φ need not be proper, we prove that for every compact set $Q \subset X_\rho$, there exists $C_Q > 0$ such that

$$\Phi^{-1}(Q) \subset \{(x, x_-, s, v) : 0 \leq s < R(x), \|su + v\| \leq C_Q(R(x) + 1)\};$$

see Theorem 16.7. This support bound controls the growth of the Fourier transforms of pulled-back functions and their frequency derivatives in terms of R . The exponential moment of R makes these bounds integrable. Contraction of the stable fibers reduces the analysis to the expanding system (Λ_+, T) .

The correlation expansion. Fix a sufficiently small Hölder exponent $c > 0$. For $\xi \in \mathfrak{a}$, the normalized twisted transfer operator $\mathcal{L}_\xi$ acts on $\mathcal{F}_c(\Lambda_+, \mathbb{C})$, the space of bounded locally c -Hölder functions, by

$$\mathcal{L}_\xi H(x) = \sum_{\gamma \in \mathcal{H}} e^{-R(\gamma x)} \frac{h(\gamma x)}{h(x)} e^{i\langle \xi, K(\gamma x) \rangle} H(\gamma x), \quad x \in \Lambda_+,$$

where $h > 0$ is an eigenfunction with eigenvalue 1 of the transfer operator associated with the potential $-R$.

For ξ near zero, $\mathcal{L}_\xi$ has a simple leading eigenvalue κ_ξ . We prove that $\xi \mapsto |\kappa_\xi|$ has a negative-definite Hessian at $\xi = 0$; see Proposition 15.20. This Hessian determines the Gaussian dependence of the leading term on the transverse displacement $w \in \ker \psi$.

To analyze correlations, we adapt the spectral and Fourier methods of Dolgopyat–Nándori–Pène [23]. Fourier inversion expresses the BMS correlation as a sum over returns n of frequency integrals, which we analyze using stable projections and $\mathcal{L}_\xi^n$. Outside a fixed small neighborhood of zero, the decay of the Fourier transforms in ξ and of $\mathcal{L}_\xi^n$ in n controls the contribution; see Proposition 17.5. The high-frequency bounds rely on quantitative non-integrability of K (Section 14), which is therefore needed already in our proof of leading local mixing asymptotic. Near $\xi = 0$, we normalize the transformed correlation associated with n returns by κ_ξ^n . Spectral estimates show that the Taylor coefficients at $\xi = 0$ of this normalized expression converge faster than any inverse power of n ; see Lemma 17.16. Combining these limits with the expansion of κ_ξ^n , Fourier inversion gives sums over n of derivatives of a Gaussian determined by the quadratic term of the centered leading eigenvalue. Evaluating these sums as in [23] gives the expansion in Theorem 1.1(1). Uniform estimates for the sums, combined with exponential-moment bounds for the transverse tails, give the uniform remainder, integrable in $w \in \ker \psi$, in Theorem 1.1(2).

Remark 1.6. In the Anosov setting, exponential mixing of the translation flow on the compact based space was established by Delarue–Monclair–Sanders [21] for irreducible projective Anosov subgroups, and by Chow–Sarkar [18] for general Zariski dense Anosov subgroups outside an exceptional cone of parameters. We expect an analogous result for the non-compact base flow considered here. The symbolic, geometric, and spectral tools developed in this paper provide key ingredients toward such an extension.

Organization of the paper. Sections 2 and 3 give the Fuchsian and higher-rank preliminaries. Section 4 states the general theorem and verifies its hypotheses for the three classes above. The remaining sections form four parts.

- Part 1 develops the cusp geometry and measure estimates, constructs the coding, and proves exponential decay and expansion.
- Part 2 establishes the symbolic conjugacy, exponential moments, Hölder regularity, non-arithmeticity, and non-integrability of the return cocycle.
- Part 3 develops the spectral theory and suspension model, proves BMS local mixing, and deduces Haar local mixing, horospherical equidistribution, and orbital counting.

Notation. For nonnegative functions f and g , we write $f \ll g$ if $f \leq Cg$ for some constant $C > 0$, and $f \asymp g$ if both $f \ll g$ and $g \ll f$. For vector-valued functions, $f \approx g$ means that $f - g$ is uniformly bounded. Dependence of an implied constant is indicated by a subscript when needed.

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2. FUCHSIAN GROUPS

We use the upper half-plane model $\mathbb{H}^2 = \{x + iy \in \mathbb{C} : y > 0\}$ with metric $ds^2 = (dx^2 + dy^2)/y^2$, and denote the associated distance by d . Its ideal boundary is $\partial\mathbb{H}^2 = \widehat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$. The orientation-preserving isometry group $\mathrm{PSL}_2(\mathbb{R})$ acts on $\mathbb{H}^2 \cup \widehat{\mathbb{R}}$ by Möbius transformations.

2.1. The hyperbolic plane and its isometries. Set $o := i \in \mathbb{H}^2$. The stabilizer of o is $\mathrm{PSO}(2)$, while the stabilizer of $\infty \in \widehat{\mathbb{R}}$ is the upper triangular subgroup P . Thus $\mathbb{H}^2 \simeq \mathrm{PSL}_2(\mathbb{R})/\mathrm{PSO}(2)$ and $\partial\mathbb{H}^2 \simeq \mathrm{PSL}_2(\mathbb{R})/P$. The action on the upward-pointing unit tangent vector at o identifies $\mathrm{PSL}_2(\mathbb{R})$ with the unit tangent bundle $T^1\mathbb{H}^2$. Under this identification, the geodesic flow for time t is right multiplication by $a_t := \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$.

An element $g \in \mathrm{PSL}_2(\mathbb{R})$ is *loxodromic* if it fixes exactly two points of $\partial\mathbb{H}^2$. One fixed point is attracting and the other is repelling; we denote the attracting fixed point by y_g . An element is *parabolic* if it fixes exactly one point of $\partial\mathbb{H}^2$. Every infinite-order element of a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ is either loxodromic or parabolic.

For $\xi \in \partial\mathbb{H}^2$ and $x, y \in \mathbb{H}^2$, the Busemann function is

$$\beta_\xi(x, y) := \lim_{z \rightarrow \xi} (d(x, z) - d(y, z)).$$

It satisfies the cocycle identity $\beta_\xi(x, z) = \beta_\xi(x, y) + \beta_\xi(y, z)$. A closed horoball based at ξ is a set of the form

$$\mathbf{H}(\xi, r) := \{x \in \mathbb{H}^2 : \beta_\xi(o, x) \geq r\}, \quad r \in \mathbb{R}.$$

For distinct $\xi, \eta \in \mathbb{H}^2 \cup \partial\mathbb{H}^2$, we write $[\xi, \eta] \subset \mathbb{H}^2$ for the geodesic segment, ray, or line joining them. For distinct $\xi, \eta \in \partial\mathbb{H}^2$, their Gromov product based at o is

$$\mathcal{G}(\xi, \eta) := \frac{1}{2}(\beta_\xi(o, x) + \beta_\eta(o, x)), \quad x \in [\xi, \eta],$$

which is independent of the choice of x . We set $\mathcal{G}(\xi, \xi) = +\infty$. We use a visual metric on $\partial\mathbb{H}^2$ based at o , which is bi-Lipschitz equivalent to $(\xi, \eta) \mapsto e^{-\mathcal{G}(\xi, \eta)}$.

For $y \in \mathbb{H}^2$ and $r > 0$, let $B(y, r) := \{z \in \mathbb{H}^2 : d(z, y) < r\}$. For $x, y \in \mathbb{H}^2$, the shadow of $B(y, r)$ seen from x is

$$O_r(x, y) := \{\xi \in \partial\mathbb{H}^2 : [x, \xi] \cap B(y, r) \neq \emptyset\}.$$

The triangle inequality gives

$$\sup_{\xi \in O_r(x, y)} |\beta_\xi(x, y) - d(x, y)| \leq 2r.$$

The Hopf parametrization is the homeomorphism

$$\mathrm{PSL}_2(\mathbb{R}) \rightarrow ((\partial\mathbb{H}^2 \times \partial\mathbb{H}^2) \setminus \mathrm{diag}) \times \mathbb{R}, \quad g \mapsto (g^+, g^-, \beta_{g^+}(o, go)),$$

where $g^+ := g(\infty)$ and $g^- := g(0)$. In these coordinates, left multiplication by $h \in \mathrm{PSL}_2(\mathbb{R})$ takes the form

$$h(\xi, \eta, s) = (h\xi, h\eta, s + \beta_\xi(h^{-1}o, o)),$$

and the geodesic flow is translation: $(\xi, \eta, s)a_t = (\xi, \eta, s + t)$.

A discrete subgroup $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ is called a Fuchsian group. Its limit set $\Lambda(\Gamma) \subset \partial\mathbb{H}^2$ is the set of accumulation points of Γo . In Hopf coordinates, the nonwandering set of the geodesic flow on $T^1(\Gamma \backslash \mathbb{H}^2)$ is identified with

$$(2.1) \quad \Omega_\Gamma := \Gamma \backslash \tilde{\Omega}_\Gamma, \quad \tilde{\Omega}_\Gamma := ((\Lambda(\Gamma) \times \Lambda(\Gamma)) \setminus \mathrm{diag}) \times \mathbb{R} \subset T^1(\mathbb{H}^2).$$

2.2. Finitely generated Fuchsian groups. Throughout the paper, all Fuchsian groups are assumed to be torsion-free. Let $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ be a non-elementary Fuchsian group. A point $\xi \in \partial\mathbb{H}^2$ is a *conical limit point* of Γ if there exist $r > 0$ and a sequence of distinct elements $\gamma_n \in \Gamma$ such that $\xi \in O_r(o, \gamma_n o)$ for every n . A point $\xi \in \partial\mathbb{H}^2$ is a *parabolic limit point* if it is fixed by a parabolic element of Γ . We denote the sets of conical and parabolic limit points by $\Lambda_c(\Gamma)$ and $\Lambda_p(\Gamma)$, respectively.

The convex core of $\Gamma \backslash \mathbb{H}^2$ is $\mathrm{core}(\Gamma \backslash \mathbb{H}^2) := \Gamma \backslash \mathrm{hull} \Lambda(\Gamma)$, where $\mathrm{hull} \Lambda(\Gamma) \subset \mathbb{H}^2$ is the convex hull of $\Lambda(\Gamma)$. We call Γ *geometrically finite* if $\Lambda(\Gamma) = \Lambda_c(\Gamma) \sqcup \Lambda_p(\Gamma)$, or equivalently, its convex core has finite area. A Fuchsian group is geometrically finite if and only if it is finitely generated; see, for example, [52, 53, 8].

Suppose that Γ is geometrically finite and $\Lambda_p(\Gamma) \neq \emptyset$. There exists a Γ -invariant family $\mathcal{H}$ of pairwise disjoint closed horoballs, one based at each parabolic limit point, such that Γ acts cocompactly on $\mathrm{hull} \Lambda(\Gamma) \setminus \bigcup_{H \in \mathcal{H}} \mathrm{int}(H)$. Its compact quotient is called the thick part of the convex core; the complementary components are cusp neighborhoods.

3. RELATIVELY ANOSOV REPRESENTATIONS AND BMS FLOWS

3.1. Symmetric spaces and boundary geometry. Let G be a connected semisimple real linear algebraic group. Fix a Cartan involution θ of G , and use the same notation for its differential on $\mathfrak{g} := \mathrm{Lie}(G)$. Let $K := G^\theta$ be the corresponding maximal compact subgroup. The Cartan decomposition is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigenspaces of θ , respectively.

Choose a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$, and set $A := \exp \mathfrak{a}$. Let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the inner product and norm on $\mathfrak{a}$ induced by the Killing form. Fix a closed positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$, and set $A^+ := \exp \mathfrak{a}^+$. The Cartan decomposition $G = KA^+K$ defines the Cartan projection $\mu : G \rightarrow \mathfrak{a}^+$ by $g \in K(\exp \mu(g))K$.

Let $X := G/K$ be the associated Riemannian symmetric space, with basepoint $o := K$. We equip X with the G -invariant Riemannian metric induced by the Killing form. Its distance satisfies $d(go, ho) = \|\mu(g^{-1}h)\|$ for $g, h \in G$. For $x = go$ and $y = ho$, we also write $\mu(x) := \mu(g)$ and $\mu(x^{-1}y) := \mu(g^{-1}h)$; these are independent of the representatives by K -bi-invariance.

We will use the following estimate on the Cartan projection throughout.

Lemma 3.1 ([2, Lemma 4.6]). *For every compact subset $Q \subset G$, there exists $C = C(Q) > 0$ such that, for all $g \in G$,*

$$\sup_{q_1, q_2 \in Q} \|\mu(q_1 g q_2) - \mu(g)\| \leq C.$$

For $\alpha \in \mathfrak{a}^*$, define

$$\mathfrak{g}_\alpha := \{Z \in \mathfrak{g} : [H, Z] = \alpha(H)Z \text{ for every } H \in \mathfrak{a}\}.$$

The nonzero α for which $\mathfrak{g}_\alpha \neq \{0\}$ form the restricted root system Φ . Let Φ^+ and Π denote the positive and simple restricted roots determined by $\mathfrak{a}^+$. Set

$$\mathfrak{n} := \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, \quad N := \exp \mathfrak{n}, \quad M := Z_K(A).$$

Then $P := MAN$ is the corresponding minimal parabolic subgroup, and $G = KAN$ is the Iwasawa decomposition. The Furstenberg boundary is $\mathcal{F} := G/P$, which we identify with K/M . Let $W := N_K(A)/M$ be the restricted Weyl group. Choose a representative $w_0 \in N_K(A)$ of its longest element. Set $\check{N} := w_0 N w_0^{-1}$, $\check{\mathfrak{n}} := \text{Lie}(\check{N})$, and $\check{P} := w_0 P w_0^{-1} = M A \check{N}$. The opposition involution $\mathfrak{i} := -\text{Ad}_{w_0}|_{\mathfrak{a}}$ preserves $\mathfrak{a}^+$ and satisfies $\mu(g^{-1}) = \mathfrak{i} \mu(g)$. We also write $\mathfrak{i}$ for the involution on $\mathfrak{a}^*$ defined by $\mathfrak{i} \varphi := \varphi \circ \mathfrak{i}$.

Two flags $\xi, \eta \in \mathcal{F}$ are transverse if $(\xi, \eta) \in G \cdot (P, w_0 P)$, where G acts diagonally. We denote the set of transverse pairs by $\mathcal{F}^{(2)} := G \cdot (P, w_0 P)$. For $g \in G$, write $g^+ := gP$ and $g^- := g w_0 P$.

Every $g \in G$ has a unique real Jordan decomposition $g = g_h g_e g_u$ into commuting hyperbolic, elliptic, and unipotent components. Here g_h , g_e , and g_u are conjugate to elements of A^+ , K , and N , respectively. The Jordan projection $\lambda(g) \in \mathfrak{a}^+$ is the unique vector such that g_h is conjugate to $\exp \lambda(g)$. It satisfies $\lambda(g) = \lim_{n \rightarrow \infty} \frac{1}{n} \mu(g^n)$. In particular, $\lambda(g^{-1}) = \mathfrak{i} \lambda(g)$.

An element $g \in G$ is called loxodromic if $\lambda(g) \in \text{int } \mathfrak{a}^+$. Equivalently, $g = h(am)h^{-1}$ for some $a \in \text{int } A^+$, $m \in M$, and $h \in G$. In this case, g^{-1} is also loxodromic, and the unique attracting fixed flags of g and g^{-1} are $y_g := hP$ and $y_{g^{-1}} := h w_0 P$, respectively.

An element g is unipotent if $g = g_u$. We call g weakly unipotent if $g_h = e$ and $g_u \neq e$.

Definition 3.2. For a torsion-free discrete subgroup $\Gamma < \text{PSL}_2(\mathbb{R})$, a representation $\rho : \Gamma \rightarrow G$ is *type-preserving* if it maps loxodromic elements to loxodromic elements and parabolic elements to weakly unipotent elements.

For $g \in G$ and $k \in K$, define the Iwasawa cocycle $\sigma(g, k) \in \mathfrak{a}$ by $gk \in K(\exp \sigma(g, k))N$. For $\xi = kP \in \mathcal{F}$ and $g, h \in G$, the $\mathfrak{a}$ -valued Busemann map is

$$(3.1) \quad \beta_\xi(g, h) := \sigma(g^{-1}, k) - \sigma(h^{-1}, k).$$

This is independent of the choice of k , and satisfies the cocycle identity and the equivariance relation $\beta_{g_0 \xi}(g_0 g, g_0 h) = \beta_\xi(g, h)$ for $g_0 \in G$. It descends to $\mathcal{F} \times X \times X$, where we use the same notation.

Lemma 3.3. *If a weakly unipotent element $g \in G$ fixes $\xi \in \mathcal{F}$, then $\beta_\xi(e, g) = 0$.*

Proof. Choose $k \in K$ with $\xi = kP$ and set $v := k^{-1} g k \in P$. The Levi projection $P \rightarrow MA$ preserves Jordan components. Since g is weakly unipotent, $v \in MN$, so $g^{-1} k = k v^{-1} \in KN$. It follows that $\sigma(g^{-1}, k) = 0$ and $\beta_\xi(e, g) = 0$. $\square$

For $(\xi, \eta) \in \mathcal{F}^{(2)}$, choose $g \in G$ with $(\xi, \eta) = (g^+, g^-)$ and define the $\mathfrak{a}$ -valued Gromov product by

$$(3.2) \quad \mathcal{G}(\xi, \eta) := \frac{1}{2} (\beta_\xi(e, g) + \mathfrak{i} \beta_\eta(e, g)).$$

This does not depend on the choice of g and satisfies that for every $g \in G$ and $(\xi, \eta) \in \mathcal{F}^{(2)}$,

$$(3.3) \quad 2\mathcal{G}(g\xi, g\eta) = 2\mathcal{G}(\xi, \eta) + \beta_{g\xi}(e, g) + \mathfrak{i} \beta_{g\eta}(e, g).$$

The Hopf parametrization identifies G/M with $\mathcal{F}^{(2)} \times \mathfrak{a}$ by

$$gM \mapsto (g^+, g^-, \beta_{g^+}(e, g)).$$

In these coordinates, left multiplication by $h \in G$ takes the form

$$h(\zeta, \eta, v) = (h\zeta, h\eta, v + \beta_\zeta(h^{-1}, e)),$$

while right multiplication by $\exp(w)$, $w \in \mathfrak{a}$, sends (ζ, η, v) to $(\zeta, \eta, v + w)$.

3.2. Limit sets and growth indicators. Let $\Gamma < G$ be a discrete subgroup. We say that a sequence $x_n \in X$ converges to $\xi \in \mathcal{F}$ if $x_n = k_n a_n o$, where $k_n \in K$, $a_n \in A^+$, $k_n P \rightarrow \xi$, and $\alpha(\log a_n) \rightarrow +\infty$ for every $\alpha \in \Pi$.

Definition 3.4. The limit set $\Lambda(\Gamma) \subset \mathcal{F}$ consists of the points that arise as limits of sequences in Γo in the above sense.

The limit cone of Γ is the asymptotic cone of $\mu(\Gamma)$:

$$\mathcal{L}_\Gamma := \left\{ \lim_{n \rightarrow \infty} t_n \mu(\gamma_n) \in \mathfrak{a}^+ : \gamma_n \in \Gamma, t_n \rightarrow 0^+ \right\}.$$

If Γ is Zariski dense, $\mathcal{L}_\Gamma$ is the smallest closed cone containing $\lambda(\Gamma)$; it is convex and has nonempty interior [2].

The growth indicator $\psi_\Gamma : \mathfrak{a} \rightarrow [-\infty, \infty)$ is defined by $\psi_\Gamma(0) = 0$ and, for $u \neq 0$,

$$\psi_\Gamma(u) := \|u\| \inf_{\mathcal{C} \ni u} \tau_{\mathcal{C}},$$

where $\mathcal{C}$ ranges over open cones containing u , and $\tau_{\mathcal{C}}$ is the abscissa of convergence of the series $\sum_{\gamma \in \Gamma, \mu(\gamma) \in \mathcal{C}} e^{-s\|\mu(\gamma)\|}$. Since $\mu(\gamma^{-1}) = i(\mu(\gamma))$, we have $\psi_\Gamma \circ i = \psi_\Gamma$. When Γ is Zariski dense, Quint [63, 65] proved that ψ_Γ is upper semicontinuous and concave, is positive on $\text{int } \mathcal{L}_\Gamma$, and satisfies $\mathcal{L}_\Gamma = \{u \in \mathfrak{a} : \psi_\Gamma(u) \geq 0\}$.

3.3. Weakly unipotent elements and Cartan growth. A unipotent element $u \in G$ is called *regular* if it is contained in a unique minimal real parabolic subgroup of G , and *algebraically regular* if it is contained in a unique Borel subgroup of $G_{\mathbb{C}}$. Every algebraically regular unipotent element is regular: its unique complex Borel subgroup is defined over $\mathbb{R}$, so G is quasi-split and its minimal real parabolic subgroups complexify to Borel subgroups. When G is split, the two notions coincide; see [73].

An $\mathfrak{sl}_2$ -triple in $\mathfrak{g}$ is a triple (E, H, F) , with $E \neq 0$, satisfying $[H, E] = 2E$, $[H, F] = -2F$, and $[E, F] = H$.

Lemma 3.5 (Cartan growth of weakly unipotent elements). *Let $g \in G$ be weakly unipotent. Then $\log g_u$ belongs to an $\mathfrak{sl}_2$ -triple $(\log g_u, H, F)$ in $\mathfrak{g}$ whose semisimple element H is G -conjugate to a vector $v_g \in \mathfrak{a}^+ \setminus \{0\}$. The triple can be chosen so that the associated homomorphism $\Phi : \text{SL}_2(\mathbb{R}) \rightarrow G$ satisfies*

$$\Phi \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = g_u, \quad \Phi \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} = e^{\frac{t}{2}H}, \quad g_e \Phi(h) = \Phi(h) g_e \quad (h \in \text{SL}_2(\mathbb{R})).$$

Moreover, $\mu(\Phi(a_t)) = \frac{t}{2}v_g + O_\Phi(1)$ for $t \geq 0$, and

$$(3.4) \quad \mu(g^n) = \mu(g_u^n) + O_g(1) = (\log |n|)v_g + O_g(1) \quad (|n| \rightarrow \infty).$$

The vector v_g depends only on g_u and satisfies $i(v_g) = v_g$. If g_u is regular, then $v_g \in \text{int } \mathfrak{a}^+$.

Proof. Set $u := g_u$ and $E := \log u$. Since g_e is semisimple, its centralizer is reductive. Moreover, g_e commutes with u , so $E \in \text{Lie } Z_G(g_e)$. Applying the real Jacobson–Morozov theorem in this reductive Lie algebra [7, Chapter VIII, Section 11, no. 2, Proposition 2 and its corollary] gives an $\mathfrak{sl}_2$ -triple (E, H, F) . Its associated homomorphism $\Phi : \text{SL}_2(\mathbb{R}) \rightarrow G$ has image in $Z_G(g_e)$ and satisfies $\Phi \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = u$. Finite-dimensional $\mathfrak{sl}_2$ representation theory, applied to the adjoint action on $\mathfrak{g}$, shows that $\text{ad } H$ is diagonalizable over $\mathbb{R}$ with integer eigenvalues. Thus H is hyperbolic and is G -conjugate to a unique vector $v_g \in \mathfrak{a}^+$. Since $E \neq 0$ and $[H, E] = 2E$, we have $H \neq 0$, and hence $v_g \neq 0$.

Since $a_t = \exp(\frac{t}{2} \operatorname{diag}(1, -1))$, we have $\Phi(a_t) = \exp(\frac{t}{2} H)$. Conjugating H to v_g and applying Lemma 3.1 gives

$$\mu(\Phi(a_t)) = \frac{t}{2} v_g + O_\Phi(1) \quad (t \geq 0).$$

For each integer $n \neq 0$, the Cartan decomposition in $\operatorname{SL}_2(\mathbb{R})$ gives

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = k_n a_{t_n} \ell_n, \quad k_n, \ell_n \in \operatorname{SO}(2), \quad t_n = 2 \log |n| + O(1).$$

Since $\Phi(\operatorname{SO}(2))$ is compact, Lemma 3.1 yields

$$\mu(u^n) = \mu(\Phi(a_{t_n})) + O_\Phi(1) = (\log |n|) v_g + O_\Phi(1).$$

The powers of g_e are relatively compact, and $g^n = g_e^n u^n$. Another application of Lemma 3.1 therefore gives (3.4). Consequently,

$$v_g = \lim_{n \rightarrow \infty} \frac{\mu(g^n)}{\log n} = \lim_{n \rightarrow \infty} \frac{\mu(u^n)}{\log n},$$

so v_g depends only on u and is independent of the chosen triple. Applying (3.4) to positive and negative powers and using $\mu(g^{-n}) = i \mu(g^n)$ gives $i(v_g) = v_g$.

Finally, suppose that u is regular. Let P_H be the real parabolic subgroup whose Lie algebra is the sum of the nonnegative eigenspaces of $\operatorname{ad} H$. Since $[H, E] = 2E$, the element $u = \exp E$ belongs to the unipotent radical of P_H . Every minimal real parabolic subgroup contained in P_H contains this radical and hence contains u . If P_H were not minimal, there would be more than one such subgroup, contradicting the regularity of u . Thus P_H is minimal, which is equivalent to $v_g \in \operatorname{int} \mathfrak{a}^+$. $\square$

3.4. Relatively Anosov representations and Morse embeddings. Let $\Gamma < \operatorname{PSL}_2(\mathbb{R})$ be a non-elementary, finitely generated, Fuchsian group with parabolic elements. For a representation $\rho : \Gamma \rightarrow G$, write $\Gamma_\rho := \rho(\Gamma)$, $\Lambda_\rho := \Lambda(\Gamma_\rho)$, $\mathcal{L}_\rho := \mathcal{L}_{\Gamma_\rho}$, and $\psi_\rho := \psi_{\Gamma_\rho}$.

Definition 3.6. A representation $\rho : \Gamma \rightarrow G$ is *relatively Borel Anosov* if it satisfies the following:

- (1) For every sequence of distinct $\gamma_n \in \Gamma$ and every $\alpha \in \Pi$, $\alpha(\mu(\rho(\gamma_n))) \rightarrow +\infty$.
- (2) Any two distinct points of Λ_ρ are transverse.
- (3) There is a ρ -equivariant homeomorphism $\Lambda(\Gamma) \rightarrow \Lambda_\rho$.

The homeomorphism in (3) is unique and is called the *boundary map* of ρ .

This is the specialization to the present Fuchsian setting of the definition in [14, Section 4.3]. The condition (1) implies that ρ is discrete and faithful. Moreover, relative Anosovness implies that $\rho(\gamma)$ is weakly unipotent for every parabolic $\gamma \in \Gamma$; see [12, Theorems 1.1(3) and B.5].

For brevity, we refer to relatively Borel Anosov representations simply as relatively Anosov representations throughout the paper.

Set $Y := \operatorname{hull} \Lambda(\Gamma) \subset \mathbb{H}^2$ and assume, after conjugating Γ if necessary, that $o = i \in Y$. We also write $o = K \in X$. For $p = gK, q = hK \in X$, write $\mu(p^{-1}q) := \mu(g^{-1}h)$. This is well-defined by the K -bi-invariance of the Cartan projection. For a map $f : Y \rightarrow X$, let $\mathcal{L}_f$ be the asymptotic cone of $\{\mu(f(x)^{-1}f(y)) : x, y \in Y\}$.

Definition 3.7 (Morse embedding). A map $f : Y \rightarrow X$ is a *Morse embedding* if it is a quasi-isometric embedding¹ and $\mathcal{L}_f \subset \text{int } \mathfrak{a}^+ \cup \{0\}$. We call $\mathcal{L}_f$ its *Morse limit cone*.

Equivalently, by the Kapovich–Leeb–Porti Morse lemma [36, Theorem 5.16 and Corollary 5.28], the images of geodesic segments in Y are Π -Morse quasigeodesics with uniform constants; see also [22, Section 4].

For $G = \text{SL}_n(\mathbb{R})$, the existence of an equivariant Morse embedding follows from [78, Corollary 1.14] and [79, Proposition 11.3]. We construct a continuous embedding for general G and identify its Morse limit cone with $\mathcal{L}_\rho$.

Proposition 3.8 (Equivariant Morse embedding). *Let $\rho : \Gamma \rightarrow G$ be a Zariski dense, relatively Anosov representation. Then there exists a continuous ρ -equivariant Morse embedding $f : Y \rightarrow X$ such that $f(o) = o$ and $\mathcal{L}_f = \mathcal{L}_\rho$. Moreover, f extends continuously to $Y \cup \Lambda(\Gamma)$, and its restriction to $\Lambda(\Gamma)$ is the boundary map of ρ .*

Proof. Choose representatives $\xi_0, \dots, \xi_\ell$ of the Γ -orbits in $\Lambda_p(\Gamma)$, and set $\Gamma_i := \text{Stab}_\Gamma(\xi_i)$. Choose a Γ -invariant family $\mathcal{H}$ of pairwise disjoint closed horoballs based at $\Lambda_p(\Gamma)$, with $\mathbf{H}_i$ based at ξ_i , all avoiding o . Let $Y_{\text{th}} := Y - \bigcup_{\mathbf{H} \in \mathcal{H}} \text{int}(\mathbf{H})$; then $\Gamma \backslash Y_{\text{th}}$ is compact.

Step 1: Construction of f . Write $n_s := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$. For each i , choose a primitive generator $h_i \in \Gamma_i$ and $b_i \in \text{PSL}_2(\mathbb{R})$ such that $b_i(\infty) = \xi_i$ and $b_i n_1 b_i^{-1} = h_i$. By relative Anosovness, $\rho(h_i)$ is weakly unipotent. Let $g_i := \rho(h_i)$, $e_i := (g_i)_e$ and $u_i := (g_i)_u$. By Lemma 3.5, choose a homomorphism $\Phi_i : \text{SL}_2(\mathbb{R}) \rightarrow G$ with $\Phi_i(n_1) = u_i$ whose image commutes with e_i , and let $v_i := v_{g_i} \in \mathfrak{a}^+ \setminus \{0\}$ be the associated vector. For every $\alpha \in \Pi$, relative Anosovness and (3.4) give

$$\alpha(\mu(g_i^n)) = (\log n)\alpha(v_i) + O(1) \rightarrow +\infty.$$

Consequently, $\alpha(v_i) > 0$ for every $\alpha \in \Pi$, so $v_i \in \text{int } \mathfrak{a}^+$. Moreover, since $g_i^n \in \Gamma_\rho$, (3.4) gives $v_i \in \mathcal{L}_\rho$.

Since the compact subgroups $\overline{\langle e_i \rangle}$ and $\Phi_i(\text{SO}(2))$ commute, we can choose $o_i \in X$ fixed by both subgroups, and choose $q_i \in G$ with $q_i o = o_i$. Write $\mathbf{H}_i = b_i\{z : \text{Im } z \geq e^{T_i}\}$ for some $T_i \in \mathbb{R}$ and define

$$f_i(b_i n_s a_t i) := \Phi_i(n_s a_t) o_i \quad (s \in \mathbb{R}, t \geq T_i).$$

Since e_i fixes o_i and commutes with $\Phi_i(\text{SL}_2(\mathbb{R}))$, it fixes the image of f_i pointwise. Thus, for $z = b_i n_s a_t i$,

$$f_i(h_i z) = \Phi_i(n_{s+1} a_t) o_i = u_i f_i(z) = e_i u_i f_i(z) = \rho(h_i) f_i(z).$$

Since $\Gamma_i = \langle h_i \rangle$, the map f_i is $\rho|_{\Gamma_i}$ -equivariant. Restricting to $\mathbf{H}_i \cap Y$, we extend to all cusp regions in Y by $f(\gamma z) := \rho(\gamma) f_i(z)$. Precise invariance of the horoballs makes this well-defined. We extend f continuously and ρ -equivariantly over Y_{th} , retaining its cusp-boundary values and prescribing $f(\gamma o) = \rho(\gamma) o$. To do so, lift a finite triangulation of $\Gamma \backslash Y_{\text{th}}$ compatible with the cusp boundaries and having the image of o as a vertex. Choose images for representatives of the remaining vertex orbits and extend equivariantly. Extend over representatives of the edge orbits using path connectedness of X , and over representatives of the triangle orbits using simple connectedness. The action is free, so these extensions propagate equivariantly and agree on common faces. The lifted triangulation and the family of cusp regions are locally finite, so the glueing lemma gives a continuous map on Y .

¹A map $f : (Z_1, d_1) \rightarrow (Z_2, d_2)$ is a *quasi-isometric embedding* if there exist $C_1 \geq 1$ and $C_2 \geq 0$ such that $C_1^{-1} d_1(z, z') - C_2 \leq d_2(f(z), f(z')) \leq C_1 d_1(z, z') + C_2$ ($z, z' \in Z_1$).

Step 2: Quasi-isometry and root estimates. We first claim that, whenever $x, y \in Y$ lie in a horoball γH_i , with $\gamma \in \Gamma$,

$$(3.5) \quad \mu(f(x)^{-1}f(y)) = \frac{1}{2}d_{\mathbb{H}^2}(x, y)v_i + O(1),$$

with an error uniform in x, y, γ , and i . By equivariance, it suffices to consider $x, y \in H_i \cap Y$. Write $x = b_i g o$ and $y = b_i h o$, where $g = n_s a_t, h = n_{s'} a_{t'} \in \mathrm{SL}_2(\mathbb{R})$. If $r = d_{\mathbb{H}^2}(x, y)$, then $g^{-1}h \in \mathrm{SO}(2) a_r \mathrm{SO}(2)$. By Lemmas 3.1 and 3.5, $\mu(\Phi_i(g^{-1}h)) = \frac{r}{2}v_i + O(1)$. As $f_i(x) = \Phi_i(g)q_i o$ and $f_i(y) = \Phi_i(h)q_i o$, we have $\mu(f_i(x)^{-1}f_i(y)) = \mu(q_i^{-1}\Phi_i(g^{-1}h)q_i) = \frac{r}{2}v_i + O(1)$. Choose a compact fundamental set $D \subset Y_{\mathrm{th}}$ for the Γ -action. Suppose that $d_{\mathbb{H}^2}(x, y) \leq 1$. If x, y lie in the same horoball in $\mathcal{H}$, the preceding estimate bounds $d_X(f(x), f(y))$ uniformly. Otherwise, $[x, y]$ meets Y_{th} . After translating by an element of Γ , both endpoints lie in the closed 1-neighborhood of D . This neighborhood is compact in Y , so continuity and equivariance again give a uniform bound on $d_X(f(x), f(y))$. Subdividing an arbitrary geodesic segment into pieces of length at most 1 now gives

$$(3.6) \quad d_X(f(x), f(y)) \leq A d_{\mathbb{H}^2}(x, y) + B \quad (x, y \in Y)$$

for some $A, B > 0$.

For the lower bound, we will use the relative Anosov hypothesis to prove

$$(3.7) \quad \alpha(\mu(f(x)^{-1}f(y))) \geq a d_{\mathbb{H}^2}(x, y) - b \quad (x, y \in Y, \alpha \in \Pi)$$

for some $a, b > 0$. Fix $\alpha \in \Pi$, and choose an irreducible proximal Tits representation $\tau_\alpha : G \rightarrow \mathrm{SL}(V_\alpha)$ whose highest weight is a positive multiple of the fundamental weight ω_α [76]. Equip V_α with a Euclidean inner product for which $\tau_\alpha(K)$ acts orthogonally and $\tau_\alpha(A)$ acts by positive self-adjoint operators. Let $\|\cdot\|_\alpha$ denote the associated norm. For $x \in Y$, choose $g_x \in G$ with $g_x o = f(x)$ and set $\|v\|_{\alpha, x} := \|\tau_\alpha(g_x^{-1})v\|_\alpha$ for $v \in V_\alpha$. Since $\tau_\alpha(K)$ preserves $\|\cdot\|_\alpha$, this definition is independent of the choice of g_x . These norms form a continuous, $\tau_\alpha \circ \rho$ -equivariant family. This family is canonical in the sense of [12, Section 3.1]. Set $|v|_{\alpha, i} := \|\tau_\alpha(q_i^{-1})v\|_\alpha$. Since e_i and $\Phi_i(\mathrm{SO}(2))$ fix $q_i o$, this norm is invariant under their images by τ_α . On $H_i \cap Y$, our construction gives

$$\|v\|_{\alpha, x} = |\tau_\alpha(\Phi_i(n_s a_t)^{-1})v|_{\alpha, i}, \quad x = b_i n_s a_t i \in H_i \cap Y.$$

In the coordinates given by b_i , this is the canonical family associated with the cusp homomorphism $\tau_\alpha \circ \Phi_i$. Thus evaluating $\|\cdot\|_{\alpha, x}$ at the basepoints of geodesics gives a canonical family on the geodesic-flow space.

Since $\rho(\Gamma)$ is Zariski dense, $\tau_\alpha \circ \rho$ is irreducible, and the ρ -boundary map induces a continuous, transverse, equivariant line-hyperplane limit map. By [12, Corollary 6.3(1)], $\tau_\alpha \circ \rho$ is relatively P_1 -Anosov, where P_1 denotes the stabilizer of a line in V_α . Let $E_\alpha(\xi)$ and $F_\alpha(\xi)$ denote the line and hyperplane components of its limit map. By [12, Corollary 6.2], there exist $C_\alpha \geq 1$ and $c_\alpha > 0$ such that, for every unit-speed geodesic $\ell : \mathbb{R} \rightarrow Y$ with endpoints $\xi_\pm \in \Lambda(\Gamma)$,

$$\frac{\|v\|_{\alpha, \ell(t)}}{\|w\|_{\alpha, \ell(t)}} \leq C_\alpha e^{-c_\alpha(t-s)} \frac{\|v\|_{\alpha, \ell(s)}}{\|w\|_{\alpha, \ell(s)}} \quad (s \leq t)$$

for all nonzero $v \in E_\alpha(\xi_+)$ and $w \in F_\alpha(\xi_-)$. Thus, when passing from the norm at $\ell(t)$ to the norm at $\ell(s)$, the stretching factor on $E_\alpha(\xi_+)$ exceeds every stretching factor on $F_\alpha(\xi_-)$

by at least $C_\alpha^{-1}e^{c_\alpha(t-s)}$. Writing σ_j for the j -th singular value, we obtain

$$\alpha(\mu(f(\ell(s))^{-1}f(\ell(t)))) = \log \frac{\sigma_1(\tau_\alpha(g_{\ell(s)}^{-1}g_{\ell(t)}))}{\sigma_2(\tau_\alpha(g_{\ell(s)}^{-1}g_{\ell(t)}))} \geq c_\alpha(t-s) - \log C_\alpha.$$

Any two points $x, y \in Y$ lie within a uniform distance D of points $\ell(s), \ell(t)$ on a common complete geodesic with endpoints in $\Lambda(\Gamma)$; see [79, Proposition 3.5 and Lemma B.3]. Orienting ℓ so that $s \leq t$, we have $t-s \geq d_{\mathbb{H}^2}(x, y) - 2D$. By (3.6), the corresponding image points are also uniformly close. Applying Lemma 3.1 and taking uniform constants over the finite set Π proves (3.7). Since $\alpha(v) \leq \|\alpha\|_* \|v\|$, this implies $d_X(f(x), f(y)) \geq a'd_{\mathbb{H}^2}(x, y) - b'$ for some $a', b' > 0$. Together with (3.6), this proves that f is a quasi-isometric embedding.

Step 3: The Morse property and the limit cone. Combining (3.7) with (3.6) gives

$$\alpha(\mu(f(x)^{-1}f(y))) \geq \frac{a}{A} \|\mu(f(x)^{-1}f(y))\| - b - \frac{aB}{A}.$$

This implies $\mathcal{L}_f \subset \text{int } \mathfrak{a}^+ \cup \{0\}$, and hence f is a Morse embedding.

Equivariance and $f(o) = o$ imply that $\Gamma_\rho(o) = f(\Gamma o) \subset f(Y)$, and hence $\mathcal{L}_\rho \subset \mathcal{L}_f$. For the reverse inclusion, let $\mathcal{C}$ be any closed convex cone such that $\mathcal{L}_\rho \setminus \{0\} \subset \text{int } \mathcal{C}$ and $\mathcal{C} \subset \text{int } \mathfrak{a}^+ \cup \{0\}$. Then $\mu(\Gamma_\rho) \setminus \mathcal{C}$ is bounded. If x, y lie in a horoball γH_i , the cusp estimate and $v_i \in \mathcal{L}_\rho$ show that $\mu(f(x)^{-1}f(y))$ lies in a uniform neighborhood of $\mathcal{C}$. If x and y do not lie in a common horoball, then $[x, y]$ meets Y_{th} . Let x' and y' be its first and last intersection points with Y_{th} , in the orientation from x to y . Since f is Morse, coarse additivity [22, Corollary 4.12] gives

$$\mu(f(x)^{-1}f(y)) = \mu(f(x)^{-1}f(x')) + \mu(f(x')^{-1}f(y')) + \mu(f(y')^{-1}f(y)) + O(1).$$

The first and third terms each lie within a uniform distance of one of the rays $\mathbb{R}_{\geq 0}v_i$, all of which lie in $\mathcal{C}$. By cocompactness of Γ on Y_{th} , (3.6), equivariance, and Lemma 3.1, the middle term lies within a uniform distance of $\mu(\Gamma_\rho)$ and hence of $\mathcal{C}$.

Since $\mathcal{C}$ is convex, the sum lies in a uniform neighborhood of $\mathcal{C}$. This implies $\mathcal{L}_f \subset \mathcal{C}$. As $\mathcal{C}$ can be chosen arbitrarily close to $\mathcal{L}_\rho$, we conclude that $\mathcal{L}_f = \mathcal{L}_\rho$. Finally, the boundary extension theorem [36, Theorem 1.4] gives a continuous extension of f whose boundary restriction is a ρ -equivariant homeomorphism $\Lambda(\Gamma) \rightarrow \Lambda_\rho$. By uniqueness, this is the boundary map of ρ . $\square$

Conversely, an equivariant Morse embedding implies relative Anosovness by [35, Theorem 8.3]. Thus, for Zariski dense representations of Γ , relative Anosovness is equivalent to the existence of an equivariant Morse embedding.

3.5. Geometric estimates. For the remainder of this section, fix a Zariski dense, relatively Anosov representation $\rho : \Gamma \rightarrow G$ and a continuous ρ -equivariant Morse embedding $f : Y \rightarrow X$ supplied by Proposition 3.8, so that $f(o) = o$ and $\mathcal{L}_f = \mathcal{L}_\rho$. We also denote its boundary extension $\Lambda(\Gamma) \rightarrow \Lambda_\rho$ by f . Let $\Lambda_p(\Gamma_\rho) := f(\Lambda_p(\Gamma))$. The Morse property and equivariance give

$$\mathcal{L}_\rho \subset \mathcal{L}_f \subset \text{int } \mathfrak{a}^+ \cup \{0\}.$$

We repeatedly use the following:

Lemma 3.9. *For every $\varphi \in \mathfrak{a}^*$ positive on $\mathcal{L}_\rho \setminus \{0\}$, there exist $a \geq 1$ and $b \geq 0$ such that, for all $x, y \in Y$,*

$$a^{-1}d(x, y) - b \leq \varphi(\mu(f(x)^{-1}f(y))) \leq a d(x, y) + b.$$

Proof. Since $\mathcal{L}_f = \mathcal{L}_\rho$, positivity of φ and compactness of the unit directions in $\mathcal{L}_f$ give $c > 1$ and $C \geq 0$ such that

$$c^{-1} \|\mu(f(x)^{-1}f(y))\| - C \leq \varphi(\mu(f(x)^{-1}f(y))) \leq c \|\mu(f(x)^{-1}f(y))\|$$

for all $x, y \in Y$. Since $\|\mu(f(x)^{-1}f(y))\| = d_X(f(x), f(y))$, the assertion now follows from the quasi-isometry bounds for f . $\square$

Lemma 3.10 (Coarse additivity, [22, Corollary 4.12]). *There exists $D_0 > 0$ such that, for all $x, z \in Y$ and $y \in [x, z]$,*

$$\|\mu(f(x)^{-1}f(z)) - \mu(f(x)^{-1}f(y)) - \mu(f(y)^{-1}f(z))\| \leq D_0.$$

We distinguish the hyperbolic shadows $O_r(x, y)$ from shadows in the symmetric space $X = G/K$. Following [47, Section 5], for $x, y \in X$ and $r > 0$, define

$$\mathcal{O}_r(x, y) := \{gkP \in \mathcal{F} : k \in K, \quad gk(\text{int } A^+)o \cap B_X(y, r) \neq \emptyset\},$$

where $x = go$ and $B_X(y, r)$ is the open ball in X centered at y . This definition is independent of the choice of g with $go = x$. Thus $\mathcal{O}_r(x, y)$ consists of the flags whose Weyl chambers based at x meet $B_X(y, r)$. The shadows are G -equivariant: $h\mathcal{O}_r(x, y) = \mathcal{O}_r(hx, hy)$ for $h \in G$.

Lemma 3.11 (Busemann–Cartan comparison). *There exists $\kappa > 0$ such that, for every $r > 0$, $x, y \in X$, and $\xi \in \mathcal{O}_r(x, y)$,*

$$(3.8) \quad \|\beta_\xi(x, y) - \mu(x^{-1}y)\| \leq \kappa r.$$

Moreover, for every $r > 0$, there exists $C = C(r) > 0$ such that, for all $x, y \in Y$ and $\zeta \in O_r(x, y) \cap \Lambda(\Gamma)$,

$$\|\beta_{f(\zeta)}(f(x), f(y)) - \mu(f(x)^{-1}f(y))\| \leq C.$$

Proof. The first assertion is [47, Lemma 5.7]. For the second, the Morse property implies that $f([x, \zeta])$ lies in a uniform neighborhood of the Weyl chamber based at $f(x)$ with endpoint $f(\zeta)$. If $\zeta \in O_r(x, y)$, then $[x, \zeta]$ meets $B(y, r)$. Since f is a quasi-isometric embedding, there is a uniform $c_0 \geq 1$ such that $f(\zeta) \in \mathcal{O}_{c_0(r+1)}(f(x), f(y))$. Applying (3.8) proves the claim. $\square$

The two types of shadows are comparable after a uniform change of radius.

Lemma 3.12 (Comparison of shadows). *There exists $c \geq 1$ such that, for all $x, y \in Y$ and $r \geq c$,*

$$f(O_{c^{-1}r}(x, y) \cap \Lambda(\Gamma)) \subset \mathcal{O}_r(f(x), f(y)) \cap \Lambda_\rho \subset f(O_{cr}(x, y) \cap \Lambda(\Gamma)).$$

Proof. The Morse argument in the proof of Lemma 3.11 gives the first inclusion for sufficiently large c ; compare [47, Proposition 5.14]. Suppose that $f(\zeta) \in \mathcal{O}_r(f(x), f(y))$ for $\zeta \in \Lambda(\Gamma)$. Let z be the nearest-point projection of y to $[x, \zeta]$. Hyperbolicity implies that z lies within a uniform distance of both $[x, y]$ and $[y, \zeta]$. Set $u := \mu(f(x)^{-1}f(z))$ and $v := \mu(f(z)^{-1}f(y))$. By Lemmas 3.1 and 3.10,

$$\mu(f(x)^{-1}f(y)) = u + v + O(1).$$

By Lemma 3.11 and the cocycle identity,

$$\beta_{f(\zeta)}(f(x), f(y)) = \beta_{f(\zeta)}(f(x), f(z)) - \beta_{f(\zeta)}(f(y), f(z)) = u - v + O(1).$$

All error terms are uniform in x, y, ζ . It follows from (3.8) that $\|v + \mathbf{i}v\| \leq \kappa r + O(1)$. Since $v, \mathbf{i}v \in \mathfrak{a}^+$, there is a constant $c_1 > 0$ such that $\|v\| \leq c_1\|v + \mathbf{i}v\|$ for all $v \in \mathfrak{a}^+$. The quasi-isometric embedding property of f therefore gives $d(y, z) \leq c_1(r + 1)$, by increasing c_1 if necessary. Increasing c so that $c > 2c_1 + 1$, we obtain, for $r \geq c$, $d(y, z) \leq c_1(r + 1) < cr$. Thus $\zeta \in O_{cr}(x, y)$, proving the second inclusion. $\square$

The proof of [47, Lemma 6.6] uses the Morse property and hyperbolicity of the domain, and therefore applies to f .

Lemma 3.13. *There exists $C > 0$ such that, for any distinct $\zeta, \eta \in \Lambda(\Gamma)$, if z is the nearest-point projection of o to $[\zeta, \eta]$, then*

$$\left\| \mathcal{G}(f(\zeta), f(\eta)) - \frac{1}{2}(\mu(f(z)) + \mathbf{i}\mu(f(z))) \right\| \leq C.$$

The following estimate relates the Gromov product to the Cartan projection along peripheral orbits.

Lemma 3.14 ([43, Proposition 4.3]). *Let $p \in \Lambda_{\mathbb{P}}(\Gamma_{\rho})$, and set $\mathbf{P} := \text{Stab}_{\Gamma_{\rho}}(p)$. For every compact subset $Q \subset \Lambda_{\rho} \setminus \{p\}$, there exists $C = C(p, Q) > 0$ such that*

$$\sup_{\eta \in Q, g \in \mathbf{P}} \left\| \mathcal{G}(p, g\eta) - \frac{1}{2}\mu(g) \right\| \leq C.$$

Finally, we record a local regularity estimate for the $\mathfrak{a}$ -valued Busemann cocycle. Fix a Riemannian distance d_{Riem} on $\mathcal{F}$.

Lemma 3.15 (Local regularity). *For every $r > 0$, there exists $C = C(r) > 0$ such that, for all $g \in \Gamma_{\rho}$ and $\xi, \eta \in \mathcal{O}_r(go, o) \cap \Lambda_{\rho}$,*

$$\|\beta_{\xi}(e, g) - \beta_{\eta}(e, g)\| \leq C d_{\text{Riem}}(\xi, \eta).$$

Proof. Enlarging r if necessary, we may apply Lemma 3.12. Therefore the estimates in [15, Propositions 4.7, 5.3, and 5.11] imply the assertion outside a finite subset of Γ_{ρ} . For each fixed g , the map $\xi \mapsto \beta_{\xi}(e, g)$ is smooth on the compact manifold $\mathcal{F}$, and hence Lipschitz. Enlarging C includes the finitely many remaining elements. $\square$

The action of Γ_{ρ} on Λ_{ρ} is a convergence group action, being conjugate to the action of Γ on $\Lambda(\Gamma)$. Together with Lemma 3.12, this gives the following convergence properties of ambient shadows.

Lemma 3.16 (Convergence of shadows). *Every sequence of pairwise distinct $g_n \in \Gamma_{\rho}$ has a subsequence and points $\xi_+, \xi_- \in \Lambda_{\rho}$ with the following properties:*

- (1) $g_n \eta \rightarrow \xi_+$ locally uniformly on $\Lambda_{\rho} \setminus \{\xi_-\}$.
- (2) For every $r > 0$, if $\eta_n \in \mathcal{O}_r(o, g_n o) \cap \Lambda_{\rho}$, then $\eta_n \rightarrow \xi_+$.
- (3) If $r_n \rightarrow \infty$ and $\eta_n \in \Lambda_{\rho} \setminus \mathcal{O}_{r_n}(g_n o, o)$, then $\eta_n \rightarrow \xi_+$.

The same assertions hold for g_n^{-1} with ξ_+ and ξ_- interchanged. In particular, for each fixed $r > 0$,

$$\text{diam}_{d_{\text{Riem}}}(\mathcal{O}_r(o, go) \cap \Lambda_{\rho}) \rightarrow 0 \quad (g \rightarrow \infty \text{ in } \Gamma_{\rho}).$$

Proof. Put $\widehat{g}_n := \rho^{-1}(g_n)$. After passing to a subsequence, $\widehat{g}_n o \rightarrow \zeta_+$ and $\widehat{g}_n^{-1} o \rightarrow \zeta_-$ in $Y \cup \Lambda(\Gamma)$. Set $\xi_{\pm} := f(\zeta_{\pm})$. The convergence action on $\partial \mathbb{H}^2$ and equivariance of f give (1). Let c be the constant in Lemma 3.12. For (2), enlarge r to at least c ; the preimages of η_n then lie in hyperbolic shadows of a fixed radius centered at $\widehat{g}_n o$, and hence tend to ζ_+ . For

(3), the first inclusion of that lemma gives $f^{-1}(\eta_n) \notin O_{r_n/c}(\widehat{g}_n o, o)$ for all large n . Thus the rays from $\widehat{g}_n o$ to $f^{-1}(\eta_n)$ avoid balls about o whose radii tend to infinity, so $f^{-1}(\eta_n) \rightarrow \zeta_+$. Continuity of f proves (2) and (3). The same argument applies to the inverse sequence. The diameter assertion follows from (2) by a subsequence argument. $\square$

3.6. Patterson–Sullivan and BMS measures. For $\psi \in \mathfrak{a}^*$, a Borel probability measure ν on $\mathcal{F}$ is called (Γ_ρ, ψ) -conformal if, for every $\gamma \in \Gamma_\rho$,

$$\frac{d\gamma_*\nu}{d\nu}(\xi) = e^{\psi(\beta_\xi(e, \gamma))} \quad \text{for } \nu\text{-almost every } \xi \in \mathcal{F}.$$

If ν is supported on Λ_ρ , we call it a (Γ_ρ, ψ) -Patterson–Sullivan measure.

For $\psi \in \mathfrak{a}^*$, let $\delta_\psi = \delta_\psi(\Gamma_\rho)$ be the abscissa of convergence of $\sum_{\gamma \in \Gamma_\rho} e^{-s\psi(\mu(\gamma))}$.

Theorem 3.17 ([14]). *If $\delta_\psi = 1$, there exists a unique (Γ_ρ, ψ) -Patterson–Sullivan measure ν_ψ . Moreover, ν_ψ is Γ_ρ -ergodic and atomless.*

Recall that ψ is tangent to ψ_ρ if $\psi \geq \psi_\rho$ and equality holds at some nonzero vector. In the present setting, this is equivalent to $\delta_\psi = 1$; see [40, 14]. Moreover, ψ_ρ is strictly concave between non-collinear vectors in $\mathcal{L}_\rho$ [66, 46]. The parametrization of tangent forms in [46, Corollary 1.3], together with the preceding theorem, gives the following.

Corollary 3.18. *For every $u \in \text{int } \mathcal{L}_\rho$, there exists a unique $\psi_u \in \mathfrak{a}^*$ tangent to ψ_ρ at u . Let ν_u be its Patterson–Sullivan measure from Theorem 3.17. The assignments $u \mapsto \psi_u \mapsto \nu_u$ give bijections among unit vectors in $\text{int } \mathcal{L}_\rho$, tangent linear forms, and Patterson–Sullivan measures on Λ_ρ .*

Using the Hopf coordinates, we set

$$\Lambda_\rho^{(2)} := (\Lambda_\rho \times \Lambda_\rho) \cap \mathcal{F}^{(2)}, \quad \Omega_\rho := \Gamma_\rho \backslash (\Lambda_\rho^{(2)} \times \mathfrak{a}) \subset X_\rho := \Gamma_\rho \backslash G/M.$$

Fix a Lebesgue measure $\text{Leb}_\mathfrak{a}$ on $\mathfrak{a}$. For a tangent form ψ , the form $\psi \circ i$ is also tangent, since ψ_ρ is invariant under i . The measure

$$(3.9) \quad d\tilde{\mathfrak{m}}_\psi(\zeta, \eta, v) = e^{2\psi(\mathcal{G}(\zeta, \eta))} d\nu_\psi(\zeta) d\nu_{\psi \circ i}(\eta) d\text{Leb}_\mathfrak{a}(v)$$

is Γ_ρ -invariant and induces an A -invariant Radon measure $\mathfrak{m}_\psi$ on X_ρ , supported on Ω_ρ . We call $\mathfrak{m}_\psi$ the Bowen–Margulis–Sullivan measure associated with ψ .

3.7. The base flow and reparametrization. Let $\psi \in \mathfrak{a}^*$ be tangent to ψ_ρ and positive on $\mathcal{L}_f \setminus \{0\}$. Then $\delta_\psi = 1$, and ψ is proper on $\mu(\Gamma_\rho)$ by [14, Theorem 10.1]. Consider the action of Γ_ρ on $\tilde{\Omega}_\psi := \Lambda_\rho^{(2)} \times \mathbb{R}$ given by

$$g(\xi, \eta, s) = (g\xi, g\eta, s + \psi(\beta_\xi(g^{-1}, e))).$$

This action is properly discontinuous, and hence the quotient

$$\Omega_\psi := \Gamma_\rho \backslash \tilde{\Omega}_\psi$$

is locally compact and Hausdorff [45]. Translation in the $\mathbb{R}$ -coordinate defines a flow on Ω_ψ , which we also denote by a_t . Let m_ψ be the translation-invariant Radon measure on Ω_ψ induced by

$$d\tilde{m}_\psi(\xi, \eta, s) = e^{2\psi(\mathcal{G}(\xi, \eta))} d\nu_\psi(\xi) d\nu_{\psi \circ i}(\eta) ds.$$

Theorem 3.19 (Finiteness and mixing [42, 6]). *The measure m_ψ is finite and nonzero, and the translation flow on (Ω_ψ, m_ψ) is strongly mixing. In particular, it is ergodic.*

Until the end of Part 3, we replace m_ψ and $\mathbf{m}_\psi$ simultaneously by

$$(3.10) \quad m_\psi(\Omega_\psi)^{-1}m_\psi \quad \text{and} \quad m_\psi(\Omega_\psi)^{-1}\mathbf{m}_\psi,$$

respectively, retaining the same notation. The probability measure m_ψ is the unique invariant probability measure of maximal entropy [42]. The projection $(\xi, \eta, v) \mapsto (\xi, \eta, \psi(v))$ is Γ_ρ -equivariant and induces a trivial $\ker \psi$ -bundle $\pi_\psi : \Omega_\rho \rightarrow \Omega_\psi$; see [45, Section 10]. Let $\text{Leb}_{\ker \psi}$ be the unique Lebesgue measure on $\ker \psi$ such that, for one (and hence every) vector $u \in \mathfrak{a}$ satisfying $\psi(u) = 1$, the map $\mathbb{R} \times \ker \psi \rightarrow \mathfrak{a}$, $(s, v) \mapsto su + v$, sends $ds d\text{Leb}_{\ker \psi}$ to $d\text{Leb}_{\mathfrak{a}}$. Under a trivialization respecting translations by $\ker \psi$, we have

$$(3.11) \quad \mathbf{m}_\psi = m_\psi \otimes \text{Leb}_{\ker \psi}.$$

When $\psi = \psi_u$, we also write $m_u^{\text{BMS}} := \mathbf{m}_{\psi_u}$, subject to the temporary normalization in (3.10). We use the same notation for the lift to $\Gamma_\rho \backslash G$ obtained using Haar probability measure on M .

The following is the hyperbolic-plane analogue of the reparametrization construction in [42, Section 6]. We give a direct proof using the continuous Morse embedding f .

Theorem 3.20 (Reparametrization). *There exist a ρ -equivariant homeomorphism $\Phi : \tilde{\Omega}_\Gamma \rightarrow \tilde{\Omega}_\psi$ and a continuous cocycle $\mathbf{t} : \tilde{\Omega}_\Gamma \times \mathbb{R} \rightarrow \mathbb{R}$ such that, for every $v \in \tilde{\Omega}_\Gamma$ and $s \in \mathbb{R}$,*

$$(\Phi(v)^+, \Phi(v)^-) = (f(v^+), f(v^-)), \quad \Phi(va_s) = \Phi(v)a_{\mathbf{t}(v,s)}.$$

For every $v \in \tilde{\Omega}_\Gamma$, the map $s \mapsto \mathbf{t}(v, s)$ is an increasing homeomorphism of $\mathbb{R}$ fixing the origin. Moreover, there exists $c > 1$ such that

$$c^{-1}|s| - c \leq |\mathbf{t}(v, s)| \leq c|s| + c \quad (v \in \tilde{\Omega}_\Gamma, s \in \mathbb{R}).$$

Proof. For $v \in \tilde{\Omega}_\Gamma$, let $\pi(v) \in Y$ be its basepoint and set $q(v) := \psi(\beta_{f(v^+)}(o, f(\pi(v))))$. By Lemma 3.11, the quasi-isometry property of f , and positivity of ψ on $\mathcal{L}_f \setminus \{0\}$, there exist $a, A > 0$ and $B \geq 0$ such that

$$as - B \leq q(va_s) - q(v) \leq As + B \quad (s \geq 0),$$

uniformly in v . Following the averaging argument in [9, proof of Lemma 4.3], choose $T > B/a$ and set $Q(v) := T^{-1} \int_0^T q(va_r) dr$. Then

$$\frac{d}{ds}Q(va_s) = \frac{q(va_{s+T}) - q(va_s)}{T} \in [a - B/T, A + B/T].$$

Define $\Phi(v) := (f(v^+), f(v^-), Q(v))$ and $\mathbf{t}(v, s) := Q(va_s) - Q(v)$. For $\gamma \in \Gamma$, the Busemann cocycle identity gives $q(\gamma v) - q(v) = \psi(\beta_{f(v^+)}(\rho(\gamma)^{-1}, e))$. This difference is constant along the flow orbit of v , so the same identity holds with Q in place of q . Hence Φ is ρ -equivariant. The flow relation and cocycle identity follow from the definitions. Since $a - B/T > 0$, the derivative bounds show that Q maps each such orbit increasingly and bijectively onto $\mathbb{R}$. Hence Φ is bijective. Continuity of Q and the uniform positive lower bound on its derivative along flow orbits give continuity of the inverse in Hopf coordinates. The same derivative bounds give the asserted properties of $\mathbf{t}$. $\square$

The construction of Theorem 3.20 uses only positivity of the linear form on $\mathcal{L}_f \setminus \{0\}$, not its tangency to the growth indicator. We will also use it for such forms.

4. GENERAL LOCAL MIXING THEOREMS AND EXAMPLES

We prove the local mixing theorems of the introduction under the following more general hypotheses.

Theorem 4.1 (Local mixing for relatively Anosov representations). *Let $\rho : \Gamma \rightarrow G$ be a Zariski dense relatively Anosov representation satisfying the peripheral root-balance condition defined below. Then the conclusions of Theorems 1.1 and 1.3 hold for every $u \in \text{int } \mathcal{L}_\rho$. The orbital counting conclusion of Corollary 1.2 also holds.*

Definition 4.2 (Peripheral root-balance condition). A relatively Anosov representation $\rho : \Gamma \rightarrow G$ satisfies the *peripheral root-balance condition* if, for every parabolic $\gamma \in \Gamma$,

$$(4.1) \quad \liminf_{n \rightarrow \infty} \frac{\min_{\alpha \in \Pi} \alpha(\mu(\rho(\gamma)^n))}{\max_{\alpha \in \Pi} \alpha(\mu(\rho(\gamma)^n))} > \frac{1}{2}.$$

By relative Anosovness, the denominator is eventually positive.

We describe three classes to which the theorem applies. We first establish a Cartan-growth estimate.

Let $v_{\text{pr}} \in \text{int } \mathfrak{a}^+$ be the unique vector satisfying

$$(4.2) \quad \alpha(v_{\text{pr}}) = 2 \quad (\alpha \in \Pi).$$

We call $\mathbb{R}_{>0}v_{\text{pr}}$ the principal direction. Since i permutes the simple roots, $i(v_{\text{pr}}) = v_{\text{pr}}$.

Lemma 4.3 (Principal Cartan direction). *Let $g \in G$ be weakly unipotent. If g_u is algebraically regular, then $v_g = v_{\text{pr}}$. In particular,*

$$\mu(g^n) = (\log |n|)v_{\text{pr}} + O_g(1) \quad (|n| \rightarrow \infty).$$

Proof. Let $u := g_u$. Since u is regular, Lemma 3.5 provides an $\mathfrak{sl}_2$ -triple (E, H, F) with $E = \log u$ and, after conjugation, $H = v_g \in \text{int } \mathfrak{a}^+$. The identity $[H, E] = 2E$ then implies that $E \in \mathfrak{n}$. Since u is algebraically regular, G is quasi-split, so $P_{\mathbb{C}}$ is a Borel subgroup of $G_{\mathbb{C}}$. Choose a maximal torus of $P_{\mathbb{C}}$ containing $A_{\mathbb{C}}$. Algebraic regularity implies that E has a nonzero component in every absolute simple-root space of the nilpotent radical; see [73]. Projecting $[H, E] = 2E$ to these root spaces gives $\tilde{\alpha}(H) = 2$ for every absolute simple root $\tilde{\alpha}$. Since every restricted simple root is the restriction of such an absolute simple root, $\alpha(H) = 2$ for every $\alpha \in \Pi$. Thus $H = v_{\text{pr}}$. $\square$

4.1. Positive representations. In this subsection, assume that G is split over $\mathbb{R}$. We recall the notion of positivity on $\mathcal{F} = G/P$.

For each $\alpha \in \Pi$, choose a nonzero vector $X_\alpha \in \mathfrak{g}_\alpha$ and set $x_\alpha(t) := \exp(tX_\alpha)$ for $t \in \mathbb{R}$. These choices determine the positivity structure used below. Let s_α denote the corresponding simple reflection. For a reduced expression $w_0 = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_k}}$, set

$$N_{>0} := \left\{ x_{\alpha_{i_1}}(t_1) \cdots x_{\alpha_{i_k}}(t_k) : t_1, \dots, t_k > 0 \right\}.$$

This is Lusztig's totally positive semigroup; it is independent of the reduced expression [50].

A quadruple (F_1, F_2, F_3, F_4) of flags is called *positive* if there exist $h \in G$ and $u, v \in N_{>0}$ such that

$$(hF_1, hF_2, hF_3, hF_4) = (P, w_0P, uw_0P, vvw_0P).$$

This agrees with the usual definition of a positive quadruple; see [32, Definition 2.13 and Proposition 2.17].

Definition 4.4. A representation $\rho : \Gamma \rightarrow G$ is called *positive* if there exists a ρ -equivariant map $\xi_p : \Lambda_p(\Gamma) \rightarrow \mathcal{F}$ sending every cyclically ordered quadruple of pairwise distinct points to a positive quadruple of flags.

Theorem 4.5. *Every type-preserving positive representation $\rho : \Gamma \rightarrow G$ is relatively Anosov. Moreover, for every parabolic $\gamma \in \Gamma$, the unipotent part $\rho(\gamma)_u$ is algebraically regular. In particular, ρ satisfies the peripheral root-balance condition.*

Proof. Relative Anosovness follows from [28, Theorem 6.17]; see also [12, Theorems 1.4 and 9.2] for $G = \mathrm{SL}_d(\mathbb{R})$.

Every simple reflection occurs in a reduced expression for w_0 , so every simple-root coordinate of an element of $N_{>0}$ is a nonempty sum of positive parameters. Thus every element of $N_{>0}$ has a nonzero component in every simple root group of $N/[N, N]$ and is algebraically regular by [73, Lemma 3.2 and Theorem 3.3].

Let $\gamma \in \Gamma$ be parabolic with fixed point $p \in \Lambda_p(\Gamma)$, and choose $x \in \Lambda_p(\Gamma) \setminus \{p\}$. Algebraic regularity of the unipotent part is preserved under inversion, so we may replace γ by γ^{-1} and assume that $(p, \gamma^{-1}x, x, \gamma x)$ is cyclically ordered. Set $g := \rho(\gamma)$, $F_1 := \xi_p(p)$, and $F_2 := \xi_p(x)$. Positivity and equivariance imply that $(F_1, g^{-1}F_2, F_2, gF_2)$ is a positive quadruple.

By the definition of a positive quadruple, after conjugation we may assume that $F_1 = P$, $F_2 = w_0P$, $g^{-1}w_0P = v^{-1}w_0P$ and $gw_0P = uw_0P$ for some $u, v \in N_{>0}$. Since $gP = P$ and $u^{-1}g(w_0P) = w_0P$, we obtain $u^{-1}g \in P \cap \check{P} = MA$, so $g = u\ell$ for some $\ell \in MA$. Since g is weakly unipotent, ℓ has trivial hyperbolic part. As G is split, every element of MA is semisimple, and its elliptic elements belong to M . Thus $\ell = m \in M$.

Since m fixes w_0P , we deduce $(m^{-1}um)^{-1}w_0P = v^{-1}w_0P$. The action of N on Nw_0P is free, so $m^{-1}um = v$. In $N/[N, N]$, every simple-root coordinate of both u and v is positive. Conjugation by $m \in M$ acts on each one-dimensional simple root space by a sign; hence all these signs are positive. Thus $\mathrm{Ad} m$ is the identity on every simple root space. Since $m \in K$, it is also the identity on the opposite simple root spaces. These root spaces generate $\mathfrak{g}$, so m belongs to the center $Z(G)$. By uniqueness of the real Jordan decomposition, $g_u = u$ and $g_e = m$. Since $u \in N_{>0}$, the element g_u is algebraically regular. By Lemma 4.3, the peripheral root-balance ratio therefore converges to 1. $\square$

4.2. Split simple groups of low rank. For split simple groups of rank at most three, relative Borel Anosovness alone implies peripheral root balance.

Proposition 4.6. *Suppose that G is split and simple with $\mathrm{rank}_{\mathbb{R}} G \leq 3$. Let $\rho : \Gamma \rightarrow G$ be relatively Anosov. Then, for every parabolic $\gamma \in \Gamma$, there exists $c_\gamma \in \{1, \frac{1}{2}\}$ such that*

$$\mu(\rho(\gamma)^n) = c_\gamma(\log |n|)v_{\mathrm{pr}} + O_\gamma(1) \quad (|n| \rightarrow \infty).$$

The value $c_\gamma = \frac{1}{2}$ can occur only when G has type A_2 . In particular, ρ satisfies the peripheral root-balance condition.

Proof. Fix a parabolic $\gamma \in \Gamma$ and set $g := \rho(\gamma)$. By relative Borel Anosovness, g is weakly unipotent. Let $v_g \in \mathfrak{a}^+$ be a vector conjugate to the semisimple element of an $\mathfrak{sl}_2$ -triple containing $\log g_u$. Conjugate the triple so that its semisimple element is $H = v_g$. By Lemma 3.5, $\mu(g^n) = (\log |n|)v_g + O_g(1)$. Since $\alpha(\mu(g^n)) \rightarrow +\infty$ for every $\alpha \in \Pi$, we have $v_g \in \mathrm{int} \mathfrak{a}^+$.

We determine the possibilities for H . For the classical types, consider the standard module of the complexified Lie algebra. A Jordan block of size d of $\log g_u$ contributes the eigenvalues $d - 1, d - 3, \dots, 1 - d$ of H .

In type A_r , with $r \leq 3$, the condition $H \in \text{int } \mathfrak{a}^+$ requires these eigenvalues to be distinct. Among partitions of $r + 1 \leq 4$, this leaves only the single-block partition, together with $(2, 1)$ when $r = 2$. A single block gives $\alpha(H) = 2$ for every $\alpha \in \Pi$. For $(2, 1)$, the eigenvalues are $1, 0, -1$, so both simple roots take the value 1. Thus $H = v_{\text{pr}}$ or, in the latter case, $H = \frac{1}{2}v_{\text{pr}}$.

In types B_r and C_r , with $r = 2$ or 3 , interiority forces the eigenvalues of H on the standard module to be distinct. In type B_r , blocks of even size are excluded because they occur with even multiplicity, while two odd blocks would both contribute the eigenvalue 0. In type C_r , odd blocks are excluded because interiority also rules out the eigenvalue 0, while two even blocks would both contribute 1 and -1 . Thus there is a single Jordan block, of size $2r + 1$ or $2r$, respectively, and $H = v_{\text{pr}}$.

The remaining type is G_2 . Its only nilpotent weighted Dynkin diagram with all labels positive is $(2, 2)$; see [20, Table 2]. Since these labels are the values $\alpha(H)$, we obtain $H = v_{\text{pr}}$. $\square$

4.3. Self-joinings of rank-one representations. For representations $\rho = (\rho_1, \dots, \rho_k)$ into products of real-rank-one groups with discrete projections, type preservation is equivalent to type preservation of each projection. Indeed, if ρ is type-preserving, each ρ_i preserves loxodromic elements and hence is faithful, since every nontrivial normal subgroup of Γ contains a loxodromic element. For parabolic γ , the element $\rho_i(\gamma)$ has trivial hyperbolic part; faithfulness and discreteness exclude ellipticity, so it is parabolic. The converse follows from the componentwise Jordan decomposition.

Proposition 4.7. *Let $G = \prod_{i=1}^k G_i$, where each G_i has real rank one, and let $\rho : \Gamma \rightarrow G$ be a Zariski dense type-preserving representation whose projections $\rho_i : \Gamma \rightarrow G_i$ have geometrically finite images. For every parabolic $\gamma \in \Gamma$, conjugate the unipotent parts $u_i := \rho_i(\gamma)_u$ into $N_i := N \cap G_i$. Suppose that $u_i \in [N_i, N_i]$ either for every i or for no i . Then ρ is relatively Anosov and satisfies the peripheral root-balance condition. More precisely,*

$$\mu(\rho(\gamma)^n) = c_\gamma(\log |n|)v_{\text{pr}} + O_\gamma(1) \quad (|n| \rightarrow \infty),$$

where $c_\gamma = \frac{1}{2}$ if $u_i \in [N_i, N_i]$ for every i , and $c_\gamma = 1$ otherwise.

Proof. Since $\rho_i(\Gamma)$ is geometrically finite, the generalization of Tukia's isomorphism theorem in [19, Theorems 1.4(i) and 1.5] therefore gives a ρ_i -equivariant boundary homeomorphism $\xi_i : \Lambda(\Gamma) \rightarrow \Lambda(\rho_i(\Gamma))$. The product map $(\xi_1, \dots, \xi_k)$ satisfies the boundary-map criterion of [12, Corollary B.8], and hence ρ is relatively Anosov.

Fix a parabolic $\gamma \in \Gamma$ and set $g := \rho(\gamma)$. By Lemma 3.5, choose an $\mathfrak{sl}_2$ -triple (E, H, F) with $E = \log g_u$. Since every simple root diverges along $\mu(g^n)$, the Cartan asymptotic in that lemma gives $v_g \in \text{int } \mathfrak{a}^+$. Conjugate the triple so that $H = v_g$. The identity $[H, E] = 2E$ then implies that $E \in \mathfrak{n}$. Write $E = (E_i)$ and $H = (H_i)$.

Each nontrivial u_i lies in a unique minimal real parabolic subgroup of G_i . Consequently, any two conjugations placing u_i in N_i differ by an element of $P_i := P \cap G_i$, which normalizes $[N_i, N_i]$. The assumed membership condition therefore still holds after the preceding conjugation of the triple.

Let α_i be the simple restricted root of G_i . For $\mathfrak{n}_i := \text{Lie}(N_i)$, we have $[\mathfrak{n}_i, \mathfrak{n}_i] = \mathfrak{g}_{i, 2\alpha_i}$ and $\mathfrak{n}_i / [\mathfrak{n}_i, \mathfrak{n}_i] \simeq \mathfrak{g}_{i, \alpha_i}$, where $\mathfrak{g}_{i, 2\alpha_i} := \{0\}$ if $2\alpha_i$ is not a root. If $u_i \notin [N_i, N_i]$ for every i , the image $\overline{E}_i$ of E_i in this quotient is nonzero. Projecting $[H_i, E_i] = 2E_i$ to the quotient gives $\alpha_i(H_i)\overline{E}_i = 2\overline{E}_i$. Thus $\alpha_i(H_i) = 2$ for every i , so $H = v_{\text{pr}}$.

If $u_i \in [N_i, N_i]$ for every i , then $E_i \in \mathfrak{g}_{i, 2\alpha_i} \setminus \{0\}$. Hence $2\alpha_i(H_i)E_i = [H_i, E_i] = 2E_i$, so $\alpha_i(H_i) = 1$ for every i and $H = \frac{1}{2}v_{\text{pr}}$. The claimed asymptotic follows from Lemma 3.5. In either case, all simple roots take the same positive value on H , so the root-balance ratio converges to 1. $\square$

For $G_i = \text{SO}(n_i, 1)^\circ$, with $n_i \geq 2$, the groups N_i are abelian. The peripheral condition is then automatic, and $c_\gamma = 1$ for every parabolic γ .

Part 1. Cusp geometry and Markov coding

For the rest of the paper, let $\rho : \Gamma \rightarrow G$ be a Zariski dense relatively Anosov representation satisfying the peripheral root-balance condition of Definition 4.2.

We first remove compact normal factors. Let G_c be the maximal connected compact normal subgroup of G , and let $q : G \rightarrow G/G_c$ be the quotient map. Since $G_c \subset M = K \cap P$, the spaces G/P , G/K , and G/M identify with the corresponding spaces for G/G_c . The Cartan and Jordan projections, restricted roots, and growth indicator are unchanged. Thus $q \circ \rho$ is Zariski dense and relatively Anosov, and satisfies the same peripheral root-balance condition. It therefore suffices to prove the results for G/G_c . For the remainder of the proof, we work in this quotient and retain the notation G and ρ ; in particular, G has no compact factors.

Fix a continuous ρ -equivariant Morse embedding $f : Y \rightarrow X$ constructed in Proposition 3.8, so that $f(o) = o$ and $\mathcal{L}_f = \mathcal{L}_\rho$. We also denote its boundary extension $\Lambda(\Gamma) \rightarrow \Lambda_\rho$ by f . Set $\Lambda_c(\Gamma_\rho) := f(\Lambda_c(\Gamma))$ and $\Lambda_p(\Gamma_\rho) := f(\Lambda_p(\Gamma))$. Choose representatives $\xi_0, \dots, \xi_\ell$ of the Γ -orbits in $\Lambda_p(\Gamma)$. Let $\mathcal{H}$, H_i , $0 \leq i \leq \ell$ and Y_{th} be as in the proof of Proposition 3.8. Let

$$p_i := f(\xi_i), \quad P_i := \rho(\Gamma_{\xi_i}), \quad \Gamma_{\xi_i} := \text{Stab}_\Gamma(\xi_i).$$

After conjugating Γ in $\text{PSL}_2(\mathbb{R})$ and Γ_ρ in G , we assume

$$\xi_0 = \infty, \quad f(\infty) = p_0 = P, \quad 0 \in \Lambda(\Gamma), \quad f(0) = w_0 P.$$

Choose compact intervals $\widehat{\Delta}_{\xi_i} \subset \partial\mathbb{H}^2$ ($0 \leq i \leq \ell$), each a fundamental set for the action of Γ_{ξ_i} on $\partial\mathbb{H}^2 \setminus \{\xi_i\}$, with endpoints in $\Lambda_p(\Gamma)$, such that

$$(4.3) \quad \{\xi_i : 0 \leq i \leq \ell\} \cap \bigcup_{i=0}^{\ell} \widehat{\Delta}_{\xi_i} = \emptyset, \quad \widehat{\Delta}_{\xi_i} \cap \widehat{\Delta}_{\xi_j} = \emptyset \quad (i \neq j).$$

We consider the following compact fundamental domain for the P_i -action on $\Lambda_\rho \setminus \{p_i\}$:

$$\Delta_i := f(\widehat{\Delta}_{\xi_i} \cap \Lambda(\Gamma)) \subset \Lambda_\rho \setminus \{p_i\},$$

Set $\partial\Delta_i := f(\partial\widehat{\Delta}_{\xi_i})$ and $\text{int } \Delta_i := \Delta_i \setminus \partial\Delta_i$. Set

$$\Delta := \bigcup_{i=0}^{\ell} \Delta_i$$

5. THE VISUAL PREMETRIC AND ITS BLOW-UP

Throughout this section and Section 6, fix a linear form $\chi \in \mathfrak{a}^*$ satisfying

$$(5.1) \quad \chi \circ \mathbf{i} = \chi, \quad \chi > 0 \quad \text{on } \mathcal{L}_\rho \setminus \{0\}.$$

In this section we introduce the visual premetric d_χ on Λ_ρ , compare it with the spherical metric on $\partial\mathbb{H}^2$, and study its blow-up at the parabolic limit point p_0 .

Definition 5.1. For $(\xi, \eta) \in \mathcal{F}^{(2)}$, set

$$d_\chi(\xi, \eta) := e^{-\chi(\mathcal{G}(\xi, \eta))}.$$

Since $\mathcal{G}(\eta, \xi) = i\mathcal{G}(\xi, \eta)$, the identity $\chi \circ i = \chi$ makes d_χ symmetric. The cocycle identity gives, for every $(\xi, \eta) \in \mathcal{F}^{(2)}$ and $g \in G$,

$$(5.2) \quad d_\chi(g\xi, g\eta) = d_\chi(\xi, \eta) e^{-\frac{1}{2}\chi(\beta_{g\xi}(e, g) + \beta_{g\eta}(e, g))}.$$

5.1. Restriction to the limit set. By Lemmas 3.9 and 3.13, d_χ extends continuously to $\Lambda_\rho \times \Lambda_\rho$ by setting $d_\chi(\xi, \xi) = 0$. Thus d_χ is a continuous premetric² on Λ_ρ . By compactness, it induces the usual topology on Λ_ρ .

Lemma 5.2. *The diameter of (Λ_ρ, d_χ) is finite.*

Proof. Let $\xi, \eta \in \Lambda(\Gamma)$ be distinct, and let z be a nearest point projection of o to the geodesic $[\xi, \eta] \subset \mathbb{H}^2$. By Lemma 3.13, $\chi(\mathcal{G}(f(\xi), f(\eta))) \approx \chi(\mu(f(z)))$. Since $\mathcal{L}_f = \mathcal{L}_\rho$, (5.1) and the definition of $\mathcal{L}_f$ imply that $\chi(\mu(f(z)))$ is bounded below uniformly in $z \in Y$. This proves the assertion. $\square$

5.2. Shadows and d_χ -balls. For $r > 0$ and $\xi \in \Lambda_\rho$, set

$$B_\chi(\xi, r) := \{\eta \in \Lambda_\rho : d_\chi(\xi, \eta) < r\}.$$

For $\xi \in \Lambda_\rho$, let $\widehat{z}_\xi : [0, \infty) \rightarrow Y$ be the unit-speed geodesic ray from o to $f^{-1}(\xi)$, and put $z_\xi(t) := f(\widehat{z}_\xi(t))$. Thus z_ξ is a uniformly regular quasigeodesic ray in X with flag endpoint ξ , with constants independent of ξ . Applying [43, Proposition 4.12 and (8.3)] to the hyperbolic rays $\widehat{z}_\xi$ and then using Lemma 3.12, we obtain the following comparison between ambient shadows and d_χ -balls.

Proposition 5.3. *The following hold.*

- (1) *For every sufficiently large $r > 0$, there exists $c > 1$ such that, for every $\xi \in \Lambda_\rho$ and $t \geq 0$,*

$$B_\chi(\xi, c^{-1}e^{-\chi(\mu(z_\xi(t)))}) \subset \mathcal{O}_r(o, z_\xi(t)) \cap \Lambda_\rho \subset B_\chi(\xi, ce^{-\chi(\mu(z_\xi(t)))}).$$

- (2) *For every $c > 0$ and every sufficiently large $r > 0$, depending on c , there exists $t_0 = t_0(c, r) > 0$ such that, for every $\xi \in \Lambda_\rho$, $t \geq 0$, and $s \geq t + t_0$,*

$$\mathcal{O}_r(o, z_\xi(s)) \cap \Lambda_\rho \subset B_\chi(\xi, ce^{-\chi(\mu(z_\xi(t)))}) \subset \mathcal{O}_r(o, z_\xi(t)) \cap \Lambda_\rho.$$

Remark 5.4. Without the symmetry assumption on χ , the radial scale in this proposition must be replaced by $e^{-\frac{1}{2}(\chi + \chi \circ i)(\mu(z_\xi(t)))}$.

5.3. Blow-up of d_χ . Set $P_0 = \text{Stab}_{\Gamma_\rho}(p_0)$. Let

$$\mathcal{U}_{p_0} := \{\xi \in \mathcal{F} : (\xi, p_0) \in \mathcal{F}^{(2)}\}$$

be the open Bruhat cell transverse to p_0 .

Definition 5.5. For $\xi, \eta \in \mathcal{U}_{p_0}$ with $(\xi, \eta) \in \mathcal{F}^{(2)}$, define $D_\chi := D_{\chi, p_0}$ by

$$(5.3) \quad D_\chi(\xi, \eta) := d_\chi(\xi, \eta) e^{\chi(\mathcal{G}(\xi, p_0))} e^{\chi(\mathcal{G}(\eta, p_0))} = \frac{d_\chi(\xi, \eta)}{d_\chi(\xi, p_0) d_\chi(\eta, p_0)}$$

and set $D_\chi(\xi, \xi) = 0$ for $\xi \in \mathcal{U}_{p_0}$.

²We call a symmetric nonnegative function on $S \times S$ a *premetric* if it vanishes precisely on the diagonal.

Remark 5.6. For $G = \mathrm{SO}(n, 1)^\circ$, use the upper half-space model of $\mathbb{H}^n$ with curvature -1 , basepoint $o = (0, 1)$, and $p_0 = \infty$. Let $H \in \mathfrak{a}^+$ satisfy $\exp(tH)o = (0, e^t)$. For $\xi, \eta \in \mathbb{R}^{n-1}$, we have

$$d_\chi(\xi, \eta) = \left(\frac{\|\xi - \eta\|}{\sqrt{1 + \|\xi\|^2} \sqrt{1 + \|\eta\|^2}} \right)^{\chi(H)}, \quad d_\chi(\xi, \infty) = (1 + \|\xi\|^2)^{-\chi(H)/2}.$$

Consequently, D_χ is a positive power of the Euclidean metric: $D_\chi(\xi, \eta) = \|\xi - \eta\|^{\chi(H)}$.

The cocycle identity (3.3) and Lemma 3.3, together with $\chi = \chi \circ \mathbf{i}$, give the following transformation properties. The uniform comparison follows by continuity and compactness.

Lemma 5.7 (Transformation of D_χ). *Let $g \in G$. For pairwise transverse $\xi, \eta, g^{-1}p_0 \in \mathcal{F}$, we have*

$$D_\chi(g\xi, g\eta) = d_\chi(\xi, \eta) e^{\chi(\mathcal{G}(\xi, g^{-1}p_0))} e^{\chi(\mathcal{G}(\eta, g^{-1}p_0))} e^{\chi(\beta_{p_0}(e, g))}.$$

If $p := g^{-1}p_0 \in \Lambda_\rho$, then for every compact $Q \subset \Lambda_\rho \setminus \{p, p_0\}$, there exists $C = C(Q, g) \geq 1$ such that

$$C^{-1}D_\chi(\xi, \eta) \leq D_\chi(g\xi, g\eta) \leq CD_\chi(\xi, \eta) \quad (\xi, \eta \in Q).$$

Moreover, D_χ is $\mathbf{P}_0$ -invariant:

$$(5.4) \quad D_\chi(g\xi, g\eta) = D_\chi(\xi, \eta) \quad (g \in \mathbf{P}_0, \xi, \eta \in \Lambda_\rho \setminus \{p_0\}).$$

5.4. Metric properties of D_χ . By (5.3), the restriction of D_χ to $\Lambda_\rho \setminus \{p_0\}$ is a continuous premetric inducing the usual topology. On every compact, pairwise transverse subset of $\mathcal{U}_{p_0}$, the premetrics d_χ and D_χ are bi-Lipschitz equivalent. We also observe:

Lemma 5.8 (Comparison of the visual and blown-up premetrics). *For every $\xi_0 \in \Lambda_\rho \setminus \{p_0\}$, there exists $c = c(\xi_0) \geq 1$ such that, for every $\xi \in \Lambda_\rho \setminus \{p_0\}$,*

$$\frac{c^{-1}}{1 + D_\chi(\xi, \xi_0)^2} \leq d_\chi(\xi, p_0)^2 \leq \frac{c}{1 + D_\chi(\xi, \xi_0)^2}.$$

Proof. By (5.3), $(1 + D_\chi(\xi, \xi_0)^2)d_\chi(\xi, p_0)^2 = d_\chi(\xi, p_0)^2 + \frac{d_\chi(\xi, \xi_0)^2}{d_\chi(\xi_0, p_0)^2}$. The right-hand side is continuous and strictly positive on the compact set Λ_ρ , since $p_0 \neq \xi_0$. It is therefore bounded above and away from zero, which proves the claim. $\square$

Via f , we regard the Euclidean metric d_{Euc} on $\mathbb{R} = \partial\mathbb{H}^2 \setminus \{\infty\}$ as a metric on $\Lambda_\rho \setminus \{p_0\}$. The following shows that D_χ and d_{Euc} are quasimetrically equivalent on $\Lambda_\rho \setminus \{p_0\}$.

Lemma 5.9. *There exist $a, C \geq 1$ such that, for all distinct $\xi, \eta_1, \eta_2 \in \Lambda_\rho \setminus \{p_0\}$,*

$$\frac{D_\chi(\xi, \eta_1)}{D_\chi(\xi, \eta_2)} \leq C \max \left\{ \left(\frac{d_{\mathrm{Euc}}(\xi, \eta_1)}{d_{\mathrm{Euc}}(\xi, \eta_2)} \right)^a, \left(\frac{d_{\mathrm{Euc}}(\xi, \eta_1)}{d_{\mathrm{Euc}}(\xi, \eta_2)} \right)^{1/a} \right\}.$$

Proof. Let $\sigma : [0, \infty) \rightarrow \mathrm{hull} \Lambda(\Gamma)$ be a geodesic ray ending at ∞ . For $i = 1, 2$, let u_i be a nearest point projection of $\sigma(t)$ to $[\xi, \eta_i]$, where t is sufficiently large. The equivariant forms of Lemmas 3.11 and 3.13 give

$$\log D_\chi(\xi, \eta_i) \approx -\chi(\mu(f(\sigma(t))^{-1}f(u_i))) + \chi(\beta_{p_0}(o, f(\sigma(t)))),$$

with an additive error independent of ξ, η_i, t . Choose $v_i \in [\sigma(t), \xi]$ at uniformly bounded distance from u_i . By coarse additivity,

$$(5.5) \quad \log \frac{D_\chi(\xi, \eta_1)}{D_\chi(\xi, \eta_2)} \approx \varepsilon \chi(\mu(f(v_1)^{-1} f(v_2))),$$

where $\varepsilon \in \{\pm 1\}$ is determined by the order of v_1, v_2 on $[\sigma(t), \xi]$. The corresponding calculation in $\mathbb{H}^2$ gives, with the same sign,

$$(5.6) \quad \log \frac{d_{\text{Euc}}(\xi, \eta_1)}{d_{\text{Euc}}(\xi, \eta_2)} \approx \varepsilon d(v_1, v_2).$$

Set $u := \log \frac{d_{\text{Euc}}(\xi, \eta_1)}{d_{\text{Euc}}(\xi, \eta_2)}$ and $v := \log \frac{D_\chi(\xi, \eta_1)}{D_\chi(\xi, \eta_2)}$. By Lemma 3.9 and (5.5)–(5.6), there exist uniform constants $a \geq 1$ and $c \geq 0$ such that $v \leq au + c$ if $u \geq 0$, and $v \leq a^{-1}u + c$ if $u \leq 0$. Indeed, these bounds follow by considering the two possible orders of v_1, v_2 ; if $\varepsilon u < 0$, (5.6) makes $d(v_1, v_2)$ uniformly bounded, and enlarging c covers this case. Exponentiating gives the claim with $C = e^c$. $\square$

Corollary 5.10 (Quasi-triangle inequality). *There exists $N_0 \geq 1$ such that, for all $\xi, \eta, \zeta \in \Lambda_\rho \setminus \{p_0\}$,*

$$D_\chi(\xi, \zeta) \leq N_0 (D_\chi(\xi, \eta) + D_\chi(\eta, \zeta)).$$

Proof. The assertion is immediate if two points coincide. Otherwise, note $d_{\text{Euc}}(\xi, \zeta) \leq 2 \max\{d_{\text{Euc}}(\xi, \eta), d_{\text{Euc}}(\eta, \zeta)\}$. Apply Lemma 5.9 at ξ or ζ , according to which term realizes the maximum. Using symmetry of D_χ , we obtain $D_\chi(\xi, \zeta) \leq 2^a C \max\{D_\chi(\xi, \eta), D_\chi(\eta, \zeta)\}$. Thus we may take $N_0 = 2^a C$. $\square$

6. CUSP HEIGHT AND GEOMETRIC ESTIMATES

We define cusp heights, compare them to Cartan projections, and prove the separation and inversion estimates used in the coding.

6.1. Height. Each P_0 -orbit $\mathfrak{D}$ in $\Lambda_P(\Gamma_\rho) \setminus \{p_0\}$ meets Δ_0 in one or two points; fix a representative $p_\mathfrak{D} \in \mathfrak{D} \cap \Delta_0$. Choose $\gamma_\mathfrak{D} \in \Gamma_\rho$ such that $\gamma_\mathfrak{D}^{-1} p_\mathfrak{D} \in \{p_0, \dots, p_\ell\}$.

Definition 6.1. The *height function* $h = h_{\Gamma_\rho, p_0, \Delta_0} : \Lambda_P(\Gamma_\rho) \setminus \{p_0\} \rightarrow \mathbb{R}_{>0}$ is the P_0 -invariant function defined by

$$h(p) := e^{-\chi(\beta_{p_\mathfrak{D}}(e, \gamma_\mathfrak{D}))} \quad \text{where } \mathfrak{D} = P_0 p.$$

In particular, if $p \in \text{int } \Delta_0$ and $\gamma^{-1} p \in \{p_0, \dots, p_\ell\}$, then $p_\mathfrak{D} = p$, so that

$$h(p) = e^{-\chi(\beta_p(e, \gamma))}.$$

Lemma 6.2. *The definition of h is independent of the choice of $\gamma_\mathfrak{D}$. Different choices of the representatives $p_\mathfrak{D} \in \mathfrak{D} \cap \Delta_0$ change h by a uniformly bounded multiplicative factor.*

Proof. Two choices of $\gamma_\mathfrak{D}$ differ by right multiplication by an element fixing $\gamma_\mathfrak{D}^{-1} p_\mathfrak{D}$. Independence follows from the cocycle identity and Lemma 3.3. By proper discontinuity, only finitely many $u \in P_0$ satisfy $u\Delta_0 \cap \Delta_0 \neq \emptyset$. The cocycle identity and compactness of Δ_0 therefore give the uniform multiplicative bound under changes of $p_\mathfrak{D}$. $\square$

If $\gamma^{-1} p \in \{p_0, \dots, p_\ell\}$ and $\tilde{\gamma} p = p_\mathfrak{D}$, then

$$(6.1) \quad h(p) = e^{-\chi(\beta_{\tilde{\gamma} p}(e, \tilde{\gamma}))}.$$

Lemma 6.3 (Height comparison on compact sets). *For every compact set $Q \subset \Lambda_\rho \setminus \{p_0\}$,*

$$h(p) \asymp_Q e^{-\chi(\beta_p(e, \gamma))}$$

whenever $p \in Q \cap \Lambda_p(\Gamma_\rho)$ and $\gamma \in \Gamma_\rho$ satisfy $\gamma^{-1}p \in \{p_0, \dots, p_\ell\}$. In particular, the comparison is uniform for $p \in \Delta$.

Proof. Choose $\tilde{\gamma} \in P_0$ with $\tilde{\gamma}p = p_\Delta \in \Delta_0$. Proper discontinuity implies that the set $\{u \in P_0 : uQ \cap \Delta_0 \neq \emptyset\}$ is finite. By (6.1) and the cocycle identity,

$$-\log h(p) = \chi(\beta_p(e, \gamma)) + \chi(\beta_p(\tilde{\gamma}^{-1}, e)) = \chi(\beta_p(e, \gamma)) + O_Q(1).$$

This proves the assertion. $\square$

6.2. Top representatives. For $p \in \Gamma_\rho p_i \cap \Delta$, choose an element $\gamma_p \in \Gamma_\rho$ such that

$$(6.2) \quad p = \gamma_p p_i \quad \text{and} \quad \gamma_p^{-1} p_i \in \Delta_i.$$

We call γ_p a top representative of p and set $x_p := \gamma_p^{-1} p_i$. There are at most two choices, differing, when distinct, by right multiplication by a generator of P_i . We write $\hat{\gamma} := \rho^{-1}(\gamma) \in \Gamma$ for $\gamma \in \Gamma_\rho$.

Lemma 6.4. *There exists $r > 0$ such that, for every $p \in \Lambda_p(\Gamma_\rho) \cap \Delta$,*

$$p \in \mathcal{O}_r(o, \gamma_p o), \quad x_p \in \mathcal{O}_r(o, \gamma_p^{-1} o).$$

Proof. Suppose that the first claim does not hold. By Lemma 3.12, there exist i and a sequence $f^{-1}(p_n) = \hat{\gamma}_{p_n} \xi_i$ such that $f^{-1}(p_n) \notin \mathcal{O}_n(o, \hat{\gamma}_{p_n} o)$ for all $n \geq 1$. This implies that $\hat{\gamma}_{p_n}^{-1} o \rightarrow \xi_i$. Since $f^{-1}(p_n)$ ranges over a compact set disjoint from ξ_i , the point o lies within a uniform distance of the geodesic $[f^{-1}(p_n), \xi_i]$. Applying $\hat{\gamma}_{p_n}^{-1}$, we find that $\hat{\gamma}_{p_n}^{-1} o$ lies within a uniform distance of $[\xi_i, y_n]$, where $y_n = \hat{\gamma}_{p_n}^{-1} \xi_i$ which also ranges over a fixed compact subset of $\Lambda(\Gamma) \setminus \{\xi_i\}$. Thus ξ_i is a conical limit point, contradicting the fact that ξ_i is parabolic. The second claim can be shown similarly. $\square$

Lemma 6.5. *For every $p \in \Lambda_p(\Gamma_\rho) \cap \Delta$,*

$$h(p) \asymp e^{-\chi(\mu(\gamma_p))}.$$

Proof. By Lemmas 3.11, 6.3 and 6.4, $h(p) \asymp e^{-\chi(\beta_p(e, \gamma_p))} \asymp e^{-\chi(\mu(\gamma_p))}$. $\square$

6.3. Height functions based at the other cusps. Set $g_0 = e$ and for $1 \leq i \leq \ell$, choose

$$g_i \in K \quad \text{such that} \quad g_i p_i = p_0.$$

Define the height function h_i on $g_i(\Lambda_p(\Gamma_\rho) \setminus \{p_i\})$ by

$$h_i(g_i p) := h_{g_i \Gamma_\rho g_i^{-1}, p_0, g_i \Delta_i}(g_i p).$$

Equivalently, for each P_i -orbit $\mathfrak{D}$ in $\Lambda_p(\Gamma_\rho) \setminus \{p_i\}$, fix $p_\mathfrak{D}^{(i)} \in \mathfrak{D} \cap \Delta_i$. If $\gamma_\mathfrak{D}^{-1} p_\mathfrak{D}^{(i)} \in \{p_0, \dots, p_\ell\}$, then

$$\log h_i(g_i p) = -\chi(\beta_{g_i p_\mathfrak{D}^{(i)}}(e, g_i \gamma_\mathfrak{D} g_i^{-1})), \quad p \in \mathfrak{D}.$$

Lemma 6.6. *There exists $C > 1$ such that, for every $0 \leq i \leq \ell$ and $p \in \Lambda_p(\Gamma_\rho) \cap \Delta$,*

$$C^{-1} \leq \frac{h_i(g_i p)}{h(p)} \leq C.$$

Proof. Choose $\gamma \in \Gamma_\rho$ with $\gamma^{-1}p \in \{p_0, \dots, p_\ell\}$, and choose $\tilde{\gamma}_i \in \mathbf{P}_i$ with $\tilde{\gamma}_i p = p_{\mathfrak{D}_i}^{(i)}$, where $\mathfrak{D}_i = \mathbf{P}_i p$. The cocycle identity gives

$$\beta_{g_i p_{\mathfrak{D}_i}^{(i)}}(e, g_i \tilde{\gamma}_i \gamma g_i^{-1}) = \beta_p(e, \gamma) + \beta_{\gamma^{-1}p}(e, g_i^{-1}) + \beta_p(\tilde{\gamma}_i^{-1} g_i^{-1}, e).$$

For each i , proper discontinuity implies that $\{\delta \in \mathbf{P}_i : \delta \Delta \cap \Delta_i \neq \emptyset\}$ is finite. Hence the elements $\tilde{\gamma}_i$ range over a fixed finite set. Since both g_i and $\tilde{\gamma}_i$ range over finite sets, the last two terms in the displayed identity are bounded. Applying the same argument for $i = 0$, with $\tilde{\gamma}_0 p = p_{\mathbf{P}_0 p}^{(0)}$, and comparing with (6.1), gives $|\log h_i(g_i p) - \log h(p)| = O(1)$ uniformly in p and i . $\square$

6.4. Separation. The height function gives a uniform lower bound separating distinct parabolic limit points.

Proposition 6.7. *There exists $0 < C_s \leq 1$ such that for all distinct $p, p' \in \Lambda_p(\Gamma_\rho) \setminus \{p_0\}$,*

$$D_\chi(p, p') \geq C_s \sqrt{h(p)h(p')}.$$

Proof. We first prove that there exists $C > 0$ such that

$$(6.3) \quad d_\chi(p, p') \geq C \sqrt{e^{-\chi(\beta_p(e, \gamma) + \beta_{p'}(e, \gamma'))}}$$

whenever $p \neq p' \in \Lambda_p(\Gamma_\rho)$ and $\gamma^{-1}p = p_k, (\gamma')^{-1}p' = p_j$ for some $0 \leq j, k \leq \ell$.

Choose an oriented geodesic σ in $\mathbb{H}^2$ from $f^{-1}(p')$ to $f^{-1}(p)$, with $\sigma(0)$ at its exit from the horoball based at $f^{-1}(p')$ in the family $\mathcal{H}$. Since $\Gamma \backslash Y_{\text{th}}$ is compact, $\sigma(0)$ lies within a uniform distance of Γo . Replacing γ and γ' within $\gamma \mathbf{P}_k$ and $\gamma' \mathbf{P}_j$, we may arrange that $\hat{\gamma}o$ and $\hat{\gamma}'o$ lie within a uniform distance of the entrance into the horoball based at $f^{-1}(p)$ and the exit from the horoball based at $f^{-1}(p')$, respectively. These replacements do not change the Busemann terms by Lemma 3.3.

Choose $g \in G$ with $g^+ = p$ and $g^- = p'$, and set $t := \chi(\beta_p(g, \gamma))$ and $t' := \chi(\beta_{p'}(g, \gamma'))$. Since $\chi \circ i = \chi$, the cocycle identity gives

$$(6.4) \quad 2\chi(\mathcal{G}(p, p')) + t + t' = \chi(\beta_p(e, \gamma) + \beta_{p'}(e, \gamma')).$$

Thus (6.3) is equivalent to a uniform lower bound for $t + t'$.

Choose $h \in \Gamma$ such that $h\sigma(0)$ lies in a fixed bounded neighborhood of o . Replacing σ by $h\sigma$ and simultaneously left translating $p, p', g, \gamma, \gamma'$ by $\rho(h)$ preserves t and t' by equivariance. We may therefore assume that $\sigma(0)$ lies in that neighborhood of o . By Lemma 3.13, $\mathcal{G}(p, p')$ is then uniformly bounded. The choices of γ, γ' and Lemma 3.12 give a uniform $r > 0$ such that

$$p \in \mathcal{O}_r(o, \gamma o), \quad p' \in \mathcal{O}_r(o, \gamma' o).$$

By Lemma 3.11 and (6.4),

$$(6.5) \quad t + t' = \chi(\mu(\gamma)) + \chi(\mu(\gamma')) + O(1).$$

Since $\chi(\mu(\delta))$ is uniformly bounded below for $\delta \in \Gamma_\rho$, this proves (6.3).

Now let $p, p' \neq p_0$ and choose $\tilde{\gamma}, \tilde{\gamma}' \in \mathbf{P}_0$ with $\tilde{\gamma}p = p_\mathfrak{D}$ and $\tilde{\gamma}'p' = p_{\mathfrak{D}'}$. Lemma 3.3 and (6.1) give

$$(6.6) \quad e^{2\chi(\mathcal{G}(p, p_0)) - \chi(\beta_p(e, \gamma))} = e^{2\chi(\mathcal{G}(p_\mathfrak{D}, p_0))} h(p),$$

and similarly for p' . Multiplying (6.3) by $e^{\chi(\mathcal{G}(p, p_0)) + \chi(\mathcal{G}(p', p_0))}$ therefore yields

$$D_\chi(p, p') \geq C e^{\chi(\mathcal{G}(p_\mathfrak{D}, p_0)) + \chi(\mathcal{G}(p_{\mathfrak{D}'}, p_0))} \sqrt{h(p)h(p')}.$$

Since $p_{\mathfrak{D}}, p_{\mathfrak{D}'} \in \Delta_0$ and Δ_0 is compact and disjoint from p_0 , the exponential factor has a uniform positive lower bound. This proves the proposition. $\square$

6.5. Ball flipping. The first estimate below shows that γ^{-1} maps a punctured D_χ -ball of radius r around $p = \gamma p_0$ into the complement of a ball around $\gamma^{-1}p_0$ of radius comparable to $h(p)/r$. The second gives the corresponding two-point estimate.

Lemma 6.8. *Let $\gamma \in \Gamma_\rho \setminus \mathbf{P}_0$ and set $p := \gamma p_0$. For all $x, y \in \Lambda_\rho \setminus \{p_0, p\}$,*

$$D_\chi(\gamma^{-1}x, \gamma^{-1}p_0) \asymp \frac{h(p)}{D_\chi(x, p)}, \quad D_\chi(\gamma^{-1}x, \gamma^{-1}y) \asymp \frac{h(p)D_\chi(x, y)}{D_\chi(x, p)D_\chi(y, p)},$$

with constants independent of γ, x, y .

Proof. Choose $\tilde{\gamma} \in \mathbf{P}_0$ such that $\tilde{\gamma}p = p_{\mathfrak{D}} \in \Delta_0$. Lemma 3.3, (3.3), and (6.1) give $e^{2\chi(\mathcal{G}(p, p_0)) - \chi(\beta_p(e, \gamma))} = e^{2\chi(\mathcal{G}(p_{\mathfrak{D}}, p_0))} h(p) \asymp h(p)$, uniformly, since Δ_0 is compact and $p_0 \notin \Delta_0$. Both estimates now follow from Lemma 5.7 and (5.3). $\square$

7. LOCAL BEHAVIOR OF PATTERSON–SULLIVAN MEASURES

For the rest of the paper, fix $\psi \in \mathfrak{a}^*$ tangent to ψ_ρ , and set

$$\psi^- := \psi \circ i, \quad \bar{\psi} := \frac{\psi + \psi^-}{2}.$$

Since ψ_ρ is i -invariant, the form ψ^- is also tangent and

$$(7.1) \quad \varphi > 0 \quad \text{on } \mathcal{L}_\rho \setminus \{0\} \quad (\varphi \in \{\psi, \psi^-, \bar{\psi}\}).$$

Since $\bar{\psi} \circ i = \bar{\psi}$, we use the geometric constructions and notation of Sections 5 and 6 with $\chi = \bar{\psi}$. Let $\nu := \nu_{\bar{\psi}}$ be the unique $(\Gamma_\rho, \bar{\psi})$ -Patterson–Sullivan probability measure on Λ_ρ ; thus $\delta_{\bar{\psi}}(\Gamma_\rho) = 1$. For $r > 0$ and $\xi \in \Lambda_\rho \setminus \{p_0\}$, write

$$\mathcal{B}_{\bar{\psi}}(\xi, r) := \{\eta \in \Lambda_\rho \setminus \{p_0\} : D_{\bar{\psi}}(\xi, \eta) < r\}.$$

We use the same notation for balls relative to conjugate limit sets, retaining the ambient premetric $D_{\bar{\psi}}$. For $H < \Gamma_\rho$, let $\delta_{\bar{\psi}}(H)$ be the abscissa of convergence of $\sum_{\gamma \in H} e^{-s\bar{\psi}(\mu(\gamma))}$.

Theorem 7.1 ([14, Theorem 1.1 and Corollary 7.2]). *We have $\sum_{\gamma \in \Gamma_\rho} e^{-\psi(\mu(\gamma))} = \infty$ and $\delta_\psi(\mathbf{P}_i) < 1$ for every $0 \leq i \leq \ell$.*

Lemma 7.2. *There exists $C > 0$ such that, for all distinct $\xi, \eta \in \Lambda_\rho$,*

$$|(\psi - \bar{\psi})(\mathcal{G}(\xi, \eta))| \leq C.$$

Proof. Let z be the nearest-point projection of o to the geodesic joining $f^{-1}(\xi)$ and $f^{-1}(\eta)$. By Lemma 3.13, $\mathcal{G}(\xi, \eta) = \frac{1}{2}(\mu(f(z)) + i\mu(f(z))) + O(1)$. The form $\psi - \bar{\psi}$ vanishes on the first term, since $(\psi - \bar{\psi}) \circ i = -(\psi - \bar{\psi})$. $\square$

7.1. Change of measure.

Lemma 7.3. *Fix $\xi_0 \in \Lambda_\rho \setminus \{p_0\}$ and $D > 0$. Uniformly for $\gamma \in \mathbf{P}_0$, Borel sets $F \subset \Lambda_\rho \setminus \{p_0\}$ with $\nu(F) > 0$ and $D_{\bar{\psi}}$ -diameter at most D , and $\xi \in F$, we have*

$$(7.2) \quad \nu(F) \asymp \frac{1 + D_{\bar{\psi}}(\gamma^{-1}\xi, \xi_0)^2}{1 + D_{\bar{\psi}}(\xi, \xi_0)^2} \nu(\gamma^{-1}F).$$

Proof. For $\gamma \in P_0$, Lemma 3.3 gives $\beta_{p_0}(e, \gamma) = 0$, so (3.3) yields

$$\beta_\eta(e, \gamma) = 2\mathcal{G}(\eta, p_0) - 2\mathcal{G}(\gamma^{-1}\eta, p_0).$$

By Lemma 7.2, applying ψ or $\bar{\psi}$ to this difference changes its value by a uniformly bounded amount. Conformality and Lemma 5.8 therefore give

$$\nu(\gamma^{-1}F) = \int_F e^{\psi(\beta_\eta(e, \gamma))} d\nu(\eta) \asymp \int_F \frac{d_{\bar{\psi}}(\gamma^{-1}\eta, p_0)^2}{d_{\bar{\psi}}(\eta, p_0)^2} d\nu(\eta) \asymp \int_F \frac{1 + D_{\bar{\psi}}(\eta, \xi_0)^2}{1 + D_{\bar{\psi}}(\gamma^{-1}\eta, \xi_0)^2} d\nu(\eta).$$

By P_0 -invariance of $D_{\bar{\psi}}$, both F and $\gamma^{-1}F$ have diameter at most D . Thus Corollary 5.10 shows that the last integrand is uniformly comparable to its value at ξ . Integrating and rearranging proves (7.2). $\square$

Lemma 7.4. *Fix $c > 1$. Uniformly for all $p \in \Delta \cap \Lambda_P(\Gamma_\rho)$, where γ_p is the top representative from (6.2), and Borel sets $F \subset \mathcal{B}_{\bar{\psi}}(p, ch(p))$, we have*

$$\nu(F) \asymp_c e^{-\psi(\mu(\gamma_p))} \nu(\gamma_p^{-1}F).$$

Proof. We claim that there exists $R = R(c) > 0$ such that $\mathcal{B}_{\bar{\psi}}(p, ch(p)) \subset \mathcal{O}_R(o, \gamma_p o)$ for every $p = \gamma_p p_0 \in \Delta$. Otherwise, choose $p_n = \gamma_{p_n} p_0 \in \Delta$ and $x_n \in \mathcal{B}_{\bar{\psi}}(p_n, ch(p_n)) \setminus \mathcal{O}_n(o, \gamma_{p_n} o)$. By Lemma 6.4, $x_n \neq p_n$ for all sufficiently large n . Set $y_n := \gamma_{p_n}^{-1} x_n$ and $z_n := \gamma_{p_n}^{-1} p_0 = x_{p_n} \in \Delta_0$. Then Lemma 6.8 gives $D_{\bar{\psi}}(y_n, z_n) \gg_c 1$. Since $x_n \notin \mathcal{O}_n(o, \gamma_{p_n} o)$, we have $d(o, \gamma_{p_n} o) \geq n$, and hence $\gamma_{p_n}^{-1} \rightarrow \infty$. By Lemma 3.16(3), after passing to a subsequence, $y_n \rightarrow \zeta \in \Lambda_\rho$. By Lemma 6.4 and Lemma 3.16(2), $z_n \rightarrow \zeta$ as well. Since Δ_0 is compact and disjoint from p_0 , we have $\zeta \neq p_0$. Continuity of $D_{\bar{\psi}}$ therefore gives $D_{\bar{\psi}}(y_n, z_n) \rightarrow 0$, a contradiction.

Now let $p = \gamma_p p_i \in \Delta$ for an arbitrary i . For fixed c , (5.3) and Lemma 5.2 imply that, when $h(p)$ is sufficiently small, the ball $\mathcal{B}_{\bar{\psi}}(p, ch(p))$ lies in a fixed compact neighborhood of Δ disjoint from all the points $p_0, \dots, p_\ell$. Hence Lemmas 5.7 and 6.6 give a uniform $C > 1$ such that

$$g_i \mathcal{B}_{\bar{\psi}}(p, ch(p)) \subset \mathcal{B}_{\bar{\psi}}(g_i p, C c h_i(g_i p)).$$

Apply the preceding argument in $\Gamma_i = g_i \Gamma_\rho g_i^{-1}$. The element $g_i \gamma_p g_i^{-1}$ sends p_0 to $g_i p$, and its inverse sends p_0 into $g_i \Delta_i$. Thus the ball on the right is contained in $\mathcal{O}_R(o, (g_i \gamma_p g_i^{-1}) o)$ for some $R = R(c)$, uniform over the finitely many cusp types. Since $g_i \in K$, pulling back by g_i gives $\mathcal{B}_{\bar{\psi}}(p, ch(p)) \subset \mathcal{O}_R(o, \gamma_p o)$.

By Lemma 6.5 and (7.1), only finitely many top representatives γ_p occur when $h(p)$ is bounded below by a fixed positive constant. Increasing R therefore handles the remaining points p and proves the claim for every cusp type.

The claim and Lemma 3.11 give $\beta_\xi(e, \gamma_p) = \mu(\gamma_p) + O_c(1)$ uniformly for $\xi \in \mathcal{B}_{\bar{\psi}}(p, ch(p))$. The result follows from conformality: $\nu(\gamma_p^{-1}F) = \int_F e^{\psi(\beta_\xi(e, \gamma_p))} d\nu(\xi)$. $\square$

7.2. Cusp depth and global shadows. A horoball in $\mathcal{H}$ has type i if it is a Γ -translate of H_i ; a parabolic point $p \in \Lambda_P(\Gamma_\rho)$ has type i if $p \in \Gamma_\rho p_i$.

For a primitive generator u_i of P_i , put $v_i := v_{u_i}$. By Lemma 3.5, $i(v_i) = v_i$ and $v_i \in \text{int } \mathfrak{a}^+$. Set

$$\kappa_i := \frac{1}{2} \psi(v_i) = \frac{1}{2} \bar{\psi}(v_i) > 0.$$

For each $\varphi \in \{\psi, \bar{\psi}\}$, the same lemma gives $\varphi(\mu(u_i^n)) = 2\kappa_i \log |n| + O(1)$ as $|n| \rightarrow \infty$. Since $P_i = \langle u_i \rangle$, the Poincaré series $\sum_{\gamma \in P_i} e^{-s\varphi(\mu(\gamma))}$ has the same abscissa of convergence as $\sum_{n \geq 1} n^{-2s\kappa_i}$. Consequently,

$$\delta_i := \delta_\psi(P_i) = \delta_{\bar{\psi}}(P_i) = \frac{1}{2\kappa_i} < 1,$$

where the last inequality follows from Theorem 7.1.

The cusp calculation (3.5) gives

$$(7.3) \quad \varphi(\mu(f(x)^{-1}f(y))) = \kappa_i d(x, y) + O(1)$$

for $\varphi \in \{\psi, \bar{\psi}\}$ whenever $x, y \in Y$ lie in the same horoball of type i , with uniform error.

Lemma 7.5 (Comparison with cusp depth). *For $\varphi \in \{\psi, \bar{\psi}\}$ and every horoball $H \in \mathcal{H}$ of type i ,*

$$\inf_{g \in \Gamma_\rho} \varphi(\mu(g^{-1}f(x))) = \kappa_i d(x, \partial H) + O(1) \quad (x \in Y \cap H).$$

For $x \in Y_{\text{th}}$, the same infimum is $O(1)$. All errors are uniform in x and H .

Proof. On Y_{th} , the assertion follows from cocompactness and Lemma 3.9. Suppose that $x \in Y \cap H$, where H has type i . Choose $z \in Y \cap \partial H$ such that $d(x, z) = d(x, \partial H) + O(1)$ and $d(z, \Gamma o) = O(1)$. The upper bound follows from (7.3) and Lemma 3.1.

For the lower bound, fix $g \in \Gamma_\rho$ and let $z \in [\hat{g}o, x] \cap \partial H$. By Lemmas 3.9 and 3.10 and (7.3),

$$\varphi(\mu(g^{-1}f(x))) \geq \kappa_i d(z, x) - C \geq \kappa_i d(x, \partial H) - C.$$

Taking the infimum over g proves the claim. $\square$

Theorem 7.6 (Global shadow lemma, [43, Theorems 1.4 and 5.1]). *For every sufficiently large fixed $R > 1$, the following estimates hold uniformly for $\zeta \in \Lambda(\Gamma)$ and $x \in [o, \zeta]$. If x lies in a horoball $H \in \mathcal{H}$ of type i , then*

$$(7.4) \quad \nu(\mathcal{O}_R(o, f(x))) \asymp e^{-\psi(\mu(f(x))) + (2\delta_i - 1)\kappa_i d(x, \partial H)}.$$

If $x \in Y_{\text{th}}$, then $\nu(\mathcal{O}_R(o, f(x))) \asymp e^{-\psi(\mu(f(x)))}$.

Proof. Apply the cited global shadow lemma using Lemmas 3.12 and 7.5. The polynomial term is absent because the peripheral groups are cyclic. $\square$

For $p \in \Lambda_p(\Gamma_\rho)$, let $H_p \in \mathcal{H}$ be the horoball based at $f^{-1}(p)$, and let $t_p \geq 0$ be the entrance time of $\hat{z}_p$ into H_p . If p has type i , choose $g_p \in \Gamma_\rho$ such that $g_p p_i = p$ and $\hat{g}_p o$ lies within a uniform distance of $\hat{z}_p(t_p)$. Writing $P_p := \text{Stab}_{\Gamma_\rho}(p)$, we have $\delta_\psi(P_p) = \delta_{\bar{\psi}}(P_p) = \delta_i$.

For $t \geq t_p$, $d(\hat{z}_p(t), \partial H_p) = t - t_p$. Coarse additivity, Lemma 3.1, and (7.3) therefore give, for $\varphi \in \{\psi, \bar{\psi}\}$,

$$(7.5) \quad \begin{aligned} \varphi(\mu(z_p(t))) &= \varphi(\mu(g_p)) + \kappa_i(t - t_p) + O(1), \\ \varphi(\mu(g_p^{-1}z_p(t))) &= \kappa_i(t - t_p) + O(1). \end{aligned}$$

Corollary 7.7. *For every sufficiently large fixed $R > 1$, uniformly in $p \in \Lambda_p(\Gamma_\rho)$ of type i and $t \geq t_p$,*

$$(7.6) \quad \nu(\mathcal{O}_R(o, z_p(t))) \asymp e^{-\psi(\mu(g_p)) + 2(\delta_i - 1)\psi(\mu(g_p^{-1}z_p(t)))} \asymp e^{-\psi(\mu(g_p)) - 2(1 - \delta_i)\kappa_i(t - t_p)}.$$

Proof. Substitute (7.5), with $\varphi = \psi$, into (7.4). $\square$

7.3. Doubling and reverse doubling.

Proposition 7.8 (Doubling and reverse doubling). *(1) For every $C > 1$,*

$$\nu(\mathcal{B}_{\bar{\psi}}(\xi, Cr)) \asymp_C \nu(\mathcal{B}_{\bar{\psi}}(\xi, r))$$

uniformly for $\xi \in \Delta$ and $0 < r \leq 1/C$.

(2) There exists $C_0 > 1$ such that, for every $\xi \in \Delta$ and $0 < r < 1/C_0$,

$$2\nu(\mathcal{B}_{\bar{\psi}}(\xi, r)) \leq \nu(\mathcal{B}_{\bar{\psi}}(\xi, C_0 r)).$$

Proof. Fix a sufficiently large R and set $\sigma := \max_i |2\delta_i - 1| < 1$. Let $\zeta \in \Lambda(\Gamma)$ and $x, y \in [o, \zeta]$ with $d(o, x) \leq d(o, y)$.

Suppose first that x, y lie in the same horoball $\mathbf{H}$ of type i . By Lemma 3.10, (7.3), and (7.4),

$$\log \frac{\nu(\mathcal{O}_R(o, f(x)))}{\nu(\mathcal{O}_R(o, f(y)))} = \kappa_i d(x, y) - (2\delta_i - 1)\kappa_i (d(y, \partial\mathbf{H}) - d(x, \partial\mathbf{H})) + O(1).$$

Since distance to $\partial\mathbf{H}$ is 1-Lipschitz, this lies between $(1-\sigma)\kappa_i d(x, y) - K$ and $(1+\sigma)\kappa_i d(x, y) + K$ for a uniform K .

Otherwise, let z be the exit point of $[x, y]$ from the horoball $\mathbf{H}$ of type i containing x , and let z' be its entrance point into the horoball $\mathbf{H}'$ of type j containing y . If $x \in Y_{\text{th}}$, set $z = x$; if $y \in Y_{\text{th}}$, set $z' = y$. Omit the corresponding cusp terms in these cases. Coarse additivity and (7.3) give

$$\psi(\mu(f(x)^{-1}f(y))) = \kappa_i d(x, z) + \psi(\mu(f(z)^{-1}f(z'))) + \kappa_j d(z', y) + O(1).$$

Apply Lemma 3.9 to the middle term and use

$$d(x, \partial\mathbf{H}) \leq d(x, z), \quad d(y, \partial\mathbf{H}') \leq d(z', y).$$

Since $\sigma < 1$, $\min_i \kappa_i > 0$, and $d(x, z) + d(z, z') + d(z', y) = d(x, y)$, the global shadow estimates give, after enlarging K to cover both cases,

$$(7.7) \quad K^{-1}d(x, y) - K \leq \log \frac{\nu(\mathcal{O}_R(o, f(x)))}{\nu(\mathcal{O}_R(o, f(y)))} \leq Kd(x, y) + K.$$

Coarse additivity and Lemma 3.9 also bound $\bar{\psi}(\mu(f(y))) - \bar{\psi}(\mu(f(x)))$ above and below by positive multiples of $d(x, y)$, up to uniform additive errors. Together with Proposition 5.3 and (7.7), this gives power bounds for ratios of $d_{\bar{\psi}}$ -ball masses. Here continuity and divergence of $t \mapsto \bar{\psi}(\mu(z_{\xi}(t)))$ allow prescribed sufficiently small scales to be realized in order along each ray. The uniform comparison of $d_{\bar{\psi}}$ and $D_{\bar{\psi}}$ near Δ transfers these bounds to $\mathcal{B}_{\bar{\psi}}$ -balls. Thus there exist $L_1 \geq 1$, $0 < \beta_- \leq \beta_+$, and $0 < r_0 \leq 1$ such that, for every $\xi \in \Delta$ and $0 < r \leq s \leq r_0$,

$$(7.8) \quad L_1^{-1}(r/s)^{\beta_+} \leq \frac{\nu(\mathcal{B}_{\bar{\psi}}(\xi, r))}{\nu(\mathcal{B}_{\bar{\psi}}(\xi, s))} \leq L_1(r/s)^{\beta_-}.$$

Taking $s = Cr$ proves (1) when $Cr \leq r_0$; compactness covers the remaining radii. For (2), choose $C_0 > 1/r_0$ such that $L_1(C_0 r_0)^{-\beta_-} \leq 1/2$. Apply the upper bound in (7.8) with $s = \min\{C_0 r, r_0\}$, noting that $r/s \leq (C_0 r_0)^{-1}$ whenever $0 < r < 1/C_0$. $\square$

We fix a choice of C_0 satisfying (2) for use in the geometric partition construction below.

Corollary 7.9. *There exist $c, \beta > 0$ such that, for every $\xi \in \Delta$ and $0 < r < 1$,*

$$\nu(\mathcal{B}_{\bar{\psi}}(\xi, r)) \geq cr^{\beta}.$$

Proof. Since ν has full support and Δ is compact, $\inf_{\xi \in \Delta} \nu(\mathcal{B}_{\bar{\psi}}(\xi, r_0)) > 0$. Apply (7.8) with $s = r_0$ to obtain the estimate for $r \leq r_0$, with $\beta = \beta_+$. The remaining radii follow by monotonicity. $\square$

7.4. Parabolic ball estimates.

Corollary 7.10. *Uniformly for $p \in \Delta \cap \Lambda_p(\Gamma_\rho)$ and $0 < r \leq h(p)$, we have*

$$(7.9) \quad \nu(\mathcal{B}_{\bar{\psi}}(p, r)) \asymp e^{-\psi(\mu(g_p))} \left(\frac{r}{h(p)} \right)^{2(1-\delta_\psi(P_p))}.$$

Proof. Fix $p \in \Delta \cap \Lambda_p(\Gamma_\rho)$ of type i . By the choice of g_p and Lemmas 3.11 and 6.3, $h(p) \asymp e^{-\bar{\psi}(\mu(g_p))}$. Thus (7.5) gives

$$r_p(t) := e^{-\bar{\psi}(\mu(z_p(t)))} \asymp h(p) e^{-\kappa_i(t-t_p)} \quad (t \geq t_p).$$

Since $d_{\bar{\psi}}$ and $D_{\bar{\psi}}$ are uniformly comparable near Δ , Proposition 5.3 gives, for sufficiently large fixed R , constants $T, t_0 > 0$ such that, for every $t \geq t_p + T$,

$$(7.10) \quad \mathcal{O}_R(o, z_p(t+t_0)) \cap \Lambda_\rho \subset \mathcal{B}_{\bar{\psi}}(p, r_p(t)) \subset \mathcal{O}_R(o, z_p(t)) \cap \Lambda_\rho.$$

Applying (7.6) to the two shadows gives

$$\nu(\mathcal{B}_{\bar{\psi}}(p, r_p(t))) \asymp e^{-\psi(\mu(g_p))} \left(\frac{r_p(t)}{h(p)} \right)^{2(1-\delta_i)}.$$

The function r_p is continuous, tends to zero, and satisfies $r_p(t_p + T) \asymp h(p)$. Hence these radii cover $(0, ch(p)]$ for some uniform $c > 0$. Proposition 7.8(1) and compactness extend the estimate to the remaining radii $ch(p) < r \leq h(p)$. $\square$

8. RECURRENCE OF THE TRANSLATION FLOW

We describe the stable and unstable leaves of the translation flow on $\Omega_{\bar{\psi}}$ and prove the uniform recurrence property used in the coding construction. We then show that a return to a fixed compact set forces the forward endpoint to lie near a parabolic limit point whose height is comparable to the scale determined by the return time.

8.1. Stable and unstable leaves. Recall that $\tilde{\Omega}_{\bar{\psi}} = \Lambda_\rho^{(2)} \times \mathbb{R}$ and $\Omega_{\bar{\psi}} = \Gamma_\rho \backslash \tilde{\Omega}_{\bar{\psi}}$.

For $g \in G$ with $(g^+, g^-) \in \Lambda_\rho^{(2)}$, write

$$[g] := (g^+, g^-, \bar{\psi}(\beta_{g^+}(e, g))) \in \tilde{\Omega}_{\bar{\psi}}.$$

For $x = [g] = (x^+, x^-, s_x) \in \tilde{\Omega}_{\bar{\psi}}$, set $s_x^- := \bar{\psi}(\beta_{g^-}(e, g))$. Since $\bar{\psi} \circ i = \bar{\psi}$, we have $s_x^- = 2\bar{\psi}(\mathcal{G}(x^+, x^-)) - s_x$, so s_x^- depends only on x .

For $x, y \in \tilde{\Omega}_{\bar{\psi}}$, define

$$d_+(x, y) := e^{(s_x + s_y)/2} d_{\bar{\psi}}(x^+, y^+), \quad d_-(x, y) := e^{(s_x^- + s_y^-)/2} d_{\bar{\psi}}(x^-, y^-).$$

They are nonnegative, symmetric, Γ_ρ -invariant and satisfy that for all $t \in \mathbb{R}$,

$$d_+(xa_t, ya_t) = e^t d_+(x, y), \quad d_-(xa_t, ya_t) = e^{-t} d_-(x, y).$$

For $x = [g]$, set

$$W^-(x) := \{(x^+, \eta, s_x) : \eta \in \Lambda_\rho \setminus \{x^+\}\} = [gN] \cap \tilde{\Omega}_{\bar{\psi}},$$

$$W^+(x) := \{(\xi, x^-, 2\bar{\psi}(\mathcal{G}(\xi, x^-)) - s_x^-) : \xi \in \Lambda_\rho \setminus \{x^-\}\} = [g\tilde{N}] \cap \tilde{\Omega}_{\bar{\psi}}.$$

These leaves are stable and unstable for the translation flow in the following sense: by [44, Proposition 8.6], there exists a metric d inducing the topology of $\Omega_{\bar{\psi}}$ such that, as $t \rightarrow +\infty$,

$$d(\Gamma_{\rho} x a_t, \Gamma_{\rho} y a_t) \rightarrow 0 \quad \text{for } y \in W^-(x) \quad \text{and} \quad d(\Gamma_{\rho} x a_{-t}, \Gamma_{\rho} y a_{-t}) \rightarrow 0 \quad \text{for } y \in W^+(x).$$

For $r > 0$, set

$$W^{\pm}(x, r) := \{y \in W^{\pm}(x) : d_{\pm}(x, y) < r\}.$$

These sets form a neighborhood basis on the corresponding leaves.

8.2. Local product structure. For $x, y \in \tilde{\Omega}_{\bar{\psi}}$ with $(y^+, x^-) \in \mathcal{F}^{(2)}$, the intersection $W^+(x) \cap W^-(y a_{\mathbb{R}})$ consists of a unique point, denoted by $\langle x, y \rangle$. Explicitly,

$$\langle x, y \rangle = (y^+, x^-, s_y + \vartheta(x, y)), \quad \text{where } \vartheta(x, y) := 2\bar{\psi}(\mathcal{G}(y^+, x^-)) - s_x^- - s_y.$$

That is, $\vartheta(x, y)$ is the unique time t such that $\langle x, y \rangle \in W^-(y a_t)$. The maps ϑ and $\langle \cdot, \cdot \rangle$ are continuous wherever (y^+, x^-) is transverse and, by (3.3), are respectively Γ_{ρ} -invariant and Γ_{ρ} -equivariant. Since $\vartheta(x, x) = 0$ and $\langle x, x \rangle = x$, continuity and compactness imply that, for every compact $Q \subset \tilde{\Omega}_{\bar{\psi}}$,

$$|\vartheta(x, y)| \rightarrow 0, \quad d_+(x, \langle x, y \rangle) \rightarrow 0, \quad d_-(y a_{\vartheta(x, y)}, \langle x, y \rangle) \rightarrow 0$$

uniformly as (x, y) approaches the diagonal in $Q \times Q$.

We choose the compact section

$$\tilde{\Delta}_0 := \{(\xi, p_0, 0) : \xi \in \Delta_0\} \subset \tilde{\Omega}_{\bar{\psi}}$$

of the flow lines with negative endpoint p_0 and positive endpoint in Δ_0 .

Lemma 8.1. *For every $r > 0$, the set $\bigcup_{x \in \tilde{\Delta}_0} W^+(x, r)$ is relatively compact in $\tilde{\Omega}_{\bar{\psi}}$.*

Proof. By compactness of Δ_0 , the formula for d_+ shows that $d_+(x, y) \rightarrow \infty$ as $y^+ \rightarrow p_0$, uniformly for $x \in \tilde{\Delta}_0$ and $y \in W^+(x)$. Thus $d_+(x, y) < r$ keeps y^+ in a fixed compact subset of $\Lambda_{\rho} \setminus \{p_0\}$. Finally, $s_y = 2\bar{\psi}(\mathcal{G}(y^+, p_0) - \mathcal{G}(\xi, p_0))$ is uniformly bounded by continuity; since $y^- = p_0$, the claim follows. $\square$

8.3. Uniform recurrence. For $\delta \in \mathbb{R}$, let

$$W_{p_0, \delta}^- := \{(p_0, \eta, \delta) : \eta \in \Lambda_{\rho} \setminus \{p_0\}\} \subset \tilde{\Omega}_{\bar{\psi}}$$

denote the stable leaf with positive endpoint p_0 and $\mathbb{R}$ -coordinate δ .

Lemma 8.2. *Let $Q \subset \Omega_{\bar{\psi}}$ be compact and let $\delta \in \mathbb{R}$. There exists $T_* = T_*(Q, \delta) > 0$ such that, for every $x \in \tilde{\Omega}_{\bar{\psi}}$ with $\Gamma_{\rho} x \in Q$, there is $0 < t < T_*$ for which*

$$W^+(x, 1) a_t \cap \Gamma_{\rho} W_{p_0, \delta}^- \neq \emptyset.$$

Proof. Applying Theorem 3.19 to a tangent multiple of $\bar{\psi}$ and rescaling time, we obtain a mixing invariant probability measure of full support on $\Omega_{\bar{\psi}}$.

Choose a compact set $\tilde{Q} \subset \tilde{\Omega}_{\bar{\psi}}$ whose image contains Q , and choose $0 < \varepsilon < 1/4$ so that $\bigcup_{x' \in W^+(x, \varepsilon)} W^+(x', \varepsilon) \subset W^+(x, 1)$ for all $x \in \tilde{Q}$. After decreasing ε if necessary, choose a relatively compact product neighborhood $\mathcal{V}$ of a point in $W_{p_0, \delta}^-$ such that every $y_0 \in \mathcal{V}$ can be written as $y_0 = y_1 a_u$, where $z \in W_{p_0, \delta}^-$, $y_1 \in W^+(z, \varepsilon)$, and $|u| < \varepsilon$, with z, y_1 ranging over fixed compact sets. We may also arrange that, whenever $w \in W^-(y_1, \varepsilon)$, the bracket $\langle w, z \rangle$ is defined and satisfies $|\vartheta(w, z)| < 1$ and $d_+(w, \langle w, z \rangle) < 1$.

Cover $\tilde{Q}$ by finitely many sufficiently small product neighborhoods $\mathcal{U}_1, \dots, \mathcal{U}_k$. Their images and that of $\mathcal{V}$ are nonempty open sets in $\Omega_{\tilde{\psi}}$. Mixing therefore gives a common $L > 2$, with $e^{-(L-2\varepsilon)} < \varepsilon$, such that every $\mathcal{U}_j$ meets $\Gamma_\rho \mathcal{V} a_{-L}$.

For $x \in \tilde{Q}$, choose j with $x \in \mathcal{U}_j$, and choose $y_0 = y_1 a_u \in \mathcal{V}$ and $\gamma \in \Gamma_\rho$ such that $\gamma y_0 a_{-L} \in \mathcal{U}_j$. Then there exist $|\sigma| < \varepsilon$ and

$$x' \in W^+(x, \varepsilon) \cap W^-(\gamma y_0 a_{-L+\sigma}, \varepsilon).$$

Put $T := L - u - \sigma$ and $w := \gamma^{-1}(x' a_T)$. Equivariance and the scaling relation for d_- give $w \in W^-(y_1, \varepsilon e^{-T}) \subset W^-(y_1, \varepsilon)$. Set $\theta := \vartheta(w, z)$ and $q_0 := \gamma \langle w, z \rangle$. By the choice of $\mathcal{V}$ and the definition of the bracket, $|\theta| < 1$ and $q_0 \in W^+(x' a_T, 1) \cap W^-(\gamma z a_\theta)$.

Set $q := q_0 a_{-\theta}$ and $t := T - \theta$. Then $q \in \gamma W_{p_0, \delta}^-$, while

$$q a_{-t} = q_0 a_{-T} \in W^+(x', e^{-T}) \subset W^+(x', \varepsilon) \subset W^+(x, 1).$$

Thus q belongs to the required intersection. Since $0 < t < L + 2$, we may take $T_* := L + 2$. Equivariance extends the conclusion to every lift of a point of Q . $\square$

The next lemma converts uniform recurrence into a parabolic approximation with controlled height.

Lemma 8.3. *For every $\tau > 1$, there exists $C > 1$ with the following property. For every compact $Q \subset \Omega_{\tilde{\psi}}$, there exists $U_0 = U_0(\tau, Q) > 0$ such that, if $x = (x^+, p_0, 0) \in \tilde{\Delta}_0$ and $\Gamma_\rho x a_{n \log \tau} \in Q$ for some $n \in \mathbb{N}$, then there exists $p \in \Lambda_p(\Gamma_\rho) \setminus \{p_0\}$ satisfying*

$$(8.1) \quad D_{\tilde{\psi}}(p, x^+) \leq C \tau^{-n}, \quad e^{-U_0} \tau^{-n} \leq h(p) \leq \tau^{-n}.$$

Proof. Fix $\tau > 1$ and a parameter $\delta \in \mathbb{R}$ to be chosen below. Let $T_* = T_*(Q, \delta)$ be given by Lemma 8.2. Applying that lemma to $x a_{n \log \tau}$, choose $\gamma \in \Gamma_\rho$, $z \in W_{p_0, \delta}^-$, and $0 < t < T_*$ such that

$$\gamma z \in W^+(x a_{n \log \tau}, 1) a_t.$$

Set $y := \gamma z a_{-n \log \tau - t}$ and $p := y^+ = \gamma p_0$. Then $y \in W^+(x, \tau^{-n})$. By Lemma 8.1, all such y lie in a fixed relatively compact set, independent of Q, n, t, δ . Consequently, p lies in a fixed compact subset of $\Lambda_\rho \setminus \{p_0\}$ and s_y is uniformly bounded.

Since $s_x = 0$, the definition of d_+ gives $e^{s_y/2} d_{\tilde{\psi}}(x^+, p) = d_+(x, y) < \tau^{-n}$. The uniform comparison of $d_{\tilde{\psi}}$ and $D_{\tilde{\psi}}$ on this compact set therefore gives $D_{\tilde{\psi}}(x^+, p) \leq C \tau^{-n}$. It follows that $s_y = \tilde{\psi}(\beta_p(e, \gamma)) + \delta - n \log \tau - t$. By Lemma 6.3 and the bound on s_y , there is a constant $c > 1$, independent of Q and δ , such that

$$c^{-1} e^\delta \tau^{-n} e^{-T_*} \leq h(p) \leq c e^\delta \tau^{-n}.$$

Choose $\delta := -\log(2c)$ and then set $U_0 := T_*(Q, \delta) + \log c - \delta$. This gives (8.1). $\square$

9. CONSTRUCTION OF THE GEOMETRIC PARTITION

For a small parameter $\eta > 0$ to be chosen below, we construct neighborhoods $J_{p, \eta}$ of parabolic points $p \in \Delta$, comparable to $D_{\tilde{\psi}}$ -balls of radius $\eta h(p)$. We remove selected neighborhoods from each Δ_i in successive height ranges, starting with the largest heights. The residual set $\mathcal{Z}_{i, n}(\eta)$ consists of the points of Δ_i remaining after n steps. We prove that its measure decays exponentially in n (Proposition 9.8).

When discussing intervals and their order, we use the Euclidean coordinate transported by f on $\Lambda_\rho \setminus \{p_0\}$. For nonempty subsets $A, B \subset \Lambda_\rho \setminus \{p_0\}$, set

$$D_{\bar{\psi}}(A, B) := \inf_{a \in A, b \in B} D_{\bar{\psi}}(a, b), \quad \mathcal{B}_{\bar{\psi}}(A, r) := \{x \in \Lambda_\rho \setminus \{p_0\} : D_{\bar{\psi}}(x, A) < r\}.$$

Let $C \geq 1$ be the constant in Lemma 5.9.

Corollary 9.1 (Quasi-monotonicity of $D_{\bar{\psi}}$). *For any $\xi_1, \xi_2, \xi_3 \in \Lambda_\rho \setminus \{p_0\}$ satisfying $d_{\text{Euc}}(\xi_1, \xi_2) \leq d_{\text{Euc}}(\xi_1, \xi_3)$ and $d_{\text{Euc}}(\xi_2, \xi_3) \leq d_{\text{Euc}}(\xi_1, \xi_3)$, one has*

$$D_{\bar{\psi}}(\xi_1, \xi_2) \leq C D_{\bar{\psi}}(\xi_1, \xi_3) \quad \text{and} \quad D_{\bar{\psi}}(\xi_2, \xi_3) \leq C D_{\bar{\psi}}(\xi_1, \xi_3).$$

Proof. Each of the two relevant Euclidean distance ratios is at most one. The upper bound in Lemma 5.9, applied at the appropriate endpoint, gives both assertions. $\square$

Recall from (4.3) that $\Delta = \bigcup_{i=0}^\ell \Delta_i$.

Lemma 9.2. *There exists $C_{\text{fd}} > 1$ such that, for every $0 < \eta < 1$, there are a finite-index subgroup $P'_0 = P'_0(\eta) < P_0$ and a compact fundamental domain $R_{0,\eta} \subset \Lambda_\rho \setminus \{p_0\}$ for its action satisfying*

$$(9.1) \quad \mathcal{B}_{\bar{\psi}}(\Delta, \eta^{-1}) \subset R_{0,\eta} \subset \mathcal{B}_{\bar{\psi}}(x, C_{\text{fd}}\eta^{-1}) \quad (x \in \Delta).$$

Proof. Choose a generator h_0 of P_0 acting by $x \mapsto x + 1$, and normalize $\hat{\Delta}_\infty = [0, 1]$. By (5.4), $D_{\bar{\psi}}$ is invariant under translation by h_0 . Let a, b be the left and right endpoints of $\overline{\mathcal{B}_{\bar{\psi}}(\Delta, \eta^{-1})}$ and put $n := \lfloor a \rfloor$ and $m := \lfloor b \rfloor$. Since this set contains Δ_0 , we have $m - n \geq 1$. Define

$$P'_0 := \langle h_0^{m-n} \rangle, \quad R_{0,\eta} := [n, m] \cap \Lambda(\Gamma).$$

Then $R_{0,\eta}$ is a compact fundamental domain for P'_0 and contains $\mathcal{B}_{\bar{\psi}}(\Delta, \eta^{-1})$.

For the upper bound, note $D_{\bar{\psi}}(a, \Delta), D_{\bar{\psi}}(b, \Delta) \leq \eta^{-1}$. Hence Corollary 5.10 gives $D_{\bar{\psi}}(x, a) + D_{\bar{\psi}}(x, b) \ll \eta^{-1}$ uniformly for $x \in \Delta$ and $0 < \eta < 1$. As $x \in [a, b]$, Corollary 9.1 therefore yields

$$D_{\bar{\psi}}(x, \xi) \ll \eta^{-1} \quad (x \in \Delta, \xi \in [a, b] \cap \Lambda(\Gamma)).$$

Since $a - n, m - b < 1$, $D_{\bar{\psi}}$ -diameters for $[n, a] \cap \Lambda(\Gamma)$ and $[b, m] \cap \Lambda(\Gamma)$ are uniformly bounded. Applying Corollary 5.10 extends the preceding bound to every $\xi \in R_{0,\eta}$. Choosing $C_{\text{fd}} > 1$ larger than the implied constant proves (9.1). $\square$

The proof applies without change to each of the finitely many conjugated systems

$$\Gamma_i := g_i \Gamma_\rho g_i^{-1}, \quad g_i P_i g_i^{-1} < \Gamma_i,$$

using $g_i \Delta_i$ as the compact set of centers. Hence, after taking the maximum of finitely many constants, the same C_{fd} works for all $0 \leq i \leq \ell$. For each i and η , fix once and for all a compact fundamental domain $R_{i,\eta}$ constructed as in Lemma 9.2 and a finite-index subgroup $P'_{i,\eta} < P_i$ such that $R_{i,\eta}$ is a fundamental domain for $g_i P'_{i,\eta} g_i^{-1}$ and

$$(9.2) \quad \mathcal{B}_{\bar{\psi}}(g_i x, \eta^{-1}) \subset R_{i,\eta} \subset \mathcal{B}_{\bar{\psi}}(g_i x, C_{\text{fd}}\eta^{-1}) \quad (x \in \Delta_i).$$

9.1. Building blocks of the coding. Fix $0 < \eta < \eta_*$, where $\eta_* < 1/4$ will be decreased a finite number of times below. Until η is fixed in Proposition 9.8, we retain it in the notation. For each parabolic point $p \in \Delta$, we construct a compact neighborhood $J_{p,\eta}$ of p that will serve as a building block of the coding. Let $p \in \Delta \cap \Lambda_p(\Gamma_\rho)$ have type i , with top representative γ_p as in (6.2); so $p = \gamma_p p_i$ and $x_p = \gamma_p^{-1} p_i \in \Delta_i$.

Here and below, complements are taken in $\Lambda_\rho \setminus \{p_0\}$ and closures in Λ_ρ .

Definition 9.3 (J_p and N_p). For $p \in \Delta \cap \Gamma_\rho p_i$, define

$$(9.3) \quad \begin{aligned} J_p = J_{p,\eta} &:= \overline{\Lambda_\rho \setminus \gamma_p g_i^{-1} R_{i,\eta}} = \overline{\gamma_p g_i^{-1} R_{i,\eta}^c}, \\ N_p &:= \{\gamma_1 \in \mathbf{P}_i : \gamma_p \gamma_1 \Delta_i \subset J_p\} = \{\gamma_1 \in \mathbf{P}_i : \gamma_1 \Delta_i \subset \overline{g_i^{-1} R_{i,\eta}^c}\}. \end{aligned}$$

The set $J_{p,\eta}$ is the image of the closed neighborhood $\overline{R_{i,\eta}^c}$ of p_0 under $\gamma_p g_i^{-1}$, which sends p_0 to p .

For $\gamma_1 \in N_p$, the invariance (5.4) applied in the conjugate system gives

$$(9.4) \quad D_{\bar{\psi}}(g_i(\gamma_p \gamma_1)^{-1} p_i, g_i \Delta_i) = D_{\bar{\psi}}(g_i x_p, g_i \gamma_1 \Delta_i) \geq \eta^{-1}.$$

The last inequality follows from (9.2), since $\mathcal{B}_{\bar{\psi}}(g_i x_p, \eta^{-1}) \subset R_{i,\eta}$.

Lemma 9.4 (Peripheral tiling of J_p). For fixed η , $0 \leq i \leq \ell$, and $p \in \Delta \cap \Gamma_\rho p_i$, we have:

- (1) The complement $\mathbf{P}_i \setminus N_p$ is finite.
- (2) There is a countable set $\mathcal{E}_p \subset \{p\} \cup \bigcup_{\gamma_1 \in \mathbf{P}_i} \gamma_p \gamma_1 \partial \Delta_i$ such that

$$J_p \setminus \mathcal{E}_p = \bigsqcup_{\gamma_1 \in N_p} \gamma_p \gamma_1 \text{int } \Delta_i.$$

In particular, $\nu(J_p \Delta \cup_{\gamma_1 \in N_p} \gamma_p \gamma_1 \Delta_i) = 0$ where Δ denotes the symmetric difference.

Proof. All but finitely many tiles lie in $\overline{R_{i,\eta}^c}$, so the definition of N_p implies that $\mathbf{P}_i \setminus N_p$ is finite. The map $\gamma_p g_i^{-1}$ sends each tile $g_i \gamma_1 \Delta_i$ to $\gamma_p \gamma_1 \Delta_i$ and sends p_0 to p . This gives the asserted decomposition, with $\mathcal{E}_p$ consisting of p and the endpoints of these images. $\square$

We compare $J_{p,\eta}$ with $D_{\bar{\psi}}$ -balls centered at p whose radii are proportional to $\eta h(p)$.

Lemma 9.5. There exist $\eta_* > 0$, $0 < C_1 < 1$, and $C_2 > 1$ such that, for every $0 < \eta < \eta_*$, $0 \leq i \leq \ell$, and $p \in \Delta \cap \Gamma_\rho p_i$, we have

$$\begin{aligned} \mathcal{B}_{\bar{\psi}}\left(g_i p, \frac{C_1 \eta h_i(g_i p)}{C_{\text{fd}}}\right) &\subset g_i J_{p,\eta} \subset \mathcal{B}_{\bar{\psi}}\left(g_i p, \frac{\eta h_i(g_i p)}{C_1}\right), \\ \mathcal{B}_{\bar{\psi}}\left(p, \frac{\eta h(p)}{C_2}\right) &\subset J_{p,\eta} \subset \mathcal{B}_{\bar{\psi}}(p, C_2 \eta h(p)). \end{aligned}$$

Proof. Fix i and $p \in \Delta \cap \Gamma_\rho p_i$. Recall that $\alpha_p := g_i \gamma_p g_i^{-1}$ satisfies $\alpha_p p_0 = g_i p$ and $\alpha_p^{-1} p_0 = g_i x_p \in g_i \Delta_i$. By Lemma 6.8, applied to Γ_i ,

$$(9.5) \quad D_{\bar{\psi}}(\alpha_p^{-1} y, g_i x_p) \asymp \frac{h_i(g_i p)}{D_{\bar{\psi}}(y, g_i p)} \quad (y \in g_i \Lambda_\rho \setminus \{p_0, g_i p\}).$$

If $y \in g_i J_{p,\eta} \setminus \{g_i p\}$, then $\alpha_p^{-1} y \in \overline{R_{i,\eta}^c}$. Since $x_p \in \Delta_i$, the first inclusion in (9.2) gives $D_{\bar{\psi}}(\alpha_p^{-1} y, g_i x_p) \geq \eta^{-1}$. Together with (9.5), this implies $D_{\bar{\psi}}(y, g_i p) \ll \eta h_i(g_i p)$. Conversely, if $y \in g_i \Lambda_\rho \setminus (g_i J_{p,\eta} \cup \{p_0\})$, then $\alpha_p^{-1} y \in R_{i,\eta}$, so the second inclusion in (9.2) gives

$D_{\bar{\psi}}(\alpha_p^{-1}y, g_i x_p) < C_{\text{fd}}\eta^{-1}$. Hence $D_{\bar{\psi}}(y, g_i p) \gg \frac{\eta h_i(g_i p)}{C_{\text{fd}}}$. Choosing $C_1 > 0$ sufficiently small proves the first pair of inclusions. By Lemmas 6.5 and 6.6, $h_i(g_i p) \asymp h(p) \ll 1$ uniformly. For sufficiently small η_* , Lemma 5.7 therefore gives uniform bi-Lipschitz bounds on the relevant neighborhoods. Pulling back the first pair of inclusions by g_i^{-1} yields the second pair, with a uniform constant $C_2 > 1$. $\square$

9.2. The inductive partition. Let $\nu_i := (g_i)_*\nu$; this is the unique (Γ_i, ψ) -Patterson-Sullivan measure on $g_i\Lambda_\rho$. Apply Corollary 9.1 to each conjugated representation $\gamma \mapsto g_i\rho(\gamma)g_i^{-1}$, with the same form $\bar{\psi}$ and distinguished point $g_i p_i = p_0$. Choose $c \geq 1$ larger than all the resulting constants.

Lemma 9.6. *There exists $H > c^2$ such that, for every $0 \leq i \leq \ell$, $p \in \Lambda_p(\Gamma_i) \setminus \{p_0\}$, and $0 < r \leq h_i(p)$,*

$$(9.6) \quad \nu_i(\mathcal{B}_{\bar{\psi}}(p, r/\sqrt{H})) \leq \frac{1}{2} \nu_i(\mathcal{B}_{\bar{\psi}}(p, r/c)).$$

Proof. To apply Corollary 7.10, first move the center p into the compact fundamental domain: choose $\gamma \in g_i P_i g_i^{-1}$ such that $q := \gamma^{-1}p \in g_i \Delta_i$. By invariance, $h_i(q) = h_i(p)$ and $\gamma^{-1}\mathcal{B}_{\bar{\psi}}(p, s) = \mathcal{B}_{\bar{\psi}}(q, s)$. Since $r \leq h_i(q) \ll 1$, the balls under consideration have uniformly bounded diameter. Applying Lemma 7.3 in Γ_i to both balls, with the same center p , gives

$$\frac{\nu_i(\mathcal{B}_{\bar{\psi}}(p, r/\sqrt{H}))}{\nu_i(\mathcal{B}_{\bar{\psi}}(p, r/c))} \asymp \frac{\nu_i(\mathcal{B}_{\bar{\psi}}(q, r/\sqrt{H}))}{\nu_i(\mathcal{B}_{\bar{\psi}}(q, r/c))}.$$

By Corollary 7.10 applied in Γ_i , the right-hand side is $\ll (c/\sqrt{H})^a$ for some uniform $a > 0$. Hence taking H sufficiently large proves the claim. $\square$

Fix H as in Lemma 9.6 and set

$$h_n := H^{-n} \quad (n \geq 0).$$

For $q = g_i p$ with $p \in \Delta$, write $J_{\Gamma_i, q, \eta} := g_i J_{p, \eta}$. Take the constants C_2 and C_s in Lemma 9.5 and proposition 6.7 uniformly over the finitely many groups Γ_i , and decrease η_* if necessary so that $4C_2\eta_* < 1$.

Definition 9.7. For $0 \leq i \leq \ell$ and $0 < \eta < \eta_*$, set $\mathcal{Z}_{i,0}(\eta) := \Delta_i$. For $n \geq 1$, define inductively

$$P_{i,n}(\eta) := \left\{ p \in \Delta_i \cap \Lambda_p(\Gamma_\rho) : h_n < \eta h_i(g_i p) \leq h_{n-1}, \mathcal{B}_{\bar{\psi}}\left(g_i p, \frac{h_{n-1}}{4\eta}\right) \subset g_i \mathcal{Z}_{i,n-1}(\eta) \right\},$$

where $\mathcal{B}_{\bar{\psi}}(g_i p, r) := \{q \in g_i \Lambda_\rho \setminus \{p_0\} : D_{\bar{\psi}}(g_i p, q) < r\}$. Thus $P_{i,n}(\eta)$ consists of the centers of the sets $J_{p,\eta}$ removed from Δ_i at stage n . Set

$$D_{i,n}(\eta) := \bigcup_{p \in P_{i,n}(\eta)} J_{p,\eta}, \quad \mathcal{Z}_{i,n}(\eta) := \mathcal{Z}_{i,n-1}(\eta) \setminus D_{i,n}(\eta).$$

For $i = 0$, abbreviate $P_n(\eta) := P_{0,n}(\eta)$, $D_n(\eta) := D_{0,n}(\eta)$, and $\mathcal{Z}_n(\eta) := \mathcal{Z}_{0,n}(\eta)$.

For $p \in P_{i,n}(\eta)$, the ball condition implies $p \in \mathcal{Z}_{i,n-1}(\eta)$. By Lemma 9.5, applied to Γ_i ,

$$g_i J_{p,\eta} \subset \mathcal{B}_{\bar{\psi}}(g_i p, C_2 \eta h_i(g_i p)) \subset \mathcal{B}_{\bar{\psi}}\left(g_i p, \frac{h_{n-1}}{4\eta}\right) \subset g_i \mathcal{Z}_{i,n-1}(\eta).$$

Thus $D_{i,n}(\eta) \subset \mathcal{Z}_{i,n-1}(\eta)$, and all three sets lie in Δ_i .

Moreover, Proposition 6.7, applied to Γ_i , gives $D_{\bar{\psi}}(g_i p, g_i p') > C_s h_n / \eta$ for distinct $p, p' \in P_{i,n}(\eta)$. Since $g_i \Delta_i$ is compact, $P_{i,n}(\eta)$ is finite. Hence $D_{i,n}(\eta)$ is a finite union of the sets $J_{p,\eta}$. The following proposition will be proved in Section 10.

Proposition 9.8 (Exponential decay of residual mass). *For all sufficiently small $\eta > 0$, there exists $\varepsilon_0 > 0$ such that, for every $0 \leq i \leq \ell$ and sufficiently large n ,*

$$\nu(\mathcal{Z}_{i,n}(\eta)) \leq (1 - \varepsilon_0)^n.$$

We take $\eta > 0$ sufficiently small for the preceding proposition and all requirements above, allowing further reductions until its final choice in the proof of Lemma 11.5. We suppress η from the notation:

$$P_{i,n} := P_{i,n}(\eta), \quad J_p := J_{p,\eta}, \quad D_{i,n} := D_{i,n}(\eta), \quad \mathcal{Z}_{i,n} := \mathcal{Z}_{i,n}(\eta),$$

and write

$$P_n := P_{0,n}, \quad D_n := D_{0,n}, \quad \mathcal{Z}_n := \mathcal{Z}_{0,n}.$$

We also abbreviate $R_i := R_{i,\eta}$ and $P'_i := P'_{i,\eta}$.

9.3. Boundary conventions. By construction, each J_p has the form $J_p = f(\hat{J}_p \cap \Lambda(\Gamma))$ for a closed interval $\hat{J}_p \subset \partial \mathbb{H}^2$. Define $\partial J_p := f(\partial \hat{J}_p \cap \Lambda(\Gamma))$. This set contains at most two points and is ν -null. For p of type i , we have $g_i \gamma_p^{-1} \partial J_p = \partial R_i$.

For $n \geq 0$, define

$$(9.7) \quad \partial \mathcal{Z}_n := \partial \Delta_0 \cup \bigcup_{k=1}^n \bigcup_{p \in P_k} \partial J_p,$$

where the union is empty when $n = 0$.

We call the sets $\partial \Delta_0$ and ∂J_p the *boundary pieces*; a piece may contain two endpoints. Thus $\partial \mathcal{Z}_n$ denotes their union, rather than a family of connected components. The separation statement in Lemma 10.6 identifies the pieces relevant to the residual set $\mathcal{Z}_n$.

10. EXPONENTIAL DECAY OF THE RESIDUAL MASS

Proposition 9.8 follows from the case $i = 0$ stated below. Indeed, applying this case to each Γ_i with measure $\nu_i = (g_i)_* \nu$ gives the desired estimate, since $\nu_i(g_i \mathcal{Z}_{i,n}(\eta)) = \nu(\mathcal{Z}_{i,n}(\eta))$.

Proposition 10.1. *For every sufficiently small $\eta > 0$, there exists $\varepsilon_0 > 0$ such that, for all sufficiently large n , $\nu(\mathcal{Z}_n) \leq (1 - \varepsilon_0)^n$.*

We classify parabolic limit points according to their proximity to the boundary pieces of the residual sets $\mathcal{Z}_{i,n}(\eta)$. The resulting geometric estimates, combined with annular mass loss and recurrence of the reparametrized flow, show that $D_{n+1} \cup \dots \cup D_{n+N}$ removes a fixed proportion of the residual mass, yielding uniform contraction over blocks of N generations.

For $n \geq 1$, define

$$Q_n := \left\{ p \in \Lambda_p(\Gamma_\rho) \setminus \{p_0\} : \begin{array}{l} h_n < \eta h(p) \leq h_{n-1}, p \notin P_n \\ \mathcal{B}_{\bar{\psi}}(p, \eta h(p)) \cap \mathcal{Z}_{n-1} \neq \emptyset \end{array} \right\}.$$

Set

$$r(p, n) := \sqrt{\eta h(p) h_n}, \quad Q'_n := \{p \in Q_n : \mathcal{B}_{\bar{\psi}}(p, r(p, n)) \cap \partial \mathcal{Z}_{n-1} \neq \emptyset\}.$$

By Definition 6.1 and Lemma 6.5, the height h is uniformly bounded. Since $\mathcal{B}_{\bar{\psi}}(p, \eta h(p))$ meets $\mathcal{Z}_{n-1} \subset \Delta_0$ for every $p \in Q_n$, all the sets Q_n lie, for sufficiently small η , in a fixed compact neighborhood of Δ_0 avoiding $p_0, \dots, p_\ell$. On this neighborhood, Lemma 6.3,

applied both to Γ_ρ and to the conjugated groups $\Gamma_i = g_i \Gamma_\rho g_i^{-1}$, together with $g_i \in K$, gives $h_i(g_i p) \asymp h(p)$. The maps g_i also satisfy uniform bi-Lipschitz bounds there by Lemma 5.7. We use both facts below without further mention.

10.1. A previous-generation boundary piece. We first consider $p \in Q'_n$ for which $\mathcal{B}_{\bar{\psi}}(p, r(p, n))$ meets ∂J_q for some $q \in \bigcup_{k < n} P_k$. Write $q = \gamma_q p_i$, where γ_q is as in (6.2), and $x_q = \gamma_q^{-1} p_i \in \Delta_i$.

Lemma 10.2. *There exists $C > 1$ such that*

$$\eta h(p) \leq h_{n-1} \leq C \eta^3 h(q) \text{ and } C^{-1} \eta h_i(g_i p) \leq h_{n-1} \leq C \eta^3 h_i(g_i q).$$

Proof. It suffices to prove the first statement; the height comparison $h_i(g_i p) \asymp h(p)$ then gives the second. Since $p \in Q_n$, we have $h_n < \eta h(p) \leq h_{n-1}$, showing the first inequality. By Proposition 6.7, we have

$$(10.1) \quad D_{\bar{\psi}}(p, q) > C_s \sqrt{h(p)h(q)} > C_s \sqrt{h_n h(q) \eta^{-1}}.$$

On the other hand, since $D_{\bar{\psi}}(p, \partial J_q) < r(p, n)$, we have for some $x \in \partial J_q$ that

$$D_{\bar{\psi}}(p, q) \leq N_0(D_{\bar{\psi}}(p, x) + D_{\bar{\psi}}(x, q)) \leq N_0(r(p, n) + C_2 \eta h(q))$$

by Corollary 5.10, where the second inequality follows by Lemma 9.5. Since $p \in Q_n$ and $q \in P_1 \cup \dots \cup P_{n-1}$, we have $r(p, n) \leq \sqrt{h_{n-1} h_n} \leq h_{n-1} < \eta h(q)$. Therefore, we have

$$D_{\bar{\psi}}(p, q) \leq N_0(1 + C_2) \eta h(q).$$

Together with (10.1), we have $C_s \sqrt{h_{n-1} h(q) (\mathbb{H} \eta)^{-1}} = C_s \sqrt{h_n h(q) \eta^{-1}} \leq N_0(1 + C_2) \eta h(q)$, and hence $h_{n-1} \leq C \eta^3 h(q)$ with $C := \frac{N_0^2(1+C_2)^2 \mathbb{H}}{C_s^2}$. $\square$

Lemma 10.3. *There exists $C > 1$ such that*

$$\mathcal{B}_{\bar{\psi}}(g_i p, h_i(g_i p)) \subset \mathcal{B}_{\bar{\psi}}(g_i q, C \eta h_i(g_i q)) \setminus \mathcal{B}_{\bar{\psi}}(g_i q, C^{-1} \eta h_i(g_i q)).$$

Proof. Let $x \in \mathcal{B}_{\bar{\psi}}(g_i p, h_i(g_i p))$. Then by Corollary 5.10, Lemma 5.7, and Lemma 10.2, there exist $c', c > 0$, depending only on $\bar{\psi}$, such that

$$(10.2) \quad \begin{aligned} D_{\bar{\psi}}(g_i q, x) &\leq N_0(D_{\bar{\psi}}(g_i q, g_i p) + D_{\bar{\psi}}(g_i p, x)) \leq N_0(D_{\bar{\psi}}(g_i q, g_i p) + h_i(g_i p)) \\ &\leq N_0(c' D_{\bar{\psi}}(q, p) + h_i(g_i p)) \leq N_0(c' D_{\bar{\psi}}(q, p) + c \eta^2 h_i(g_i q)). \end{aligned}$$

Since $p \in Q_n$, we have

$$D_{\bar{\psi}}(p, \partial J_q) < r(p, n) = \sqrt{\eta h(p) h_n} \leq \sqrt{h_{n-1} h_n} \leq h_{n-1}.$$

By Lemma 9.5, Corollary 5.10, and Lemma 10.2, there exists $c'' > 0$ such that

$$D_{\bar{\psi}}(q, p) \leq N_0(C_2 \eta h(q) + h_{n-1}) \leq N_0 c'' \eta h_i(g_i q).$$

Combining this with (10.2) yields the inclusion

$$\mathcal{B}_{\bar{\psi}}(g_i p, h_i(g_i p)) \subset \mathcal{B}_{\bar{\psi}}(g_i q, C \eta h_i(g_i q)) \quad \text{for some } C > 1.$$

Now by Corollary 5.10, we have

$$\begin{aligned} D_{\bar{\psi}}(g_i q, g_i(\partial J_q)) &\leq N_0^2(D_{\bar{\psi}}(g_i q, x) + D_{\bar{\psi}}(x, g_i p) + D_{\bar{\psi}}(g_i p, g_i(\partial J_q))) \\ &\leq N_0^2(D_{\bar{\psi}}(g_i q, x) + h_i(g_i p) + D_{\bar{\psi}}(g_i p, g_i(\partial J_q))). \end{aligned}$$

Again, by the bi-Lipschitz property of g_i (Lemma 5.7),

$$D_{\bar{\psi}}(g_i p, g_i(\partial J_q)) \leq c' D_{\bar{\psi}}(p, \partial J_q) < c' r(p, n) \leq c' h_{n-1}.$$

By Lemma 10.2, this implies that for some uniform constant $\tilde{c} > 0$,

$$h_i(g_i p) + D_{\bar{\psi}}(g_i p, g_i(\partial J_q)) \leq \tilde{c} \eta^2 h_i(g_i q).$$

Since $D_{\bar{\psi}}(g_i q, g_i(\partial J_q)) \geq C_1 \eta h_i(g_i q) / C_{\text{fd}}$ by Lemma 9.5, we finally have that

$$\left(\frac{C_1}{C_{\text{fd}} N_0^2} - \tilde{c} \eta \right) \eta h_i(g_i q) \leq D_{\bar{\psi}}(g_i q, x).$$

Since $x \in \mathcal{B}_{\bar{\psi}}(g_i p, h_i(g_i p))$ is arbitrary, it follows that

$$\mathcal{B}_{\bar{\psi}}(g_i p, h_i(g_i p)) \cap \mathcal{B}_{\bar{\psi}}(g_i q, C^{-1} \eta h_i(g_i q)) = \emptyset$$

for some $C > 1$. This completes the proof. $\square$

Lemma 10.4. *We have*

$$h_i(g_i \gamma_q^{-1} p) \asymp \frac{h_i(g_i p)}{\eta^2 h_i(g_i q)}.$$

Proof. By the definition of h_i , the cocycle identity, and Lemmas 3.11 and 3.14,

$$h_i(g_i \gamma_q^{-1} p) \asymp \frac{h_i(g_i q) h_i(g_i p)}{d_{\bar{\psi}}(g_i p, g_i q)^2}.$$

The points p and q lie in a fixed compact neighborhood of Δ_0 avoiding $p_0, \dots, p_\ell$. Hence $g_i p$ and $g_i q$ lie in a fixed compact subset of $g_i \Lambda_\rho \setminus \{p_0\}$, on which $d_{\bar{\psi}} \asymp D_{\bar{\psi}}$. Taking $x = g_i p$ in Lemma 10.3 therefore gives $d_{\bar{\psi}}(g_i p, g_i q) \asymp \eta h_i(g_i q)$. Substitution proves the lemma. $\square$

Define

$$(10.3) \quad \tilde{r}_{p,n} := \sqrt{\frac{h_n h_i(g_i \gamma_q^{-1} p)}{\eta h_i(g_i q)}}.$$

By Lemma 10.4 and the assumption that $p \in Q_n$, we have

$$(10.4) \quad \tilde{r}_{p,n} \ll \frac{\sqrt{\eta h_n h_i(g_i p)}}{\eta^2 h_i(g_i q)} \ll \frac{h_i(g_i p)}{\eta h_i(g_i q)} \ll \eta h_i(g_i \gamma_q^{-1} p).$$

In particular, $\tilde{r}_{p,n} \ll \eta$, since $h_i(g_i \gamma_q^{-1} p)$ is invariant under $g_i P_i g_i^{-1}$ and is bounded above on a fixed fundamental domain.

Lemma 10.5. *There exists $C_3 > 1$ such that every $p' \in \Lambda_\rho$ with $D_{\bar{\psi}}(g_i \gamma_q^{-1} p', g_i \gamma_q^{-1} \partial J_q) \leq \frac{\tilde{r}_{p,n}}{C_3}$ satisfies $D_{\bar{\psi}}(p', \partial J_q) \leq h_n$.*

Proof. Choose $x \in \partial J_q$ realizing the distance in the hypothesis. By Lemma 5.7,

$$(10.5) \quad D_{\bar{\psi}}(g_i p', g_i x) \asymp \frac{D_{\bar{\psi}}(g_i \gamma_q^{-1} p', g_i \gamma_q^{-1} x)}{D_{\bar{\psi}}(g_i \gamma_q^{-1} p', g_i x_q) D_{\bar{\psi}}(g_i x_q, g_i \gamma_q^{-1} x)} h_i(g_i q).$$

Here we use Lemmas 6.4 to 6.6 to obtain $e^{\bar{\psi}(\beta_{p_0}(e, g_i \gamma_q g_i^{-1}))} \asymp e^{-\bar{\psi}(\mu(\gamma_q))} \asymp h_i(g_i q)$.

Since $g_i \gamma_q^{-1} \partial J_q = \partial R_i$, (9.2) gives $D_{\bar{\psi}}(g_i x_q, g_i \gamma_q^{-1} x) \asymp \eta^{-1}$. The hypothesis and (10.4) give $D_{\bar{\psi}}(g_i \gamma_q^{-1} p', g_i \gamma_q^{-1} x) \leq \tilde{r}_{p,n} / C_3 \ll \eta / C_3$. Thus Corollary 5.10 implies $D_{\bar{\psi}}(g_i \gamma_q^{-1} p', g_i x_q) \asymp \eta^{-1}$ for sufficiently large C_3 . Substituting these bounds into (10.5) and using (10.4), we obtain

$$D_{\bar{\psi}}(g_i p', g_i x) \ll \frac{\eta^2 h_i(g_i q)}{C_3} \tilde{r}_{p,n} \ll \frac{\eta h_i(g_i p)}{C_3}.$$

Since $x \in \partial J_q \subset \Delta_0$ and the preceding bound is $O(\eta)$, p' lies in a fixed compact neighborhood of Δ_0 where the inverse coordinate change is uniformly bi-Lipschitz. Using Lemma 5.7 and $\eta h(p) \leq h_{n-1} = Hh_n$, we obtain $D_{\bar{\psi}}(p', x) \ll Hh_n/C_3$. Increasing C_3 proves the lemma. $\square$

10.2. Separation of boundary pieces. We next estimate the separation of the sets J_p , $p \in P_n$, from one another and from $\mathcal{Z}_{n-1}^c$.

Lemma 10.6. *After decreasing η_* , for every $n \geq 1$ and distinct $p, p' \in P_n$,*

$$D_{\bar{\psi}}(J_p, J_{p'}) > \frac{C_s}{2\eta N_0^2} h_n \quad \text{and} \quad D_{\bar{\psi}}(J_p, \mathcal{Z}_{n-1}^c) > \frac{C_s}{2\eta N_0^2} h_n.$$

Consequently, the sets J_p , $p \in \bigcup_{n \geq 1} P_n$, are pairwise disjoint.

Proof. For $x \in J_p$ and $y \in J_{p'}$, Proposition 6.7 and Lemma 9.5 give

$$C_s \sqrt{h(p)h(p')} < N_0^2 (D_{\bar{\psi}}(x, y) + C_2 \eta h(p) + C_2 \eta h(p')).$$

Since $\eta h(p), \eta h(p') \in (h_n, h_{n-1}]$, it follows that $D_{\bar{\psi}}(x, y) > h_n \left(\frac{C_s}{\eta N_0^2} - 2HC_2 \right)$, which is at least the claimed lower bound for small η .

If $x \in \mathcal{Z}_{n-1}^c$ and $y \in J_p$, then the definition of P_n and Lemma 9.5 give $\frac{h_{n-1}}{4\eta} \leq N_0 (D_{\bar{\psi}}(x, y) + C_2 \eta h(p))$. The same choice of η_* gives the second estimate. If $p \in P_j$, $p' \in P_k$, and $j < k$, then $J_p \subset \mathcal{Z}_{k-1}^c$, so the second estimate applied to $J_{p'}$ proves cross-generation disjointness. $\square$

Lemma 10.7. *For every $p \in Q'_n$, the ball $\mathcal{B}_{\bar{\psi}}(p, r(p, n))$ meets exactly one of the sets $\partial\Delta_0$ and ∂J_q , $q \in \bigcup_{k=1}^{n-1} P_k$.*

Proof. The ball meets at least one of these sets by definition. If it met two distinct sets E and F , then Corollary 5.10 would give $D_{\bar{\psi}}(E, F) \leq 2N_0 r(p, n) \leq 2N_0 h_{n-1}$. On the other hand, Lemma 10.6 gives $D_{\bar{\psi}}(E, F) \geq C_s h_{n-1} / (2\eta N_0^2)$, since $h_j \geq h_{n-1}$ for $j \leq n-1$. For sufficiently small η , these bounds contradict each other. $\square$

10.3. Equivalence classes. We construct the sets $C(p)$, the radii $r_{p,m}$, and the neighborhoods $B_{p,m}$ inductively. At stage $n \geq 1$, consider $p \in Q_n \setminus \bigcup_{k < n} Q''_k$, where the union is empty when $n = 1$.

Case 1: $p \in Q'_n$. By Lemma 10.7, $\mathcal{B}_{\bar{\psi}}(p, r(p, n))$ meets exactly one of $\partial\Delta_0$ and ∂J_q , $q \in \bigcup_{k < n} P_k$.

Case 1-1: $D_{\bar{\psi}}(p, \partial\Delta_0) < r(p, n)$. Set

$$C(p) := \{ \gamma_1 p : \gamma_1 \in P_0, \gamma_1 \mathcal{B}_{\bar{\psi}}(p, r(p, n)) \cap \partial\Delta_0 \neq \emptyset \}.$$

For $p' \in C(p)$ and $m \geq n$, define

$$r_{p',m} := \sqrt{\eta h_m h(p')}, \quad B_{p',m} := \mathcal{B}_{\bar{\psi}}(p', r_{p',m}).$$

Case 1-2: $D_{\bar{\psi}}(p, \partial J_q) < r(p, n)$. Write $q = \gamma_q p_i$ and use $\tilde{r}_{p,n}$ and C_3 from (10.3) and Lemma 10.5. If $\mathcal{B}_{\bar{\psi}}(g_i \gamma_q^{-1} p, \tilde{r}_{p,n}/C_3)$ meets $g_i \gamma_q^{-1} \partial J_q$, set

$$(10.6) \quad C(p) := \{ (g_i \gamma_q^{-1})^{-1} \gamma_1 g_i \gamma_q^{-1} p : \gamma_1 \in g_i P'_i g_i^{-1}, \gamma_1 \mathcal{B}_{\bar{\psi}}(g_i \gamma_q^{-1} p, \tilde{r}_{p,n}/C_3) \cap g_i \gamma_q^{-1} \partial J_q \neq \emptyset \}.$$

Otherwise, set $C(p) := \{p\}$. For $p' \in C(p)$ and $m \geq n$, define

$$r_{p',m} := \frac{1}{C_3} \sqrt{\frac{h_m h_i(g_i \gamma_q^{-1} p')}{\eta h_i(g_i q)}}, \quad B_{p',m} := (g_i \gamma_q^{-1})^{-1} \mathcal{B}_{\bar{\psi}}(g_i \gamma_q^{-1} p', r_{p',m}).$$

Case 2. For every remaining point $p \in Q_n \setminus \bigcup_{k < n} Q_k''$ not covered by the classes above, set $C(p) := \{p\}$ and define $r_{p,m}$ and $B_{p,m}$ as in Case 1-1.

In each case, $r_{p',m} = r_{p,m}$ for $p' \in C(p)$, by invariance of the corresponding height under the peripheral group. For each member p' of a constructed class $C(p)$, we write $C(p') := C(p)$; the consistency of these assignments is proved in Lemma 10.11.

Finally, set

$$Q_n'' := \bigcup_{p \in Q_n \setminus \bigcup_{k < n} Q_k''} C(p).$$

Then $\bigcup_{j \leq n} Q_j \subset \bigcup_{j \leq n} Q_j''$. A class constructed at the n -th step is said to have *generation n* . We call $\partial \Delta_0$ in Case 1-1 and ∂J_q in Case 1-2 the *associated boundary* of the class. Classes constructed in Case 2 have no associated boundary.

10.4. Uniformity among equivalence classes.

Lemma 10.8. *There exists $C_4 > 1$ such that for any $p \in Q_n \setminus \bigcup_{k \leq n-1} Q_k''$ and $p' \in C(p)$, we have*

$$C_4^{-1} \leq \frac{h(p)}{h(p')} \leq C_4.$$

The assertion is immediate for singleton classes. In Case 1-1, $p' \in P_0 p$ and h is P_0 -invariant by Definition 6.1, so $h(p') = h(p)$. It therefore remains to consider Case 1-2, with associated boundary ∂J_q , where $q = \gamma_q p_i$. We first establish the following estimate.

Lemma 10.9. *There exists $C > 1$ such that for any $p \in Q_n' \setminus \bigcup_{k \leq n-1} Q_k''$ satisfying the hypothesis of Case 1-2 (i.e., $\mathcal{B}_{\bar{\psi}}(p, r(p, n)) \cap \partial J_q \neq \emptyset$ for some $q \in P_1 \cup \dots \cup P_{n-1}$), any $p' \in C(p)$ and any $m \geq n$, we have*

$$\begin{aligned} \mathcal{B}_{\bar{\psi}}(g_i \gamma_q^{-1} p', r_{p',m}) &\subset \mathcal{B}_{\bar{\psi}}(g_i x_q, C\eta^{-1}) \setminus \mathcal{B}_{\bar{\psi}}(g_i x_q, (C\eta)^{-1}); \\ g_i B_{p',m} &\subset \mathcal{B}_{\bar{\psi}}(g_i q, C\eta h_i(g_i q)) \setminus \mathcal{B}_{\bar{\psi}}(g_i q, C^{-1}\eta h_i(g_i q)). \end{aligned}$$

Proof. Since $g_i \gamma_q^{-1} \partial J_q = \partial R_i$, (9.2) gives $D_{\bar{\psi}}(g_i x_q, z) \asymp \eta^{-1}$ for every $z \in g_i \gamma_q^{-1} \partial J_q$.

In the first alternative of Case 1-2, (10.6) and Lemma 5.7 imply that $\mathcal{B}_{\bar{\psi}}(g_i \gamma_q^{-1} p', r_{p',n})$ meets $g_i \gamma_q^{-1} \partial J_q$. In the singleton alternative, $p' = p$, and Lemmas 6.8 and 10.3 give $D_{\bar{\psi}}(g_i \gamma_q^{-1} p, g_i x_q) \asymp \eta^{-1}$. By Definition 6.1, h_i is $g_i P_i g_i^{-1}$ -invariant. Together with (10.6), this gives $h_i(g_i \gamma_q^{-1} p') = h_i(g_i \gamma_q^{-1} p)$. Hence, by the definitions of the radii and (10.4), $r_{p',m} \leq r_{p',n} = r_{p,n} = \tilde{r}_{p,n}/C_3 \ll \eta$.

Thus Corollary 5.10 implies $D_{\bar{\psi}}(z, g_i x_q) \asymp \eta^{-1}$ throughout $\mathcal{B}_{\bar{\psi}}(g_i \gamma_q^{-1} p', r_{p',m})$. This proves the first inclusion. For every $z \in \mathcal{B}_{\bar{\psi}}(g_i \gamma_q^{-1} p', r_{p',m})$, Lemma 6.8 gives

$$D_{\bar{\psi}}((g_i \gamma_q g_i^{-1})z, g_i q) \asymp \frac{h_i(g_i q)}{D_{\bar{\psi}}(z, g_i x_q)} \asymp \eta h_i(g_i q).$$

Since $(g_i \gamma_q g_i^{-1}) \mathcal{B}_{\bar{\psi}}(g_i \gamma_q^{-1} p', r_{p',m}) = g_i B_{p',m}$ by definition, the second inclusion follows. $\square$

Proof of Lemma 10.8. It remains to consider Case 1-2. Since $h_i(g_i q) \ll 1$, Lemma 10.9 places p and p' in a fixed compact neighborhood of Δ_0 avoiding the cusp representatives, after decreasing η . In particular, (5.3) gives $d_{\bar{\psi}}(g_i p', g_i q) \asymp D_{\bar{\psi}}(g_i p', g_i q)$.

The computation in the proof of Lemma 10.4, together with Lemma 10.9, therefore gives

$$(10.7) \quad h_i(g_i \gamma_q^{-1} p') \asymp \frac{h_i(g_i q) h_i(g_i p')}{d_{\bar{\psi}}(g_i p', g_i q)^2} \asymp \frac{h_i(g_i p')}{\eta^2 h_i(g_i q)}.$$

By the definition of $C(p)$, $h_i(g_i \gamma_q^{-1} p') = h_i(g_i \gamma_q^{-1} p)$. Combining (10.7) with Lemma 10.4 yields $h_i(g_i p') \asymp h_i(g_i p)$.

Finally, Lemma 6.3, applied to the original and conjugated systems, and $g_i \in K$ give $h(p') \asymp h_i(g_i p')$ and $h(p) \asymp h_i(g_i p)$ on these compact neighborhoods. Hence $h(p') \asymp h(p)$, as required. $\square$

10.5. Size of the neighborhoods $B_{p,m}$.

Lemma 10.10. *There exists $C_5 > 1$ such that for any $p \in Q_n''$ and any $m \geq n$,*

$$\mathcal{B}_{\bar{\psi}}(p, \sqrt{\eta h(p) h_m} / C_5) \subset B_{p,m} \subset \mathcal{B}_{\bar{\psi}}(p, C_5 \sqrt{\eta h(p) h_m}).$$

Proof. The assertion follows directly from the definitions in Cases 1-1 and 2. Consider Case 1-2, with associated boundary ∂J_q , where $q = \gamma_q p_i$ and $x_q = \gamma_q^{-1} p_i$.

By Lemma 10.9, p lies in a fixed compact neighborhood of Δ_0 avoiding the cusp representatives. On these neighborhoods, Lemmas 5.7 and 6.3 give $h_i(g_i p) \asymp h(p)$ and uniform bi-Lipschitz bounds for g_i . For $x \in \mathcal{B}_{\bar{\psi}}(g_i \gamma_q^{-1} p, r_{p,m})$, Lemmas 6.8 and 10.9 give

$$(10.8) \quad D_{\bar{\psi}}(g_i p, (g_i \gamma_q g_i^{-1} x)) \asymp \eta^2 h_i(g_i q) D_{\bar{\psi}}(g_i \gamma_q^{-1} p, x).$$

Moreover, the definition of $r_{p,m}$ and (10.7) imply

$$(10.9) \quad r_{p,m} \asymp \frac{\sqrt{\eta h_i(g_i p) h_m}}{\eta^2 h_i(g_i q)}.$$

Since $(g_i \gamma_q g_i^{-1}) \mathcal{B}_{\bar{\psi}}(g_i \gamma_q^{-1} p, r_{p,m}) = g_i B_{p,m}$, these estimates give a uniform $C > 1$ such that $g_i B_{p,m} \subset \mathcal{B}_{\bar{\psi}}(g_i p, C \sqrt{\eta h_i(g_i p) h_m})$. Pulling back by g_i and using $h_i(g_i p) \asymp h(p)$ proves the second inclusion.

To prove the first inclusion, take $x \in \mathcal{B}_{\bar{\psi}}(g_i p, \sqrt{\eta h_i(g_i p) h_m} / C)$. By Lemma 10.8 and $m \geq n$, $\sqrt{\eta h_i(g_i p) h_m} \ll h_n$. Since $q \in \bigcup_{j < n} P_j$, we also have $h_n \ll \eta h_i(g_i q)$. Thus Lemma 10.9 and corollary 5.10 imply $D_{\bar{\psi}}(x, g_i q) \asymp \eta h_i(g_i q)$ for sufficiently large C . Applying Lemma 6.8 again, we obtain

$$(10.10) \quad D_{\bar{\psi}}(g_i \gamma_q^{-1} p, g_i \gamma_q^{-1} g_i^{-1} x) \asymp \frac{h_i(g_i q) D_{\bar{\psi}}(g_i p, x)}{D_{\bar{\psi}}(g_i p, g_i q) D_{\bar{\psi}}(x, g_i q)} \ll \frac{\sqrt{\eta h_i(g_i p) h_m}}{C \eta^2 h_i(g_i q)} \asymp \frac{r_{p,m}}{C}.$$

Increasing C makes the last quantity smaller than $r_{p,m}$, so $x \in g_i B_{p,m}$. Pulling back the two inclusions by g_i and using $h_i(g_i p) \asymp h(p)$ proves the lemma. $\square$

10.6. Well-definedness of equivalence classes.

Lemma 10.11. *Any two equivalence classes are equal or disjoint.*

Proof. Suppose that $C(p') \cap C(p'') \neq \emptyset$. First consider the case when $C(p') = \{p'\}$. If $C(p'')$ is also a singleton, the assertion is immediate. Otherwise, let k and n be the generations at which $C(p')$ and $C(p'')$ are defined. Since $p' \in C(p'') \subset Q_n''$, the construction gives $k \leq n$.

If the associated boundary of $C(p'')$ is $\partial\Delta_0$, then $p' \in P_0p''$. The P_0 -invariance of h (Definition 6.1) gives $h(p') = h(p'')$, hence $k = n$. Moreover, $\mathcal{B}_{\bar{\psi}}(p', r(p', n))$ meets $\partial\Delta_0$. Thus $C(p')$ is also defined in Case 1-1, and Lemma 5.7 gives $C(p') = C(p'')$.

Suppose instead that the associated boundary is ∂J_q , where $q = \gamma_q p_i$. By Lemma 10.2 and Lemma 10.8, $h(q) \gg h_k/\eta^3$. For sufficiently small η , this implies $q \in \bigcup_{j < k} P_j$. By (10.6) and Lemma 5.7, the ball centered at $g_i \gamma_q^{-1} p'$ with radius $\tilde{r}_{p'', n}/C_3$ meets $g_i \gamma_q^{-1} \partial J_q$. Hence Lemma 10.5 gives

$$D_{\bar{\psi}}(p', \partial J_q) \leq h_n \leq h_k < r(p', k).$$

By Lemma 10.7, $C(p')$ is therefore constructed in Case 1-2 with associated boundary ∂J_q . The $g_i P_i g_i^{-1}$ -invariance of h_i (Definition 6.1) gives $\tilde{r}_{p', k} \geq \tilde{r}_{p', n} = \tilde{r}_{p'', n}$. Thus (10.6) implies $C(p'') \subset C(p')$, contradicting the singleton assumption.

It remains to consider two classes containing at least two points. Let their generations be $m \leq n$, their associated boundaries be ∂_1 and ∂_2 , and choose $p \in C(p') \cap C(p'')$. The definitions and Lemma 10.5 give $D_{\bar{\psi}}(p, \partial_1) \leq h_{m-1}$ and $D_{\bar{\psi}}(p, \partial_2) \leq h_{n-1}$. By Lemma 10.8, $h(p') \asymp h(p'')$, so $h_{m-1} \asymp h_{n-1}$. If $\partial_1 \neq \partial_2$, then Lemma 10.6 and Corollary 5.10 yield

$$\frac{C_s h_{n-1}}{2\eta N_0^2} \leq D_{\bar{\psi}}(\partial_1, \partial_2) \leq N_0(h_{m-1} + h_{n-1}) \ll h_{n-1},$$

a contradiction for small η . Thus the associated boundaries agree.

If the common boundary is $\partial\Delta_0$, then $h(p') = h(p) = h(p'')$, so $m = n$. Both classes are consequently defined by the same P_0 -orbit and the same radius, and hence are equal.

If the common boundary is ∂J_q , then $g_i \gamma_q^{-1} p'$ and $g_i \gamma_q^{-1} p''$ lie in the same $g_i P'_i g_i^{-1}$ -orbit. Their h_i -heights are therefore equal, and $m \leq n$ gives $\tilde{r}_{p'', n} \leq \tilde{r}_{p', m}$. Thus (10.6) gives $C(p'') \subset C(p')$. In particular, $p'' \in Q''_m$, so the construction forces $m = n$. The defining radii are then equal, and therefore $C(p') = C(p'')$. $\square$

10.7. Auxiliary sets in $\mathcal{Z}_n$. By construction and Lemma 10.11, the sets Q''_n are pairwise disjoint. Define

$$B_n := \mathcal{Z}_n \cap \bigcup_{j \leq n} \bigcup_{p \in Q''_j} B_{p, n}, \quad A_n := \mathcal{Z}_n \setminus B_n.$$

For $p = \gamma p_i \in \Lambda_p(\Gamma_\rho)$, the horoball based at $f^{-1}(p)$ whose boundary contains $\rho^{-1}(\gamma P_i) o$ is called the *horoball for p* . This does not depend on the choice of γ .

Let $\Phi : \tilde{\Omega}_\Gamma \rightarrow \tilde{\Omega}_{\bar{\psi}}$ be the reparametrization from Theorem 3.20. For $0 < \varepsilon < 1$, let $\mathcal{C}_\varepsilon$ consist of those $x \in \tilde{\Omega}_{\bar{\psi}}$ for which there exists $p \in \Lambda_p(\Gamma_\rho)$ such that $(\Phi^{-1}(x a_t))(0)$ lies in the interior of the horoball for p whenever $|t| \leq -\log \varepsilon$.

For each fixed ε , the set $\mathcal{C}_\varepsilon$ is open and Γ_ρ -invariant. Let $Q \subset \Omega_{\bar{\psi}}$ be the image of the compact set $\Gamma \setminus (T^1 Y_{\text{th}} \cap \tilde{\Omega}_\Gamma)$ under the homeomorphism induced by Φ . Every point outside the image of $\mathcal{C}_\varepsilon$ meets Q within flow time $|t| \leq -\log \varepsilon$. Hence $\Gamma_\rho \setminus (\tilde{\Omega}_{\bar{\psi}} \setminus \mathcal{C}_\varepsilon)$ is compact.

Lemma 10.12. *There exists $C > 0$ such that, for every sufficiently small $\varepsilon > 0$, if $x = (x^+, p_0, 0)$ satisfies $x^+ \in \mathcal{Z}_0$ and $x a_{n \log H} \in \mathcal{C}_{C\varepsilon}$ for some $n \geq 1$, then there exists $p \in \Lambda_p(\Gamma_\rho) \setminus \{p_0\}$ such that*

$$h_n < \varepsilon h(p), \quad D_{\bar{\psi}}(x^+, p) < \sqrt{\varepsilon h(p) h_n}.$$

Proof. Let $0 < C < 1$ be chosen below, set $T := n \log H$, and choose p as in the definition of $\mathcal{C}_{C\varepsilon}$. The set $\Phi^{-1}(\{(\xi, p_0, 0) : \xi \in \mathcal{Z}_0\})$ is compact. Hence Theorem 3.20 gives a uniform

upper bound for the forward exit time from the horoball for p_0 . Since $T \geq 0$, we have $p \neq p_0$ for sufficiently small ε .

We also record that p lies in a fixed compact subset of $\Lambda_\rho \setminus \{p_0\}$. Indeed, place $f^{-1}(p_0)$ at ∞ in the upper half-plane model. The horoballs for $p \neq p_0$ have uniformly bounded Euclidean diameters, since they are uniformly bounded enlargements of the horoballs in $\mathcal{H}$. As $f^{-1}(x^+)$ ranges over a bounded subset of $\mathbb{R}$, the basepoints of horoballs meeting the vertical geodesic from ∞ to $f^{-1}(x^+)$ also remain bounded.

Write $p = \gamma p_i$. Since $x^- = p_0 \neq p$, there is an entrance time $T - t_1$, with $t_1 > -\log(C\varepsilon)$. Set $\sigma_1 := \Phi^{-1}(xa_{T-t_1})$ and choose $\gamma_1 \in \gamma P_i$ so that $\hat{\gamma}_1 o$ is uniformly close to $\sigma_1(0)$. The entrance point is uniformly close to both rays $[o, f^{-1}(p)]$ and $[o, f^{-1}(x^+)]$. Thus Lemma 3.12 and Lemma 3.11 give $\bar{\psi}(\beta_{x^+}(e, \gamma_1)) = \bar{\psi}(\beta_p(e, \gamma_1)) + O(1)$. Moreover, equivariance and continuity of Φ imply that the $\mathbb{R}$ -coordinate of $\gamma_1^{-1}\Phi(\sigma_1)$ is uniformly bounded. Consequently,

$$T - t_1 = \bar{\psi}(\beta_{x^+}(e, \gamma_1)) + O(1) = \bar{\psi}(\beta_p(e, \gamma_1)) + O(1).$$

By Lemma 6.3, $h_n \asymp e^{-t_1}h(p) \ll C\varepsilon h(p)$, which proves the first inequality when C is sufficiently small.

If $x^+ = p$, the second inequality is immediate. Otherwise, there is an exit time $T + t_2$, with $t_2 > -\log(C\varepsilon)$. Set $\sigma_2 := \Phi^{-1}(xa_{T+t_2})$ and choose $\gamma_2 \in \gamma P_i$ so that $\hat{\gamma}_2 o$ is uniformly close to $\sigma_2(0)$. As above, $T + t_2 = \bar{\psi}(\beta_{x^+}(e, \gamma_2)) + O(1)$. The exit point is uniformly close to $[f^{-1}(x^+), f^{-1}(p)]$. Hence Lemma 3.13 gives $\mathcal{G}(\gamma_2^{-1}x^+, p_i) = O(1)$. The cocycle identity, $\bar{\psi} \circ i = \bar{\psi}$, and Lemma 6.3 therefore yield

$$2\bar{\psi}(\mathcal{G}(x^+, p)) = \bar{\psi}(\beta_{x^+}(e, \gamma_2)) + \bar{\psi}(\beta_p(e, \gamma_2)) + O(1) = T + t_2 - \log h(p) + O(1).$$

Since x^+ and p lie in a fixed compact set avoiding p_0 , (5.3) now gives

$$D_{\bar{\psi}}(x^+, p) \asymp d_{\bar{\psi}}(x^+, p) \asymp \sqrt{e^{-t_2}h(p)h_n} \ll \sqrt{C\varepsilon h(p)h_n}.$$

Choosing C sufficiently small proves the second inequality. $\square$

Lemma 10.13. *There exists $C_6 > 0$ such that if $x = (x^+, p_0, 0) \in \tilde{\Omega}_{\bar{\psi}}$ satisfies $x^+ \in \mathcal{Z}_n$ and $xa_{n \log H} \in \mathcal{C}_{C_6 \eta}$ for some $n \in \mathbb{N}$, then $x^+ \in B_n$.*

Proof. Let C be the constant in Lemma 10.12, and choose $0 < C_6 < C \min\{1, C_2^{-1}, C_5^{-2}\}$. Applying that lemma with $\varepsilon = C_6 \eta / C$, we obtain $p \in \Lambda_p(\Gamma_\rho) \setminus \{p_0\}$ such that

$$h_n < \eta h(p), \quad D_{\bar{\psi}}(x^+, p) < \min \left\{ \frac{\eta h(p)}{C_2}, \frac{\sqrt{\eta h(p)h_n}}{C_5} \right\}.$$

Since $h(p) \ll 1$, for sufficiently small η there is $1 \leq k \leq n$ with $h_k < \eta h(p) \leq h_{k-1}$. The ball $\mathcal{B}_{\bar{\psi}}(p, \eta h(p))$ contains $x^+ \in \mathcal{Z}_n \subset \mathcal{Z}_{k-1}$, so $p \in P_k \cup Q_k$.

If $p \in P_k$, then Lemma 9.5 gives $x^+ \in J_p$, contradicting $x^+ \in \mathcal{Z}_n$. Thus $p \in Q_k \subset \bigcup_{j \leq k} Q_j''$. Since $D_{\bar{\psi}}(x^+, p) < \sqrt{\eta h(p)h_n}/C_5$, Lemma 10.10 gives $x^+ \in B_{p,n}$. Together with $x^+ \in \mathcal{Z}_n$, this yields $x^+ \in B_n$. $\square$

10.8. Separation across generations.

Lemma 10.14. *For every $n \geq 1$,*

$$P_{n+1} \cap \bigcup_{k \leq n} Q_k'' = \emptyset.$$

Proof. Suppose that $p \in P_{n+1} \cap C(p')$, where $C(p')$ is defined at generation $m \leq n$. If $C(p')$ is a singleton or is associated with $\partial\Delta_0$, then $h(p) = h(p')$, using the P_0 -invariance of h in the latter case (Definition 6.1). This contradicts the disjoint height ranges of P_{n+1} and Q_m .

Thus $C(p')$ is a non-singleton class in Case 1-2, with associated boundary ∂J_q for some $q \in \bigcup_{j \leq m} P_j$. By Lemma 10.5, $D_{\bar{\psi}}(p, \partial J_q) \leq h_m$. On the other hand, $\mathcal{B}_{\bar{\psi}}(p, h_n/(4\eta)) \subset \mathcal{Z}_n$ and $\partial J_q \subset J_q \subset \mathcal{Z}_n^c$. Together with Lemma 10.8, this gives

$$\frac{h_n}{4\eta} \leq D_{\bar{\psi}}(p, \partial J_q) \leq h_m < \eta h(p') \leq C_4 \eta h(p) \leq C_4 h_n,$$

a contradiction for sufficiently small η . $\square$

Lemma 10.15. *There exists $0 < C_7 < 1$ such that, for every $n \geq 1$, $p \in P_{n+1} \cup Q''_{n+1}$, and $q \in \bigcup_{j \leq n} Q''_j$,*

$$D_{\bar{\psi}}(p, B_{q,n}) \geq C_7 \eta^{-1} h_n.$$

In particular, $D_{\bar{\psi}}(p, B_n) \geq C_7 \eta^{-1} h_n$.

Proof. By Lemma 10.11 and Lemma 10.14, $p \neq q$. The height ranges and Lemma 10.8 give

$$h(p) \geq \frac{h_n}{\mathbb{H}C_4\eta}, \quad h(q) \geq \frac{h_n}{C_4\eta}.$$

By Proposition 6.7, Lemma 10.10, and Corollary 5.10,

$$D_{\bar{\psi}}(p, B_{q,n}) \geq \frac{C_s}{N_0} \sqrt{h(p)h(q)} - C_5 \sqrt{\eta h(q)h_n} \geq \frac{C_s}{2N_0} \sqrt{h(p)h(q)} \geq \frac{C_s}{2N_0 C_4 \sqrt{\mathbb{H}}} \frac{h_n}{\eta},$$

where the second inequality holds for sufficiently small η , since $\sqrt{\eta h_n/h(p)} \leq \eta \sqrt{\mathbb{H}C_4}$. Thus we may take $C_7 := C_s/(2N_0 C_4 \sqrt{\mathbb{H}})$. The assertion for B_n follows from its definition. $\square$

Recall that $D_{n+1} = \bigcup_{p \in P_{n+1}} J_p$ and $\mathcal{Z}_{n+1} = \mathcal{Z}_n \setminus D_{n+1}$.

Lemma 10.16. *For sufficiently small η and every $n \geq 1$, the following hold:*

- (1) $D_{n+1} \cap B_{q,n} = \emptyset$ for every $q \in \bigcup_{j \leq n} Q''_j$. In particular, $D_{n+1} \cap B_n = \emptyset$.
- (2) $B_{p,n+1} \cap B_{q,n} = \emptyset$ whenever $p \in Q''_{n+1}$ and $q \in \bigcup_{j \leq n} Q''_j$. In particular, $B_{p,n+1} \cap B_n = \emptyset$ for every $p \in Q''_{n+1}$.
- (3) $A_{n+1} = (A_n \setminus (D_{n+1} \cup B_{n+1})) \sqcup (A_{n+1} \cap B_n)$.
- (4) $A_n \cap B_{n+1} = \mathcal{Z}_{n+1} \cap \bigcup_{p \in Q''_{n+1}} B_{p,n+1}$.
- (5) $A_{n+1} \cap B_n = B_n \setminus B_{n+1}$.

Moreover, $B_{p,m} \cap \mathcal{Z}_m = B_{p,m} \cap \mathcal{Z}_n$ whenever $p \in Q''_n$ and $m \geq n$.

Proof. For $p \in P_{n+1}$, Lemma 9.5 gives $J_p \subset \mathcal{B}_{\bar{\psi}}(p, C_2 h_n)$. For $p \in Q''_{n+1}$, Lemma 10.8 and Lemma 10.10 give $B_{p,n+1} \subset \mathcal{B}_{\bar{\psi}}(p, C_5 \sqrt{C_4} h_n)$. For sufficiently small η , both radii are smaller than $C_7 h_n/\eta$, so (1) and (2) follow from Lemma 10.15.

By the definitions, $A_{n+1} \setminus B_n = A_n \setminus (D_{n+1} \cup B_{n+1})$. Partitioning A_{n+1} according to membership in B_n proves (3).

The sets $B_{p,m}$ decrease with m . Thus, for $q \in \bigcup_{j \leq n} Q''_j$, $B_{q,n+1} \cap \mathcal{Z}_{n+1} \subset B_n$. The neighborhoods $B_{p,n+1}$, $p \in Q''_{n+1}$, are disjoint from B_n by (2). Splitting the defining union for B_{n+1} into these two groups proves (4).

By (1), $B_n \subset \mathcal{Z}_{n+1}$. Hence $A_{n+1} \cap B_n = (\mathcal{Z}_{n+1} \setminus B_{n+1}) \cap B_n = B_n \setminus B_{n+1}$, proving (5).

Finally, let $p \in Q''_n$ and $m \geq n$. For every $n \leq k < m$, nesting and (1) give $B_{p,m} \subset B_{p,k}$ and $B_{p,k} \cap D_{k+1} = \emptyset$. Thus $B_{p,m}$ is disjoint from all sets removed between $\mathcal{Z}_n$ and $\mathcal{Z}_m$, proving the last assertion. $\square$

10.9. Transfer of mass between B_n and A_{n+1} . The sets A_n and B_n partition $\mathcal{Z}_n$. The following lemma shows that a uniform fraction of the mass of B_n passes to A_{n+1} .

Lemma 10.17. *There exists $C_8 > 0$, independent of η , such that, for every sufficiently small η and every $n \geq 1$,*

$$\nu(B_n \cap A_{n+1}) \geq C_8 \nu(B_n).$$

Proof. We first show that for all $p, p' \in \bigcup_{j \leq n} Q_j''$, $p \neq p'$,

$$(10.11) \quad B_{p,n} \cap B_{p',n} = \emptyset.$$

For points from different generations, this follows from Lemma 10.16(2), since $B_{p,s+1} \subset B_{p,s}$. If $p, p' \in Q_j''$, then Lemma 10.8 gives $h(p), h(p') \asymp h_j/\eta$, so Proposition 6.7 yields $D_{\bar{\psi}}(p, p') \gg h_j/\eta$. An intersection of $B_{p,j}$ and $B_{p',j}$ would instead give $D_{\bar{\psi}}(p, p') \ll h_j$ by Lemma 10.10 and Corollary 5.10. This is impossible for small η , proving (10.11).

We also record a consequence of (9.6). The ball estimates in Corollary 7.10, together with Lemma 7.3, give

$$(10.12) \quad \nu_i \left(\mathcal{B}_{\bar{\psi}}(z, r/c) \setminus \mathcal{B}_{\bar{\psi}}(z, r/\sqrt{H}) \right) \geq \frac{1}{2} \nu_i(\mathcal{B}_{\bar{\psi}}(z, r/c)) \gg \nu_i(\mathcal{B}_{\bar{\psi}}(z, r))$$

uniformly for $z \in \Lambda_p(\Gamma_i) \setminus \{p_0\}$ and $0 < r \leq h_i(z)$.

Fix a class $C(p)$ of generation $m \leq n$, and set $M_s := \sum_{p' \in C(p)} \nu(B_{p',s} \cap \mathcal{Z}_m)$ for $s \geq m$. By the last assertion of Lemma 10.16, $\mathcal{Z}_m$ may be replaced by $\mathcal{Z}_s$ in this definition. Together with (10.11) and Lemma 10.16(2), (5), this reduces the proof to $M_n - M_{n+1} \gg M_n$, uniformly over the classes.

We first record the separation needed below. Every center in $C(p)$ lies outside P_m . For singleton classes this follows from $p \in Q_m$. Otherwise, the class definition and Lemma 10.5 give $D_{\bar{\psi}}(p', \partial \mathcal{Z}_{m-1}) \ll h_m$ for every $p' \in C(p)$. If $p' \in P_m$, then, by definition, $\mathcal{B}_{\bar{\psi}}(p', \frac{h_{m-1}}{4\eta}) \subset \mathcal{Z}_{m-1}$. Since $\partial \mathcal{Z}_{m-1} \subset \overline{\mathcal{Z}_{m-1}^c}$, this implies $D_{\bar{\psi}}(p', \partial \mathcal{Z}_{m-1}) \geq \frac{h_{m-1}}{4\eta} = \frac{H}{4\eta} h_m$. For sufficiently small η , this contradicts the preceding bound $D_{\bar{\psi}}(p', \partial \mathcal{Z}_{m-1}) \ll h_m$.

The heights of centers in $C(p)$ and P_m are comparable to h_m/η , so Proposition 6.7 separates distinct centers by a distance $\gg h_m/\eta$. Since the radii in Lemma 9.5 and Lemma 10.10 are $O(h_m)$, every $B_{p',m}$ is disjoint from D_m . In either boundary case, the class centers are within $O(h_m)$ of the associated boundary. Thus Lemma 10.6 also keeps each $B_{p',m}$, $p' \in C(p)$, disjoint from every J_q removed before generation m , except the associated J_q in Case 1-2; in Case 1-2, it also keeps each $B_{p',m}$ inside Δ_0 .

Suppose first that $C(p)$ is constructed in Case 1-1. The preceding separation gives $B_{p',s} \cap \mathcal{Z}_m = B_{p',s} \cap \Delta_0$ for $p' \in C(p)$ and $s \geq m$. For each $p' \in C(p)$, let $u_{p'} \in P_0$ satisfy $p' = u_{p'} p$. For small η , $B_{p,m}$ lies in the two adjacent fundamental intervals sharing the endpoint it meets. The maps $u_{p'}$ carry these intervals onto Δ_0 . Thus, for every Borel set $E \subset B_{p,m}$, the sets $E \cap u_{p'}^{-1} \Delta_0$, $p' \in C(p)$, partition E up to endpoints. Applying Lemma 7.3 with $E = B_{p,n}$ and $\bar{E} = B_{p,n} \setminus B_{p,n+1}$ gives

$$M_n \asymp \nu(B_{p,n}), \quad M_n - M_{n+1} \asymp \nu(B_{p,n} \setminus B_{p,n+1}).$$

Since $r_{p,n+1} = r_{p,n}/\sqrt{H}$ and $r_{p,n} \leq r_{p,m} < \eta h(p) \leq h(p)$, Lemma 9.6 gives $M_n - M_{n+1} \gg M_n$.

Now suppose that $C(p)$ is constructed in Case 2. Write $r := r_{p,n}$. The neighborhood $B_{p,m}$ misses $\partial \mathcal{Z}_{m-1}$. Since $r \leq r_{p,m}$, Corollary 9.1 implies that $\mathcal{B}_{\bar{\psi}}(p, r/c)$ is contained in $\mathcal{Z}_{m-1}$ if $p \in \mathcal{Z}_{m-1}$, and is disjoint from $\mathcal{Z}_{m-1}$ otherwise. In the latter case, $\sqrt{H} > c$ implies

$M_{n+1} = 0$. In the former case, separation from D_m gives $\mathcal{B}_{\bar{\psi}}(p, r/c) \subset \mathcal{Z}_m$. Consequently, (10.12) yields

$$M_n - M_{n+1} \geq \nu \left(\mathcal{B}_{\bar{\psi}}(p, r/c) \setminus \mathcal{B}_{\bar{\psi}}(p, r/\sqrt{H}) \right) \gg \nu(\mathcal{B}_{\bar{\psi}}(p, r)) \geq M_n.$$

Finally, consider Case 1-2, with associated boundary ∂J_q , where $q = \gamma_q p_i$. Set $T := g_i \gamma_q^{-1}$ and $\mathcal{U}_s := \mathcal{B}_{\bar{\psi}}(Tp, r_{p,s})$. Since $\nu_i = (g_i)_* \nu$ and $g_i \in K$, Lemma 10.9 and Lemma 7.4, applied in the conjugated coordinates, give

$$\nu(E) \asymp e^{-\psi(\mu(\gamma_q))} \nu_i(TE) \quad (E \subset B_{p',m} \text{ Borel}, p' \in C(p)).$$

By separation, for every $p' \in C(p)$ and $s \geq m$,

$$T(B_{p',s} \cap \mathcal{Z}_m) = \mathcal{B}_{\bar{\psi}}(Tp', r_{p',s}) \cap R_i,$$

up to ν_i -null endpoints. Moreover, $r_{p,n+1} = r_{p,n}/\sqrt{H}$ and $r_{p,n} \leq r_{p,m} \ll \eta h_i(Tp)$ by (10.4), so $r_{p,n} \leq h_i(Tp)$ for small η .

In the first alternative of Case 1-2, for each $p' \in C(p)$, choose $u_{p'} \in g_i P'_i g_i^{-1}$ with $Tp' = u_{p'} Tp$. Then $TB_{p',s} = u_{p'} \mathcal{U}_s$. For small η , $\mathcal{U}_m$ lies in two adjacent translates of R_i sharing the endpoint it meets. The class definition selects precisely the peripheral maps carrying these intervals onto R_i . Consequently, the sets

$$u_{p'}^{-1} T(B_{p',s} \cap \mathcal{Z}_m) = \mathcal{U}_s \cap u_{p'}^{-1} R_i, \quad p' \in C(p),$$

partition $\mathcal{U}_s$ up to null endpoints. Apply this with $s = n$ and $s = n+1$. By Lemma 10.9, all these portions lie at $D_{\bar{\psi}}$ -distance comparable to η^{-1} from $g_i x_q$, while their diameters are uniformly bounded by (10.4). Thus Lemma 7.3 gives uniform measure comparisons under these peripheral maps. Applying these comparisons separately to the ball and the annulus gives

$$M_n \asymp e^{-\psi(\mu(\gamma_q))} \nu_i(\mathcal{U}_n), \quad M_n - M_{n+1} \asymp e^{-\psi(\mu(\gamma_q))} \nu_i(\mathcal{U}_n \setminus \mathcal{U}_{n+1}).$$

It follows from Lemma 9.6 that $M_n - M_{n+1} \gg M_n$.

In the singleton alternative, $\mathcal{U}_m$ misses ∂R_i . Write $r := r_{p,n}$. By Corollary 9.1, $\mathcal{B}_{\bar{\psi}}(Tp, r/c)$ is contained in R_i if $Tp \in R_i$, and is disjoint from R_i otherwise. In the latter case, $\sqrt{H} > c$ gives $M_{n+1} = 0$. In the former case, separation gives

$$T^{-1}(\mathcal{B}_{\bar{\psi}}(Tp, r/c) \setminus \mathcal{U}_{n+1}) \subset \mathcal{Z}_m.$$

Hence (10.12) and the change-of-measure comparison give $M_n - M_{n+1} \gg e^{-\psi(\mu(\gamma_q))} \nu_i(\mathcal{U}_n) \gg M_n$.

The implied lower bound is uniform over all classes. Summing over them proves the lemma. $\square$

10.10. Non-concentration of PS measures near partition boundaries. For any $r > 0$, set

$$(10.13) \quad N_r(\Delta_0) := \{x \in \Delta_0 : D_{\bar{\psi}}(x, \partial \Delta_0) \leq r\}$$

where $\partial \Delta_0 = f(\partial \widehat{\Delta}_\infty)$.

Lemma 10.18. *There exists $0 < \varepsilon_0 < 1$ such that for any $0 < \varepsilon < \varepsilon_0$, there exists $\lambda = \lambda(\varepsilon) \in (0, 1)$ so that*

$$\nu(N_{\varepsilon r}(\Delta_0)) \leq \lambda \nu(N_r(\Delta_0)) \quad \text{for all } 0 < r < 1.$$

Moreover, $\lambda(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Proof. Write $\partial\Delta_0 = \{x_1, x_2\}$, where $x_2 = \gamma x_1$ for a generator $\gamma \in \mathbf{P}_0$. Let $C_0 > 1$ be as in Proposition 7.8(2). Choose $0 < r_* < C_0^{-1}$ so that each $\mathcal{B}_{\bar{\psi}}(x_i, r_*)$ lies in the two fundamental intervals adjacent to x_i . For $0 < r < r_*$, the peripheral translations satisfy

$$\gamma(\mathcal{B}_{\bar{\psi}}(x_1, r) \setminus \Delta_0) \subset \mathcal{B}_{\bar{\psi}}(x_2, r) \cap \Delta_0, \quad \gamma^{-1}(\mathcal{B}_{\bar{\psi}}(x_2, r) \setminus \Delta_0) \subset \mathcal{B}_{\bar{\psi}}(x_1, r) \cap \Delta_0.$$

Thus Lemma 5.7 and Lemma 7.3 give

$$\nu(\mathcal{B}_{\bar{\psi}}(x_1, r)) + \nu(\mathcal{B}_{\bar{\psi}}(x_2, r)) \ll \nu(N_r(\Delta_0)) \quad (0 < r < r_*).$$

Iterating Proposition 7.8(2), we obtain

$$\nu(\mathcal{B}_{\bar{\psi}}(x_i, C_0^{-k}r)) \leq 2^{-k}\nu(\mathcal{B}_{\bar{\psi}}(x_i, r)) \quad (0 < r < r_*, k \geq 1).$$

If $C_0^{-k-1} \leq \varepsilon < C_0^{-k}$, then $N_{\varepsilon r}(\Delta_0)$ is contained in $\bigcup_{i=1}^2 \mathcal{B}_{\bar{\psi}}(x_i, C_0^{-k}r)$. Consequently, $\nu(N_{\varepsilon r}(\Delta_0)) \ll 2^{-k}\nu(N_r(\Delta_0))$ uniformly for $0 < r < r_*$.

For $r_* \leq r < 1$, monotonicity gives

$$\frac{\nu(N_{\varepsilon r}(\Delta_0))}{\nu(N_r(\Delta_0))} \leq \frac{\nu(N_{\varepsilon}(\Delta_0))}{\nu(N_{r_*}(\Delta_0))} \rightarrow 0 \quad (\varepsilon \rightarrow 0),$$

since $\nu(\partial\Delta_0) = 0$ by Theorem 3.17. Combining the two ranges proves the lemma. $\square$

We will use the following counting result from [43, Proposition 5.2], which is based on [5, Proposition 7.2].

Proposition 10.19. *For any $\xi \in \Lambda_p(\Gamma_\rho)$ and all sufficiently large T ,*

$$\#\{\gamma \in \mathbf{P}_\xi : \bar{\psi}(\mu(\gamma)) \leq T\} \asymp e^{\delta_\psi(\mathbf{P}_\xi)T}.$$

For $p \in \Delta \cap \Lambda_p(\Gamma_\rho)$ and $r > 0$, define

$$(10.14) \quad N_r(\partial J_p) := \{x \in J_p^c : D_{\bar{\psi}}(x, \partial J_p) \leq r\}.$$

When $p = \gamma_p p_0$, we define $N_r(\partial(\gamma_p^{-1} J_p))$ in the same way.

Lemma 10.20. *Let $C > 1$. For every sufficiently small $\eta > 0$, there exists $c = c(\eta) \in (0, 1)$ such that for all $p \in \Delta \cap \Lambda_p(\Gamma_\rho)$ and $0 < r < h(p)$,*

$$\nu(N_{C\eta r}(\partial J_p)) \leq c\nu(N_r(\partial J_p)).$$

Moreover, $c(\eta) \rightarrow 0$ as $\eta \rightarrow 0$.

Proof. Suppose that $p \in \Gamma_\rho p_i$ and write $N_s^{(i)} := N_s(\partial J_{\Gamma_i, g_i p})$. Choose fixed constants $0 < a < 1$ and $L \geq 1$ so that the neighborhoods of radius ar and ar/L used below lie in compact sets where the coordinate changes are bi-Lipschitz with constant L . For sufficiently small η , Lemma 5.7 and $\nu_i = (g_i)_* \nu$ then give $\nu(N_{C\eta r}(\partial J_p)) \leq \nu_i(N_{LC\eta r}^{(i)})$ and $\nu_i(N_{ar/L}^{(i)}) \leq \nu(N_r(\partial J_p))$. By Lemma 6.6, we may also choose a so that $r_i := ar/L < h_i(g_i p)$. Since $LC\eta r = (L^2 C/a)\eta r_i$, it suffices to prove the assertion at the cusp p_0 in each conjugated system, after adjusting the fixed constant C . We give the proof in the original notation, with $p = \gamma_p p_0$. Write $x_p := \gamma_p^{-1} p_0$ and set $\beta := C\sqrt{\eta} < 1$. We first estimate $\nu(N_{\beta r}(\partial J_p))/\nu(N_r(\partial J_p))$ in three ranges of r .

Suppose first that $r \leq \eta h(p)$. By Lemma 9.5 and corollary 5.10, every $x \in N_r(\partial J_p)$ satisfies $D_{\bar{\psi}}(x, p) \asymp \eta h(p)$. The ball-flipping estimates therefore give $D_{\bar{\psi}}(\gamma_p^{-1} x, x_p) \asymp \eta^{-1}$ and

$$D_{\bar{\psi}}(\gamma_p^{-1} x, \gamma_p^{-1} \partial J_p) \asymp \frac{D_{\bar{\psi}}(x, \partial J_p)}{\eta^2 h(p)}.$$

Applying these estimates in both directions, we obtain constants $c_1, c_2 > 0$ such that

$$(10.15) \quad N_{c_1 r / (\eta^2 h(p))}(\gamma_p^{-1} \partial J_p) \subset \gamma_p^{-1} N_r(\partial J_p) \subset N_{c_2 r / (\eta^2 h(p))}(\gamma_p^{-1} \partial J_p).$$

For the first inclusion, choose c_1 sufficiently small that points in the left-hand neighborhood remain at distance comparable to η^{-1} from x_p , and apply the inverse ball-flipping estimate. Moreover, Lemma 7.4 gives

$$(10.16) \quad \nu(\gamma_p^{-1} N_r(\partial J_p)) \asymp e^{\psi(\mu(\gamma_p))} \nu(N_r(\partial J_p)).$$

These estimates also hold with βr in place of r .

Case A1: $r \leq \eta^2 h(p)$. The endpoints of $\gamma_p^{-1} J_p$ are peripheral translates of the endpoints of Δ_0 . By (10.15), the part near each endpoint $z \in \gamma_p^{-1} \partial J_p$ is contained in

$$\left\{ x \in (\gamma_p^{-1} J_p)^c : D_{\bar{\psi}}(x, z) \leq \frac{c_2 r}{\eta^2 h(p)} \right\}.$$

Since $r \leq \eta^2 h(p)$, each of these endpoint neighborhoods has uniformly bounded diameter and lies at distance comparable to η^{-1} from x_p . Thus Lemma 7.3, applied to the two endpoint neighborhoods as in the proof of Lemma 10.18, gives a fixed $x_0 \in \partial \Delta_0$ and constants $a, b > 0$ such that, for $t = r$ and $t = \beta r$,

$$\eta^2 \nu \left(\mathcal{B}_{\bar{\psi}} \left(x_0, \frac{at}{\eta^2 h(p)} \right) \right) \ll \nu(\gamma_p^{-1} N_t(\partial J_p)) \ll \eta^2 \nu \left(\mathcal{B}_{\bar{\psi}} \left(x_0, \frac{bt}{\eta^2 h(p)} \right) \right).$$

Together with (10.16), Proposition 7.8(2) yields $\nu(N_{\beta r}(\partial J_p)) \leq \omega(\beta) \nu(N_r(\partial J_p))$, where $\omega(t) \rightarrow 0$ as $t \rightarrow 0$. The bounded range of radii is handled as in Lemma 10.18.

Case A2: $\eta^{3/2} h(p) < r \leq \eta h(p)$. Write $\delta_0 := \delta_{\bar{\psi}}(\mathbf{P}_0)$. For a generator u of $\mathbf{P}_0$, Lemmas 3.3, 5.2 and 5.7 and Lemma 3.14 give $D_{\bar{\psi}}(x_0, u^j x_0) \asymp e^{\bar{\psi}(\mu(u^j))/2}$ for large $|j|$. Hence Proposition 10.19 gives a tile count comparable to $T^{2\delta_0}$ at $D_{\bar{\psi}}$ -scale T .

For $\eta^2 h(p) \leq t \leq \eta h(p)$, apply this count with $T = t/(\eta^2 h(p))$ to the endpoint neighborhoods in (10.15). For large T , quasi-monotonicity gives both an upper covering count and a comparable lower count of full tiles contained in $N_{c_1 t / (\eta^2 h(p))}(\gamma_p^{-1} \partial J_p)$. The bounded range of T follows from the endpoint estimates used in Case A1. For the upper bound, use the peripheral tiles that meet $\gamma_p^{-1} N_t(\partial J_p)$. The preceding distance estimate and the uniformly bounded diameters of the tiles place every such tile at distance comparable to η^{-1} from x_p . For the lower bound, use the full tiles contained in $N_{c_1 t / (\eta^2 h(p))}(\gamma_p^{-1} \partial J_p)$ in (10.15); the choice of c_1 keeps every point of that neighborhood at distance comparable to η^{-1} from x_p . Thus Lemma 7.3 gives measure comparable to η^2 for every tile used in either estimate. Consequently,

$$\nu(\gamma_p^{-1} N_t(\partial J_p)) \asymp \eta^2 \left(\frac{t}{\eta^2 h(p)} \right)^{2\delta_0}.$$

Both $t = r$ and $t = \beta r$ belong to this range, since $\beta r > C\eta^2 h(p)$. Using (10.16), we obtain $\nu(N_{\beta r}(\partial J_p)) \ll \beta^{2\delta_0} \nu(N_r(\partial J_p))$.

Case B: $\sqrt{\eta} h(p) \leq r < h(p)$. Here $\eta h(p) \leq \beta r$. By Lemma 9.5 and corollaries 5.10 and 9.1, there are fixed constants $0 < a < 1 < b$ such that

$$\mathcal{B}_{\bar{\psi}}(p, ar) \subset J_p \cup N_r(\partial J_p), \quad J_p \cup N_{\beta r}(\partial J_p) \subset \mathcal{B}_{\bar{\psi}}(p, b\beta r).$$

Take η sufficiently small that $b\beta < 1$. Since $N_{\beta r}(\partial J_p) \subset N_r(\partial J_p)$ and both sets are disjoint from J_p ,

$$\frac{\nu(N_{\beta r}(\partial J_p))}{\nu(N_r(\partial J_p))} \leq \frac{\nu(J_p \cup N_{\beta r}(\partial J_p))}{\nu(J_p \cup N_r(\partial J_p))} \leq \frac{\nu(\mathcal{B}_{\bar{\psi}}(p, b\beta r))}{\nu(\mathcal{B}_{\bar{\psi}}(p, ar))} \ll \beta^{2(1-\delta_0)}$$

by Corollary 7.10. The exponent is positive by Theorem 7.1.

The three cases give a common $c(\eta) \rightarrow 0$ such that $\nu(N_{\beta r}(\partial J_p)) \leq c(\eta)\nu(N_r(\partial J_p))$ throughout the indicated ranges. Since $C\eta r \leq \beta r$, this proves the desired estimate in those ranges.

In either remaining range, $\eta^2 h(p) < r \leq \eta^{3/2} h(p)$ or $\eta h(p) < r < \sqrt{\eta} h(p)$, set $s := \sqrt{\eta} r$. Then s belongs to Case A1 or Case A2, respectively, and $C\eta r = \beta s$. Therefore,

$$\nu(N_{C\eta r}(\partial J_p)) = \nu(N_{\beta s}(\partial J_p)) \leq c(\eta)\nu(N_s(\partial J_p)) \leq c(\eta)\nu(N_r(\partial J_p)).$$

For sufficiently small η , we also have $c(\eta) < 1$. □

10.11. Boundary-neighborhood estimates. Let

$$C_B = 2cN_0C_2C_0 + 4C_5 + 3N_0^2C_5\sqrt{C_4}H.$$

Here C_0, C_2, C_4 , and C_5 are the constants from Proposition 7.8, Lemma 9.5, Lemma 10.8, and Lemma 10.10, respectively. The constant c was fixed before Lemma 9.6.

Set

$$\mathcal{Z}'_n = \{x \in \mathcal{Z}_n : D_{\bar{\psi}}(x, \partial \mathcal{Z}_n) \leq C_B h_n\}.$$

Lemma 10.21. *There exists $C_9 > 0$, depending only on η , such that for all sufficiently large n ,*

$$\nu(\mathcal{Z}'_n) \leq C_9 \nu(\mathcal{Z}_n);$$

in particular, $\nu(A_n \cap \mathcal{Z}'_n) \leq C_9 \nu(\mathcal{Z}_n)$. Moreover, $C_9 \rightarrow 0$ as $\eta \rightarrow 0$.

Proof. Choose $0 < a < \min\{1, C_s/(4N_0^3)\}$ and set $r_n := ah_n/\eta$. For $p \in P_1 \cup \dots \cup P_n$, we have $h_n < \eta h(p)$, so $r_n < h(p)$. For sufficiently large n , we also have $r_n < 1$.

Since $C_B h_n = (C_B/a)\eta r_n$, Lemmas 10.18 and 10.20 give a common $c = c(\eta) > 0$, with $c(\eta) \rightarrow 0$ as $\eta \rightarrow 0$, such that

$$\begin{aligned} \nu(N_{C_B h_n}(\Delta_0)) &\leq c \nu(N_{r_n}(\Delta_0)), \\ \nu(N_{C_B h_n}(\partial J_p)) &\leq c \nu(N_{r_n}(\partial J_p)) \quad (p \in P_1 \cup \dots \cup P_n). \end{aligned}$$

By Lemma 10.6 and the choice of a , the sets $N_{r_n}(\Delta_0)$ and $N_{r_n}(\partial J_p)$, $p \in P_1 \cup \dots \cup P_n$, are pairwise disjoint subsets of $\mathcal{Z}_n$. Hence

$$\nu(\mathcal{Z}'_n) \leq \nu(N_{C_B h_n}(\Delta_0)) + \sum_{p \in P_1 \cup \dots \cup P_n} \nu(N_{C_B h_n}(\partial J_p)) \leq c \nu(\mathcal{Z}_n).$$

Taking $C_9 := c$ proves the lemma. □

Lemma 10.22. *There exists $0 < C_{10} < 1$ such that for all $n \geq 1$,*

$$\nu(A_n \cap (D_{n+1} \cup B_{n+1})) \leq C_{10} \nu(A_n) + \nu(A_n \cap \mathcal{Z}'_n).$$

Proof. Set

$$Q'''_{n+1} := \{p \in Q''_{n+1} : B_{p,n+1} \cap ((A_n \cap B_{n+1}) \setminus \mathcal{Z}'_n) \neq \emptyset\}.$$

For $p \in Q'''_{n+1}$, choose $x \in B_{p,n+1} \cap A_n \cap B_{n+1}$ with $x \notin \mathcal{Z}'_n$. By Lemmas 10.8 and 10.10, $D_{\bar{\psi}}(x, p) < C_5 \sqrt{C_4} h_n$.

We claim that the class containing p is constructed in Case 2. Otherwise, its construction and Lemma 10.5 give $D_{\bar{\psi}}(p, \partial \mathcal{Z}_n) \leq h_n$. The quasi-triangle inequality then implies $D_{\bar{\psi}}(x, \partial \mathcal{Z}_n) \leq N_0(C_5\sqrt{C_4} + 1)h_n < C_B h_n$, contrary to $x \notin \mathcal{Z}'_n$. Thus $p \in Q_{n+1} \setminus Q'_{n+1}$, $C(p) = \{p\}$, and $B_{p,n+1} = \mathcal{B}_{\bar{\psi}}(p, r(p, n+1)) \subset \mathcal{B}_{\bar{\psi}}(p, h_n)$.

For $p \in P_{n+1}$, Lemma 9.5 gives $J_p \subset \mathcal{B}_{\bar{\psi}}(p, C_2 h_n)$. Together with Lemma 10.16(4), these inclusions show that

$$(A_n \cap (D_{n+1} \cup B_{n+1})) \setminus \mathcal{Z}'_n \subset \bigcup_{p \in P_{n+1} \cup Q'''_{n+1}} \mathcal{B}_{\bar{\psi}}(p, C_2 h_n).$$

Set $R_n := C_B h_n / (2cN_0)$. We next show that $\mathcal{B}_{\bar{\psi}}(p, R_n) \subset A_n$ for every $p \in P_{n+1} \cup Q'''_{n+1}$. By Lemma 10.15, $D_{\bar{\psi}}(p, B_n) \geq C_7 h_n / \eta > R_n$ for sufficiently small η , so these balls avoid B_n . For $p \in P_{n+1}$, their inclusion in $\mathcal{Z}_n$ follows from the defining condition $\mathcal{B}_{\bar{\psi}}(p, h_n / (4\eta)) \subset \mathcal{Z}_n$.

For $p \in Q'''_{n+1}$, use the point x chosen above, which now satisfies $D_{\bar{\psi}}(x, p) < h_n$. If $z \in \mathcal{B}_{\bar{\psi}}(p, R_n) \setminus \mathcal{Z}_n$, then Corollaries 5.10 and 9.1 give

$$C_B h_n < D_{\bar{\psi}}(x, \partial \mathcal{Z}_n) \leq cD_{\bar{\psi}}(x, z) \leq cN_0(h_n + R_n) < C_B h_n,$$

since $C_B > 2cN_0$. This contradiction proves the required inclusion in A_n .

By Proposition 6.7, the balls $\mathcal{B}_{\bar{\psi}}(p, R_n)$, $p \in P_{n+1} \cup Q'''_{n+1}$, are pairwise disjoint for sufficiently small η . Their centers lie in Δ_0 , so Lemma 6.5 and $h_{n+1} < \eta h(p)$ give $h_n \ll \eta$ whenever the index set is nonempty. Thus $C_2 h_n < 1/C_0$ after decreasing η . Since $R_n \geq C_0 C_2 h_n$ by the choice of C_B , Proposition 7.8(2) yields

$$\begin{aligned} \nu((A_n \cap (D_{n+1} \cup B_{n+1})) \setminus \mathcal{Z}'_n) &\leq \sum_{p \in P_{n+1} \cup Q'''_{n+1}} \nu(\mathcal{B}_{\bar{\psi}}(p, C_2 h_n)) \\ &\leq \frac{1}{2} \sum_{p \in P_{n+1} \cup Q'''_{n+1}} \nu(\mathcal{B}_{\bar{\psi}}(p, R_n)) \leq \frac{1}{2} \nu(A_n). \end{aligned}$$

Adding $\nu(A_n \cap \mathcal{Z}'_n)$ proves the lemma with $C_{10} = 1/2$. $\square$

Lemma 10.23. *There exist $N \in \mathbb{N}$ and $C_{11} > 0$, depending on η , such that, for all sufficiently large n ,*

$$\nu\left(\bigcup_{j=1}^N D_{n+j}\right) \geq C_{11} \nu(A_n \setminus \mathcal{Z}'_n).$$

Proof. Set $A'_n := A_n \setminus \mathcal{Z}'_n$. Let $U_0 > 0$ be given by Lemma 8.3 for $\tau = \mathbb{H}$ and the compact set $Q := \Gamma_\rho(\bar{\Omega}_{\bar{\psi}} \setminus \mathcal{C}_{C_6 \eta}) \subset \Omega_{\bar{\psi}}$. Set $k := \lfloor -\log \eta / \log \mathbb{H} \rfloor$ and $L := \lceil U_0 / \log \mathbb{H} \rceil$, and choose $N > 2k + L + 2$. We will show that there exists $C' > 1$ such that

$$(10.17) \quad \text{for every } x \in A'_n, \text{ there is } p \in \bigcup_{j=1}^N P_{n+j} \text{ with } D_{\bar{\psi}}(x, p) < C' h_n.$$

We first consider a point $y \in A_m$ satisfying $D_{\bar{\psi}}(y, \partial \mathcal{Z}_m) > 2h_m / \eta$. We claim that there exists $q \in P_{m+1} \cup \dots \cup P_{m+k+L+1}$ with $D_{\bar{\psi}}(y, q) \ll h_m$.

Indeed, Lemma 10.13 implies that $\Gamma_\rho(y, p_0, 0) a_{m \log \mathbb{H}} \in Q$. By Lemma 8.3, there is a parabolic point p such that $D_{\bar{\psi}}(y, p) \leq C h_m$ and $e^{-U_0} h_m \leq h(p) \leq h_m$, where C is independent of η . For sufficiently small η , Corollary 9.1 and the distance of y from $\partial \mathcal{Z}_m$ imply that $p \in \mathcal{Z}_m$.

Set $j := \lfloor -\log(\eta h(p)) / \log \mathbb{H} \rfloor$. Then $m + k \leq j \leq m + k + L$, $h_{j+1} < \eta h(p) \leq h_j$, and $h_j / \eta < \mathbb{H} h_m$. If $p \in P_{j+1}$, take $q = p$. If $p \notin \mathcal{Z}_j$, then $p \in J_q$ for some $q \in P_{m+1} \cup \dots \cup P_j$. By Lemma 9.5 and corollary 5.10, $D_{\bar{\psi}}(y, q) \leq N_0(C + C_2)h_m$.

It remains to consider $p \in \mathcal{Z}_j \setminus P_{j+1}$. The definition of P_{j+1} and Corollary 9.1 give $D_{\bar{\psi}}(p, \partial \mathcal{Z}_j) \leq ch_j/(4\eta)$. On the other hand, $D_{\bar{\psi}}(p, \partial \mathcal{Z}_m) \geq 2h_m/(N_0\eta) - Ch_m$. Choose $z \in \partial \mathcal{Z}_j$ such that

$$D_{\bar{\psi}}(p, z) \leq ch_j/(4\eta).$$

Since $h_j/\eta < Hh_m$, the lower bound for $D_{\bar{\psi}}(p, \partial \mathcal{Z}_m)$ implies $z \notin \partial \mathcal{Z}_m$ for sufficiently small η . Thus there is $q \in P_{m+1} \cup \dots \cup P_j$ with $D_{\bar{\psi}}(p, \partial J_q) \leq ch_j/(4\eta)$. Another application of Lemma 9.5 and corollary 5.10 gives

$$D_{\bar{\psi}}(y, q) \leq N_0Ch_m + N_0^2 \left(\frac{ch_j}{4\eta} + C_2h_m \right) \ll h_m.$$

This proves the auxiliary claim.

Now fix $x \in A'_n$ and put $m := n + k$. Recall that $\eta \leq H^{-k} < H\eta$.

If $x \notin \mathcal{Z}_m$, then $x \in J_p$ for some $p \in P_{n+1} \cup \dots \cup P_m$. By Lemma 9.5, $D_{\bar{\psi}}(x, p) < C_2\eta h(p) \leq C_2h_n$.

Suppose next that $x \in \mathcal{Z}_m$ and $D_{\bar{\psi}}(x, \partial \mathcal{Z}_m) < 3N_0h_m/\eta$. Since $3N_0h_m/\eta < 3N_0Hh_n < C_Bh_n$ and $x \notin \mathcal{Z}'_n$, there is $p \in P_{n+1} \cup \dots \cup P_m$ with $D_{\bar{\psi}}(x, \partial J_p) < 3N_0h_m/\eta$. Hence $D_{\bar{\psi}}(x, p) \leq N_0(3N_0h_m/\eta + C_2h_n) \ll h_n$.

We may therefore assume that $x \in \mathcal{Z}_m$ and $D_{\bar{\psi}}(x, \partial \mathcal{Z}_m) \geq 3N_0h_m/\eta$. If $x \in A_m$, the auxiliary claim gives the required point in $P_{m+1} \cup \dots \cup P_{m+k+L+1}$.

It remains to consider $x \in B_m$. Since $x \notin B_n$ and the sets $B_{p,t}$ decrease with t , there is $p \in Q''_{n+1} \cup \dots \cup Q''_m$ such that $x \in B_{p,m}$. By Lemmas 10.8 and 10.10,

$$(10.18) \quad B_{p,m} \subset \mathcal{B}_{\bar{\psi}} \left(p, C_5 \sqrt{C_4 h_n h_m} \right).$$

In particular, any two points of $B_{p,m}$ are at $D_{\bar{\psi}}$ -distance $O(\sqrt{h_n h_m})$. Since $\sqrt{h_n h_m} \leq \sqrt{\eta} h_m/\eta$, Corollary 9.1 and the distance of x from $\partial \mathcal{Z}_m$ imply that $B_{p,m} \subset \mathcal{Z}_m$ for sufficiently small η .

As in the proof of Lemma 10.17, the annular estimate Lemma 9.6, applied in the coordinates defining $B_{p,m}$, gives $\nu(B_{p,m} \setminus B_{p,m+1}) > 0$. Choose y in this difference. By (10.11) and Lemma 10.16(2),(5), $y \in B_m \setminus B_{m+1} \subset A_{m+1}$. Moreover, for sufficiently small η

$$D_{\bar{\psi}}(y, \partial \mathcal{Z}_m) \geq \frac{1}{N_0} D_{\bar{\psi}}(x, \partial \mathcal{Z}_m) - D_{\bar{\psi}}(x, y) > \frac{2h_m}{\eta}$$

If $D_{\bar{\psi}}(y, \partial \mathcal{Z}_{m+1}) > 2h_{m+1}/\eta$, apply the auxiliary claim at level $m+1$. It gives $q \in P_{m+2} \cup \dots \cup P_{m+k+L+2}$ with $D_{\bar{\psi}}(y, q) \ll h_{m+1}$. Otherwise, the preceding lower bound shows that $D_{\bar{\psi}}(y, \partial J_q) \leq 2h_{m+1}/\eta$ for some $q \in P_{m+1}$. Then Lemma 9.5 and corollary 5.10 give $D_{\bar{\psi}}(y, q) \ll h_{m+1}/\eta + h_m \ll h_n$, since $h_{m+1}/\eta \leq h_n$.

In either case, $q \in \bigcup_{j=1}^N P_{n+j}$ and $D_{\bar{\psi}}(x, q) \leq N_0(D_{\bar{\psi}}(x, y) + D_{\bar{\psi}}(y, q)) \ll h_n$. This proves (10.17) after choosing C' sufficiently large.

Finally, for every $p \in \bigcup_{j=1}^N P_{n+j}$, Lemma 9.5 gives $\mathcal{B}_{\bar{\psi}}(p, h_{n+N}/C_2) \subset J_p$, and these sets J_p are pairwise disjoint by Lemma 10.6. Applying Proposition 7.8(1) with the fixed dilation factor $C'C_2H^N$, we obtain, for all sufficiently large n ,

$$\nu \left(\bigcup_{j=1}^N D_{n+j} \right) \geq \sum_p \nu \left(\mathcal{B}_{\bar{\psi}}(p, h_{n+N}/C_2) \right) \geq C_{11} \sum_p \nu \left(\mathcal{B}_{\bar{\psi}}(p, C'h_n) \right) \geq C_{11} \nu(A'_n),$$

where both sums range over $p \in \bigcup_{j=1}^N P_{n+j}$. The last inequality follows from (10.17). $\square$

10.12. Proof of Proposition 10.1. By Lemma 10.16(1), (3), (5), Lemma 10.17, Lemma 10.21, and Lemma 10.22,

$$\nu(A_n) \geq (1 - C_9 - C_{10})\nu(A_{n-1}) + (C_8 - C_9)\nu(B_{n-1}).$$

Since $C_9 \rightarrow 0$ as $\eta \rightarrow 0$, choose η so small that $2C_9 < \min\{1 - C_{10}, C_8\}$, and put $m := \min\{1 - C_9 - C_{10}, C_8 - C_9\} > C_9$. As $\mathcal{Z}_{n-1} = A_{n-1} \sqcup B_{n-1}$ and $\mathcal{Z}_n \subset \mathcal{Z}_{n-1}$, we have $\nu(A_n) \geq m\nu(\mathcal{Z}_n)$. Thus Lemma 10.21 gives

$$\nu(A_n - \mathcal{Z}'_n) \geq \nu(A_n) - \nu(\mathcal{Z}'_n) \geq (m - C_9)\nu(\mathcal{Z}_n).$$

By Lemma 10.23, there exist $N \geq 1$ and $c_0 \in (0, 1)$ such that, for all sufficiently large n , $\nu\left(\bigcup_{j=1}^N D_{n+j}\right) \geq c_0\nu(\mathcal{Z}_n)$. Since $\mathcal{Z}_{n+N} = \mathcal{Z}_n \setminus \bigcup_{j=1}^N D_{n+j}$, $\nu(\mathcal{Z}_{n+N}) \leq (1 - c_0)\nu(\mathcal{Z}_n)$. Iteration gives $\nu(\mathcal{Z}_n) \ll (1 - c_0)^{n/N}$. Choose $\varepsilon_0 > 0$ so that $(1 - c_0)^{1/N} < 1 - \varepsilon_0 < 1$. Then $\nu(\mathcal{Z}_n) \leq (1 - \varepsilon_0)^n$ for all sufficiently large n , proving Proposition 10.1. $\square$

11. THE INDUCED EXPANDING MAP

We now use the geometric partition to construct an expanding map with countably many branches. We first define T_0 on the sets $\gamma_p\gamma_1\Delta_i$ appearing in the decompositions of the sets J_p . We then successively take first-return maps until we obtain a return map on Δ_0 . At each step, we prove that the return time is finite almost everywhere and establish exponential summability of the branches. Finally, we take a fixed power of the return map so that all return vectors lie uniformly in the prescribed positive cone.

Let η be sufficiently small as in Proposition 9.8, allowing further reduction until its final choice after Lemma 11.5. We suppress η from the notation. Set

$$\tilde{\Lambda}' := \bigcup_{j=0}^{\ell} \bigcup_{n \geq 1} \bigcup_{p \in P_{j,n}} J_p, \quad \tilde{\Lambda} := \tilde{\Lambda}' \setminus \Lambda_p(\Gamma_\rho).$$

By Lemma 9.4,

$$\tilde{\Lambda} = \bigcup_{i,j=0}^{\ell} \bigcup_{n \geq 1} \bigcup_{p \in \Gamma_\rho p_i \cap P_{j,n}} \bigcup_{\gamma_1 \in N_p} \gamma_p\gamma_1(\Delta_i \setminus \Lambda_p(\Gamma_\rho)).$$

For each j , the construction gives

$$\Delta_j \setminus \bigcup_{n \geq 1} \bigcup_{p \in P_{j,n}} J_p = \bigcap_{n \geq 0} \mathcal{Z}_{j,n}.$$

By Proposition 9.8, $\nu(\mathcal{Z}_{j,n}) \rightarrow 0$ as $n \rightarrow \infty$. Hence $\tilde{\Lambda}'$ has full ν -measure in Δ . Since $\Lambda_p(\Gamma_\rho)$ is countable and ν is atomless by Theorem 3.17, removing the parabolic fixed points leaves $\tilde{\Lambda}$ with full ν -measure in Δ .

Define $T_0 : \tilde{\Lambda} \rightarrow \Delta$ by

$$T_0(x) = (\gamma_p\gamma_1)^{-1}x \quad \text{if } x \in \tilde{\Lambda} \cap \gamma_p\gamma_1\Delta_i.$$

For $0 \leq i, j \leq \ell$, let $\mathcal{H}_0(i, j)$ be the set of elements $\gamma_p\gamma_1$ with $p \in \Gamma_\rho p_i \cap P_{j,n}$ for some $n \geq 1$ and $\gamma_1 \in N_p$. Each $\gamma \in \mathcal{H}_0(i, j)$ maps Δ_i onto a cell $\gamma\Delta_i \subset \Delta_j$, on which T_0 acts as γ^{-1} wherever it is defined. We call γ an inverse branch of T_0 . Thus i records the domain of the inverse branch, and j records the set containing its image.

Set

$$\tilde{\Lambda}_+ := \bigcap_{m \geq 0} T_0^{-m}(\tilde{\Lambda}).$$

This is precisely the set of points whose iterates T_0^m are defined for every $m \geq 1$, and it has full ν -measure in Δ by quasi-invariance of ν . By construction, T_0 maps $\tilde{\Lambda}_+$ into itself.

The inverse branches also preserve this set: if $x \in \tilde{\Lambda}_+ \cap \Delta_i$ and $\gamma \in \mathcal{H}_0(i, j)$, then $\gamma x \in \tilde{\Lambda}_+ \cap \Delta_j$ and $T_0(\gamma x) = x$. Hence all iterates of γx are defined, so $\gamma x \in \tilde{\Lambda}_+ \cap \Delta_j$.

The transition graph of T_0 is the directed graph with vertex set $\{0, \dots, \ell\}$ and an edge $j \rightarrow i$ whenever $\mathcal{H}_0(i, j) \neq \emptyset$. Since this graph is finite, we can choose a nonempty set I_0 of indices such that no directed edge leaves I_0 and the graph on I_0 is strongly connected: every vertex can be reached from every other by a directed path within I_0 . Thus T_0 maps $\tilde{\Lambda}_+ \cap \bigcup_{i \in I_0} \Delta_i$ into itself. We may assume the index 0 belongs to I_0 .

Definition 11.1 (First returns to Δ_0). Let Λ_+ be the set of points $x \in \tilde{\Lambda}_+ \cap \Delta_0$ such that $T_0^m x \in \Delta_0$ for infinitely many $m \geq 1$. For $x \in \Lambda_+$, let $n(x) := \min\{m \geq 1 : T_0^m x \in \Delta_0\}$.

Proposition 11.2. *The set Λ_+ has full ν -measure in Δ_0 .*

We prove this by removing one index other than 0 at a time and taking the first-return map to the remaining domains. After $|I_0| - 1$ steps, only Δ_0 remains.

Set $X_0 := \tilde{\Lambda}_+ \cap \bigcup_{i \in I_0} \Delta_i$, and retain the notation T_0 for its restriction to X_0 . Then T_0 maps X_0 into itself, and $\gamma(X_0 \cap \Delta_i) \subset X_0 \cap \Delta_j$ for every $i, j \in I_0$ and $\gamma \in \mathcal{H}_0(i, j)$.

Suppose inductively that we have defined an index set I_k containing 0 and a map $T_k : X_k \rightarrow X_k$, where $X_k \subset \bigcup_{i \in I_k} \Delta_i$ has full ν -measure in that union. For $i, j \in I_k$, let $\mathcal{H}_k(i, j)$ denote the group elements defining the inverse branches of T_k from Δ_i to Δ_j . Assume that, for every $\gamma \in \mathcal{H}_k(i, j)$ and $y \in X_k \cap \Delta_i$, we have $\gamma y \in X_k \cap \Delta_j$ and $T_k(\gamma y) = y$. The transition graph of T_k has vertex set I_k and an edge $j \rightarrow i$ whenever $\mathcal{H}_k(i, j) \neq \emptyset$. Assume that this graph is strongly connected. If $I_k = \{0\}$, the construction stops.

Otherwise, put $\ell_k := \max I_k > 0$ and $J_k := I_k \setminus \{\ell_k\}$, and set $Y_{k+1} := X_k \cap \bigcup_{i \in J_k} \Delta_i$. For $x \in Y_{k+1}$, define its first-return time by

$$n_{k+1}(x) := \inf\{m \geq 1 : T_k^m x \in Y_{k+1}\},$$

with $\inf \emptyset = \infty$. Set

$$X_{k+1} := \{x \in Y_{k+1} : T_k^m x \in Y_{k+1} \text{ for infinitely many } m \geq 1\},$$

and define $T_{k+1}(x) := T_k^{n_{k+1}(x)} x$ for $x \in X_{k+1}$. Put $I_{k+1} := J_k$. After its first return, a point of X_{k+1} still returns infinitely often to Y_{k+1} . Hence T_{k+1} maps X_{k+1} into itself.

The inverse branches of T_{k+1} are compositions of inverse branches of T_k along paths that start and end in J_k and whose intermediate vertices are all ℓ_k . If such a composition γ has domain Δ_i and $y \in X_{k+1} \cap \Delta_i$, then γy first returns to Y_{k+1} at y . Hence γy also returns infinitely often, so $\gamma y \in X_{k+1}$ and $T_{k+1}(\gamma y) = y$.

Lemma 11.3 (Connectivity of the transition graphs). *For every k , the transition graph of T_k is strongly connected. If $I_k \neq \{0\}$, there is an edge from ℓ_k to a vertex of J_k .*

Proof. The assertion holds for $k = 0$ by the choice of I_0 . Suppose that the transition graph of T_k is strongly connected and $I_k \neq \{0\}$. Since J_k is nonempty, a path from ℓ_k to J_k contains an edge from ℓ_k to a vertex of J_k .

Given $i, j \in J_k$, choose a directed path from i to j in the graph of T_k and divide it at successive visits to J_k . Each resulting segment has all its intermediate vertices equal to ℓ_k , so it defines a branch of T_{k+1} . These branches give a directed path from i to j in the transition graph of T_{k+1} . Thus this graph is strongly connected, completing the induction. $\square$

The following proposition, proved below, shows that the restriction to X_{k+1} removes only a null set.

Proposition 11.4. *For every k with $I_k \neq \{0\}$, the set X_{k+1} has full ν -measure in Y_{k+1} .*

11.1. Estimates for the initial branch map. Recall that

$$(11.1) \quad \mathcal{H}_0(i, j) = \bigcup_{n \geq 1} \{\gamma_p \gamma_1 : p \in \Gamma_\rho p_i \cap P_{j,n}, \gamma_1 \in N_p\}, \quad i, j \in I_0.$$

Write $\mathcal{H}_0 := \bigcup_{i,j \in I_0} \mathcal{H}_0(i, j)$ for the collection of all inverse branches of T_0 .

For $i \in I_0$, define the following neighborhood of p_i in Λ_ρ :

$$(11.2) \quad U_i := \Lambda_\rho \setminus g_i^{-1} \mathcal{B}_{\bar{\psi}} \left(g_i \Delta_i, \frac{1}{2N_0\eta} \right).$$

Lemma 11.5. *For every $\eta > 0$ sufficiently small, we have $\gamma^{-1}U_j \subset U_i$ for all $\gamma \in \mathcal{H}_0(i, j)$.*

Proof. Write $\gamma = \gamma_p \gamma_1$, so that $\gamma p_i = p \in \Delta_j$. Since Δ_j is compact and disjoint from p_j , we may choose a neighborhood W_j of p_j so that $\overline{W_j} \cap \Delta_j = \emptyset$. As $\eta \rightarrow 0$, the sets U_j shrink to p_j . Decreasing η if necessary, we may therefore assume $U_j \subset W_j$. Compactness and (5.3) now give $\varepsilon > 0$ such that, independent of η ,

$$(11.3) \quad D_{\bar{\psi}}(g_i x, g_i p) \geq \varepsilon \quad (x \in U_j \setminus \{p_i\}),$$

uniformly over the finitely many pairs i, j and all such p .

Fix $x \in U_j$. By (9.4), applied in the conjugated system, $D_{\bar{\psi}}(g_i \gamma^{-1} p_i, g_i \Delta_i) \geq \eta^{-1}$. If $x = p_i$, this already gives $\gamma^{-1}x \in U_i$. Otherwise, Lemma 6.8, applied to $g_i \gamma g_i^{-1}$, together with (11.3) and the uniform bound $h_i(g_i p) \ll 1$, gives $D_{\bar{\psi}}(g_i \gamma^{-1} x, g_i \gamma^{-1} p_i) \leq C$, where C is independent of γ , x , and η . Therefore, decreasing $\eta > 0$ sufficiently small that $\eta \leq (2N_0 C)^{-1}$, Corollary 5.10 gives

$$D_{\bar{\psi}}(g_i \gamma^{-1} x, g_i \Delta_i) \geq \frac{1}{N_0 \eta} - C \geq \frac{1}{2N_0 \eta}.$$

This means that $\gamma^{-1}x \in U_i$. □

For the rest of the paper, we fix $\eta > 0$ sufficiently small to satisfy all the preceding requirements, including those of Lemma 11.5.

Lemma 11.6. *There exists $r > 0$ such that, for every $n \geq 1$, $i_0, \dots, i_n \in I_0$, and $\gamma_m \in \mathcal{H}_0(i_{m-1}, i_m)$ ($1 \leq m \leq n$), the product $g := \gamma_n \cdots \gamma_1$ satisfies $g \Delta_{i_0} \subset \mathcal{O}_r(o, go)$.*

Proof. Suppose otherwise. For each $k \geq 1$, choose a product $g_k := \gamma_{k,n_k} \cdots \gamma_{k,1}$ as in the statement and $x_k \in \Delta_{i_{k,0}}$ such that $g_k x_k \notin \mathcal{O}_k(o, g_k o)$, or equivalently, $x_k \notin \mathcal{O}_k(g_k^{-1} o, o)$. It follows that $g_k \rightarrow \infty$ and we may therefore assume that the g_k are pairwise distinct, $i_{k,0} = i$, $i_{k,n_k} = j$, and $x_k \rightarrow x \in \Delta_i$.

By Lemma 3.16(1) and (3), after passing to a further subsequence, there exists $y \in \Lambda_\rho$ such that $g_k^{-1} \zeta \rightarrow x$ for every $\zeta \in \Lambda_\rho \setminus \{y\}$. Repeated application of Lemma 11.5 gives $g_k^{-1} U_j \subset U_i$. Choose $\zeta \in U_j \setminus \{y\}$; then $g_k^{-1} \zeta \in U_i$ and $g_k^{-1} \zeta \rightarrow x$. As U_i is closed, $x \in U_i$, contradicting $U_i \cap \Delta_i = \emptyset$. □

Proposition 11.7. *There exists $\varepsilon > 0$ such that*

$$\sum_{\gamma \in \mathcal{H}_0} e^{-(1-\varepsilon)\psi(\mu(\gamma))} < \infty.$$

Proof. By Lemma 9.5 and the proof of Lemma 7.4, applied to the finitely many conjugated systems, there exists $r > 0$ such that, for every $i \in I_0$, $p \in \Delta \cap \Gamma_\rho p_i$, and $\gamma_1 \in N_p$,

$$(11.4) \quad \gamma_p \gamma_1 \Delta_i \subset J_p \subset \mathcal{O}_r(o, \gamma_p o).$$

After increasing r , we also have

$$(11.5) \quad \gamma_1 \Delta_i \subset \mathcal{O}_r(o, \gamma_1 o).$$

Indeed, otherwise equivariance and Lemma 3.16(3), applied to a sequence in P_i , would force a sequence in Δ_i to converge to p_i , contradicting compactness of Δ_i and $p_i \notin \Delta_i$.

Now fix $j \in I_0$ and restrict to $p \in P_{j,n} \cap \Gamma_\rho p_i$, with $n \geq 1$ and $i \in I_0$. For $\gamma_1 \in N_p$ and $x \in \Delta_i$, the cocycle identity gives

$$\beta_{\gamma_p \gamma_1 x}(e, \gamma_p \gamma_1) = \beta_{\gamma_p \gamma_1 x}(e, \gamma_p) + \beta_{\gamma_1 x}(e, \gamma_1).$$

By (11.4), (11.5), and Lemma 3.11, the two terms on the right are $\mu(\gamma_p) + O(1)$ and $\mu(\gamma_1) + O(1)$, respectively, with uniform errors. By Lemma 11.6, $\gamma_p \gamma_1 \Delta_i \subset \mathcal{O}_r(o, \gamma_p \gamma_1 o)$. Hence Lemma 3.11 and conformality of ν give $\nu(\gamma_p \gamma_1 \Delta_i) \asymp e^{-\psi(\mu(\gamma_p \gamma_1))}$. Combining this with the preceding cocycle estimate, we obtain

$$(11.6) \quad e^{-\psi(\mu(\gamma_p \gamma_1))} \asymp \nu(\gamma_p \gamma_1 \Delta_i) \asymp e^{-\psi(\mu(\gamma_p))} e^{-\psi(\mu(\gamma_1))}.$$

By Theorem 7.1, choose $\varepsilon > 0$ small enough that $1 - \varepsilon > \delta_\psi(P_i)$ for every i . For fixed η , the set N_p depends only on the type i . Both sums below are finite and strictly positive, since N_p is cofinite in P_i . As there are only finitely many types, we obtain, uniformly in p ,

$$\sum_{\gamma_1 \in N_p} e^{-(1-\varepsilon)\psi(\mu(\gamma_1))} \asymp \sum_{\gamma_1 \in N_p} e^{-\psi(\mu(\gamma_1))}.$$

Together with (11.6) and Lemma 9.4, this gives

$$(11.7) \quad \sum_{\gamma_1 \in N_p} e^{-(1-\varepsilon)\psi(\mu(\gamma_p \gamma_1))} \asymp e^{\varepsilon\psi(\mu(\gamma_p))} \sum_{\gamma_1 \in N_p} \nu(\gamma_p \gamma_1 \Delta_i) = e^{\varepsilon\psi(\mu(\gamma_p))} \nu(J_p).$$

By (7.1), $\psi^-(\mu(\gamma))$ is bounded below uniformly on Γ_ρ , so $\psi(\mu(\gamma)) \leq 2\bar{\psi}(\mu(\gamma)) + O(1)$. For $p \in P_{j,n}$, Lemmas 6.5 and 6.6 and the definition of $P_{j,n}$ give $\bar{\psi}(\mu(\gamma_p)) \leq n \log H + O(1)$, with the fixed parameter η absorbed in the error. Hence, by (11.1) and (11.7),

$$\sum_{i \in I_0} \sum_{\gamma \in \mathcal{H}_0(i, j)} e^{-(1-\varepsilon)\psi(\mu(\gamma))} \ll \sum_{n \geq 1} H^{2\varepsilon n} \sum_{p \in P_{j,n}} \nu(J_p) \ll \sum_{n \geq 1} H^{2\varepsilon n} \nu(\mathcal{Z}_{j,n-1}) \ll \sum_{n \geq 1} (H^{2\varepsilon}(1 - \varepsilon_0))^n.$$

The last estimate follows from Proposition 9.8. Taking ε smaller if necessary so that $H^{2\varepsilon}(1 - \varepsilon_0) < 1$ proves the desired summability for every $j \in I_0$, and hence for $\mathcal{H}_0$. $\square$

11.2. The inductive argument. We now prove the estimates needed at each step of the induction. Fix k with $I_k \neq \{0\}$. For $I, J \subset I_k$, write $\mathcal{H}_k(I, J) := \bigcup_{i \in I, j \in J} \mathcal{H}_k(i, j)$, and set $\mathcal{H}_k := \mathcal{H}_k(I_k, I_k)$.

Lemma 11.8. *There exists $\varepsilon_k > 0$ such that, for every $n \geq 1$,*

$$\sum_{\gamma_1, \dots, \gamma_n \in \mathcal{H}_k(\ell_k, \ell_k)} e^{-\psi(\mu(\gamma_1 \dots \gamma_n))} \ll (1 - \varepsilon_k)^n.$$

Proof. Set $\mathcal{J} := \mathcal{H}_k(\ell_k, \ell_k)$ and $E := \Delta_{\ell_k} \setminus \bigcup_{\gamma \in \mathcal{J}} \gamma \Delta_{\ell_k}$. By Lemma 11.3, there exist $i \in J_k = I_k \setminus \{\ell_k\}$ and $\gamma \in \mathcal{H}_k(i, \ell_k)$. The set $\gamma \Delta_i$ is disjoint from $\bigcup_{\delta \in \mathcal{J}} \delta \Delta_{\ell_k}$ up to a ν -null set. Since $\nu(\Delta_i) > 0$ and ν is quasi-invariant, $\nu(E) \geq \nu(\gamma \Delta_i) > 0$.

Each branch of T_k is a composition of branches of T_0 . By conformality and Lemmas 3.11 and 11.6, there exists $C > 1$ such that, for $g = e$ or any finite product of elements of $\mathcal{J}$, and every Borel $A \subset \Delta_{\ell_k}$,

$$(11.8) \quad C^{-1} e^{-\psi(\mu(g))} \nu(A) \leq \nu(gA) \leq C e^{-\psi(\mu(g))} \nu(A).$$

Set $\varepsilon_k := C^{-2} \nu(E) / \nu(\Delta_{\ell_k}) \in (0, 1)$. Then $\nu(gE) \geq \varepsilon_k \nu(g \Delta_{\ell_k})$. The sets $g \gamma \Delta_{\ell_k}$, $\gamma \in \mathcal{J}$, are disjoint up to ν -null sets, and their complement in $g \Delta_{\ell_k}$ is gE . Thus their total measure is at most $(1 - \varepsilon_k) \nu(g \Delta_{\ell_k})$. Iterating this estimate and using the lower bound in (11.8), we obtain

$$\sum_{\gamma_1, \dots, \gamma_n \in \mathcal{J}} e^{-\psi(\mu(\gamma_1 \cdots \gamma_n))} \leq \frac{C}{\nu(\Delta_{\ell_k})} \sum_{\gamma_1, \dots, \gamma_n \in \mathcal{J}} \nu(\gamma_1 \cdots \gamma_n \Delta_{\ell_k}) \leq C(1 - \varepsilon_k)^n.$$

□

We next estimate the return time to Y_{k+1} .

Proposition 11.9. *There exists $\varepsilon_k > 0$ such that, for every $n \geq 1$,*

$$\nu(\{x \in Y_{k+1} : n_{k+1}(x) > n + 1\}) \ll (1 - \varepsilon_k)^n.$$

In particular, X_{k+1} has full ν -measure in Y_{k+1} .

Proof. Set $\mathcal{J} := \mathcal{H}_k(\ell_k, \ell_k)$. Every $x \in Y_{k+1}$ with $n_{k+1}(x) > n + 1$ belongs to a set $\gamma_0 g \Delta_{\ell_k}$, where $\gamma_0 \in \mathcal{H}_k(\ell_k, J_k)$ and $g = \gamma_1 \cdots \gamma_n$ with $\gamma_m \in \mathcal{J}$.

Applying conformality and Lemmas 3.11 and 11.6 to γ_0 , and (11.8) to g , gives

$$\nu(\gamma_0 g \Delta_{\ell_k}) \ll \frac{\nu(\gamma_0 \Delta_{\ell_k})}{\nu(\Delta_{\ell_k})} \nu(g \Delta_{\ell_k}) \ll \nu(\gamma_0 \Delta_{\ell_k}) e^{-\psi(\mu(g))}.$$

The sets $\gamma_0 \Delta_{\ell_k}$, with $\gamma_0 \in \mathcal{H}_k(\ell_k, J_k)$, are disjoint up to null sets, so their measures sum to at most 1. Every point with $n_{k+1}(x) > n + 1$ lies in a set $\gamma_0 \gamma_1 \cdots \gamma_n \Delta_{\ell_k}$, where $\gamma_0 \in \mathcal{H}_k(\ell_k, J_k)$ and $\gamma_1, \dots, \gamma_n \in \mathcal{J}$. Summing the preceding estimate over these choices gives

$$\begin{aligned} \nu(\{x \in Y_{k+1} : n_{k+1}(x) > n + 1\}) &\ll \left(\sum_{\gamma_0 \in \mathcal{H}_k(\ell_k, J_k)} \nu(\gamma_0 \Delta_{\ell_k}) \right) \left(\sum_{\gamma_1, \dots, \gamma_n \in \mathcal{J}} e^{-\psi(\mu(\gamma_1 \cdots \gamma_n))} \right) \\ &\ll (1 - \varepsilon_k)^n \end{aligned}$$

by Lemma 11.8. Letting $n \rightarrow \infty$ shows that the set of points in Y_{k+1} that never return to Y_{k+1} has measure zero. Since $Y_{k+1} \setminus X_{k+1} \subset \bigcup_{m \geq 0} T_k^{-m}(\{y \in Y_{k+1} : n_{k+1}(y) = \infty\})$, X_{k+1} has full ν -measure in Y_{k+1} . □

The inverse branches of T_{k+1} are

$$\mathcal{H}_{k+1} = \mathcal{H}_k(J_k, J_k) \cup \bigcup_{n \geq 0} \mathcal{H}_k(\ell_k, J_k) \mathcal{H}_k(\ell_k, \ell_k)^n \mathcal{H}_k(J_k, \ell_k),$$

where the zeroth power is $\{e\}$.

Proposition 11.10. *Assume that, for some $\varepsilon > 0$, $\sum_{\gamma \in \mathcal{H}_k} e^{-(1-\varepsilon)\psi(\mu(\gamma))} < \infty$. Then there exists $\varepsilon_1 > 0$ such that $\sum_{\gamma \in \mathcal{H}_{k+1}} e^{-(1-\varepsilon_1)\psi(\mu(\gamma))} < \infty$.*

Proof. Decreasing ε if necessary, we may assume that $0 < \varepsilon < 1$. Set $\mathcal{J} := \mathcal{H}_k(\ell_k, \ell_k)$. The cocycle identity and Lemmas 3.11 and 11.6 give a constant $C > 0$ such that

$$\psi(\mu(\gamma_1 \cdots \gamma_n)) \geq \sum_{m=1}^n \psi(\mu(\gamma_m)) - Cn$$

for every $\gamma_1, \dots, \gamma_n \in \mathcal{J}$. The hypothesis therefore implies that, for some $B \geq 1$,

$$\sum_{\gamma_1, \dots, \gamma_n \in \mathcal{J}} e^{-(1-\varepsilon)\psi(\mu(\gamma_1 \cdots \gamma_n))} \leq B^n.$$

Choose $0 < \theta < 1$ small enough that $(1 - \varepsilon_k)^{1-\theta} B^\theta < 1$, where ε_k is given by Lemma 11.8. Since $1 - \theta\varepsilon = (1 - \theta) + \theta(1 - \varepsilon)$, Hölder's inequality with exponents $1/(1 - \theta)$ and $1/\theta$ gives

$$\begin{aligned} & \sum_{\gamma_1, \dots, \gamma_n \in \mathcal{J}} e^{-(1-\theta\varepsilon)\psi(\mu(\gamma_1 \cdots \gamma_n))} \\ (11.9) \quad & \leq \left(\sum_{\gamma_1, \dots, \gamma_n \in \mathcal{J}} e^{-\psi(\mu(\gamma_1 \cdots \gamma_n))} \right)^{1-\theta} \left(\sum_{\gamma_1, \dots, \gamma_n \in \mathcal{J}} e^{-(1-\varepsilon)\psi(\mu(\gamma_1 \cdots \gamma_n))} \right)^\theta \\ & \ll ((1 - \varepsilon_k)^{1-\theta} B^\theta)^n, \end{aligned}$$

where the last inequality follows from Lemma 11.8 and the preceding bound. Set $\varepsilon_1 = \theta\varepsilon$. Then

$$(11.10) \quad \sum_{n \geq 0} \sum_{\gamma_1, \dots, \gamma_n \in \mathcal{J}} e^{-(1-\varepsilon_1)\psi(\mu(\gamma_1 \cdots \gamma_n))} < \infty.$$

Every branch in $\mathcal{H}_{k+1} \setminus \mathcal{H}_k(J_k, J_k)$ has the form $\gamma_0 g \gamma'_0$, where $\gamma_0 \in \mathcal{H}_k(\ell_k, J_k)$, $\gamma'_0 \in \mathcal{H}_k(J_k, \ell_k)$, and g is a product of elements of $\mathcal{J}$, possibly empty. The same cocycle argument gives

$$\mu(\gamma_0 g \gamma'_0) = \mu(\gamma_0) + \mu(g) + \mu(\gamma'_0) + O(1),$$

with a uniform error. Consequently, the sum of $e^{-(1-\varepsilon_1)\psi(\mu(\gamma_0 g \gamma'_0))}$ over these branches is bounded by a constant times the product of the sums over γ_0 , g , and γ'_0 . The sum over g is finite by (11.10). Since $\varepsilon_1 < \varepsilon$ and $\psi(\mu(\gamma))$ is uniformly bounded below by (7.1), the hypothesis implies that $\sum_{\gamma \in \mathcal{H}_k} e^{-(1-\varepsilon_1)\psi(\mu(\gamma))} < \infty$. This bounds both outer sums and the sum over $\mathcal{H}_k(J_k, J_k)$, proving the proposition. $\square$

11.3. Proofs of Proposition 11.4 and Proposition 11.2.

Proof. Starting with X_0 , apply Proposition 11.9 at each stage. It follows inductively that X_{k+1} has full ν -measure in Y_{k+1} , and hence in $\bigcup_{i \in I_{k+1}} \Delta_i$. This proves Proposition 11.4. Moreover, Propositions 11.7 and 11.10 show that, at every stage k , there exists $\varepsilon > 0$ such that $\sum_{\gamma \in \mathcal{H}_k} e^{-(1-\varepsilon)\psi(\mu(\gamma))} < \infty$.

Let $r := |I_0| - 1$, so that $I_r = \{0\}$. Every point of Λ_+ returns to Δ_0 infinitely often and therefore belongs to every X_k . Thus $\Lambda_+ \subset X_r$. Conversely, the iterates of T_r give successive returns under T_0 to Δ_0 at strictly increasing times. Hence $X_r \subset \Lambda_+$, and therefore $X_r = \Lambda_+$. Since X_r has full ν -measure in Δ_0 , this proves Proposition 11.2. The map T_r is the first-return map of T_0 to Δ_0 . $\square$

We henceforth write $T_{\text{ret}} := T_r$ and $\mathcal{H}_{\text{ret}} := \mathcal{H}_r$. Thus $T_{\text{ret}}(x) = T_0^{n(x)}x$ for $x \in \Lambda_+$, and $\mathcal{H}_{\text{ret}}$ is its collection of inverse branches.

11.4. A fixed iterate with positive roof. We will choose $n_0 \geq 1$ so that the return vectors of $T := T_{\text{ret}}^{n_0}$ satisfy the required cone condition and scalar lower bound. In terms of T_0 ,

$$T(x) = T_0^{N_{n_0}(x)}(x), \quad N_{n_0}(x) := \sum_{j=0}^{n_0-1} n(T_{\text{ret}}^j x).$$

Each first-return branch is represented by a product $\gamma_1 \cdots \gamma_n$, where $\gamma_m \in \mathcal{H}_0(i_m, i_{m-1})$, $i_0 = i_n = 0$, and $i_m \neq 0$ for $0 < m < n$. For $x \in \Lambda_+$, the corresponding sequence of branches is uniquely determined by its first $n(x)$ iterates under T_0 .

The sets $\gamma\Lambda_+$, $\gamma \in \mathcal{H}_{\text{ret}}$, are the first-return cylinders. Until Lemma 11.12 is proved, $\mathcal{H}_{\text{ret}}^m$ denotes the set of ordered m -tuples of branches; in expressions such as $\gamma\Lambda_+$, γ denotes their group product. For every $m \geq 1$,

$$\Lambda_+ = \bigsqcup_{\gamma \in \mathcal{H}_{\text{ret}}^m} \gamma\Lambda_+.$$

Since Δ_0 is the closure of its relative interior, Λ_+ is dense in Δ_0 . Thus $\gamma\Delta_0$ is the closure of $\gamma\Lambda_+$. The following lemma places these closures in the relative interior of Δ_0 .

Lemma 11.11. *For every $b \in \mathcal{H}_0(i, j)$, $b\Delta_i \subset \text{int}_{\Lambda_\rho}(\Delta_j)$. Consequently, $\gamma\Delta_0 \subset \text{int}_{\Lambda_\rho}(\Delta_0)$ for every nonempty product γ of elements of $\mathcal{H}_{\text{ret}}$.*

Proof. Let $b \in \mathcal{H}_0(i, j)$. Then $b\Delta_i \subset J_p$ for some $p \in P_{j,n}$. Set $q := g_j p$ and work in the conjugated system Γ_j . Since $\eta h_j(q) \leq h_{n-1}$ and $\eta < 1/(4C_2)$, we have $C_2 \eta h_j(q) < h_{n-1}/(4\eta)$. Hence Lemma 9.5 and definition 9.7 give

$$g_j(b\Delta_i) \subset g_j J_p \subset \mathcal{B}_{\bar{\psi}}\left(q, \frac{h_{n-1}}{4\eta}\right) \subset g_j \Delta_j.$$

The ball in this display is open in $g_j \Lambda_\rho$. Applying g_j^{-1} proves the first assertion.

In a product defining a first-return branch, the leftmost factor belongs to $\mathcal{H}_0(i, 0)$ for some i , and the remaining factors map Δ_0 into Δ_i . Its image is therefore contained in the relative interior of Δ_0 . Any further composition of first-return branches has image contained in that of its leftmost factor, proving the final assertion. $\square$

Lemma 11.12. *The set $\mathcal{H}_{\text{ret}}$ freely generates a semigroup.*

Proof. Suppose that $\gamma_1 \cdots \gamma_n = \gamma'_1 \cdots \gamma'_m$. The common image of Λ_+ is a nonempty subset of both $\gamma_1 \Lambda_+$ and $\gamma'_1 \Lambda_+$. Since distinct first-return branches have disjoint images, $\gamma_1 = \gamma'_1$. Cancelling and repeating identifies the factors up to the shorter length.

By Lemma 11.11, every nonempty product maps Δ_0 into $\text{int}_{\Lambda_\rho}(\Delta_0)$. Since Δ_0 has nonempty relative boundary, this image is a proper subset of Δ_0 , so the product cannot be the identity. The two lengths must therefore agree, proving the lemma. $\square$

Recall that $U_0 = \Lambda_\rho \setminus \mathcal{B}_{\bar{\psi}}\left(\Delta_0, \frac{1}{2N_0\eta}\right)$. Repeated application of Lemma 11.5 gives $\gamma^{-1}U_0 \subset U_0$ for every nonempty product γ of elements of $\mathcal{H}_{\text{ret}}$.

Lemma 11.13. *There exists $r > 0$ such that $U_0 \subset \mathcal{O}_r(\gamma o, o)$ for every $\gamma \in \bigcup_{m \geq 1} \mathcal{H}_{\text{ret}}^m$.*

Proof. By Lemma 11.6, there exists $r > 0$ such that

$$(11.11) \quad \gamma\Delta_0 \subset \mathcal{O}_r(o, \gamma o) \quad (\gamma \in \bigcup_{m \geq 1} \mathcal{H}_{\text{ret}}^m).$$

Suppose the assertion fails. For each $k \geq 1$, choose $x_k \in U_0$ and $\gamma_k \in \bigcup_{m \geq 1} \mathcal{H}_{\text{ret}}^m$ such that $x_k \notin \mathcal{O}_k(\gamma_k o, o)$. Then $d(o, \gamma_k o) \geq k$. Passing to a subsequence, assume that the γ_k are pairwise distinct and $x_k \rightarrow x \in U_0$.

Fix $z \in \Delta_0$. By (11.11) and Lemma 3.16(2) and (3), after passing to a further subsequence, $\gamma_k z \rightarrow x$. Since $\gamma_k z \in \Delta_0$ and Δ_0 is compact, $x \in \Delta_0$, contradicting $U_0 \cap \Delta_0 = \emptyset$. $\square$

Lemma 11.14. *Let $\mathcal{D}_0 \subset \mathfrak{a}$ be compact, and let $\mathcal{C}_0 \subset \mathfrak{a}^+$ be a closed cone such that $\mathcal{L}_\rho \setminus \{0\} \subset \text{int } \mathcal{C}_0$. For every $C_0 > 0$, there exists $n_0 \geq 1$ such that, for every $n \geq n_0$ and every $\gamma = \gamma_1 \cdots \gamma_n$ with $\gamma_j \in \mathcal{H}_{\text{ret}}$:*

- (1) $\beta_x(e, \gamma) + \mathcal{D}_0 \subset \mathcal{C}_0$ for every $x \in \Lambda_+ \cap \gamma \Delta_0$;
- (2) $\varphi(\beta_x(e, \gamma)) \geq C_0$ for every $x \in \Lambda_+ \cap \gamma \Delta_0$ and $\varphi \in \{\psi, \bar{\psi}\}$;
- (3) $\varphi(\beta_y(\gamma, e)) \geq C_0$ for every $y \in U_0$ and $\varphi \in \{\psi, \bar{\psi}\}$.

Proof. By (11.11) and Lemmas 3.11 and 11.13,

$$\beta_x(e, \gamma) = \mu(\gamma) + O(1), \quad \beta_y(\gamma, e) = i\mu(\gamma) + O(1),$$

uniformly for the products γ and points x, y in the statement.

By the definition of $\mathcal{L}_\rho$, every accumulation point of $\mu(\gamma)/\|\mu(\gamma)\|$ as $\|\mu(\gamma)\| \rightarrow \infty$ belongs to $\mathcal{L}_\rho$. Since $\mathcal{D}_0$ is compact, the hypothesis on $\mathcal{C}_0$ and positivity of ψ, ψ^- , and $\bar{\psi}$ on $\mathcal{L}_\rho \setminus \{0\}$ give all three conclusions whenever $\|\mu(\gamma)\|$ is sufficiently large. By discreteness, only finitely many elements of Γ_ρ have Cartan projection below this norm bound. By Lemma 11.12, each such element has at most one expression as a product of elements of $\mathcal{H}_{\text{ret}}$. The lengths of these exceptional products are therefore bounded. Choosing n_0 larger than these lengths proves the lemma. $\square$

11.5. The final expanding map. We now fix n_0 and the map T . Since Δ_0 is compact and disjoint from p_0 , there is a compact set $\mathcal{D}_0 \subset \mathfrak{a}$ such that

$$(11.12) \quad 2\mathcal{G}(x, p_0) - 2\mathcal{G}(y, p_0) \in \mathcal{D}_0 \quad (x, y \in \Delta_0).$$

Choose a closed cone $\mathcal{C}_0 \subset \text{int } \mathfrak{a}^+ \cup \{0\}$ such that $\mathcal{L}_\rho \setminus \{0\} \subset \text{int } \mathcal{C}_0$ and $\psi, \psi^-, \bar{\psi} > 0$ on $\mathcal{C}_0 \setminus \{0\}$. Apply Lemma 11.14 with

$$C_0 = C_R := 1 + 4 \max_{\varphi \in \{\psi, \bar{\psi}\}} \sup_{x \in \Delta_0} |\varphi(\mathcal{G}(x, p_0))|.$$

Fix the resulting n_0 and define

$$T := T_{\text{ret}}^{n_0} : \Lambda_+ \rightarrow \Lambda_+.$$

This is our final expanding map, whose inverse branches form $\mathcal{H} := \mathcal{H}_{\text{ret}}^{n_0}$. By Lemma 11.12, $\mathcal{H}$ freely generates a semigroup. Write $\langle \mathcal{H} \rangle_+ := \bigcup_{n \geq 0} \mathcal{H}^n$, with $\mathcal{H}^0 := \{e\}$.

Part 2. The return cocycle

We establish the symbolic coding and the exponential moments and uniform distortion estimates of the return cocycle. We then prove Zariski density of the return semigroup, non-arithmeticity of the return periods, and quantitative non-integrability.

12. SYMBOLIC CODING AND COCYCLE ESTIMATES

We identify the expanding map $T : \Lambda_+ \rightarrow \Lambda_+$ with the one-sided full shift on its countable family $\mathcal{H}$ of inverse branches. We then define the return cocycle and prove the moment and regularity estimates needed for the spectral analysis.

Equip the countable alphabet $\mathcal{H}$ with the discrete topology, and let $\Sigma^+ := \mathcal{H}^{\mathbb{N}}$. For distinct $x, y \in \Sigma^+$, set $n(x, y) := \min\{j \geq 1 : x_j \neq y_j\}$ and define

$$(12.1) \quad d_{\Sigma^+}(x, y) := 2^{-n(x, y)}, \quad d_{\Sigma^+}(x, x) := 0.$$

The one-sided shift is $\sigma(x_1, x_2, \dots) := (x_2, x_3, \dots)$. Since all transitions are allowed, this full shift is topologically mixing and has the big-images-and-preimages (BIP) property: any fixed symbol can precede and follow every symbol; see [71, 70]. For $n \geq 1$, write $\Sigma_n^+ := \mathcal{H}^n$ for the words of length n . The cylinder of $\omega = (\omega_1, \dots, \omega_n) \in \Sigma_n^+$ is

$$[\omega] := \{x \in \Sigma^+ : x_j = \omega_j \text{ for } 1 \leq j \leq n\}.$$

12.1. Conjugacy with full shift. We retain the notation Δ_0 , Λ_+ , T , and $\mathcal{H}$ from Section 11. Thus

$$\Lambda_+ = \bigsqcup_{\gamma \in \mathcal{H}} \gamma \Delta_0, \quad T|_{\gamma \Delta_0} = \gamma^{-1},$$

and, more generally,

$$(12.2) \quad \Lambda_+ = \bigsqcup_{\gamma \in \mathcal{H}^n} \gamma \Delta_0 \quad (n \geq 1).$$

The compact set $\gamma \Delta_0$ is the closure of the corresponding branch cylinder.

Lemma 12.1. *For every $(\gamma_1, \gamma_2, \dots) \in \Sigma^+$, the compact sets $\gamma_1 \cdots \gamma_n \Delta_0$, $n \geq 1$, form a decreasing sequence whose intersection consists of a single point. Moreover, this point belongs to Λ_+ .*

Proof. The sets are nonempty, compact, and nested because every inverse branch maps Δ_0 into itself. Put $g_n := \gamma_1 \cdots \gamma_n$. By Lemma 11.12, the elements g_n are distinct. By Lemma 11.6, there exists $r > 0$, independent of n , such that

$$g_n \Delta_0 \subset \mathcal{O}_r(o, g_n o) \cap \Lambda_\rho.$$

Hence Lemma 3.16 gives $\text{diam}_{d_{\text{Riem}}}(g_n \Delta_0) \rightarrow 0$. The intersection therefore consists of a single point x .

Since $x \in \mathcal{O}_r(o, g_n o)$ for every n , Lemma 3.12 shows that $f^{-1}(x)$ is a conical limit point of Γ . In particular, $x \notin \Lambda_\rho(\Gamma_\rho)$.

Expand each γ_n into its finite word of basic T_0 -branches. The inclusions $x \in g_n \Delta_0$ show that every intermediate image of x belongs to the corresponding basic cell. Each such image is nonparabolic, since the set of parabolic fixed points is Γ_ρ -invariant. It therefore belongs to $\tilde{\Lambda}$, and the prescribed T_0 -iterate is defined. Each γ_n records n_0 successive returns to Δ_0 , so this orbit returns to Δ_0 infinitely often. Thus $x \in \Lambda_+$, and $T^n x = g_n^{-1} x$ for every $n \geq 1$. $\square$

Define

$$\Psi : \Sigma^+ \rightarrow \Lambda_+$$

by letting $\Psi((\gamma_j)_{j \geq 1})$ be the unique point in $\bigcap_{n \geq 1} \gamma_1 \cdots \gamma_n \Delta_0$.

Proposition 12.2. *The map $\Psi : \Sigma^+ \rightarrow \Lambda_+$ is a homeomorphism and*

$$\Psi \circ \sigma = T \circ \Psi.$$

Proof. For each $x \in \Lambda_+$, the branch partition determines a unique sequence (γ_n) such that $T^{n-1}x \in \gamma_n \Lambda_+$ for every $n \geq 1$. Then $x \in \gamma_1 \cdots \gamma_n \Delta_0$ for every n , so $x = \Psi((\gamma_n))$ by Lemma 12.1. Conversely, the proof of that lemma shows that the image of each sequence follows its prescribed branches. Thus Ψ is a bijection and $\Psi \circ \sigma = T \circ \Psi$.

Fix $s = (\gamma_1, \gamma_2, \dots) \in \Sigma^+$ and put $g_n := \gamma_1 \cdots \gamma_n$. Sequences agreeing with s in their first n entries have images in $g_n \Delta_0$. Since the diameters of these sets tend to zero, Ψ is continuous.

To prove continuity of Ψ^{-1} , put $x := \Psi(s)$. By Lemma 11.11,

$$x \in g_n \gamma_{n+1} \Delta_0 \subset g_n \text{int}_{\Lambda_\rho}(\Delta_0).$$

The branch definitions give $\Lambda_+ \cap g_n \Delta_0 = g_n \Lambda_+ = \Psi([\gamma_1, \dots, \gamma_n])$. Thus $\Lambda_+ \cap g_n \text{int}_{\Lambda_\rho}(\Delta_0)$ is a neighborhood of x contained in the image of this symbolic cylinder. Since symbolic cylinders form a basis, Ψ^{-1} is continuous. $\square$

Recall the constant C_R used in the choice of n_0 in Section 11.5.

Lemma 12.3 (Uniform branch contraction). *For every $n \geq 1$, $\gamma \in \mathcal{H}^n$, and $x, y \in \Lambda_+$,*

$$(12.3) \quad d_{\bar{\psi}}(\gamma x, \gamma y) \leq e^{-C_R n} d_{\bar{\psi}}(x, y).$$

Proof. For $\gamma \in \mathcal{H}$, (5.2) and Lemma 11.14(2) give

$$d_{\bar{\psi}}(\gamma x, \gamma y) = d_{\bar{\psi}}(x, y) e^{-\frac{1}{2} \bar{\psi}(\beta_{\gamma x}(e, \gamma) + \beta_{\gamma y}(e, \gamma))} \leq e^{-C_R} d_{\bar{\psi}}(x, y).$$

Since every inverse branch maps Λ_+ into itself, iteration gives the estimate for $\gamma \in \mathcal{H}^n$. $\square$

Proposition 12.4. *There exists $C > 0$ such that, for all $s, t \in \Sigma^+$,*

$$d_{\bar{\psi}}(\Psi(s), \Psi(t)) \leq C d_{\Sigma^+}(s, t)^{\alpha_0}, \quad \alpha_0 := \frac{C_R}{\log 2}.$$

In particular, Ψ is Hölder continuous.

Proof. The assertion is immediate when $s = t$. Otherwise, put $N := n(s, t)$ so that $d_{\Sigma^+}(s, t) = 2^{-N}$. So s and t share their first $N - 1$ symbols, so Lemmas 5.2 and 12.3 give

$$d_{\bar{\psi}}(\Psi(s), \Psi(t)) \leq e^{-C_R(N-1)} \text{diam}_{d_{\bar{\psi}}}(\Lambda_\rho) \ll d_{\Sigma^+}(s, t)^{C_R / \log 2}.$$

$\square$

12.2. Return cocycle and positivity. We define the return cocycle for $T = T_{\text{ret}}^{n_0}$, which records n_0 successive first returns to Δ_0 , and establish its positivity and comparison with the Cartan projection.

Definition 12.5 (Return cocycle and roof function). For $x \in \Lambda_+$, let $\gamma \in \mathcal{H}$ be the unique element such that $x \in \gamma \Lambda_+$. Define

$$K(x) := \beta_x(e, \gamma) + 2\mathcal{G}(Tx, p_0) - 2\mathcal{G}(x, p_0), \quad R(x) := \psi(K(x)).$$

Put $K_0 := 0$ and $R_0 := 0$. For $n \geq 1$, set $K_n := \sum_{j=0}^{n-1} K \circ T^j$ and $R_n := \psi \circ K_n = \sum_{j=0}^{n-1} R \circ T^j$.

For $n \geq 1$ and $\gamma \in \mathcal{H}^n$, put

$$(12.4) \quad R_\gamma := \sup_{x \in \gamma \Lambda_+} R_n(x).$$

Lemma 12.6. *There exists $C > 0$ such that, for every $n \geq 1$, $\gamma \in \mathcal{H}^n$, and $x \in \gamma\Lambda_+$,*

$$(12.5) \quad K_n(x) = \beta_x(e, \gamma) + 2\mathcal{G}(\gamma^{-1}x, p_0) - 2\mathcal{G}(x, p_0),$$

$$(12.6) \quad \|K_n(x) - \mu(\gamma)\| \leq C, \quad |R_n(x) - \psi(\mu(\gamma))| \leq C.$$

Moreover, for every $n \geq 1$ and $x \in \Lambda_+$, $K_n(x) \in \mathcal{C}_0$ and $R_n(x) \geq n$.

Proof. Write $\gamma = \gamma_1 \cdots \gamma_n$ with $\gamma_1, \dots, \gamma_n \in \mathcal{H}$. The cocycle identity gives $\sum_{j=0}^{n-1} \beta_{T^j x}(e, \gamma_{j+1}) = \beta_x(e, \gamma)$. Since the $\mathcal{G}$ -terms telescope and $T^n x = \gamma^{-1}x$, this proves (12.5).

By (11.12), the endpoint term belongs to the fixed compact set $\mathcal{D}_0$. Together with (11.11) and Lemma 3.11, this gives (12.6). Since γ is a word of length $nm_0 \geq n_0$ in $\mathcal{H}_{\text{ret}}$, Lemma 11.14(1) also gives $K_n(x) \in \mathcal{C}_0$.

For $n = 1$, Lemma 11.14(2) and the choice of C_R give $R(x) \geq C_R - 4 \sup_{z \in \Delta_0} |\psi(\mathcal{G}(z, p_0))| \geq 1$. Summing along the orbit yields $R_n(x) \geq n$. $\square$

12.3. Exponential summability and moments. Exponential summability for $\mathcal{H}_{\text{ret}}$ also holds for $\mathcal{H} = \mathcal{H}_{\text{ret}}^{n_0}$.

Proposition 12.7 (Exponential branch summability). *There exists $\varepsilon_* \in (0, 1)$ such that, for every $0 < \varepsilon \leq \varepsilon_*$,*

$$\sum_{\gamma \in \mathcal{H}} \sup_{x \in \Lambda_+} e^{-(1-\varepsilon)R(\gamma x)} \asymp \sum_{\gamma \in \mathcal{H}} e^{-(1-\varepsilon)\psi(\mu(\gamma))} < \infty.$$

Proof. By Propositions 11.7 and 11.10, there exists $\varepsilon_* \in (0, 1)$ such that $\sum_{\gamma \in \mathcal{H}_{\text{ret}}} e^{-(1-\varepsilon_*)\psi(\mu(\gamma))} < \infty$. Since $\psi(\mu(\gamma))$ is uniformly bounded below, the same convergence holds for $0 < \varepsilon \leq \varepsilon_*$.

For $\gamma_1, \dots, \gamma_{n_0} \in \mathcal{H}_{\text{ret}}$, the cocycle identity, (11.11), and Lemma 3.11 give

$$\psi(\mu(\gamma_1 \cdots \gamma_{n_0})) = \sum_{j=1}^{n_0} \psi(\mu(\gamma_j)) + O_{n_0}(1),$$

uniformly in the letters. Hence

$$\sum_{\gamma \in \mathcal{H}} e^{-(1-\varepsilon)\psi(\mu(\gamma))} \ll \left(\sum_{\gamma \in \mathcal{H}_{\text{ret}}} e^{-(1-\varepsilon)\psi(\mu(\gamma))} \right)^{n_0} < \infty.$$

Finally, (12.6) with $n = 1$ gives $R(\gamma x) = \psi(\mu(\gamma)) + O(1)$ uniformly in γ and x , proving the comparison of the two sums. $\square$

Proposition 12.8 (Exponential moments). *(1) For every $0 < \varepsilon < 1$,*

$$\int_{\Lambda_+} e^{\varepsilon R(x)} d\nu(x) \asymp_{\varepsilon} \sum_{\gamma \in \mathcal{H}} e^{-(1-\varepsilon)\psi(\mu(\gamma))}.$$

(2) There exist $\varepsilon_0, C > 0$ such that

$$\int_{\Lambda_+} e^{\varepsilon_0 R} d\nu < \infty, \quad \nu(\{x \in \Lambda_+ : R(x) > t\}) \leq C e^{-\varepsilon_0 t} \quad (t \geq 0).$$

(3) There exists $\varepsilon_1 > 0$ such that

$$\int_{\Lambda_+} e^{\varepsilon_1 \|K\|} d\nu < \infty.$$

Proof. By conformality, (11.11), and Lemma 3.11,

$$\nu(\gamma\Lambda_+) = \int_{\Lambda_+} e^{-\psi(\beta_{\gamma x}(e, \gamma))} d\nu(x) \asymp e^{-\psi(\mu(\gamma))},$$

uniformly in $\gamma \in \mathcal{H}$, since $\nu(\Lambda_+) = \nu(\Delta_0) > 0$. Moreover, (12.6) gives

$$R(y) = \psi(\mu(\gamma)) + O(1) \quad (y \in \gamma\Lambda_+),$$

with a uniform error. Therefore, summing over the branch partition, we obtain

$$\int_{\Lambda_+} e^{\varepsilon R(y)} d\nu(y) = \sum_{\gamma \in \mathcal{H}} \int_{\gamma\Lambda_+} e^{\varepsilon R(y)} d\nu(y) \asymp_{\varepsilon} \sum_{\gamma \in \mathcal{H}} e^{\varepsilon \psi(\mu(\gamma))} \nu(\gamma\Lambda_+) \asymp_{\varepsilon} \sum_{\gamma \in \mathcal{H}} e^{-(1-\varepsilon)\psi(\mu(\gamma))}.$$

This proves (1).

Taking $\varepsilon_0 = \varepsilon_*$ and applying Proposition 12.7 gives the finite moment in (2). The tail bound follows from $\nu(\{R > t\}) \leq e^{-\varepsilon_0 t} \int_{\Lambda_+} e^{\varepsilon_0 R} d\nu$.

Finally, $K(x) \in \mathcal{C}_0$ by Lemma 12.6, and ψ is positive on $\mathcal{C}_0 \setminus \{0\}$. Thus $m_0 := \min\{\psi(v) : v \in \mathcal{C}_0, \|v\| = 1\} > 0$, so $R(x) = \psi(K(x)) \geq m_0 \|K(x)\|$. Taking $\varepsilon_1 = m_0 \varepsilon_0$ and applying (2) proves (3). $\square$

12.4. Uniform distortion and Hölder regularity. We estimate the variation of K_n at $\gamma x, \gamma y$ in terms of $d_{\bar{\psi}}(x, y)$, uniformly in n and $\gamma \in \mathcal{H}^n$.

Definition 12.9. Let $\alpha > 0$. A map $\phi : \Sigma^+ \rightarrow \mathfrak{a}$ or $\phi : \Sigma^+ \rightarrow \mathbb{R}$ is *locally α -Hölder* if there exists $C > 0$ such that

$$\|\phi(s) - \phi(t)\| \leq C d_{\Sigma^+}(s, t)^{\alpha} \quad \text{whenever } s_1 = t_1.$$

We call ϕ *locally Hölder* if this holds for some $\alpha > 0$.

Lemma 12.10. *There exist $\theta \in (0, 1]$ and $C > 0$ such that, for all $\xi, \eta \in \Lambda_{\rho}$,*

$$d_{\text{Riem}}(\xi, \eta) \leq C d_{\bar{\psi}}(\xi, \eta)^{\theta}.$$

Proof. For distinct $\xi, \eta \in \Lambda_{\rho}$, let z be the nearest-point projection of o to the geodesic joining $f^{-1}(\xi)$ and $f^{-1}(\eta)$. By [12, Corollary B.7(2)], the boundary map f is Hölder with respect to a visual metric on $\Lambda(\Gamma)$ and d_{Riem} on $\mathcal{F}$. Since the visual distance is comparable to $e^{-d(o, z)}$, there exists $a > 0$ such that $d_{\text{Riem}}(\xi, \eta) \ll e^{-ad(o, z)}$.

By Lemmas 3.9 and 3.13, we can choose $b \geq a$ such that $\bar{\psi}(\mathcal{G}(\xi, \eta)) \leq b d(o, z) + O(1)$. Thus $d_{\bar{\psi}}(\xi, \eta) \gg e^{-bd(o, z)}$, and hence $d_{\text{Riem}}(\xi, \eta) \ll d_{\bar{\psi}}(\xi, \eta)^{a/b}$. Taking $\theta = a/b$ proves the lemma. $\square$

Remark 12.11. A reverse Hölder estimate also holds: there exist $\Theta \geq 1$ and $C > 0$ such that

$$d_{\text{Riem}}(\xi, \eta) \geq C^{-1} d_{\bar{\psi}}(\xi, \eta)^{\Theta} \quad (\xi, \eta \in \Lambda_{\rho}).$$

This follows from [47, Lemma 3.11], together with Lemmas 3.9 and 3.13. We will not use this estimate.

Proposition 12.12 (Uniform distortion of the return cocycle). *There exist $c_0 \in (0, 1]$ and $C > 0$ such that, for every $n \geq 1$, $\gamma \in \mathcal{H}^n$, and $x, y \in \Lambda_+$,*

$$(12.7) \quad \begin{aligned} \|\beta_{\gamma x}(e, \gamma) - \beta_{\gamma y}(e, \gamma)\| &\leq C d_{\bar{\psi}}(x, y)^{c_0}, \\ \|K_n(\gamma x) - K_n(\gamma y)\| &\leq C d_{\bar{\psi}}(x, y)^{c_0}, \\ |R_n(\gamma x) - R_n(\gamma y)| &\leq C d_{\bar{\psi}}(x, y)^{c_0}. \end{aligned}$$

Proof. Take $c_0 = \theta$ from Lemma 12.10. By (11.11) and equivariance, $x, y \in \mathcal{O}_r(\gamma^{-1}o, o) \cap \Lambda_\rho$. Applying Lemma 3.15 to γ^{-1} and using $\beta_{\gamma x}(e, \gamma) = -\beta_x(e, \gamma^{-1})$, we obtain

$$\|\beta_{\gamma x}(e, \gamma) - \beta_{\gamma y}(e, \gamma)\| \leq C d_{\text{Riem}}(x, y) \leq C d_{\bar{\psi}}(x, y)^{c_0}.$$

The map $2\mathcal{G}(\cdot, p_0)$ is smooth on a neighborhood of the compact set Δ_0 , so its restriction to Δ_0 is Lipschitz. By Lemmas 12.3 and 12.10,

$$\begin{aligned} & 2\|\mathcal{G}(x, p_0) - \mathcal{G}(y, p_0)\| + 2\|\mathcal{G}(\gamma x, p_0) - \mathcal{G}(\gamma y, p_0)\| \\ & \ll d_{\bar{\psi}}(x, y)^{c_0} + d_{\bar{\psi}}(\gamma x, \gamma y)^{c_0} \ll d_{\bar{\psi}}(x, y)^{c_0}. \end{aligned}$$

Combining this with the first estimate and (12.5) proves the estimate for K_n . Applying ψ gives the estimate for R_n . $\square$

Corollary 12.13 (Symbolic Hölder regularity). *For every $c \in (0, c_0]$, the map $K \circ \Psi$ is locally θ_c -Hölder, where $\theta_c := cC_R/\log 2$. The same holds for $R \circ \Psi$.*

Proof. Since $d_{\bar{\psi}}$ is bounded on Λ_ρ , the estimates in Proposition 12.12 also hold with c_0 replaced by any $c \in (0, c_0]$, with a constant depending on c .

Let $s \neq t$ satisfy $s_1 = t_1$. Applying Proposition 12.12 to their common first branch and then Proposition 12.4, we obtain

$$\|K(\Psi(s)) - K(\Psi(t))\| \ll_c d_{\bar{\psi}}(\Psi(\sigma s), \Psi(\sigma t))^c \ll_c d_{\Sigma^+}(\sigma s, \sigma t)^{\theta_c} = 2^{\theta_c} d_{\Sigma^+}(s, t)^{\theta_c}.$$

Composition with ψ preserves this estimate and yields the last claim. $\square$

13. NON-ARITHMETICITY OF PERIODIC RETURN VECTORS

We prove that the periodic return vectors generate a dense additive subgroup of $\mathfrak{a}$. We first identify these vectors with Jordan projections and prove that the return semigroup is Zariski dense. The conclusion then follows from Benoist's non-arithmeticity theorem [3].

For $n \geq 1$, let $\text{Fix}(T^n) \subset \Lambda_+$ denote the fixed-point set of T^n .

Theorem 13.1 (Non-arithmeticity of return vectors). *The additive subgroup generated by*

$$\{K_n(x) : x \in \text{Fix}(T^n), n \geq 1\}$$

is dense in $\mathfrak{a}$.

Lemma 13.2. *The set Λ_+ is Zariski dense in $\mathcal{F}$.*

Proof. By Theorem 7.1, $\sum_{\gamma \in \Gamma_\rho} e^{-\psi(\mu(\gamma))} = \infty$. Hence [41, Theorem 1.4] implies that every proper subvariety of $\mathcal{F}$ is ν -null. Since Λ_+ has full ν -measure in Δ_0 by Proposition 11.2 and $\nu(\Delta_0) > 0$, it cannot be contained in a proper subvariety. $\square$

Lemma 13.3 (Periodic return vectors). *Every nontrivial element of $\langle \mathcal{H} \rangle_+$ is loxodromic. For every $n \geq 1$, the fixed points of T^n are precisely the attracting flags y_γ , $\gamma \in \mathcal{H}^n$, and $K_n(y_\gamma) = \lambda(\gamma)$.*

Proof. Let $\gamma = \gamma_1 \cdots \gamma_n \in \mathcal{H}^n$. The periodic repetition of $(\gamma_1, \dots, \gamma_n)$ defines a point $x \in \Lambda_+$ with $T^n x = x$ and $\gamma x = x$. Since $K_{kn}(x) = kK_n(x)$, applying (12.6) to γ^k gives

$$\|kK_n(x) - \mu(\gamma^k)\| \leq C \quad (k \geq 1).$$

Dividing by k and letting $k \rightarrow \infty$, we obtain $K_n(x) = \lambda(\gamma)$. By Lemma 12.6, this vector belongs to $\mathcal{C}_0$ and satisfies $\psi(K_n(x)) \geq n$. Thus $\lambda(\gamma)$ belongs to $\text{int } \mathfrak{a}^+$, so γ is loxodromic. Moreover, (11.11) gives $x \in \mathcal{O}_r(o, \gamma^k o)$ for every $k \geq 1$. By Lemma 3.16(2), $x = y_\gamma$.

Conversely, every fixed point of T^n has an itinerary obtained by repeating a word of length n . The preceding argument identifies it with the attracting flag of the corresponding element of $\mathcal{H}^n$. $\square$

Lemma 13.4. *The set $\{y_\gamma : \gamma \in \mathcal{H}^n, n \geq 1\}$ is dense in Λ_+ .*

Proof. Periodic sequences are dense in Σ^+ . Apply the homeomorphism Ψ and Lemma 13.3. $\square$

Let $\Gamma_{\mathcal{H}} < \Gamma_\rho$ be the subgroup generated by $\langle \mathcal{H} \rangle_+$.

Lemma 13.5. *No finite-index subgroup of $\Gamma_{\mathcal{H}}$ is contained in a proper parabolic subgroup of G .*

Proof. Suppose that a finite-index subgroup $H < \Gamma_{\mathcal{H}}$ is contained in gQg^{-1} , where Q is a maximal proper parabolic subgroup containing P . Then H fixes $gQ \in G/Q$.

The flags transverse to gP form a nonempty Zariski-open subset of $\mathcal{F}$. By Lemmas 13.2 and 13.4, choose $\gamma_1, \gamma_2 \in \langle \mathcal{H} \rangle_+ \setminus \{e\}$ with distinct attracting flags, both transverse to gP . Replacing them by a common positive power, we may assume that $\gamma_1, \gamma_2 \in H$. For $i = 1, 2$, transversality gives $\gamma_i^{-m}gP \rightarrow y_{\gamma_i^{-1}}$ as $m \rightarrow \infty$. Since $\gamma_i^{-m}gQ = gQ$, both flags $y_{\gamma_i^{-1}}$ project to gQ . The preimages of γ_1, γ_2 are hyperbolic. Two hyperbolic elements of a Fuchsian group that share one fixed point share both fixed points. Hence the distinct attracting flags of γ_1, γ_2 imply that their repelling flags are distinct. As these repelling flags belong to Λ_ρ , they have the form hP and hw_0P for some $h \in G$. Their images hQ and hw_0Q in G/Q are distinct because Q is proper, contradicting the preceding paragraph. $\square$

Lemma 13.6. *Let G be a connected semisimple real algebraic group, and let $L < G$ be a reductive real algebraic subgroup. Suppose that the projection of L° to every simple factor of G is non-compact. If $\{y_g \in \mathcal{F} : g \in L^\circ, \text{loxodromic}\}$ is Zariski dense in $\mathcal{F}$, then $L = G$.*

Proof. By [56], after conjugation we may assume that L° is invariant under θ . Write $\mathfrak{l} = \mathfrak{k}_L \oplus \mathfrak{p}_L$ and $K_L = L^\circ \cap K$. After a further conjugation by K , choose a maximal abelian subspace $\mathfrak{a}_L \subset \mathfrak{p}_L$ contained in $\mathfrak{a}$. Then $L^\circ = K_L \exp(\mathfrak{a}_L^+) K_L$ for a closed positive Weyl chamber $\mathfrak{a}_L^+$. For a loxodromic $g \in L^\circ$, write $g^n = k_n \exp(H_n) k_n'$ in this decomposition. Let $\mathcal{W} \subset N_K(A)$ be a finite set of representatives of the Weyl group. Choose $w_n \in \mathcal{W}$ such that $\text{Ad}(w_n^{-1})H_n = \mu(g^n)$. Along a subsequence, $w_n = w$ and $k_n \rightarrow k \in K_L$. Since $n^{-1}\mu(g^n) \rightarrow \lambda(g) \in \text{int } \mathfrak{a}^+$, we obtain $y_g = kwP \in \bigcup_{w \in \mathcal{W}} K_L \cdot (wP)$. By the hypothesis, the union $\bigcup_{w \in \mathcal{W}} K_L \cdot (wP)$ is Zariski dense.

Each $K_L \cdot (wP)$ is a compact semialgebraic submanifold, and its real Zariski closure has the same dimension. Therefore for some $w \in \mathcal{W}$, $K_L \cdot wP$ must have dimension $\dim \mathcal{F}$. That orbit is open and closed in the connected space $\mathcal{F}$, so $K_L \cdot wP = \mathcal{F}$. Thus K_L acts transitively on $\mathcal{F} = K/M$, and $K = K_L M$.

Write $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$, where $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$ are the simple ideals. By the non-compactness assumption, $\mathfrak{p}_L$ projects nontrivially to every $\mathfrak{p}_i$. We can therefore choose $X \in \mathfrak{p}_L$ whose projections to all $\mathfrak{p}_i$ are nonzero. Conjugating by K_L , we obtain $H \in \mathfrak{a}_L$ whose projections H_i are still nonzero. Since M centralizes H and $K = K_L M$,

$$\text{Ad}(K)H = \text{Ad}(K_L)H \subset \mathfrak{p}_L.$$

Let V be the linear span of $\text{Ad}(K)H$. For each i , the action of $\mathfrak{k}_i$ on $\mathfrak{p}_i$ is nontrivial and irreducible. Choose $Z_i \in \mathfrak{k}_i$ with $[Z_i, H_i] \neq 0$. Since V is $\mathfrak{k}$ -invariant and distinct simple ideals commute, $0 \neq [Z_i, H] = [Z_i, H_i] \in V \cap \mathfrak{p}_i$. Since $V \cap \mathfrak{p}_i$ is a nonzero $\mathfrak{k}_i$ -invariant

subspace of $\mathfrak{p}_i$, irreducibility gives $\mathfrak{p}_i \subset V$. Thus $\mathfrak{p} \subset V \subset \mathfrak{l}$. As G has no compact factors, $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$, so $\mathfrak{l} = \mathfrak{g}$. Connectedness of G now gives $L = G$. $\square$

Corollary 13.7. *The semigroup $\langle \mathcal{H} \rangle_+$ is Zariski dense in G .*

Proof. Let L be the Zariski closure of $\langle \mathcal{H} \rangle_+$. By [4, Lemma 6.15], L is an algebraic subgroup containing $\Gamma_{\mathcal{H}}$. By Lemma 13.3, every nontrivial element of $\langle \mathcal{H} \rangle_+$ is loxodromic and hence the projection of L° to every simple factor of G is non-compact.

Since L/L° is finite, $\Gamma_{\mathcal{H}} \cap L^\circ$ has finite index in $\Gamma_{\mathcal{H}}$. If L° were nonreductive, then it should be contained in a proper parabolic subgroup of G , contrary to Lemma 13.5. Hence L is reductive. For any $g \in \langle \mathcal{H} \rangle_+$, there is n such that $g^n \in L^\circ$ and $y_g = y_{g^n}$. By Lemmas 13.2 and 13.4 and Lemma 13.6, $L = G$. $\square$

Since inversion is a biregular map of G , $\langle \mathcal{H} \rangle_+^{-1}$ is also Zariski dense.

Proof of Theorem 13.1. By Lemma 13.3,

$$\{K_n(x) : x \in \text{Fix}(T^n), n \geq 1\} = \{\lambda(\gamma) : \gamma \in \langle \mathcal{H} \rangle_+ \setminus \{e\}\}.$$

By Corollary 13.7, $\langle \mathcal{H} \rangle_+$ is Zariski dense. Therefore [3, Proposition, p. 34] implies that $\{\lambda(\gamma) : \gamma \in \langle \mathcal{H} \rangle_+ \setminus \{e\}\}$ generates a dense subgroup of $\mathfrak{a}$; see also [58] for a dynamical proof. $\square$

14. QUANTITATIVE NON-INTEGRABILITY OF THE RETURN COCYCLE

We prove the quantitative non-integrability estimate for K used in Proposition 17.5. The proof uses Zariski density of $\langle \mathcal{H} \rangle_+$ and the peripheral root-balance condition (4.1).

For $C > 0$, set $\mathcal{H}_C := \{\gamma \in \mathcal{H} : R_\gamma < C\}$, where R_γ is defined in (12.4). By (12.6), $R_\gamma = \psi(\mu(\gamma)) + O(1)$ uniformly in γ . Thus $\mathcal{H}_C$ is finite by Proposition 12.7.

Theorem 14.1 (Quantitative non-integrability). *There exist constants $C > 1$, $0 < s_1 \leq s_2$, $s_3 > 0$, $N_{\text{NI}} > 1$, and $\varepsilon_1, \delta_1, \eta \in (0, 1)$ with the following property. For every unit vector $v_0 \in \mathfrak{a}$, $b > 3$, and integer $n > N_{\text{NI}}$, there exist $\gamma_1, \gamma_2 \in \mathcal{H}_C^n$, $\gamma \in \mathcal{H}_C$, and $x_1, x_2 \in \Lambda_+$ such that*

$$x_1 \in \gamma\Lambda_+, \quad B_{\bar{\psi}}\left(x_1, \frac{\varepsilon_1}{b^{s_1}}\right) \subset \gamma\Delta_0, \quad x_2 \in B_{\bar{\psi}}\left(x_1, \frac{\varepsilon_1}{2b^{s_1}}\right) \setminus B_{\bar{\psi}}\left(x_1, \frac{\delta_1\varepsilon_1}{2b^{s_2}}\right),$$

and

$$(14.1) \quad \left| \langle v_0, K_n(\gamma_1 x_1) - K_n(\gamma_2 x_1) \rangle - \langle v_0, K_n(\gamma_1 x_2) - K_n(\gamma_2 x_2) \rangle \right| \geq \eta d_{\bar{\psi}}(x_1, x_2)^{s_3}.$$

Moreover, writing $\gamma_k = \gamma_{k,1} \cdots \gamma_{k,n}$ with $\gamma_{k,i} \in \mathcal{H}_C$, we have

$$(14.2) \quad B_{\bar{\psi}}\left(\gamma_k x_1, \frac{\varepsilon_1}{2b^{s_1}}\right) \cup B_{\bar{\psi}}\left(\gamma_k x_2, \frac{\varepsilon_1}{2b^{s_1}}\right) \subset \gamma_{k,1}\Delta_0 \quad (k = 1, 2).$$

14.1. Root balance in the return limit cone. Let $\mathcal{L}_{\mathcal{H}}$ be the asymptotic cone of $\mu(\langle \mathcal{H} \rangle_+)$. Since $\langle \mathcal{H} \rangle_+$ is Zariski dense by Corollary 13.7, [2, Theorem 1.2(a)] gives

$$\mathcal{L}_{\mathcal{H}} = \overline{\{t\lambda(\gamma) : t \geq 0, \gamma \in \langle \mathcal{H} \rangle_+\}}.$$

Lemma 14.2 (Peripheral root balance). *There exists $v \in \mathcal{L}_{\mathcal{H}} \cap \text{int } \mathfrak{a}^+$ such that*

$$\frac{\min_{\alpha \in \Pi} \alpha(v)}{\max_{\alpha \in \Pi} \alpha(v)} > \frac{1}{2}.$$

Proof. Recall that $\mathcal{H} = \mathcal{H}_{\text{ret}}^{n_0}$, where $\mathcal{H}_{\text{ret}}$ consists of products $\eta_1 \cdots \eta_r$ satisfying

$$\eta_j \in \mathcal{H}_0(i_j, i_{j-1}), \quad i_0 = i_r = 0, \quad i_j \neq 0 \quad (0 < j < r).$$

The last factor has the form $\eta_r = \gamma_p h$, where $p \in \Gamma_\rho p_0$ and $h \in N_p \subset P_0$. By (11.1), $\gamma_p N_p \subset \mathcal{H}_0(0, i_{r-1})$, and N_p is cofinite in P_0 by (9.3).

For every nontrivial $u \in P_0$, the elements hu^m are distinct and hence belong to N_p for all sufficiently large m . Replacing the last factor of the last return word therefore shows that, for every $g \in \mathcal{H}$ and $u \in P_0 \setminus \{e\}$, there exists $m_0 = m_0(g, u)$ such that

$$(14.3) \quad gu^m \in \mathcal{H} \quad (m \geq m_0).$$

Now fix $g \in \mathcal{H}$ and $u_0 \in P_0 \setminus \{e\}$. Then u_0 is weakly unipotent and hence, by Lemmas 3.1 and 3.5,

$$\mu(gu_0^m) = \mu(u_0^m) + O_g(1) = (\log m)v_{u_0} + O(1).$$

By (14.3), $gu_0^m \in \mathcal{H}$ for all sufficiently large m . Dividing by $\log m$ therefore gives $v_{u_0} \in \mathcal{L}_{\mathcal{H}}$. Moreover, $v_{u_0} \neq 0$, and the Cartan asymptotic and (4.1) imply

$$\frac{\min_{\alpha \in \Pi} \alpha(v_{u_0})}{\max_{\alpha \in \Pi} \alpha(v_{u_0})} = \lim_{m \rightarrow \infty} \frac{\min_{\alpha \in \Pi} \alpha(\mu(u_0^m))}{\max_{\alpha \in \Pi} \alpha(\mu(u_0^m))} > \frac{1}{2}.$$

Thus $v_{u_0} \in \text{int } \mathfrak{a}^+$, and we may take $v = v_{u_0}$. $\square$

For the remainder of this section, fix a vector $v_* \in \mathcal{L}_{\mathcal{H}} \cap \text{int } \mathfrak{a}^+$ supplied by Lemma 14.2, and set

$$(14.4) \quad \theta_* := \frac{\min_{\alpha \in \Pi} \alpha(v_*)}{\max_{\alpha \in \Pi} \alpha(v_*)} \in \left(\frac{1}{2}, 1\right].$$

14.2. Quantitative non-concentration. Equip $\mathfrak{g}$ with the inner product $\langle X, Y \rangle := -B(X, \theta Y)$, where B is the Killing form, and use the induced norms. Its restriction to $\mathfrak{a}$ agrees with the inner product fixed above. Consider the root decomposition

$$(14.5) \quad \mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha \in \Phi^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}), \quad (\mathfrak{m} = \text{Lie } M).$$

This is an orthogonal decomposition which is $\text{Ad}(M)$ -invariant. For $\kappa \in (0, 1]$ and $\alpha \in \Pi$, put

$$W_\kappa(\mathfrak{g}_{-\alpha}) := \left\{ Z \in \mathfrak{n} : \|Z\| = 1, \sup_{Z' \in \mathfrak{g}_{-\alpha}, \|Z'\|=1} |\langle Z, Z' \rangle| \geq \kappa \right\}.$$

The following proposition gives an attracting fixed point y_{γ^*} such that, in local Bruhat coordinates centered there, Λ_+ escapes every hyperplane through the origin whose normal has a definite component in a negative simple-root space, with quantitative bounds at all sufficiently small scales. The powers of $d_{\tilde{\psi}}$ match the contraction rates in the corresponding root spaces; see (14.9).

Proposition 14.3 (Adapted non-concentration). *Fix $s \in (\frac{1}{2}, \theta_*)$. There exists a loxodromic element $\gamma^* \in \langle \mathcal{H} \rangle_+$ with the following property. For every fixed $g_* \in G$ satisfying $g_*^{-1} \gamma^* g_* \in A^+ M$ and $\kappa \in (0, 1]$, there exist $c_0, \delta_0, \eta_0 \in (0, 1)$ such that, for every $\alpha \in \Pi$, $Z \in W_\kappa(\mathfrak{g}_{-\alpha})$, and $\varepsilon \in (0, c_0)$, there exists $Y \in \mathfrak{n}$ with $g_* \exp(Y)P \in \Lambda_+$ and*

$$(14.6) \quad \delta_0 \varepsilon < d_{\tilde{\psi}}(y_{\gamma^*}, g_* \exp(Y)P) \frac{\alpha(\lambda(\gamma^*))}{\psi(\lambda(\gamma^*))} < \varepsilon, \quad |\langle Z, Y \rangle| > \eta_0 \varepsilon, \quad \|Y\| \leq \varepsilon^s.$$

Proof. Choose $s_0 \in (s, \theta_*)$. By (14.4), for every $\beta \in \Phi^+$ and $\alpha \in \Pi$,

$$\beta(v_*) \geq \min_{\zeta \in \Pi} \zeta(v_*) > s_0 \max_{\zeta \in \Pi} \zeta(v_*) \geq s_0 \alpha(v_*).$$

Since $\mathcal{L}_{\mathcal{H}}$ is the closure of the rays through $\lambda(\langle \mathcal{H} \rangle_+)$, we may choose $\gamma^* \in \langle \mathcal{H} \rangle_+ \setminus \{e\}$ such that

$$(14.7) \quad \beta(\lambda(\gamma^*)) \geq s_0 \alpha(\lambda(\gamma^*)) \quad (\beta \in \Phi^+, \alpha \in \Pi).$$

By Lemma 13.3, γ^* is loxodromic, and $\bar{\psi}(\lambda(\gamma^*)) > 0$ by (7.1). Fix $g_* \in G$ with $g_*^{-1}\gamma^*g_* \in A^+M$. Then

$$g_*P = y_{\gamma^*}, \quad g_*w_0P = y_{(\gamma^*)^{-1}}, \quad g_*^{-1}\gamma^*g_* \in \exp(\lambda(\gamma^*))M.$$

For every $k \geq 0$, we have

$$\begin{aligned} (\gamma^*)^k g_* \exp(V)P &= g_* \exp(\text{Ad}(g_*^{-1}(\gamma^*)^k g_*)V)P \quad (V \in \mathfrak{n}), \\ \|\text{Ad}(g_*^{-1}(\gamma^*)^k g_*)V\| &= e^{-k\beta(\lambda(\gamma^*))}\|V\| \quad (V \in \mathfrak{g}_{-\beta}, \beta \in \Phi^+). \end{aligned}$$

Since the exponential and logarithm maps of $\tilde{N}$ are polynomial, $V \mapsto g_* \exp(V)P$ is an algebraic isomorphism onto the open Bruhat cell $g_*\tilde{N}P$. By Lemma 13.2, the set $\{V \in \mathfrak{n} : g_* \exp(V)P \in \Lambda_+\}$ is Zariski dense and hence spans $\mathfrak{n}$. Choose a basis $Y_1, \dots, Y_d$ from this set and $C > 1$ such that

$$(14.8) \quad \max_j \|Y_j\| \leq C, \quad \max_j |\langle V, Y_j \rangle| \geq C^{-1}\|V\| \quad (V \in \mathfrak{n}).$$

By Lemma 13.3, $y_{\gamma^*} \in \Lambda_+$, and (12.2) gives $(\gamma^*)^k \Lambda_+ \subset \Lambda_+$ for every $k \geq 0$. By (11.11), Lemma 3.11, and Lemma 3.1,

$$\beta_{(\gamma^*)^{kz}}(e, (\gamma^*)^k) = \mu((\gamma^*)^k) + O(1) = k\lambda(\gamma^*) + O(1)$$

uniformly for $k \geq 0$ and $z \in \Lambda_+$. Applying (5.2) to y_{γ^*} and $g_* \exp(Y_j)P$, whose distance is positive, and enlarging C , we obtain

$$(14.9) \quad C^{-1}e^{-k\alpha(\lambda(\gamma^*))} \leq d_{\bar{\psi}}(y_{\gamma^*}, (\gamma^*)^k g_* \exp(Y_j)P)^{\frac{\alpha(\lambda(\gamma^*))}{\bar{\psi}(\lambda(\gamma^*))}} \leq Ce^{-k\alpha(\lambda(\gamma^*))}$$

for every $\alpha \in \Pi$, j , and $k \geq 0$. Fix $\kappa \in (0, 1]$. For $Z \in W_{\kappa}(\mathfrak{g}_{-\alpha})$, we have

$$\|\text{Ad}(g_*^{-1}(\gamma^*)^k g_*)^* Z\| \geq \kappa e^{-k\alpha(\lambda(\gamma^*))},$$

where $*$ denotes the adjoint in $\mathfrak{n}$. Combining this with (14.8) gives the first estimate below, while (14.7) gives the second:

$$(14.10) \quad \begin{aligned} \max_j |\langle Z, \text{Ad}(g_*^{-1}(\gamma^*)^k g_*)Y_j \rangle| &\geq C^{-1}\kappa e^{-k\alpha(\lambda(\gamma^*))}, \\ \max_j \|\text{Ad}(g_*^{-1}(\gamma^*)^k g_*)Y_j\| &\leq Ce^{-ks_0\alpha(\lambda(\gamma^*))}. \end{aligned}$$

Since $s_0 > s$, choose $c_0 \in (0, 1)$ so small that $C(\frac{\varepsilon}{2C})^{s_0} \leq \varepsilon^s$ for all $0 < \varepsilon < c_0$. Given $\alpha \in \Pi$, $Z \in W_{\kappa}(\mathfrak{g}_{-\alpha})$, and $\varepsilon \in (0, c_0)$, choose $k \geq 1$ so that

$$Ce^{-k\alpha(\lambda(\gamma^*))} \leq \frac{\varepsilon}{2} < Ce^{-(k-1)\alpha(\lambda(\gamma^*))}.$$

Choose j attaining the maximum in the first line of (14.10), and put $Y := \text{Ad}(g_*^{-1}(\gamma^*)^k g_*)Y_j$. Then $g_* \exp(Y)P = (\gamma^*)^k g_* \exp(Y_j)P \in \Lambda_+$. By (14.9) and the choice of k ,

$$\delta_0 \varepsilon < d_{\bar{\psi}}(y_{\gamma^*}, g_* \exp(Y)P)^{\frac{\alpha(\lambda(\gamma^*))}{\bar{\psi}(\lambda(\gamma^*))}} \leq \frac{\varepsilon}{2} < \varepsilon.$$

where $\delta_0 := (2C^2)^{-1}e^{-\max_{\beta \in \Pi} \beta(\lambda(\gamma^*))}$. Likewise, (14.10) gives

$$|\langle Z, Y \rangle| > \frac{\kappa \varepsilon}{2C^2 e^{\alpha(\lambda(\gamma^*))}} \geq \kappa \delta_0 \varepsilon, \quad \|Y\| \leq C \left(\frac{\varepsilon}{2C} \right)^{s_0} \leq \varepsilon^s.$$

Taking $\eta_0 := \kappa \delta_0 \in (0, 1)$ proves (14.6). $\square$

14.3. A geometric formula for cocycle differences. We express differences between return branches in the Bruhat coordinates associated with the attracting and repelling flags of γ^* used in Proposition 14.3.

Recall that $p_0 = P$. For $g \in G$, recall

$$[g] := (g^+, g^-, \beta_{g^+}(e, g)) \in \mathcal{F}^{(2)} \times \mathfrak{a}, \quad g^+ = gp_0, \quad g^- = gw_0p_0.$$

Lemma 14.4 ([44, Lemma 8.4]). *For $g \in G$, $n \in N$, and $h \in \check{N}$,*

$$\begin{aligned} [gn] &= (g^+, gn^-, \beta_{g^+}(e, g)), \\ [gh] &= (gh^+, g^-, \beta_{g^+}(e, g) + 2\mathcal{G}(gh^+, g^-) - 2\mathcal{G}(g^+, g^-)). \end{aligned}$$

For the rest of this section, fix $s \in (1/2, \theta_*)$ and γ^* as in Proposition 14.3. Choose $g_* \in G$ such that $g_*^{-1}\gamma^*g_* \in \exp(\lambda(\gamma^*))M$, and set

$$(14.11) \quad x_1 := g_*P = y_{\gamma^*}, \quad q_* := g_*w_0P = y_{(\gamma^*)^{-1}}.$$

For every $x \in \mathcal{F}$ transverse to q_* , let $h_x \in \check{N}$ be the unique element such that $x = g_*h_xP$. In particular, $h_{x_1} = e$, and $h_{g_*\exp(Y)P} = \exp(Y)$ for $Y \in \mathfrak{n}$.

For $(x, y) \in \mathcal{F}^{(2)}$ with x transverse to q_* and y transverse to x_1 , set $g_1 := g_*$ and successively choose $g_2 \in g_1N$, $g_3 \in g_2\check{N}$, $g_4 \in g_3N$, and $g_5 \in g_4\check{N}$ such that

$$g_2^- = y, \quad g_3^+ = x, \quad g_4^- = q_*, \quad g_5^+ = x_1.$$

Each choice is unique. For $(h, n) \in \check{N} \times N$ such that $(g_*hP, g_*nw_0P) \in \mathcal{F}^{(2)}$, apply this construction with $x = g_*hP$ and $y = g_*nw_0P$. Since g_5 and g_* have the same two flags, $g_*^{-1}g_5 \in AM$. Define $\Xi(h, n)$ to be the unique element of A satisfying

$$(14.12) \quad g_*M = g_5\Xi(h, n)M.$$

For fixed $n \in N$, write $\Xi_n(h) := \Xi(h, n)$. For $\gamma \in \langle \mathcal{H} \rangle_+ \setminus \{e\}$, let $n_\gamma \in N$ be determined by

$$q_\gamma := \gamma^{-1}p_0 = g_*n_\gamma w_0P.$$

By Lemma 11.5 and inverse iteration in U_0 , the flags p_0 , q_* , and q_γ belong to U_0 , which is disjoint from $\overline{\Lambda_+}$. They are therefore transverse to every point of $\overline{\Lambda_+}$, so h_x and $\Xi(h_x, n_\gamma)$ are defined for $x \in \Lambda_+$.

Lemma 14.5. *For $n \geq 1$, $\gamma, \eta \in \mathcal{H}^n$, and $x \in \Lambda_+$,*

$$\log \Xi(h_x, n_\gamma) - \log \Xi(h_x, n_\eta) = K_n(\gamma x) - K_n(\eta x) - K_n(\gamma x_1) + K_n(\eta x_1).$$

Proof. Apply Lemma 14.4 to the associated sequence for (x, q_γ) . Since $\log \Xi(h_x, n_\gamma) = \beta_{x_1}(e, g_*) - \beta_{x_1}(e, g_5)$, we obtain

$$\log \Xi(h_x, n_\gamma) = 2\mathcal{G}(x, q_*) - 2\mathcal{G}(x_1, q_*) - 2\mathcal{G}(x, q_\gamma) + 2\mathcal{G}(x_1, q_\gamma).$$

On the other hand, (12.5) and (3.3) give

$$K_n(\gamma x) - K_n(\gamma x_1) = 2\mathcal{G}(x, p_0) - 2\mathcal{G}(x_1, p_0) - 2\mathcal{G}(x, q_\gamma) + 2\mathcal{G}(x_1, q_\gamma).$$

Subtracting the corresponding formulas for η cancels the terms involving q_* and p_0 , respectively, and proves the identity. $\square$

Define the backward limit set

$$\Lambda_b := \overline{\{y_{\gamma^{-1}} : \gamma \in \langle \mathcal{H} \rangle_+ \setminus \{e\}\}}.$$

By Lemma 11.5, $\gamma^{-m}p_0 \in U_0$ for every $\gamma \in \langle \mathcal{H} \rangle_+ \setminus \{e\}$ and $m \geq 1$. Since p_0 is transverse to $y_\gamma \in \Lambda_+$, these points converge to $y_{\gamma^{-1}}$. Closedness of U_0 therefore gives $\Lambda_b \subset U_0$. In particular, Λ_b is disjoint from $\overline{\Lambda}_+$.

Lemma 14.6. *The set Λ_b is Zariski dense in $\mathcal{F}$, contains p_0 , and satisfies $\gamma^{-1}\Lambda_b \subset \Lambda_b$ for every $\gamma \in \langle \mathcal{H} \rangle_+$.*

Proof. Since $\langle \mathcal{H} \rangle_+^{-1}$ is Zariski dense by Corollary 13.7, [2, Section 3.6] gives the invariance and Zariski density of Λ_b . Fix $g \in \mathcal{H}$ and $u \in \mathcal{P}_0 \setminus \{e\}$. By (14.3), $gu^m \in \mathcal{H}$ for all sufficiently large m . Choose $\zeta \in \Lambda_b \setminus \{gp_0\}$. The invariance just proved gives $u^{-m}g^{-1}\zeta = (gu^m)^{-1}\zeta \in \Lambda_b$ for all sufficiently large m . Since $g^{-1}\zeta \neq p_0$ and $u \in \mathcal{P}_0$, these points converge to p_0 . Thus $p_0 \in \Lambda_b$ by closedness. $\square$

Since $\overline{\Lambda}_+$ and Λ_b are disjoint subsets of Λ_ρ , every pair in $\overline{\Lambda}_+ \times \Lambda_b$ is transverse. As $x_1 \in \Lambda_+$ and $q_* \in \Lambda_b$, the inverse image of this product under $(h, n) \mapsto (g_*hP, g_*nw_0P)$ is a compact product contained in the domain of Ξ .

Choose relatively compact open neighborhoods $\check{N}_1 \subset \check{N}$ and $N_1 \subset N$ of its two factors such that $\check{N}_1 \times \overline{N}_1$ is still contained in that domain. Then $h_x \in \check{N}_1$ for $x \in \overline{\Lambda}_+$, and $n_\gamma \in N_1$ for $\gamma \in \langle \mathcal{H} \rangle_+ \setminus \{e\}$. Moreover, $e \in \check{N}_1 \cap N_1$, since $h_{x_1} = e$ and $q_* = g_*w_0P$.

Let $\pi_a : \mathfrak{g} \rightarrow \mathfrak{a}$ be the projection associated to (14.5).

Lemma 14.7. *The map Ξ is real analytic on its domain, $\Xi_n(e) = e$, and*

$$(14.13) \quad (d(\log \circ \Xi_n))_e = \pi_a \circ \text{Ad}_{n^{-1}}|_{\mathfrak{a}}.$$

Proof. Let $(h, n) \in \check{N} \times N$ belong to the domain of Ξ . Then $(n^{-1}hP, w_0P) \in \mathcal{F}^{(2)}$, and hence $n^{-1}h \in \check{N}P$. Write $n^{-1}h = h'amn'$, with $h' \in \check{N}$, $a \in A$, $m \in M$, and $n' \in N$. The associated sequence satisfies

$$g_3 = g_1nh', \quad g_4 = g_1ha^{-1}m^{-1}, \quad g_5 = g_1a^{-1}m^{-1}.$$

Thus (14.12) gives $\Xi(h, n) = a$. The components of this decomposition are real analytic, and the A -component of n^{-1} is e , so $\Xi_n(e) = e$.

Since right multiplication by n preserves the A -component, for $Y \in \mathfrak{a}$ and $t \in \mathbb{R}$ close to 0, $\Xi_n(\exp(tY))$ is the A -component of $n^{-1}\exp(tY)n = \exp(t\text{Ad}_{n^{-1}}Y)$. The differential at e of the logarithm of the A -component is π_a , by (14.5). This proves (14.13). $\square$

By compactness,

$$(14.14) \quad C_1 := \sup_{n \in \overline{N}_1} \|\text{Ad}_{n^{-1}}\| < \infty.$$

14.4. A finite family of nondegenerate adjoint directions.

Lemma 14.8. *Let $\xi \in \Lambda_b$. For every $\varepsilon > 0$, there exists a loxodromic element $\gamma_0 \in \langle \mathcal{H} \rangle_+$ such that*

$$d_{\bar{\psi}}((\gamma\gamma_0)^{-1}p_0, \xi) < \varepsilon \quad \text{for every } \gamma \in \langle \mathcal{H} \rangle_+.$$

Proof. By the definition of Λ_b , choose $\eta \in \langle \mathcal{H} \rangle_+ \setminus \{e\}$ such that $d_{\bar{\psi}}(y_{\eta^{-1}}, \xi) < \varepsilon$. Every point of the compact set Λ_b is transverse to $y_\eta \in \Lambda_+$. Hence $\eta^{-m}y \rightarrow y_{\eta^{-1}}$ uniformly for $y \in \Lambda_b$. By continuity of $d_{\bar{\psi}}$, we may choose $m \geq 1$ such that $\sup_{y \in \Lambda_b} d_{\bar{\psi}}(\eta^{-m}y, \xi) < \varepsilon$. Set

$\gamma_0 := \eta^m$. Since $\gamma^{-1}p_0 \in \Lambda_b$ for every $\gamma \in \langle \mathcal{H} \rangle_+$ by Lemma 14.6, the assertion follows from $(\gamma\gamma_0)^{-1}p_0 = \eta^{-m}\gamma^{-1}p_0$. $\square$

For $\beta \in \Pi$, let $H_\beta \in \mathfrak{a}$ be defined by $\langle H, H_\beta \rangle = \beta(H)$ for every $H \in \mathfrak{a}$.

Lemma 14.9 ([33, Chapter VI, Section 3, Lemma 3.1]). *For every $\beta \in \Pi$, there are unit vectors $E_\beta \in \mathfrak{g}_\beta$ and $E_{-\beta} \in \mathfrak{g}_{-\beta}$ such that*

$$[E_\beta, E_{-\beta}] \in \mathbb{R}_{>0} H_\beta.$$

Fix the unit vectors $E_{\pm\beta}$ from Lemma 14.9. Since the simple roots span $\mathfrak{a}^*$, compactness of the unit sphere gives $\delta_2 > 0$ such that every unit vector $v_0 \in \mathfrak{a}$ satisfies $|\beta(v_0)| \geq \delta_2$ for some $\beta \in \Pi$.

Proposition 14.10. *There exist $C_2, C_3 > 0$, $j_0, k_0 \in \mathbb{N}$, and $\gamma'_j \in \mathcal{H}_{C_2}^{k_0}$ ($1 \leq j \leq j_0$) with the following property. For every integer $k > k_0$ and $\gamma_j \in \mathcal{H}^{k-k_0}\gamma'_j$ ($1 \leq j \leq j_0$), the following hold:*

- (1) $n_{\gamma_j} \in N_1$ for every $1 \leq j \leq j_0$;
- (2) for every unit vector $v_0 \in \mathfrak{a}$ and $\beta \in \Pi$ satisfying $|\beta(v_0)| \geq \delta_2$, there is $1 \leq j \leq j_0$ such that $|\langle v_0, \text{Ad}_{n_{\gamma_j}^{-1}} E_{-\beta} \rangle| \geq C_3$.

Proof. By Lemma 14.6, $\gamma^{-1}p_0 \in \Lambda_b$ for every $\gamma \in \langle \mathcal{H} \rangle_+$. Thus (1) follows from the choice of N_1 . Fix a unit vector $v_0 \in \mathfrak{a}$ and $\beta \in \Pi$ with $|\beta(v_0)| \geq \delta_2$. The function $n \mapsto \langle v_0, \text{Ad}_{n^{-1}} E_{-\beta} \rangle$ is polynomial in exponential coordinates on N , since $\text{Ad}_{\exp(-X)} = e^{-\text{ad } X}$ is a finite sum for $X \in \mathfrak{n}$. Its derivative at e in the direction E_β is $-\langle v_0, [E_\beta, E_{-\beta}] \rangle \neq 0$ by Lemma 14.9.

The map $n \mapsto g_*nw_0P$ is an algebraic isomorphism from N onto the open Bruhat cell transverse to x_1 , which contains Λ_b . By Lemma 14.6, we may therefore choose $n \in N_1$ such that $g_*nw_0P \in \Lambda_b$ and $\langle v_0, \text{Ad}_{n^{-1}} E_{-\beta} \rangle \neq 0$. Continuity and Lemma 14.8 then give $\gamma' \in \langle \mathcal{H} \rangle_+ \setminus \{e\}$ and $c > 0$ such that, for every unit vector v sufficiently close to v_0 ,

$$|\langle v, \text{Ad}_{\omega_{\gamma'}^{-1}} E_{-\beta} \rangle| > c \quad \text{for every } \omega \in \langle \mathcal{H} \rangle_+.$$

By compactness of the sets $\{v : \|v\| = 1, |\beta(v)| \geq \delta_2\}$, finitely many such choices give elements $\gamma'_j \in \mathcal{H}^{k_j}$ and a common lower bound $C_3 > 0$, valid after any left multiplication by an element of $\langle \mathcal{H} \rangle_+$. Fix $\vartheta \in \mathcal{H}$ and put $k_0 := \max_j k_j$. Replacing γ'_j by $\vartheta^{k_0-k_j}\gamma'_j$ preserves this property and makes every word have length k_0 . Choose $C_2 > 0$ so that all letters in these finitely many words belong to $\mathcal{H}_{C_2}$. This proves (2). $\square$

14.5. Proof of Theorem 14.1. Recall $x_1 = g_*P$ and $q_* = g_*w_0P$ from (14.11). We use the constants and branches supplied by Proposition 14.10.

Step 1: infinitesimal non-integrability. Since $q_* = g_*w_0P$, its N -coordinate is e . Also, $\pi_{\mathfrak{a}|\mathfrak{n}} = 0$. Applying Lemma 14.8 to $q_* \in \Lambda_b$ and using continuity gives $\gamma'_0 \in \mathcal{H}^{m_0}$, for some $m_0 \in \mathbb{N}$, such that

$$(14.15) \quad \|\pi_{\mathfrak{a}} \circ \text{Ad}_{\omega_{\gamma'_0}^{-1}}|_{\mathfrak{n}}\| < C_3/2 \quad \text{for every } \omega \in \langle \mathcal{H} \rangle_+.$$

Let $\gamma \in \mathcal{H}$ be the first letter of the word representing γ^* . The periodic itinerary of x_1 gives $\gamma^{-1}x_1 \in \Lambda_+$. Since $\Lambda_+ \subset \text{int}_{\Lambda_\rho}(\Delta_0)$ by Lemma 11.11, we have $x_1 \in \gamma\Lambda_+ \cap \text{int}_{\Lambda_\rho}(\gamma\Delta_0)$.

Fix $\gamma_b \in \mathcal{H}$. By Lemma 11.11, $\gamma_b^2\Delta_0$ is a compact subset of $\text{int}_{\Lambda_\rho}(\gamma_b\Delta_0)$. We may therefore choose $r_0 > 0$ such that

$$(14.16) \quad B_{\bar{\psi}}(x_1, r_0) \subset \gamma\Delta_0, \quad B_{\bar{\psi}}(\gamma_b^2\Delta_0, r_0) \subset \gamma_b\Delta_0.$$

Choose $C > 1$ so that γ, γ_b and all letters of $\gamma'_0, \gamma'_1, \dots, \gamma'_{j_0}$ belong to $\mathcal{H}_C$, and put $N_{\text{NI}} := \max\{m_0, k_0\} + 2$. Fix an integer $\ell > N_{\text{NI}}$ and define

$$(14.17) \quad \gamma_0 := \gamma_b^{\ell-m_0} \gamma'_0, \quad \gamma_j := \gamma_b^{\ell-k_0} \gamma'_j \quad (1 \leq j \leq j_0).$$

Every γ_j , including γ_0 , belongs to $\mathcal{H}_C^\ell$ and begins with γ_b^2 .

For $1 \leq j \leq j_0$, define

$$\text{BP}_j(h) := \log \Xi_{n_{\gamma_j}}(h) - \log \Xi_{n_{\gamma_0}}(h) \quad (h \in \check{N}_1).$$

Then $\text{BP}_j(e) = 0$, and Lemma 14.5 gives

$$(14.18) \quad \text{BP}_j(h_x) = K_\ell(\gamma_j x) - K_\ell(\gamma_0 x) - K_\ell(\gamma_j x_1) + K_\ell(\gamma_0 x_1) \quad (x \in \Lambda_+).$$

By (14.13),

$$(d\text{BP}_j)_e = \pi_{\mathfrak{a}} \circ (\text{Ad}_{n_{\gamma_j}^{-1}} - \text{Ad}_{n_{\gamma_0}^{-1}})|_{\check{\mathfrak{n}}}.$$

Since $\pi_{\mathfrak{a}}$ is an orthogonal projection, (14.14) gives

$$(14.19) \quad \|(d\text{BP}_j)_e\| \leq 2C_1.$$

By Proposition 14.10 and (14.15), for every unit vector $v_0 \in \mathfrak{a}$ and $\alpha \in \Pi$ with $|\alpha(v_0)| \geq \delta_2$, some $1 \leq j' \leq j_0$ satisfies

$$(14.20) \quad |\langle v_0, (d\text{BP}_{j'})_e E_{-\alpha} \rangle| \geq C_3/2.$$

Step 2: finite-scale non-integrability. Put

$$s_1 := \min \left\{ 1, \min_{\alpha \in \Pi} \frac{\bar{\psi}(\lambda(\gamma^*))}{\alpha(\lambda(\gamma^*))} \right\}, \quad s_2 := \max \left\{ 1, \max_{\alpha \in \Pi} \frac{\bar{\psi}(\lambda(\gamma^*))}{\alpha(\lambda(\gamma^*))} \right\}, \quad s_3 := 1/s_1.$$

For the fixed $s \in (1/2, \theta_*)$, let c_0, δ_0, η_0 be supplied by Proposition 14.3 for g_* and $\kappa := \min\{1, C_3/(4C_1)\}$.

Since $n_{\gamma_j} \in N_1$ for $0 \leq j \leq j_0$, analyticity of Ξ on a neighborhood of $\overline{\check{N}_1} \times \overline{N_1}$ gives $M > 0$, independent of ℓ and j , such that for Y in a fixed neighborhood of 0 in $\check{\mathfrak{n}}$,

$$(14.21) \quad \|\text{BP}_j(\exp Y) - (d\text{BP}_j)_e Y\| \leq M \|Y\|^2.$$

Choose $\varepsilon_3 \in (0, c_0)$ so small that this estimate holds whenever $\|Y\| \leq \varepsilon_3^s$, that $2\varepsilon_3^{s_1} < \min\{1, r_0\}$, and that $M\varepsilon_3^{2s-1} \leq C_3\eta_0/4$. Set

$$\varepsilon_1 := 2\varepsilon_3^{s_1}, \quad \delta_1 := \delta_0^{s_2} \varepsilon_3^{s_2-s_1}, \quad \eta := \min\{1/2, C_3\eta_0/4\}.$$

These constants are independent of ℓ .

Fix a unit vector $v_0 \in \mathfrak{a}$ and $b > 3$, and choose $\alpha \in \Pi$ with $|\alpha(v_0)| \geq \delta_2$. Let j' be given by (14.20), and put $Z := ((d\text{BP}_{j'})_e)^* v_0 \in \check{\mathfrak{n}}$. By (14.20) and (14.19),

$$|\langle Z, E_{-\alpha} \rangle| \geq C_3/2, \quad C_3/2 \leq \|Z\| \leq 2C_1.$$

Thus $Z/\|Z\| \in W_\kappa(\mathfrak{g}_{-\alpha})$. Apply Proposition 14.3 to this unit vector with $\varepsilon = \varepsilon_3/b$, and let Y be the resulting vector. Set $x_2 := g_* \exp(Y)P$. Then $h_{x_2} = \exp(Y)$ and $x_2 \in \Lambda_+$. Since $s_1 \leq \bar{\psi}(\lambda(\gamma^*))/\alpha(\lambda(\gamma^*)) \leq s_2$ and $\varepsilon < 1$, we obtain

$$(14.22) \quad \left(\frac{\delta_0 \varepsilon_3}{b} \right)^{s_2} < d_{\bar{\psi}}(x_1, x_2) < \left(\frac{\varepsilon_3}{b} \right)^{s_1}, \quad |\langle Z, Y \rangle| \geq \frac{C_3\eta_0}{2} \frac{\varepsilon_3}{b}, \quad \|Y\| \leq \left(\frac{\varepsilon_3}{b} \right)^s.$$

By (14.21) and (14.22),

$$|\langle v_0, \text{BP}_{j'}(h_{x_2}) \rangle| \geq \frac{C_3\eta_0}{2} \frac{\varepsilon_3}{b} - M \left(\frac{\varepsilon_3}{b} \right)^{2s} \geq \frac{C_3\eta_0}{4} \frac{\varepsilon_3}{b} \geq \eta d_{\bar{\psi}}(x_1, x_2)^{s_3}.$$

Here the second inequality uses $2s > 1$ and the choice of ε_3 , and the last uses $s_1 s_3 = 1$. Taking $\gamma_{j'}$ and γ_0 as the two branches in the theorem, (14.18) gives (14.1). The definitions of ε_1, δ_1 turn (14.22) into the required annulus. Also, $\varepsilon_1/b^{s_1} < r_0$, so (14.16) gives the required ball about x_1 .

Finally, every word in (14.17) begins with γ_b^2 , so $\gamma_j x_k \in \gamma_b^2 \Delta_0$. By (14.16),

$$B_{\bar{\psi}}\left(\gamma_j x_k, \frac{\varepsilon_1}{2b^{s_1}}\right) \subset \gamma_b \Delta_0 \quad (0 \leq j \leq j_0, k = 1, 2).$$

Since γ_b is the first letter of both selected branches, this proves (14.2). $\square$

Part 3. Spectral theory and BMS local mixing

15. SPECTRAL THEORY OF TRANSFER OPERATORS

We study the real and complex transfer operators for (Λ_+, T) , with weights determined by the roof function R and the return cocycle K . We apply the Ruelle–Perron–Frobenius theory for countable shifts and establish regularity directly with respect to $d_{\bar{\psi}}$. We obtain a positive eigenfunction and a spectral gap for the normalized real operator, and show that the gap persists under small complex perturbations. Finally, we compute the second-order expansion of the centered leading eigenvalue and prove that its negative Hessian at the origin is positive definite.

Definition 15.1. Let $\mathcal{L}$ be a bounded linear operator on a complex Banach space X , with spectrum $\sigma(\mathcal{L})$ and spectral radius $\rho(\mathcal{L}) := \sup\{|z| : z \in \sigma(\mathcal{L})\}$. We say that $\mathcal{L}$ has a *spectral gap* if there exist $\kappa \in \mathbb{C}$, a bounded rank-one projection $\mathcal{P}$ on X , and a bounded linear operator $\mathcal{R}$ on X such that

$$\mathcal{L} = \kappa \mathcal{P} + \mathcal{R}, \quad \mathcal{P}\mathcal{R} = \mathcal{R}\mathcal{P} = 0, \quad \rho(\mathcal{R}) < |\kappa|.$$

In this case, κ is a simple eigenvalue of $\mathcal{L}$, $\rho(\mathcal{L}) = |\kappa|$, and we call κ the *leading eigenvalue* of $\mathcal{L}$.

For a topological space X and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, let $\mathcal{C}(X, \mathbb{K})$ and $\mathcal{C}_b(X, \mathbb{K})$ denote the spaces of continuous and bounded continuous $\mathbb{K}$ -valued functions on X , respectively. When $\mathbb{K} = \mathbb{R}$, we suppress $\mathbb{R}$ from the notation.

Recall the constant $c_0 > 0$ from Proposition 12.12 and let

$$(15.1) \quad c \in (0, c_0]$$

be any number in this section and in Section 16.

Definition 15.2 (Bounded c -locally Hölder functions). For $f \in \mathcal{C}(\Lambda_+, \mathbb{C})$ and $\gamma \in \mathcal{H}$, define

$$|f|_{c, \gamma} := \sup_{x, y \in \gamma \Lambda_+, x \neq y} \frac{|f(x) - f(y)|}{d_{\bar{\psi}}(x, y)^c}.$$

Set

$$|f|_c := \sup_{\gamma \in \mathcal{H}} |f|_{c, \gamma}, \quad \|f\|_c := \|f\|_\infty + |f|_c,$$

where $\|f\|_\infty := \sup_{x \in \Lambda_+} |f(x)|$. We define

$$\mathcal{F}_c(\Lambda_+, \mathbb{C}) := \{f \in \mathcal{C}(\Lambda_+, \mathbb{C}) : \|f\|_c < \infty\}.$$

On this space, $|\cdot|_c$ is a seminorm and $\|\cdot\|_c$ is a norm. Equipped with $\|\cdot\|_c$, this is a complex Banach space. We write $\mathcal{F}_c(\Lambda_+)$ for the corresponding real Banach space.

15.1. Thermodynamic formalism for the $-R$ -potential. By Proposition 12.7, there exists $\varepsilon_0 \in (0, 1)$ such that

$$(15.2) \quad \sum_{\gamma \in \mathcal{H}} \sup_{x \in \Lambda_+} e^{-(1-\varepsilon_0)R(\gamma x)} < \infty.$$

Definition 15.3 (Real transfer operator on Λ_+). Define $\tilde{\mathcal{L}}_{-R}: \mathcal{C}_b(\Lambda_+) \rightarrow \mathcal{C}_b(\Lambda_+)$ by

$$\tilde{\mathcal{L}}_{-R}H(x) := \sum_{x' \in T^{-1}(x)} e^{-R(x')} H(x') = \sum_{\gamma \in \mathcal{H}} e^{-R(\gamma x)} H(\gamma x);$$

by (15.2), the series on the right converges absolutely and uniformly on Λ_+ .

Setting $\lambda := e^{-C_R} \in (0, 1)$, where C_R is the constant in (12.3), we have

$$(15.3) \quad d_{\bar{\psi}}(\gamma x, \gamma y) \leq \lambda^n d_{\bar{\psi}}(x, y)$$

for every $n \geq 1$, $\gamma \in \mathcal{H}^n$, and $x, y \in \Lambda_+$.

15.1.1. Ruelle–Perron–Frobenius theorem on (Σ^+, d_{Σ^+}) . Recall from Section 12.1 that $\Sigma^+ = \mathcal{H}^{\mathbb{N}}$ is equipped with the metric d_{Σ^+} and that $\sigma: \Sigma^+ \rightarrow \Sigma^+$ is the one-sided shift map. Recall also from Proposition 12.2 the homeomorphism $\Psi: \Sigma^+ \rightarrow \Lambda_+$ conjugating σ to T . Set

$$R^\diamond := R \circ \Psi \quad \text{and} \quad K^\diamond := K \circ \Psi.$$

Fix the Hölder exponent

$$(15.4) \quad \theta := -\frac{c \log \lambda}{\log 2} > 0, \quad \text{equivalently} \quad 2^{-\theta} = \lambda^c.$$

Thus, for $z \neq z'$, $d_{\Sigma^+}(z, z')^\theta = \lambda^{cn(z, z')}$ where $n(z, z')$ is as in (12.1). For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, let $\mathcal{F}_\theta(\Sigma^+, \mathbb{K})$ denote the space of bounded $\mathbb{K}$ -valued functions on Σ^+ that are locally θ -Hölder, equipped with the norm

$$\|f\|_\theta := \|f\|_\infty + \sup_{z \neq z' \in \Sigma^+, n(z, z') \geq 2} \frac{|f(z) - f(z')|}{d_{\Sigma^+}(z, z')^\theta}.$$

The potential $-R^\diamond$ has the following properties:

- (R1) The function $R^\diamond$ is locally θ -Hölder by Corollary 12.13, since $\lambda = e^{-C_R}$ and hence $\theta = cC_R/\log 2$.
- (R2) The potential $-R^\diamond$ is summable by Proposition 12.7:

$$\sum_{\gamma \in \mathcal{H}} \exp\left(\sup_{z \in [\gamma]} (-R^\diamond(z))\right) < \infty.$$

Therefore, the transfer operator $\tilde{\mathcal{L}}_{-R^\diamond}: \mathcal{C}_b(\Sigma^+) \rightarrow \mathcal{C}_b(\Sigma^+)$, defined by

$$\tilde{\mathcal{L}}_{-R^\diamond}F(z) := \sum_{\sigma z' = z} e^{-R^\diamond(z')} F(z'),$$

is bounded. By Proposition 12.2, this operator is conjugate to $\tilde{\mathcal{L}}_{-R}$:

$$(15.5) \quad \tilde{\mathcal{L}}_{-R^\diamond} = \Psi^* \circ \tilde{\mathcal{L}}_{-R} \circ (\Psi^{-1})^*$$

where Ψ^* denotes the pull-back by Ψ .

We recall several notions from the thermodynamic formalism for countable-state shifts. Let $f: \Sigma^+ \rightarrow \mathbb{R}$ be a locally Hölder summable potential. Its *Gurevich pressure* is

$$\Pr(f) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{\substack{\sigma^n x = x \\ x_1 = \gamma}} \exp(S_n f(x)) \right), \quad S_n f := \sum_{j=0}^{n-1} f \circ \sigma^j,$$

where $\gamma \in \mathcal{H}$ is any fixed element. Since the full shift is topologically mixing, the limit exists, possibly with value $+\infty$, and is independent of the choice of γ [71, Theorem 1]. Since f is summable, the limit is finite. The full shift also has the BIP property. A Borel probability measure μ on Σ^+ is a *Gibbs state* for f if there exists $C > 1$ such that for every $n \geq 1$ and $z \in \Sigma^+$,

$$C^{-1} \leq \frac{\mu([z_1, \dots, z_n])}{\exp(S_n f(z) - n \Pr(f))} \leq C.$$

We define the action of $\tilde{\mathcal{L}}_{-R^\diamond}^*$ on finite Borel measures on Σ^+ by duality:

$$\int F d(\tilde{\mathcal{L}}_{-R^\diamond}^* \mu) = \int \tilde{\mathcal{L}}_{-R^\diamond} F d\mu \quad (F \in \mathcal{C}_b(\Sigma^+)).$$

By (R1) and (R2), we have the following; see [54, Corollary 2.7.5] and [38, Theorem 2.4 and Corollary 2.5].

Theorem 15.4 (Real Ruelle–Perron–Frobenius theorem). *(1) There is a unique Borel probability eigenmeasure $\tilde{m}^\diamond$ of $\tilde{\mathcal{L}}_{-R^\diamond}^*$, and the corresponding eigenvalue is $e^{\Pr(-R^\diamond)}$. Moreover $\tilde{m}^\diamond$ is a Gibbs state for $-R^\diamond$.*

(2) There is a unique eigenfunction $h^\diamond \in \mathcal{F}_\theta(\Sigma^+)$ of $\tilde{\mathcal{L}}_{-R^\diamond}$ with eigenvalue $e^{\Pr(-R^\diamond)}$ normalized by $\int_{\Sigma^+} h^\diamond d\tilde{m}^\diamond = 1$. Moreover, $h^\diamond$ is bounded above and below by positive constants.

(3) The measure $\tilde{\nu}^\diamond := h^\diamond \tilde{m}^\diamond$ is the unique σ -invariant Gibbs state for $-R^\diamond$, and it is ergodic.

(4) There exist $M > 0$ and $\varrho \in (0, 1)$ such that for every $g \in \mathcal{F}_\theta(\Sigma^+)$ and $n \geq 0$,

$$(15.6) \quad \left\| e^{-n \Pr(-R^\diamond)} \tilde{\mathcal{L}}_{-R^\diamond}^n g - \left(\int_{\Sigma^+} g d\tilde{m}^\diamond \right) h^\diamond \right\|_\theta \leq M \varrho^n \|g\|_\theta.$$

Applying (15.6) to the real and imaginary parts of g shows that, after enlarging M , the same estimate holds for every $g \in \mathcal{F}_\theta(\Sigma^+, \mathbb{C})$.

15.1.2. The conformal eigenmeasures. We continue to write ν for the restriction of the Patterson–Sullivan measure to Λ_+ . Since $\nu(\Lambda_+) > 0$ and $\psi(\mathcal{G}(x, p_0))$ is uniformly bounded on Λ_+ , the measure

$$d\hat{\nu}(x) := \frac{e^{2\psi(\mathcal{G}(x, p_0))}}{\int_{\Lambda_+} e^{2\psi(\mathcal{G}(y, p_0))} d\nu(y)} d\nu(x)$$

is a well-defined probability measure on Λ_+ .

Corollary 15.5. *We have $\Psi_* \tilde{m}^\diamond = \hat{\nu}$ and $\Pr(-R^\diamond) = 0$.*

Proof. For $\gamma \in \mathcal{H}$ and $x \in \gamma \Lambda_+$, conformality and the definition of R give $\frac{d\gamma_* \hat{\nu}}{d\hat{\nu}}(x) = e^{R(x)}$ on $\gamma \Lambda_+$. Thus $\tilde{\mathcal{L}}_{-R}^* \hat{\nu} = \hat{\nu}$. By (15.5), $\Psi^* \hat{\nu}$ is a Borel probability eigenmeasure of $\tilde{\mathcal{L}}_{-R^\diamond}^*$ with eigenvalue 1. The uniqueness in Theorem 15.4(1) therefore gives $\Psi^* \hat{\nu} = \tilde{m}^\diamond$ and $e^{\Pr(-R^\diamond)} = 1$, proving both assertions. $\square$

Since $e^{-n \Pr(-R^\diamond)} = 1$, this factor will henceforth be omitted from (15.6).

15.1.3. *The eigenfunctions and spectral gap.* The following Lasota–Yorke inequality is the main input for establishing the $d_{\tilde{\Psi}}$ -regularity of the transported eigenfunction in Proposition 15.8. It can be proved by a standard argument, based on Proposition 12.12.

Lemma 15.6. *There is $C > 0$ such that, for every $f \in \mathcal{F}_c(\Lambda_+, \mathbb{C})$ and $n \in \mathbb{N}$,*

$$\|\tilde{\mathcal{L}}_{-R}^n f\|_c \leq C\|f\|_\infty + C\lambda^{cn}\|f\|_c.$$

In particular, $\tilde{\mathcal{L}}_{-R}$ preserves $\mathcal{F}_c(\Lambda_+, \mathbb{C})$ and $\mathcal{F}_c(\Lambda_+)$.

The following follows from (15.3) by the same argument as in (R1).

Lemma 15.7. *With θ as in (15.4), Ψ^* maps $\mathcal{F}_c(\Lambda_+, \mathbb{C})$ boundedly into $\mathcal{F}_\theta(\Sigma^+, \mathbb{C})$, i.e.,*

$$\|f \circ \Psi\|_\theta \ll \|f\|_c \quad \text{for } f \in \mathcal{F}_c(\Lambda_+, \mathbb{C}).$$

With $h^\diamond$ given by Theorem 15.4, set

$$(15.7) \quad h := h^\diamond \circ \Psi^{-1} : \Lambda_+ \rightarrow \mathbb{R}.$$

Proposition 15.8. (1) *We have $h \in \mathcal{F}_c(\Lambda_+)$ and h is bounded above and below by positive constants. Moreover, $\tilde{\mathcal{L}}_{-R}h = h$, and $\hat{\nu}(h) = 1$. It is the unique eigenfunction of $\tilde{\mathcal{L}}_{-R}$ in $\mathcal{F}_c(\Lambda_+)$ with eigenvalue 1 and this normalization.*

(2) *There exist $M' > 0$ and $\varrho' \in (0, 1)$ such that, for every $g \in \mathcal{F}_c(\Lambda_+, \mathbb{C})$ and $n \geq 0$,*

$$(15.8) \quad \left\| \tilde{\mathcal{L}}_{-R}^n g - \left(\int_{\Lambda_+} g d\hat{\nu} \right) h \right\|_c \leq M'(\varrho')^n \|g\|_c.$$

In particular, $\tilde{\mathcal{L}}_{-R}$ has a spectral gap on $\mathcal{F}_c(\Lambda_+, \mathbb{C})$, with leading eigenvalue 1 and rank-one spectral projection $g \mapsto \left(\int_{\Lambda_+} g d\hat{\nu} \right) h$.

Proof. To prove (1), it suffices to show that $\|h\|_c < \infty$; the remaining claims follow from (15.5), Corollary 15.5, Lemma 15.7, and Theorem 15.4. Let $\mathbf{1}$ denote the constant function 1 on Λ_+ , and set $u_n := \tilde{\mathcal{L}}_{-R}^n \mathbf{1}$ for each $n \in \mathbb{N}$. Since $\|\mathbf{1}\|_c = 1$, Lemma 15.6 gives $\|u_n\|_c \leq 2C$ for all $n \geq 1$.

On the other hand, by (15.5), (15.6), and Corollary 15.5,

$$\|u_n - h\|_\infty = \|\tilde{\mathcal{L}}_{-R}^n \mathbf{1} - h^\diamond\|_\infty \leq M\varrho^n \rightarrow 0.$$

Passing to the limit in the Hölder inequalities gives $\|h\|_c \leq 2C$.

Now to prove (2), fix $g \in \mathcal{F}_c(\Lambda_+, \mathbb{C})$ and set $D_n := \tilde{\mathcal{L}}_{-R}^n g - \hat{\nu}(g)h$. Since $\tilde{\mathcal{L}}_{-R}h = h$ by part (1),

$$(15.9) \quad D_{n+m} = \tilde{\mathcal{L}}_{-R}^m D_n \quad (n, m \geq 0).$$

By Lemma 15.7, $g \circ \Psi \in \mathcal{F}_\theta(\Sigma^+, \mathbb{C})$ and $\|g \circ \Psi\|_\theta \ll \|g\|_c$. Applying (15.6) to $g \circ \Psi$, and using Corollary 15.5 and (15.5) gives

$$(15.10) \quad \|D_n\|_\infty \ll \varrho^n \|g\|_c.$$

Here we used $\int_{\Sigma^+} g \circ \Psi d\tilde{m}^\diamond = \int_{\Lambda_+} g d\hat{\nu}$. By Lemma 15.6 and part (1), we also have

$$(15.11) \quad \|D_n\|_c \ll \|g\|_\infty + \lambda^{cn}\|g\|_c + \left| \int g d\hat{\nu} \right| \|h\|_c \ll \|g\|_c,$$

with the case $n = 0$ following directly from the definition of D_0 . Combining (15.9) with Lemma 15.6, (15.10), and (15.11), we obtain

$$\|D_{n+m}\|_c \ll \|D_n\|_\infty + \lambda^{cm}\|D_n\|_c \ll (\varrho^n + \lambda^{cm})\|g\|_c.$$

For $N \geq 1$, take $n = \lfloor N/2 \rfloor$ and $m = N - n$. It follows that (15.8) holds with any $\varrho' \in (\max\{\varrho, \lambda^c\}^{1/2}, 1)$ and a suitable M' , enlarged if necessary to include $N = 0$. Finally, define $\mathcal{P}, \mathcal{R} : \mathcal{F}_c(\Lambda_+, \mathbb{C}) \rightarrow \mathcal{F}_c(\Lambda_+, \mathbb{C})$ by

$$\mathcal{P}g := \widehat{\nu}(g)h, \quad \mathcal{R} := \widetilde{\mathcal{L}}_{-R} - \mathcal{P},$$

which are bounded. By Corollary 15.5 and part (1), $\mathcal{P}^2 = \mathcal{P}$ and $\widetilde{\mathcal{L}}_{-R}\mathcal{P} = \mathcal{P}\widetilde{\mathcal{L}}_{-R} = \mathcal{P}$. Hence $\mathcal{P}\mathcal{R} = \mathcal{R}\mathcal{P} = 0$ and $\mathcal{R}^n = \widetilde{\mathcal{L}}_{-R}^n - \mathcal{P}$ for every $n \geq 1$. By (15.8), $\|\mathcal{R}^n\| \leq M'(\varrho')^n$, so $\rho(\mathcal{R}) \leq \varrho' < 1$. This proves the spectral-gap assertion in the sense of Definition 15.1. $\square$

Let $h \in \mathcal{F}_c(\Lambda_+)$ be as in (15.7), and define the probability measure

$$(15.12) \quad d\nu_+ := h d\widehat{\nu}$$

on Λ_+ . By Proposition 15.8, both h and h^{-1} belong to $\mathcal{F}_c(\Lambda_+)$, and multiplication by either is bounded on $\mathcal{F}_c(\Lambda_+, \mathbb{C})$. Hence, by Lemma 15.6, the normalized operator

$$(15.13) \quad \mathcal{L}_0 H := h^{-1} \widetilde{\mathcal{L}}_{-R}(hH) \quad (H \in \mathcal{F}_c(\Lambda_+, \mathbb{C}))$$

is bounded on $\mathcal{F}_c(\Lambda_+, \mathbb{C})$. It is the case $a = w = 0$ of the family introduced in Definition 15.12 below. By (15.13), (15.12), Corollary 15.5, and Proposition 15.8(1),

$$(15.14) \quad \mathcal{L}_0 \mathbf{1} = \mathbf{1}, \quad \mathcal{L}_0^* \nu_+ = \nu_+.$$

Define the operators $\mathcal{P}_0, \mathcal{R}_0 : \mathcal{F}_c(\Lambda_+, \mathbb{C}) \rightarrow \mathcal{F}_c(\Lambda_+, \mathbb{C})$ by

$$(15.15) \quad \mathcal{P}_0 f := \left(\int_{\Lambda_+} f d\nu_+ \right) \mathbf{1}, \quad \mathcal{R}_0 := \mathcal{L}_0 - \mathcal{P}_0.$$

Corollary 15.9. *The operators $\mathcal{P}_0$ and $\mathcal{R}_0$ are bounded on $\mathcal{F}_c(\Lambda_+, \mathbb{C})$ and satisfy*

$$(15.16) \quad \mathcal{P}_0 \mathcal{R}_0 = \mathcal{R}_0 \mathcal{P}_0 = 0, \quad \rho(\mathcal{R}_0) < 1.$$

Proof. By (15.13), $\mathcal{L}_0$ is conjugate to $\widetilde{\mathcal{L}}_{-R}$ by multiplication by h . Multiplication by h and by h^{-1} is bounded on $\mathcal{F}_c(\Lambda_+, \mathbb{C})$, so the spectral gap in Proposition 15.8(2) transfers to $\mathcal{L}_0$. By (15.12), the conjugated spectral projection is

$$f \mapsto \left(\int_{\Lambda_+} h f d\widehat{\nu} \right) \mathbf{1} = \left(\int_{\Lambda_+} f d\nu_+ \right) \mathbf{1} = \mathcal{P}_0 f.$$

This proves the corollary. $\square$

Corollary 15.10. *The probability measure ν_+ is T -invariant and ergodic.*

Proof. By (15.12), (15.7), Corollary 15.5 and Theorem 15.4,

$$\Psi^* \nu_+ = (h \circ \Psi) \Psi^* \widehat{\nu} = h^\diamond \widetilde{m}^\diamond = \widehat{\nu}^\diamond,$$

which is σ -invariant and ergodic by Theorem 15.4(3). Since Ψ conjugates σ and T (Proposition 12.2), this finishes the proof. $\square$

Recall from Lemma 12.6 that $R \geq 1$. Since $d\nu_+/d\nu$ is bounded above and below by positive constants, Proposition 12.8 implies that $R \in L^1(\nu_+)$ and $K \in L^1(\nu_+; \mathfrak{a})$. Hence, the ψ -symbolic vector

$$(15.17) \quad \mathbf{u} := \mathbf{u}_\psi := \frac{\int_{\Lambda_+} K d\nu_+}{\int_{\Lambda_+} R d\nu_+} \in \mathfrak{a}$$

is well-defined, and $\psi(\mathbf{u}) = 1$ because $R = \psi \circ K$.

Theorem 15.11. *The symbolic direction $\mathbf{u}_\psi$ satisfies $\psi_\rho(\mathbf{u}_\psi) = \psi(\mathbf{u}_\psi) = 1$. Consequently, if $\psi = \psi_u$ for some $u \in \text{int } \mathcal{L}_\rho$, then*

$$u = \psi_u(u) \mathbf{u}_{\psi_u}.$$

Proof. Setting $\bar{K} := \int_{\Lambda_+} K d\nu_+$ and $\bar{R} := \int_{\Lambda_+} R d\nu_+$, it suffices to prove that $\psi_\rho(\bar{K}) = \psi(\bar{K})$.

The measure ν_+ is the T -invariant Gibbs probability measure for the potential $-R$, whose pressure is zero; see Theorem 15.4, Corollary 15.5, Corollary 15.10. Thus there exists $C \geq 1$ such that

$$\nu_+(\gamma\Lambda_+) \leq Ce^{-R_n(x)} \quad (n \geq 1, \gamma \in \mathcal{H}^n, x \in \gamma\Lambda_+).$$

Fix $\varepsilon > 0$ and set $E_n := \{x \in \Lambda_+ : \|K_n(x)/n - \bar{K}\| < \varepsilon\}$. By ergodicity and the Birkhoff ergodic theorem, $\nu_+(E_n) \rightarrow 1$. Let $\mathcal{W}_n \subset \mathcal{H}^n$ consist of the words whose cylinders meet E_n , and choose $x_\gamma \in \gamma\Lambda_+ \cap E_n$ for each $\gamma \in \mathcal{W}_n$. Writing $\|\psi\|_*$ for the dual norm, we have $R_n(x_\gamma) \geq n(\bar{R} - \|\psi\|_*\varepsilon)$ for all sufficiently large n . Since these cylinders cover E_n , the Gibbs estimate gives, for all sufficiently large n ,

$$(15.18) \quad \#\mathcal{W}_n \geq \frac{1}{2C} e^{n(\bar{R} - \|\psi\|_*\varepsilon)}.$$

Distinct words represent distinct group elements. Moreover, by (12.6), there is a uniform constant $C_0 > 0$ such that

$$\|\mu(\gamma) - n\bar{K}\| \leq n\varepsilon + C_0 \quad (\gamma \in \mathcal{W}_n).$$

Let $\mathcal{O}$ be any open cone containing $\bar{K}$. Choose $\varepsilon > 0$ sufficiently small that $B(\bar{K}, 2\varepsilon) \subset \mathcal{O}$ and $\bar{R} - \|\psi\|_*\varepsilon > 0$. For all sufficiently large n , every $\gamma \in \mathcal{W}_n$ then satisfies $\mu(\gamma) \in \mathcal{O}$ and $\|\mu(\gamma)\| \leq n(\|\bar{K}\| + \varepsilon) + C_0$. The bound (15.18) implies that the abscissa of convergence $\tau_{\mathcal{O}}$ of the Poincaré series $\sum_{\gamma \in \Gamma_\rho, \mu(\gamma) \in \mathcal{O}} e^{-s\|\mu(\gamma)\|}$ satisfies

$$\tau_{\mathcal{O}} \geq \frac{\bar{R} - \|\psi\|_*\varepsilon}{\|\bar{K}\| + \varepsilon}.$$

Indeed, for any positive s below this ratio, the sum over $\mathcal{W}_n$ of $e^{-s\|\mu(\gamma)\|}$ tends to infinity as $n \rightarrow \infty$, so the restricted series diverges. Letting $\varepsilon \rightarrow 0$ and then taking the infimum over $\mathcal{O}$, the definition of the growth indicator gives

$$\psi_\rho(\bar{K}) = \|\bar{K}\| \inf_{\mathcal{O} \ni \bar{K}} \tau_{\mathcal{O}} \geq \bar{R} = \psi(\bar{K}).$$

The reverse inequality follows from $\psi_\rho \leq \psi$. Homogeneity now yields $\psi_\rho(\mathbf{u}_\psi) = \psi(\mathbf{u}_\psi) = 1$.

By the strict concavity of ψ_ρ [66, 46], there exists a unique tangent direction of ψ . Hence $\psi = \psi_u$ gives $u \in \mathbb{R}_{>0}\mathbf{u}_{\psi_u}$. $\square$

15.2. Complex transfer operators. Throughout this section, we fix an inner product $\langle \cdot, \cdot \rangle_{\mathbf{u}}$ on $\mathfrak{a}$ for which the decomposition

$$\mathfrak{a} = \mathbb{R}\mathbf{u} \oplus \ker \psi$$

is orthogonal and $\langle \mathbf{u}, \mathbf{u} \rangle_{\mathbf{u}} = 1$. We write $\|\cdot\|_{\mathbf{u}}$ for the associated norm, omitting the subscript when it is clear from the context.

Recall the constant $\varepsilon_0 > 0$ from (15.2).

Definition 15.12 (Normalized complex transfer operators). For $a \in \mathbb{C}$ with $\Re a \geq -\varepsilon_0$ and $w \in \mathfrak{a}$, define the normalized transfer operator $\mathcal{L}_{a,w} : \mathcal{F}_c(\Lambda_+, \mathbb{C}) \rightarrow \mathcal{F}_c(\Lambda_+, \mathbb{C})$ by

$$\mathcal{L}_{a,w}H(x) = \frac{1}{h(x)} \sum_{\gamma \in \mathcal{H}} e^{-(1+a)R(\gamma x) + i\langle w, K(\gamma x) \rangle_{\mathfrak{u}}} h(\gamma x) H(\gamma x).$$

The defining series converges absolutely and uniformly by (15.2) and the boundedness of h and h^{-1} . Boundedness on $\mathcal{F}_c(\Lambda_+, \mathbb{C})$ follows from Proposition 12.12, (15.3), and $h, h^{-1} \in \mathcal{F}_c(\Lambda_+)$. It preserves $\mathcal{F}_c(\Lambda_+)$ when $a \in \mathbb{R}$ and $w = 0$. We abbreviate

$$\mathcal{L}_w := \mathcal{L}_{0,w}, \quad \mathcal{L}_0 := \mathcal{L}_{0,0},$$

the latter agreeing with the operator in (15.13).

Recall from (12.4) that $R_\gamma := \sup_{x \in \gamma\Lambda_+} R(x)$ for $\gamma \in \mathcal{H}$. By Lemma 12.6, $R_\gamma \geq 1$, and, after enlarging $C > 0$ if necessary,

$$(15.19) \quad |R_\gamma - \psi(\mu(\gamma))| \leq C, \quad R_\gamma - C \leq R(x) \leq R_\gamma$$

for every $\gamma \in \mathcal{H}$ and $x \in \gamma\Lambda_+$.

The following provides the estimates needed for the spectral analysis.

Lemma 15.13 (Lasota–Yorke inequalities). *There exists $C > 0$ such that for all $w \in \mathfrak{a}$, $n \in \mathbb{N}$ and $f \in \mathcal{F}_c(\Lambda_+, \mathbb{C})$,*

$$(15.20) \quad \|\mathcal{L}_w^n f\|_\infty \leq C \|f\|_\infty,$$

$$(15.21) \quad |\mathcal{L}_w^n f|_c \leq C \max\{\|w\|, 1\} \|f\|_\infty + C\lambda^{cn} \|f\|_c,$$

$$(15.22) \quad \|\mathcal{L}_w^n f\|_c \leq C \max\{\|w\|, 1\} \|f\|_\infty + C\lambda^{cn} \|f\|_c.$$

Proof. It suffices to prove (15.20) and (15.21), since (15.22) then follows after enlarging C . Fix $w \in \mathfrak{a}$, $n \in \mathbb{N}$, and $f \in \mathcal{F}_c(\Lambda_+, \mathbb{C})$. We first prove (15.20). Since $\tilde{\mathcal{L}}_{-R}^n h = h$ by Proposition 15.8(1),

$$|\mathcal{L}_w^n f(x)| \leq \frac{\|f\|_\infty}{h(x)} \sum_{\gamma \in \mathcal{H}^n} e^{-R_n(\gamma x)} h(\gamma x) = \|f\|_\infty.$$

Next we prove (15.21). Let $x, y \in \gamma_0\Lambda_+$ be distinct where $\gamma_0 \in \mathcal{H}$. We have

$$\mathcal{L}_w^n f(x) - \mathcal{L}_w^n f(y) = \sum_{\gamma \in \mathcal{H}^n} (I_1(\gamma) + I_2(\gamma) + I_3(\gamma) + I_4(\gamma))$$

where

$$I_1(\gamma) := \left(\frac{1}{h(x)} - \frac{1}{h(y)} \right) e^{-R_n(\gamma x) + i\langle w, K_n(\gamma x) \rangle_{\mathfrak{u}}} h(\gamma x) f(\gamma x),$$

$$I_2(\gamma) := \frac{1}{h(y)} \left(e^{-R_n(\gamma x) + i\langle w, K_n(\gamma x) \rangle_{\mathfrak{u}}} - e^{-R_n(\gamma y) + i\langle w, K_n(\gamma y) \rangle_{\mathfrak{u}}} \right) h(\gamma x) f(\gamma x),$$

$$I_3(\gamma) := \frac{1}{h(y)} e^{-R_n(\gamma y) + i\langle w, K_n(\gamma y) \rangle_{\mathfrak{u}}} (h(\gamma x) - h(\gamma y)) f(\gamma x),$$

$$I_4(\gamma) := \frac{1}{h(y)} e^{-R_n(\gamma y) + i\langle w, K_n(\gamma y) \rangle_{\mathfrak{u}}} h(\gamma y) (f(\gamma x) - f(\gamma y)).$$

Since $h, h^{-1} \in \mathcal{F}_c(\Lambda_+)$, the preceding summation bound gives

$$\sum_{\gamma \in \mathcal{H}^n} |I_1(\gamma)| \ll \|f\|_\infty d_{\tilde{\psi}}^-(x, y)^c.$$

Similarly, using (15.3),

$$\sum_{\gamma \in \mathcal{H}^n} |I_3(\gamma)| \ll \|f\|_\infty d_{\tilde{\psi}}(x, y)^c.$$

By the uniform distortion estimates for R_n and K_n in Proposition 12.12 and the mean value estimate for the exponential function,

$$\left| e^{-R_n(\gamma x) + i\langle w, K_n(\gamma x) \rangle_{\mathfrak{u}}} - e^{-R_n(\gamma y) + i\langle w, K_n(\gamma y) \rangle_{\mathfrak{u}}} \right| \ll \max\{\|w\|, 1\} e^{-R_n(\gamma x)} d_{\tilde{\psi}}(x, y)^c.$$

Applying the same summation argument as in the proof of (15.20), we obtain

$$\sum_{\gamma \in \mathcal{H}^n} |I_2(\gamma)| \ll \max\{\|w\|, 1\} \|f\|_\infty d_{\tilde{\psi}}(x, y)^c.$$

Finally, for each $\gamma \in \mathcal{H}^n$,

$$|f(\gamma x) - f(\gamma y)| \leq \|f\|_c d_{\tilde{\psi}}(\gamma x, \gamma y)^c \leq \lambda^{cn} \|f\|_c d_{\tilde{\psi}}(x, y)^c.$$

Hence

$$\sum_{\gamma \in \mathcal{H}^n} |I_4(\gamma)| \ll \lambda^{cn} \|f\|_c d_{\tilde{\psi}}(x, y)^c.$$

Combining these estimates and taking the supremum over all distinct x, y in a common one-cylinder proves (15.21). $\square$

Lemma 15.14. *For every $w \in \mathfrak{a}$, we have $\rho(\mathcal{L}_w) \leq 1$.*

Proof. By (15.22), $\|\mathcal{L}_w^n f\|_c \leq (C \max\{\|w\|, 1\} + C) \|f\|_c$ for all $n \geq 1$ and $f \in \mathcal{F}_c(\Lambda_+, \mathbb{C})$. The spectral-radius formula gives the assertion. $\square$

15.3. Spectral gap of complex transfer operators.

Definition 15.15. Let $w \in \mathfrak{a}$ and $0 \leq a < 2\pi$. We call $\langle w, K(\cdot) \rangle_{\mathfrak{u}}$ an *a-function* if there exists $\zeta \in \mathcal{F}_c(\Lambda_+, \mathbb{C})$ with $|\zeta| \equiv 1$ such that

$$(15.23) \quad e^{i(\langle w, K(\cdot) \rangle_{\mathfrak{u}} - a)} \zeta = \zeta \circ T.$$

The argument of [38, Proposition 2.6] gives the following:

Proposition 15.16 ([38, Proposition 2.6]). *Let $w \in \mathfrak{a}$ and $0 \leq a < 2\pi$. Then e^{ia} is an eigenvalue of $\mathcal{L}_w$ on $\mathcal{F}_c(\Lambda_+, \mathbb{C})$ if and only if $\langle w, K(\cdot) \rangle_{\mathfrak{u}}$ is an a-function.*

It follows from Proposition 15.16 and Lemma 15.14 that, if $\langle w, K(\cdot) \rangle_{\mathfrak{u}}$ is an *a-function*, then

$$(15.24) \quad \rho(\mathcal{L}_w) = 1.$$

Lemma 15.17. *Let $w \in \mathfrak{a}$ and $0 \leq a < 2\pi$. If $\langle w, K(\cdot) \rangle_{\mathfrak{u}}$ is an a-function, then*

$$\langle w, K_n(x) \rangle_{\mathfrak{u}} - an \in 2\pi\mathbb{Z} \quad \text{for every } n \geq 1 \text{ and } x \in \text{Fix}(T^n).$$

Proof. Iterating (15.23) gives $e^{i(\langle w, K_n(x) \rangle_{\mathfrak{u}} - an)} \zeta(x) = \zeta(T^n x)$. If $T^n x = x$, then $|\zeta(x)| = 1$ implies the assertion. $\square$

Proposition 15.18. *Let $w \in \mathfrak{a} \setminus \{0\}$. If $\mathcal{L}_w$ has an eigenvalue e^{ia} with $0 \leq a < 2\pi$, then $a/\pi \notin \mathbb{Q}$.*

Proof. Suppose $a/\pi \in \mathbb{Q}$. By Proposition 15.16 and Lemma 15.17, the character $v \in \mathfrak{a} \mapsto e^{i\langle w, v \rangle_{\mathfrak{u}}}$ takes only finitely many values on periods of T^n . On the other hand, by the non-arithmeticity of periods (Theorem 13.1), this implies $w = 0$, a contradiction. $\square$

Write $\mathfrak{a}_{\mathbb{C}}$ for the complexification of $\mathfrak{a}$, and extend $\langle \cdot, \cdot \rangle_{\mathfrak{u}}$ complex-bilinearly to $\mathfrak{a}_{\mathbb{C}}$. The bound $\|K(x)\| \ll R(x)$ from Lemma 12.6 and (15.2) give an open neighborhood $\tilde{\mathcal{U}} \subset \mathfrak{a}_{\mathbb{C}}$ of $\mathfrak{a}$ on which the series below converges absolutely and uniformly for every $H \in \mathcal{F}_c(\Lambda_+, \mathbb{C})$.

We extend the family $\mathcal{L}_w$, defined for $w \in \mathfrak{a}$, to $w \in \tilde{\mathcal{U}} \subset \mathfrak{a}_{\mathbb{C}}$ by

$$(15.25) \quad \mathcal{L}_w H(x) := \frac{1}{h(x)} \sum_{\gamma \in \mathcal{H}} e^{-R(\gamma x) + i\langle w, K(\gamma x) \rangle_{\mathfrak{u}}} h(\gamma x) H(\gamma x).$$

The argument in the proof of Lemma 15.13 with $n = 1$ shows that $\mathcal{L}_w$ is bounded, after shrinking $\tilde{\mathcal{U}}$ if necessary.

We now perturb the decomposition (15.16) in the parameter w .

Theorem 15.19 (Spectral decomposition near zero). *The following hold:*

- There exist an open neighborhood $\mathcal{U} \subset \mathfrak{a}_{\mathbb{C}}$ of 0 such that $w \mapsto \mathcal{L}_w$ is analytic on $\mathcal{U}$.
- For each $w \in \mathcal{U}$, $\mathcal{L}_w$ admits a spectral decomposition

$$(15.26) \quad \mathcal{L}_w = \kappa_w \mathcal{P}_w + \mathcal{R}_w,$$

where κ_w is a simple eigenvalue of $\mathcal{L}_w$ with $\kappa_0 = 1$, $\mathcal{P}_w$ is the corresponding rank-one spectral projection, and $\mathcal{R}_w$ is a bounded operator on $\mathcal{F}_c(\Lambda_+, \mathbb{C})$ satisfying:

- (1) The maps $w \mapsto \kappa_w$, $w \mapsto \mathcal{P}_w$, and $w \mapsto \mathcal{R}_w$ are analytic on $\mathcal{U}$.
- (2) There exists $\eta > 0$ such that for every $w \in \mathcal{U}$,

$$(15.27) \quad \mathcal{P}_w \mathcal{R}_w = \mathcal{R}_w \mathcal{P}_w = 0, \quad \rho(\mathcal{R}_w) < |\kappa_w| - \eta.$$

- (3) Setting $\mathcal{O} := \mathcal{U} \cap \mathfrak{a}$, we have

$$|\kappa_w| < 1 \quad \text{for every } w \in \mathcal{O} \setminus \{0\}.$$

- For every $w \in \mathcal{U}$, $\int_{\Lambda_+} \mathcal{P}_w \mathbf{1} d\nu_+ \neq 0$ and thus

$$(15.28) \quad H_w := \nu_+(\mathcal{P}_w \mathbf{1})^{-1} \mathcal{P}_w \mathbf{1} \in \mathcal{F}_c(\Lambda_+, \mathbb{C})$$

is well defined. Moreover, H_w is the unique eigenfunction of $\mathcal{L}_w$ with eigenvalue κ_w satisfying $\int_{\Lambda_+} H_w d\nu_+ = 1$, $H_0 = \mathbf{1}$; and the map $w \mapsto H_w$ is analytic on $\mathcal{U}$.

- Setting $A_w := \kappa_w^{-1} \mathcal{R}_w$, there exists $\vartheta \in (0, 1)$ such that, for every multi-index α , there is a constant $C_\alpha > 0$ for which

$$(15.29) \quad \sup_{w \in \mathcal{O}} \|\partial_w^\alpha (A_w^n)\| \leq C_\alpha \vartheta^n \quad \text{for all } n \geq 0$$

where $\|\cdot\|$ is the operator norm. In particular, there exists $C > 0$ such that $\|\mathcal{R}_0^n\| \leq C \vartheta^n$ for all $n \geq 0$.

Proof. Analyticity of $\mathcal{L}_w$. For $\gamma \in \mathcal{H}$, let $\mathcal{L}_w^\gamma$ denote the operator on $\mathcal{F}_c(\Lambda_+, \mathbb{C})$ given by the corresponding summand in (15.25). Since K is bounded on $\gamma\Lambda_+$, Proposition 12.12 shows that $w \mapsto \mathcal{L}_w^\gamma$ is analytic in operator norm.

Using $\|K(x)\| \ll R(x)$ from Lemma 12.6, choose a bounded neighborhood $\mathcal{U} \subset \tilde{\mathcal{U}}$ of 0 so small that

$$|\langle \Im(w), K(x) \rangle_{\mathfrak{u}}| \leq \frac{\varepsilon_0}{2} R(x) \quad (w \in \mathcal{U}, x \in \Lambda_+).$$

By Proposition 12.12, (15.19), (15.3), and $h, h^{-1} \in \mathcal{F}_c(\Lambda_+)$,

$$\sum_{\gamma \in \mathcal{H}} \sup_{w \in \mathcal{U}} \|\mathcal{L}_w^\gamma\| \ll \sum_{\gamma \in \mathcal{H}} e^{-(1-\varepsilon_0/2)R_\gamma} < \infty,$$

where the last inequality follows from (15.2). Thus $\mathcal{L}_w = \sum_{\gamma \in \mathcal{H}} \mathcal{L}_w^\gamma$ converges uniformly in operator norm on $\mathcal{U}$, and $w \mapsto \mathcal{L}_w$ is analytic there.

The decomposition and (1)–(3). By Corollary 15.9, 1 is a simple isolated eigenvalue of $\mathcal{L}_0$, so we may fix a contour enclosing 1 and no other point of $\sigma(\mathcal{L}_0)$. Analytic perturbation theory [37, Chapter VII, Section 1] and [61, Proposition 4.6] apply: because the unperturbed eigenvalue is simple, no branching occurs, and one obtains, after shrinking $\mathcal{U}$ if necessary, analytic maps $w \mapsto \kappa_w \in \mathbb{C}$, $w \mapsto \mathcal{P}_w$, and $w \mapsto \mathcal{R}_w$ on $\mathcal{U}$ satisfying (15.26) and the first half of (15.27), with $\mathcal{P}_w$ the rank-one eigenprojection of κ_w . Hence (1) follows. By the upper semicontinuity of the spectral radius, shrinking $\mathcal{U}$ while keeping the same contour yields the uniform spectral separation $\rho(\mathcal{R}_w) < |\kappa_w| - \eta$ on $\mathcal{U}$ in (2) for some $\eta > 0$; in particular, $\kappa_w \neq 0$.

We now prove (3), by applying [17, Lemma 6.12]. Let $w \in \mathcal{O} := \mathcal{U} \cap \mathfrak{a}$. By Lemma 15.14 we have $\rho(\mathcal{L}_w) \leq 1$, and κ_w is an eigenvalue of $\mathcal{L}_w$, so $|\kappa_w| \leq 1$. By Proposition 15.18, if $|\kappa_w| = 1$, then $\kappa_w \in \{e^{ia} : a/\pi \notin \mathbb{Q}\} \cup \{1\}$, which is totally disconnected.

To apply [17, Lemma 6.12], it remains to show that there is no nonconstant analytic curve $t \mapsto w(t)$ in $\mathcal{O}$ with $w(0) = 0$ and $\kappa_{w(t)} = 1$ for all t . If such a curve exists, then by Proposition 15.16 and Lemma 15.17 with $a = 0$, we have for each t that $\langle w(t), K_n(x) \rangle_{\mathfrak{u}} \in 2\pi\mathbb{Z}$ for every $n \geq 1$ and $x \in \text{Fix}(T^n)$. By the non-arithmeticity in Theorem 13.1, this implies $w(t) = 0$ for all t , which is a contradiction. Hence, we can apply [17, Lemma 6.12] and have that, after shrinking $\mathcal{U}$ once more, $|\kappa_w| < 1$ for every $w \in \mathcal{O} \setminus \{0\}$.

The eigenfunction. Since $\nu_+(\mathcal{P}_0 \mathbf{1}) = 1$, analyticity of $\mathcal{P}_w$ allows us to shrink $\mathcal{U}$ so that $\nu_+(\mathcal{P}_w \mathbf{1}) \neq 0$ there. Formula (15.28) therefore defines an analytic family H_w with $H_0 = \mathbf{1}$ and $\nu_+(H_w) = 1$. Uniqueness follows because the range of $\mathcal{P}_w$ is the one-dimensional eigenspace of κ_w .

The uniform bound (15.29). Since $w \mapsto \kappa_w$ is analytic on $\mathcal{U}$ with $\kappa_0 = 1$, we can shrink $\mathcal{U}$ so that $0 < |\kappa_w| < 2$ for $w \in \mathcal{U}$. As $\rho(\mathcal{R}_w) < |\kappa_w| - \eta$ for $w \in \mathcal{U}$ and $A_w := \kappa_w^{-1} \mathcal{R}_w$, there exist constants $0 < \vartheta_0 < \vartheta < 1$ such that for each $w \in \mathcal{U}$,

$$\sigma(A_w) \subset \{\zeta \in \mathbb{C} : |\zeta| \leq \vartheta_0\}.$$

After shrinking $\mathcal{U}$, we may assume that $\overline{\mathcal{U}}$ is compact and that A_w is analytic on a neighborhood of $\overline{\mathcal{U}}$, where the same spectral bound holds. Set $\mathcal{O} := \mathcal{U} \cap \mathfrak{a}$. The circle $|\zeta| = \vartheta$ is therefore a resolvent contour common to all these operators, and by Cauchy's estimates on a neighborhood of $\overline{\mathcal{U}}$, the quantity $M_\alpha := \sup_{w \in \mathcal{O}, |\zeta| = \vartheta} \|\partial_w^\alpha (\zeta I - A_w)^{-1}\|$ is finite for each α . Dunford's formula and differentiation under the integral sign give

$$\partial_w^\alpha A_w^n = \frac{1}{2\pi i} \oint_{|\zeta| = \vartheta} \zeta^n \partial_w^\alpha (\zeta I - A_w)^{-1} d\zeta.$$

Thus $\|\partial_w^\alpha A_w^n\| \leq \vartheta M_\alpha \vartheta^n$, proving (15.29). $\square$

15.4. Expansion of the leading eigenvalue. Let $\mathcal{O} \subset \mathfrak{a}$ be the neighborhood of 0 provided by Theorem 15.19, and let $\mathfrak{u} \in \mathfrak{a}$ be as in (15.17). When convenient, we write $w \in \mathfrak{a}$ in the coordinates $(s, v) \in \mathbb{R} \times \ker \psi$ determined by $w = su + v$, and we set

$$(15.30) \quad \tilde{\kappa}(w) := \kappa_w e^{-is\nu_+(R)} = \kappa_w e^{-i\psi(w)\nu_+(R)}, \quad w \in \mathcal{O}.$$

Proposition 15.20. *There is a positive definite symmetric linear operator S on $\mathfrak{a}$ such that for all w in a neighborhood of 0 in $\mathfrak{a}$,*

$$\tilde{\kappa}(w) = 1 - \frac{1}{2} \langle Sw, w \rangle_{\mathfrak{u}} + o(\|w\|_{\mathfrak{u}}^2).$$

The main point is the positive definiteness of $-D^2\tilde{\kappa}(0)$. The analogous conclusion in [62, Lemma 8(3)] follows from symmetry of a homological cocycle. That symmetry is unavailable for our $\mathfrak{a}$ -valued Busemann cocycle.

Before proving the proposition, we establish a regularity lemma to handle the unbounded return cocycle. For $w \in \mathfrak{a}$, set

$$\Phi_w(x) := \langle w, K(x) \rangle_{\mathfrak{u}} - \psi(w)\nu_+(R) \quad (x \in \Lambda_+).$$

For every $G \in \mathcal{F}_c(\Lambda_+, \mathbb{C})$ and $j \geq 1$, the series

$$\mathcal{L}_0(\Phi_w^j G)(x) := \frac{1}{h(x)} \sum_{\gamma \in \mathcal{H}} e^{-R(\gamma x)} h(\gamma x) \Phi_w(\gamma x)^j G(\gamma x)$$

converges absolutely and uniformly on Λ_+ . Indeed, $|\Phi_w| \ll_w 1 + R$ by Lemma 12.6. Using $(1+u)^j e^{-u} \ll_j e^{-(1-\varepsilon_0)u}$ for $u \geq 1$ and (15.2), we obtain

$$(15.31) \quad \sum_{\gamma \in \mathcal{H}} \sup_{x \in \Lambda_+} e^{-R(\gamma x)} |\Phi_w(\gamma x)|^j \ll_{w,j} \sum_{\gamma \in \mathcal{H}} \sup_{x \in \Lambda_+} e^{-(1-\varepsilon_0)R(\gamma x)} < \infty.$$

The claimed convergence follows since G , h , and h^{-1} are bounded.

Lemma 15.21. *For every $w \in \mathfrak{a}$, $G \in \mathcal{F}_c(\Lambda_+, \mathbb{C})$, and integer $j \geq 1$, we have $\mathcal{L}_0(\Phi_w^j G) \in \mathcal{F}_c(\Lambda_+, \mathbb{C})$. In particular, $\mathcal{L}_0 \Phi_w \in \mathcal{F}_c(\Lambda_+)$.*

Proof. By Theorem 15.19, the map $t \mapsto e^{-it\psi(w)\nu_+(R)} \mathcal{L}_{tw} G$ is analytic near 0 with values in $\mathcal{F}_c(\Lambda_+, \mathbb{C})$. Since $|e^{it\psi(w)}| = 1$ for real t , (15.31) justifies termwise differentiation of its branch formula, giving

$$(15.32) \quad \left. \frac{d^j}{dt^j} \right|_{t=0} (e^{-it\psi(w)\nu_+(R)} \mathcal{L}_{tw} G) = i^j \mathcal{L}_0(\Phi_w^j G) \quad (j \geq 1).$$

The left-hand side belongs to $\mathcal{F}_c(\Lambda_+, \mathbb{C})$ by analyticity. $\square$

Proof of Proposition 15.20. For $w \in \mathcal{O}$, recall $H_w \in \mathcal{F}_c(\Lambda_+, \mathbb{C})$ from (15.28). By Theorem 15.19 the maps $w \mapsto \kappa_w$ and $w \mapsto H_w$ are analytic on $\mathcal{O}$, and $H_0 = \mathbf{1}$.

By positivity of the branch weights, (15.14), and monotone convergence, $\mathcal{L}_0$ extends to $L^1(\nu_+)$, given almost everywhere by the same branch formula:

$$(15.33) \quad \|\mathcal{L}_0 F\|_{L^1(\nu_+)} \leq \int_{\Lambda_+} \mathcal{L}_0 |F| d\nu_+ = \|F\|_{L^1(\nu_+)}.$$

The linear term vanishes. Differentiating the identity $\kappa_w = \int_{\Lambda_+} \mathcal{L}_w(H_w) d\nu_+$ at $w = 0$, it follows from $\mathcal{L}_0^* \nu_+ = \nu_+$ and $\int_{\Lambda_+} H_w d\nu_+ = 1$ that

$$\begin{aligned} D\kappa_0(\cdot) &= \int_{\Lambda_+} (D_w \mathcal{L}_w|_{w=0})(H_0) d\nu_+ + \int_{\Lambda_+} \mathcal{L}_0(D_w H_w|_{w=0}) d\nu_+ \\ &= \int_{\Lambda_+} \mathcal{L}_0(i\langle \cdot, K \rangle_{\mathfrak{u}} H_0) d\nu_+ + D_w \Big|_{w=0} \left(\int_{\Lambda_+} H_w d\nu_+ \right) \\ &= i \left\langle \cdot, \int_{\Lambda_+} K d\nu_+ \right\rangle_{\mathfrak{u}} = i \langle \cdot, \nu_+(R) \mathfrak{u} \rangle_{\mathfrak{u}}, \end{aligned}$$

the last equality by (15.17). Hence $D\kappa_0(w) = i\nu_+(R)\psi(w)$, and therefore $(D\tilde{\kappa}(0))(\cdot) = 0$. By Taylor's theorem,

$$(15.34) \quad \tilde{\kappa}(w) = 1 + \frac{1}{2} D^2\tilde{\kappa}(0)(w, w) + o(\|w\|_{\mathfrak{u}}^2),$$

so it suffices to prove that the quadratic form

$$Q(w) := -D^2\tilde{\kappa}(0)(w, w)$$

is positive definite on $\mathfrak{a}$. Writing $Q(w) = \langle Sw, w \rangle_{\mathfrak{u}}$ for the corresponding positive definite symmetric operator S , (15.34) gives the asserted expansion.

Setup along a ray. Fix $w = su + v \in \mathfrak{a} \setminus \{0\}$ in the coordinate $\mathbb{R} \times \ker \psi$, and write the function $\Phi := \Phi_w$ as in Lemma 15.21. By (15.17), $\int_{\Lambda_+} \Phi d\nu_+ = 0$. By Proposition 12.8 and the boundedness of $d\nu_+/d\nu$, there exists $\varepsilon_1 > 0$ such that

$$(15.35) \quad \int_{\Lambda_+} e^{\varepsilon_1|\Phi|} d\nu_+ < \infty.$$

For $t \in \mathbb{R}$ with $|t|$ small, write $\tilde{\kappa}(t) := \tilde{\kappa}(tw)$ and $H_t := H_{tw}$, and define

$$(15.36) \quad \tilde{\mathcal{L}}_t(F) := \mathcal{L}_0(e^{it\Phi}F), \quad F \in \mathcal{F}_c(\Lambda_+, \mathbb{C}).$$

Then $\tilde{\mathcal{L}}_t = e^{-it\nu_+(R)s} \mathcal{L}_{tw}$, and hence $\tilde{\mathcal{L}}_t H_t = \tilde{\kappa}(t) H_t$. By Lemma 15.21, $\mathcal{L}_0 \Phi \in \mathcal{F}_c(\Lambda_+)$.

The covariance formula. The derivative identities (15.32) allow us to differentiate the eigenrelation in $\mathcal{F}_c(\Lambda_+, \mathbb{C})$. Writing $\dot{H}_0 = \frac{d}{dt}|_{t=0} H_t$ and using $\tilde{\kappa}'(0) = 0$, we obtain

$$(15.37) \quad (I - \mathcal{L}_0)\dot{H}_0 = i\mathcal{L}_0\Phi, \quad \nu_+(\dot{H}_0) = 0.$$

Both $\mathcal{L}_0\Phi$ and $\dot{H}_0$ have mean zero. The spectral gap of $\mathcal{L}_0$ therefore gives

$$(15.38) \quad \dot{H}_0 = i(1 - \mathcal{R}_0)^{-1} \mathcal{L}_0\Phi = i \sum_{n \geq 1} \mathcal{L}_0^n \Phi,$$

with convergence in $\mathcal{F}_c(\Lambda_+, \mathbb{C})$. Differentiating the eigenrelation twice and integrating against ν_+ gives

$$\tilde{\kappa}''(0) = -\nu_+(\Phi^2) + 2i\nu_+(\Phi\dot{H}_0).$$

Consequently,

$$(15.39) \quad Q(w) = \nu_+(\Phi^2) + 2 \sum_{n \geq 1} \nu_+(\Phi \mathcal{L}_0^n \Phi).$$

The series is absolutely convergent because $\|\mathcal{L}_0^n \Phi\|_{\infty}$ decays exponentially for $n \geq 1$ and $\Phi \in L^1(\nu_+)$.

Set

$$f := \Phi + (1 - \mathcal{R}_0)^{-1} \mathcal{L}_0\Phi = \sum_{k \geq 0} \mathcal{L}_0^k(\Phi).$$

Then f is real-valued and $f - \Phi \in \mathcal{F}_c(\Lambda_+)$. Since $\Phi \in L^2(\nu_+)$ by (15.35) and $\mathcal{L}_0 f = f - \Phi$ is bounded, we have $f, \mathcal{L}_0 f \in L^2(\nu_+)$. Substituting $\Phi = f - \mathcal{L}_0 f$ and $\sum_{k \geq 1} \mathcal{L}_0^k \Phi = \mathcal{L}_0 f$ into (15.39), we obtain

$$(15.40) \quad \begin{aligned} Q(w) &= \int_{\Lambda_+} (f - \mathcal{L}_0 f)^2 d\nu_+ + 2 \int_{\Lambda_+} (f - \mathcal{L}_0 f)(\mathcal{L}_0 f) d\nu_+ \\ &= \int_{\Lambda_+} f^2 - (\mathcal{L}_0 f)^2 d\nu_+ = \int_{\Lambda_+} \mathcal{L}_0(f^2) - (\mathcal{L}_0 f)^2 d\nu_+. \end{aligned}$$

In the last equality, we used $\mathcal{L}_0^* \nu_+ = \nu_+$.

Moreover, $\mathcal{L}_0 f = f - \Phi \in \mathcal{F}_c(\Lambda_+)$. Applying Lemma 15.21 to the first two terms in

$$f^2 = \Phi^2 + 2\Phi(f - \Phi) + (f - \Phi)^2,$$

and using the boundedness of $\mathcal{L}_0$ on $\mathcal{F}_c(\Lambda_+)$ for the last term, gives $\mathcal{L}_0(f^2) \in \mathcal{F}_c(\Lambda_+)$. In particular, the integrand in (15.40) is continuous.

Positivity. Since $\mathcal{L}_0$ is a positive operator on $\mathcal{C}_b(\Lambda_+)$ and $\mathcal{L}_0 \mathbf{1} = \mathbf{1}$, Jensen's inequality gives $\mathcal{L}_0(f^2) \geq (\mathcal{L}_0 f)^2$ pointwise, so $Q(w) \geq 0$. Suppose $Q(w) = 0$ for some $w \neq 0$. Since Q is a quadratic form and $\Phi_{tw} = t\Phi_w$, we may replace w by tw for small $t > 0$ and so assume $w \in \mathcal{O}$. By (15.40), the full support of ν_+ , and the continuity of $\mathcal{L}_0(f^2) - (\mathcal{L}_0 f)^2$ on Λ_+ , we have $\mathcal{L}_0(f^2) = (\mathcal{L}_0 f)^2$ on Λ_+ . Together with the definition of $\mathcal{L}_0$ whose branch weights are all strictly positive, this implies that f is constant on $T^{-1}(x)$ for each $x \in \Lambda_+$. Fix $\gamma \in \mathcal{H}$ and define $\varphi : \Lambda_+ \rightarrow \mathbb{R}$ by $\varphi(x) := f(\gamma x)$. Then $f = \varphi \circ T$ and $\varphi = \mathcal{L}_0 f \in \mathcal{F}_c(\Lambda_+)$, and hence $\Phi = \varphi \circ T - \varphi$, i.e.,

$$(15.41) \quad \langle w, K(\cdot) \rangle_{\mathbf{u}} - \nu_+(R)s = (\varphi \circ T - \varphi)(\cdot) \quad \text{on } \Lambda_+.$$

Set $\zeta := e^{i\varphi}$, so that $|\zeta| \equiv 1$. Since $\varphi \in \mathcal{F}_c(\Lambda_+)$, we have $\zeta \in \mathcal{F}_c(\Lambda_+, \mathbb{C})$. Exponentiating (15.41) now gives

$$e^{i(\langle w, K \rangle_{\mathbf{u}} - \nu_+(R)s)} \zeta = \zeta \circ T \quad \text{on } \Lambda_+,$$

so $\langle w, K(\cdot) \rangle_{\mathbf{u}}$ is an a -function in the sense of Definition 15.15, for the $a \in [0, 2\pi)$ with $\nu_+(R)s \in a + 2\pi\mathbb{Z}$. Hence $\rho(\mathcal{L}_w) = 1$ by (15.24). On the other hand, by Theorem 15.19, this is impossible unless $w = 0$. Therefore $Q(w) > 0$ for every $w \neq 0$, so Q is positive definite, completing the proof. $\square$

16. SUSPENSION MODELS AND FACTOR MAPS

We construct suspension models for the translation flow on Ω_ψ and the diagonal flow on $\Omega := \Omega_\rho$. Write $\tilde{\Omega} := \Lambda_\rho^{(2)} \times \mathfrak{a}$ and retain the projection π_ψ and BMS measures of Section 3.7. The measures have the normalization (3.10), so $m_\psi(\Omega_\psi) = 1$. Retain the splitting $\mathfrak{a} = \mathbb{R}\mathbf{u} \oplus \ker \psi$ from (15.17) and define the following functions on Λ_+ :

$$K^\circ(\cdot) := K(\cdot) - R(\cdot)\mathbf{u}, \quad K_n^\circ(\cdot) := \sum_{j=0}^{n-1} (K^\circ \circ T^j)(\cdot) = K_n(\cdot) - R_n(\cdot)\mathbf{u}, \quad K_0^\circ := 0$$

where $n \in \mathbb{N}$. The translations

$$\mathcal{W}_t(\xi, \eta, z) := (\xi, \eta, z + t\mathbf{u}), \quad \mathcal{W}_t(\xi, \eta, s) := (\xi, \eta, s + t)$$

on $\tilde{\Omega}$ and $\tilde{\Omega}_\psi$ descend to the quotients Ω and Ω_ψ , respectively. For fixed $\mathbf{w} \in \ker \psi$, we study correlations under the maps

$$q \mapsto qa_{t\mathbf{u} + \ell(t)\mathbf{w}}, \quad \ell(t) = 0 \text{ or } \sqrt{t}.$$

Recall from Section 3.7 that $\pi_\psi : \Omega \rightarrow \Omega_\psi$ is a trivial principal $\ker \psi$ -bundle. Fix a continuous section $\sigma : \Omega_\psi \rightarrow \Omega$. It gives the identification

$$\Omega_\psi \times \ker \psi \rightarrow \Omega, \quad (q, v) \mapsto \sigma(q)a_v.$$

By (3.11), for every $F \in C_c(\Omega)$,

$$(16.1) \quad \int_{\Omega} F d\mathbf{m}_\psi = \int_{\Omega_\psi} \int_{\ker \psi} F(\sigma(q)a_v) d\text{Leb}_{\ker \psi}(v) dm_\psi(q).$$

16.1. The contracting skew product and the suspension spaces. Set

$$\Lambda_- := U_0 = \Lambda_\rho \setminus \mathcal{B}_{\bar{\psi}} \left(\Delta_0, \frac{1}{2N_0\eta} \right).$$

Lemma 16.1. *The set Λ_- is a compact neighborhood of p_0 in Λ_ρ and satisfies $\Lambda_- \cap \overline{\Delta_0} = \emptyset$ and $\Lambda_b \subset \Lambda_-$. For every $\gamma \in \mathcal{H}$ and $y, y' \in \Lambda_-$,*

$$\gamma^{-1}\Lambda_- \subset \Lambda_-, \quad d_{\bar{\psi}}(\gamma^{-1}y, \gamma^{-1}y') \leq \lambda d_{\bar{\psi}}(y, y'),$$

where λ is as in (15.3).

Proof. The compact neighborhood assertion and disjointness follow from the definition. Iterating Lemma 11.5 gives $\gamma^{-1}\Lambda_- \subset \Lambda_-$. Since $\langle \mathcal{H} \rangle_+^{-1}$ is Zariski dense (Corollary 13.7), a nonempty $\langle \mathcal{H} \rangle_+^{-1}$ -invariant compact set Λ_- contains the unique $\langle \mathcal{H} \rangle_+^{-1}$ -minimal set Λ_b [2]. The contraction estimate follows from (5.2), Lemma 11.14(3), and the choice of λ in (15.3). $\square$

For $(x, x_-), (y, y_-) \in \Lambda_+ \times \Lambda_-$ with x and y belonging to a common one-cylinder, define the premetric

$$d_{\Lambda_+ \times \Lambda_-}((x, x_-), (y, y_-)) := d_{\bar{\psi}}(x, y) + d_{\bar{\psi}}(x_-, y_-).$$

Let $c \in (0, c_0]$ be any number as in (15.1). Define $\mathcal{F}_c(\Lambda_-, \mathbb{C})$ as in Definition 15.2, imposing the c -Hölder condition on all of Λ_- . Similarly, define $\mathcal{F}_c(\Lambda_+ \times \Lambda_-, \mathbb{C})$ using $d_{\Lambda_+ \times \Lambda_-}$ and imposing the Hölder condition on each $(\gamma\Lambda_+) \times \Lambda_-$, $\gamma \in \mathcal{H}$. In both spaces, use the norm $\|\cdot\|_c = \|\cdot\|_\infty + |\cdot|_c$.

Recall that $Tx = \gamma^{-1}x$ for $x \in \gamma\Lambda_+$. Define

$$\widehat{T}(x, x_-) := (Tx, \gamma^{-1}x_-) = (\gamma^{-1}x, \gamma^{-1}x_-) \quad (x \in \gamma\Lambda_+, x_- \in \Lambda_-).$$

Proposition 16.2. *There is a unique $\widehat{T}$ -invariant probability measure $\nu^\diamond$ on $\Lambda_+ \times \Lambda_-$ whose projection to Λ_+ is ν_+ , and it is ergodic. Its disintegration $\{\nu_{+,x} : x \in \Lambda_+\}$ over ν_+ can be chosen to be defined at every $x \in \Lambda_+$, and there is $C_c > 0$ with the following two properties.*

(1) For every $f \in \mathcal{F}_c(\Lambda_+ \times \Lambda_-, \mathbb{C})$,

$$(16.2) \quad \|\bar{f}\|_c \leq C_c \|f\|_c, \quad \bar{f}(x) := \int_{\Lambda_-} f(x, x_-) d\nu_{+,x}(x_-).$$

(2) For $\varphi \in \mathcal{F}_c(\Lambda_-, \mathbb{C})$ and $x, y \in \Lambda_+$ in a common one-cylinder,

$$(16.3) \quad \left| \int_{\Lambda_-} \varphi d\nu_{+,x} - \int_{\Lambda_-} \varphi d\nu_{+,y} \right| \leq C_c \|\varphi\|_\infty d_{\bar{\psi}}(x, y)^c.$$

Proof. We construct the conditional measures at every $x \in \Lambda_+$. For $\gamma \in \mathcal{H}^n$, put

$$p_\gamma^{(n)}(x) := e^{-R_n(\gamma x)} h(\gamma x) h(x)^{-1}$$

where h is given by (15.7). By (15.14), $\sum_{\gamma \in \mathcal{H}^n} p_\gamma^{(n)}(x) = 1$, and for every $\gamma_1 \in \mathcal{H}^\ell$ and $\gamma_2 \in \mathcal{H}^n$,

$$(16.4) \quad p_{\gamma_1 \gamma_2}^{(n+\ell)}(x) = p_{\gamma_2}^{(n)}(x) p_{\gamma_1}^{(\ell)}(\gamma_2 x).$$

Fix $q_- \in \Lambda_-$ and define the probability measure

$$\nu_{+,x}^{(n)} := \sum_{\gamma \in \mathcal{H}^n} p_\gamma^{(n)}(x) \delta_{\gamma^{-1}q_-}$$

where $\delta_{\gamma^{-1}q_-}$ is the Dirac mass at $\gamma^{-1}q_-$. Since $(\gamma_1\gamma_2)^{-1}q_-, \gamma_2^{-1}q_- \in \gamma_2^{-1}\Lambda_-$, by Lemma 16.1, their distance is at most $\lambda^n \text{diam}_{d_{\bar{\psi}}}(\Lambda_-)$. For every $\varphi \in C(\Lambda_-, \mathbb{C})$, (16.4), Lemma 12.10, and uniform continuity show that $\nu_{+,x}^{(n)}(\varphi)$ is Cauchy uniformly in x . Thus $\nu_{+,x}^{(n)}$ converges weakly to a Borel probability measure $\nu_{+,x}$. By (16.4), we have $\nu_{+,x}^{(n+k)} = \sum_{\gamma \in \mathcal{H}^k} p_\gamma^{(k)}(x)(\gamma^{-1})_* \nu_{+, \gamma x}^{(n)}$. Letting $n \rightarrow \infty$ gives that for every $k \geq 1$,

$$(16.5) \quad \nu_{+,x} = \sum_{\gamma \in \mathcal{H}^k} p_\gamma^{(k)}(x)(\gamma^{-1})_* \nu_{+, \gamma x}.$$

Define a Borel probability measure on $\Lambda_+ \times \Lambda_-$:

$$d\nu^\diamond(x, x_-) := d\nu_+(x) d\nu_{+,x}(x_-).$$

We first show the $\widehat{T}$ -invariance of $\nu^\diamond$. For a bounded Borel function $F : \Lambda_+ \times \Lambda_- \rightarrow \mathbb{C}$, we have from $\mathcal{L}_0^* \nu_+ = \nu_+$, as extended in (15.33), and (16.5) that

$$\int F \circ \widehat{T} d\nu^\diamond = \int_{\Lambda_+} \sum_{\gamma \in \mathcal{H}} p_\gamma^{(1)}(x) \int_{\Lambda_-} F(x, \gamma^{-1}x_-) d\nu_{+, \gamma x}(x_-) d\nu_+(x) = \int F d\nu^\diamond.$$

It is clear from the definition that $\nu^\diamond$ projects to ν_+ on Λ_+ .

To prove uniqueness, let η be any $\widehat{T}$ -invariant probability measure projecting to ν_+ . For a bounded Borel function $a : \Lambda_+ \rightarrow \mathbb{C}$ and $\varphi \in \mathcal{C}(\Lambda_-)$, put $F(x, x_-) := a(x)\varphi(x_-)$. Since η is $\widehat{T}$ -invariant and φ is uniformly continuous, Lemma 16.1 gives

$$\int F d\eta = \lim_{n \rightarrow \infty} \int_{\Lambda_+} F(\widehat{T}^n(x, q_-)) d\nu_+(x).$$

The right-hand side is independent of η . Since these product functions determine a probability measure, the invariant lift is unique. Finally, let E be an invariant set with $\nu^\diamond(E) > 0$. The normalized restriction $\nu^\diamond|_E / \nu^\diamond(E)$ projects to a T -invariant probability measure absolutely continuous with respect to ν_+ . By Corollary 15.10, this projection equals ν_+ . Uniqueness of the invariant lift gives $\nu^\diamond|_E / \nu^\diamond(E) = \nu^\diamond$, hence $\nu^\diamond(E) = 1$. Thus $\nu^\diamond$ is ergodic.

To prove (2), let $x, y \in \Lambda_+$ belong to a common one-cylinder. Applying the estimates for I_1, I_2, I_3 in the proof of Lemma 15.13 with $w = 0$ and $f = \mathbf{1}$, we obtain

$$\sum_{\gamma \in \mathcal{H}^n} |p_\gamma^{(n)}(x) - p_\gamma^{(n)}(y)| \leq C_c d_{\bar{\psi}}(x, y)^c \quad (n \geq 1);$$

hence $|\nu_{+,x}^{(n)}(\varphi) - \nu_{+,y}^{(n)}(\varphi)| \leq C_c \|\varphi\|_\infty d_{\bar{\psi}}(x, y)^c$. Letting $n \rightarrow \infty$ proves (2).

We now prove (1). For $f \in \mathcal{F}_c(\Lambda_+ \times \Lambda_-, \mathbb{C})$, define $\bar{f}$ by (16.2). Then $\|\bar{f}\|_\infty \leq \|f\|_\infty$ is clear. For x and y in a common one-cylinder, write

$$\begin{aligned} \bar{f}(x) - \bar{f}(y) &= \int_{\Lambda_-} (f(x, x_-) - f(y, x_-)) d\nu_{+,x}(x_-) \\ &\quad + \left(\int_{\Lambda_-} f(y, x_-) d\nu_{+,x}(x_-) - \int_{\Lambda_-} f(y, x_-) d\nu_{+,y}(x_-) \right). \end{aligned}$$

The first integral is bounded in absolute value by $\|f\|_\infty d_{\bar{\psi}}(x, y)^c$. The remaining difference is bounded by $C_c \|f\|_\infty d_{\bar{\psi}}(x, y)^c$, by applying (2) to $f(y, \cdot)$. Together with $\|\bar{f}\|_\infty \leq \|f\|_\infty$, this proves (1) after enlarging C_c . $\square$

Let

$$\tilde{\Lambda}_{\mathbf{a}} := \Lambda_+ \times \Lambda_- \times \mathbb{R} \times \ker \psi, \quad \tilde{\Lambda}_{\psi} := \Lambda_+ \times \Lambda_- \times \mathbb{R}.$$

Define the skew-product maps $\zeta: \tilde{\Lambda}_{\mathbf{a}} \rightarrow \tilde{\Lambda}_{\mathbf{a}}$ and $\zeta_{\psi}: \tilde{\Lambda}_{\psi} \rightarrow \tilde{\Lambda}_{\psi}$ by

$$(16.6) \quad \begin{aligned} \zeta(x, x_-, s, v) &:= (\widehat{T}(x, x_-), s - R(x), v - K^{\circ}(x)), \\ \zeta_{\psi}(x, x_-, s) &:= (\widehat{T}(x, x_-), s - R(x)). \end{aligned}$$

Let

$$\begin{aligned} \Lambda_{\mathbf{a}} &:= \{(x, x_-, s, v) \in \tilde{\Lambda}_{\mathbf{a}} : 0 \leq s < R(x)\}, \\ \Lambda_{\psi} &:= \{(x, x_-, s) \in \tilde{\Lambda}_{\psi} : 0 \leq s < R(x)\}. \end{aligned}$$

For $t \geq 0$, define the semiflow T_t on $\Lambda_{\mathbf{a}}$ by

$$(16.7) \quad T_t(x, x_-, s, v) := \zeta^n(x, x_-, s + t, v) = (\widehat{T}^n(x, x_-), s + t - R_n(x), v - K_n^{\circ}(x)),$$

where $n = n(t, x, s)$ is the unique nonnegative integer such that $R_n(x) \leq s + t < R_{n+1}(x)$. Such an integer exists because $R \geq 1$ by Lemma 12.6. This descends to the semiflow on Λ_{ψ} via the projection $\pi: \Lambda_{\mathbf{a}} \rightarrow \Lambda_{\psi}$, which we also denote by T_t . For $v \in \ker \psi$, define

$$(x, x_-, s, v_0)a_v := (x, x_-, s, v_0 + v).$$

These translations commute with T_t . On the product spaces, define

$$(16.8) \quad d\tilde{M}_{\psi} := \frac{1}{\nu_+(R)} d\nu^{\diamond} ds d\text{Leb}_{\ker \psi}(v), \quad d\tilde{M}_{\psi} := \frac{1}{\nu_+(R)} d\nu^{\diamond} ds.$$

Since $\nu^{\diamond}$ is $\widehat{T}$ -invariant (Proposition 16.2), $\tilde{M}_{\psi}$ and $\tilde{M}_{\psi}$ are invariant under ζ and ζ_{ψ} , respectively. Consider the following T_t -invariant measures for $t \geq 0$:

$$M_{\psi} := \tilde{M}_{\psi}|_{\Lambda_{\mathbf{a}}}, \quad M_{\psi} = \tilde{M}_{\psi}|_{\Lambda_{\psi}}.$$

Lemma 16.3. *The measure M_{ψ} is an ergodic probability measure for $\{T_t : t \geq 0\}$.*

Proof. Since $\nu^{\diamond}$ projects to ν_+ , $M_{\psi}(\Lambda_{\psi}) = \frac{1}{\nu_+(R)} \int_{\Lambda_+ \times \Lambda_-} R(x) d\nu^{\diamond}(x, x_-) = 1$. Every measurable set invariant under all T_t , $t \geq 0$, is the suspension of a $\widehat{T}$ -invariant set. Thus ergodicity follows from Proposition 16.2. $\square$

16.2. The factor maps. The symbolic suspensions are related to the geometric flows by explicit equivariant maps. Define the maps

$$(16.9) \quad \tilde{\Phi}: \tilde{\Lambda}_{\mathbf{a}} \rightarrow \tilde{\Omega}, \quad \tilde{\Phi}(x, x_-, s, v) := (x, x_-, su + v + 2\mathcal{G}(x, p_0)),$$

$$(16.10) \quad \tilde{\Phi}_{\psi}: \tilde{\Lambda}_{\psi} \rightarrow \tilde{\Omega}_{\psi}, \quad \tilde{\Phi}_{\psi}(x, x_-, s) := (x, x_-, s + 2\psi(\mathcal{G}(x, p_0))).$$

Restricting to the suspension domains and composing with the projections $\tilde{\Omega} \rightarrow \Omega$ and $\tilde{\Omega}_{\psi} \rightarrow \Omega_{\psi}$ yields continuous maps

$$\Phi: \Lambda_{\mathbf{a}} \rightarrow \Omega, \quad \Phi_{\psi}: \Lambda_{\psi} \rightarrow \Omega_{\psi}.$$

By (12.5) and (16.6), these maps satisfy

$$(16.11) \quad \Phi \circ T_t = \mathcal{W}_t \circ \Phi, \quad \Phi_{\psi} \circ T_t = \mathcal{W}_t \circ \Phi_{\psi}, \quad \Phi(za_v) = \Phi(z)a_v.$$

In particular,

$$\Phi((T_t z)a_{\ell(t)\mathbf{w}}) = \Phi(z)a_{tu+\ell(t)\mathbf{w}}.$$

Proposition 16.4. *We have $(\Phi_{\psi})_* M_{\psi} = m_{\psi}$.*

We first record two elementary measure-theoretic facts needed in the proof.

Lemma 16.5. *There is a Borel set $U \subset \Lambda_\psi$ such that $m_\psi(\Phi_\psi(U)) > 0$ and Φ_ψ is injective on U .*

Proof. The map $\tilde{\Phi}_\psi$ is an embedding by (16.10). Since $\inf_{x \in \Lambda_+} R(x) > 0$ by Lemma 12.6, we have $\tilde{m}_\psi(\tilde{\Phi}_\psi(\Lambda_\psi)) > 0$. Since Γ_ρ is torsion-free, there exists an open $O \subset \tilde{\Omega}_\psi$ on which $\tilde{\Omega}_\psi \rightarrow \Omega_\psi$ is injective and $\tilde{m}_\psi(O \cap \tilde{\Phi}_\psi(\Lambda_\psi)) > 0$. Setting $U := \tilde{\Phi}_\psi^{-1}(O) \cap \Lambda_\psi$ finishes the proof. $\square$

Here stable leaves are taken in Ω_ψ , rather than the auxiliary flow $\Omega_{\tilde{\psi}}$ of Section 8: two points lie on the same stable leaf if they admit lifts (x, x_-, s) and (x, y_-, s) with the same positive endpoint and $\mathbb{R}$ -coordinate. By [44, Proposition 8.6], there exists a metric d on Ω_ψ compatible with the topology of Ω_ψ such that

$$(16.12) \quad d(\mathcal{W}_t q, \mathcal{W}_t q') \rightarrow 0 \quad (t \rightarrow +\infty).$$

whenever $q, q' \in \Omega_\psi$ lie on the same stable leaf.

Lemma 16.6. *Let $Q \subset \Lambda_\psi$ be M_ψ -conull and $Q' \subset \Omega_\psi$ be m_ψ -conull. Then $\Phi_\psi(Q)$ and Q' meet a common stable leaf of Ω_ψ .*

Proof. Let U be as in Lemma 16.5 and set $V := \Phi_\psi(U)$. Also set $m := ((\Phi_\psi|_U)^{-1})_*(m_\psi|_V)$. For $p(x, x_-, s) := (x, s)$, the local product formula for BMS measure gives $p_* m \ll d\nu ds$. Since ν_+ and $\nu|_{\Lambda_+}$ are equivalent, it follows from Proposition 16.2 that $p_* m \ll p_* M_\psi$. Since Q is M_ψ -conull, by Proposition 16.2, $\nu_{+,x}(\{x_- : (x, x_-, s) \in Q\}) = 1$ for $p_* M_\psi$ -a.e. (x, s) and hence for $p_* m$ -a.e. (x, s) . Since $\Phi_\psi^{-1}(Q' \cap V)$ is m -conull, there exists $(x, x'_-, s) \in \Phi_\psi^{-1}(Q') \cap U$ such that $\nu_{+,x}(\{x_- : (x, x_-, s) \in Q\}) = 1$. Choose x_- with $(x, x_-, s) \in Q$. By (16.10), $\Phi_\psi(x, x_-, s)$ and $\Phi_\psi(x, x'_-, s) \in Q'$ lie on the same stable leaf in Ω_ψ . $\square$

Proof of Proposition 16.4. Let $f \in \mathcal{C}_c(\Omega_\psi)$. By Lemma 16.3, Theorem 3.19, and the Birkhoff ergodic theorem, there exist conull sets $Q' \subset \Omega_\psi$ and $Q \subset \Lambda_\psi$ on which, respectively,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\mathcal{W}_t q) dt = m_\psi(f), \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\Phi_\psi(T_t z)) dt = M_\psi(f \circ \Phi_\psi).$$

By Lemma 16.6, choose $z \in Q$ and $q \in Q'$ such that $\Phi_\psi(z)$ and q lie on the same stable leaf. Then by (16.11), the uniform continuity of f , and (16.12), we have

$$\frac{1}{T} \int_0^T f(\mathcal{W}_t q) - f(\mathcal{W}_t \Phi_\psi(z)) dt \rightarrow 0.$$

It follows that $m_\psi(f) = M_\psi(f \circ \Phi_\psi)$. Since this holds for every $f \in \mathcal{C}_c(\Omega_\psi)$, $(\Phi_\psi)_* M_\psi = m_\psi$. $\square$

Theorem 16.7. *We have $\Phi_* M_\psi = m_\psi$. Moreover, for every compact set $Q \subset \Omega$ and $a \geq 0$, there is $C_{Q,a} > 0$ such that*

$$\|su + v\| \leq C_{Q,a} R(x) + C_{Q,a}$$

whenever $(x, x_-, s, v) \in \tilde{\Lambda}_a$, $-a \leq s < R(x)$, and $[\tilde{\Phi}(x, x_-, s, v)] \in Q$. Here $[\cdot]$ denotes projection to Ω .

Proof. In the bundle trivialization determined by the section $\sigma : \Omega_\psi \rightarrow \Omega$, there is a unique continuous map $\tau_\sigma : \Lambda_\psi \rightarrow \ker \psi$ such that $\Phi(z, 0) = \sigma(\Phi_\psi(z))a_{\tau_\sigma(z)}$. Proposition 16.4 and (16.1) give, for $F \in \mathcal{C}_c(\Omega)$,

$$\begin{aligned} \int_{\Lambda_\mathfrak{a}} F(\Phi(z, v)) d\mathbf{M}_\psi(z, v) &= \int_{\Lambda_\psi} \int_{\ker \psi} F(\sigma(\Phi_\psi(z))a_v) dv dM_\psi(z) \\ &= \int_{\Omega_\psi} \int_{\ker \psi} F(\sigma(q)a_v) dv dm_\psi(q) = \mathbf{m}_\psi(F), \end{aligned}$$

where $dv = d\text{Leb}_{\ker \psi}(v)$. This proves $\Phi_*\mathbf{M}_\psi = \mathbf{m}_\psi$. We now prove the remaining claim. Fix a compact subset $Q \subset \Omega$ and $a \geq 0$. We fix a compact subset $\tilde{Q} \subset \tilde{\Omega}$ whose image under the projection to Ω contains Q . Let $(x, x_-, s, v) \in \tilde{\Lambda}_\mathfrak{a}$ be such that $-a \leq s < R(x)$ and $[\tilde{\Phi}(x, x_-, s, v)] \in Q$. Then there exists $\gamma \in \Gamma_\rho$ so that

$$\gamma\tilde{\Phi}(x, x_-, s, v) = (\gamma x, \gamma x_-, \mathbf{s}\mathbf{u} + v + 2\mathcal{G}(x, p_0) + \beta_x(\gamma^{-1}, e)) \in \tilde{Q}.$$

Since $\mathcal{G}(x, p_0)$ is uniformly bounded by Lemma 3.13 and $\tilde{Q}$ is compact, this implies that $\mathbf{s}\mathbf{u} + v + \beta_x(\gamma^{-1}, e)$ is uniformly bounded, depending only on Q .

By the proof of [45, Proposition 9.7] and compactness of Q , either

$$\beta_x(\gamma^{-1}, e) = \mu(\gamma) + O_Q(1) \quad \text{or} \quad \beta_x(\gamma^{-1}, e) = -\mu(\gamma^{-1}) + O_Q(1).$$

By (7.1), $\psi > 0$ on $\mathcal{L}_\rho \setminus \{0\}$, so

$$\|\mathbf{s}\mathbf{u} + v\| \approx_Q \|\beta_x(\gamma^{-1}, e)\| \ll_Q 1 + |\psi(\beta_x(\gamma^{-1}, e))| \approx_Q |s| \leq a + R(x).$$

This completes the proof. $\square$

17. BMS LOCAL MIXING AND THE CORRELATION EXPANSION

In this section, we prove local mixing for $(\Omega, \mathcal{W}_t, \mathbf{m}_\psi)$. All spaces of smooth functions in this section are understood to consist of complex-valued functions. The factor map $\Phi : (\Lambda_\mathfrak{a}, T_t, \mathbf{m}_\psi) \rightarrow (\Omega, \mathcal{W}_t, \mathbf{m}_\psi)$ from Theorem 16.7 allows us to express correlations on Ω in terms of the suspension flow T_t on $\Lambda_\mathfrak{a}$. The principal technical difficulty is that Φ involves an unbounded $\mathfrak{a}$ -coordinate: the pullback of a compactly supported function on Ω need not be compactly supported on $\Lambda_\mathfrak{a}$. Set

$$r := \text{rank}_\mathbb{R} G = \dim \mathfrak{a}, \quad c_R := \left(\int_{\Lambda_+} R d\nu_+ \right)^{1/2}.$$

Throughout this section, we omit the subscripts in $\langle \cdot, \cdot \rangle_\mathfrak{u}$ and $\| \cdot \|_\mathfrak{u}$. We use the orthogonal splitting $\mathfrak{a} = \mathbb{R}\mathbf{u} \oplus \ker \psi$, and write elements of $\mathfrak{a}$ as pairs (s, v) accordingly. Fix an orthonormal basis for $\langle \cdot, \cdot \rangle$ containing $\mathbf{u}$. Let S be the matrix in this basis of the positive definite symmetric operator in Proposition 15.20.

Write $dz := d\text{Leb}_\mathfrak{a}(z) = ds d\text{Leb}_{\ker \psi}(v)$, as in Section 3.6. For $F \in L^1(dz; \mathbb{C})$, our Fourier convention is

$$\widehat{F}(\xi) := \int_{\mathfrak{a}} e^{i\langle \xi, z \rangle} F(z) dz.$$

Frequency integrals use the dual Lebesgue measure $d\xi$, characterized by the inversion formula

$$F(z) = \frac{1}{(2\pi)^r} \int_{\mathfrak{a}} e^{-i\langle \xi, z \rangle} \widehat{F}(\xi) d\xi \quad (F \in C_c^\infty(\mathfrak{a})).$$

Let $\mathcal{G}$ be the centered Gaussian probability density on $\mathfrak{a}$ with covariance S , defined by

$$\mathcal{G}(z) := \frac{1}{\int_{\mathfrak{a}} e^{-\frac{1}{2}y^t S^{-1}y} dy} e^{-\frac{1}{2}z^t S^{-1}z}, \quad z \in \mathfrak{a}.$$

Then $\int_{\mathfrak{a}} \mathcal{G}(z) dz = 1$, $\int_{\mathfrak{a}} z \mathcal{G}(z) dz = 0$, $\int_{\mathfrak{a}} z z^t \mathcal{G}(z) dz = S$ and its Fourier transform is

$$\widehat{\mathcal{G}}(\xi) = e^{-\frac{1}{2}\xi^t S \xi}, \quad \xi \in \mathfrak{a}.$$

We sometimes write $\mathcal{G}(s, v) := \mathcal{G}(su + v)$ for $s \in \mathbb{R}$ and $v \in \ker \psi$. Fix a basis $X_1, \dots, X_{\dim G}$ of left-invariant vector fields on G , regarded also as vector fields on $\Gamma_{\rho} \backslash G$. For $f \in C_c^{\infty}(\Gamma_{\rho} \backslash G)$, $k \in \mathbb{Z}_{\geq 0}$ and $\beta \in (0, 1]$, set

$$\|f\|_{k, \beta} := \max_{|\alpha| \leq k} (\|X^{\alpha} f\|_{\infty} + |X^{\alpha} f|_{\beta}),$$

where $X^{\alpha} := X_1^{\alpha_1} \cdots X_{\dim G}^{\alpha_{\dim G}}$ and $|g|_{\beta} := \sup_{x \neq y} \frac{|g(x) - g(y)|}{d_{\text{Riem}}(x, y)^{\beta}}$. Here d_{Riem} is the quotient metric induced by a fixed left G -invariant and right K -invariant Riemannian metric on G . We also write d_{Riem} for the induced quotient metric on $\Gamma_{\rho} \backslash G/M$.

For a compact subset $\mathcal{K} \subset \Gamma_{\rho} \backslash G$, denote by $\mathcal{C}_{\mathcal{K}}^{\infty}(\Gamma_{\rho} \backslash G)^M$ the space of smooth M -invariant functions whose supports are contained in $\mathcal{K}$. For simplicity, we write $\mathcal{C}_{\mathcal{K}}^{\infty} := \mathcal{C}_{\mathcal{K}}^{\infty}(\Gamma_{\rho} \backslash G)^M$.

Theorem 17.1 (BMS local mixing). *The following assertions hold.*

- (1) *There exist bilinear forms $\mathbf{c}_j : \mathcal{C}_c^{\infty}(\Gamma_{\rho} \backslash G)^M \times \mathcal{C}_c^{\infty}(\Gamma_{\rho} \backslash G)^M \rightarrow \mathbb{C}$, $j \geq 0$, such that, for every compact set $\mathcal{K} \subset \Gamma_{\rho} \backslash G$ and $L \in \mathbb{Z}_{\geq 0}$, there exist $k \in \mathbb{N}$ and $\beta \in (0, 1)$ for which*

$$|\mathbf{c}_j(f_1, f_2)| \ll_{\mathcal{K}, L} \|f_1\|_{k, \beta} \|f_2\|_{k, \beta} \quad (0 \leq j \leq L)$$

and the following expansion holds for all $f_1, f_2 \in \mathcal{C}_{\mathcal{K}}^{\infty}$ as $t \rightarrow \infty$:

$$t^{\frac{r-1}{2}} \int_{\Gamma_{\rho} \backslash G} f_1(x a_{t\mathbf{u}}) f_2(x) d\mathbf{m}_{\psi}(x) = \sum_{j=0}^L \mathbf{c}_j(f_1, f_2) t^{-j} + O_{\mathcal{K}, L} \left(\|f_1\|_{k, \beta} \|f_2\|_{k, \beta} t^{-(L+1)} \right).$$

Moreover,

$$\mathbf{c}_0(f_1, f_2) = c_R^r \left(\int_{\mathbb{R}} \mathcal{G}(c_R s \mathbf{u}) ds \right) \mathbf{m}_{\psi}(f_1) \mathbf{m}_{\psi}(f_2).$$

- (2) *There exists $a > 0$ such that, for every compact set $\mathcal{K} \subset \Gamma_{\rho} \backslash G$, there exist $k \in \mathbb{N}$, $\beta \in (0, 1)$, and $C, T > 0$ with the following property. For all $f_1, f_2 \in \mathcal{C}_{\mathcal{K}}^{\infty}$, $t \geq T$, and $\mathbf{w} \in \ker \psi$,*

$$t^{\frac{r-1}{2}} \int_{\Gamma_{\rho} \backslash G} f_1(x a_{t\mathbf{u} + \sqrt{t}\mathbf{w}}) f_2(x) d\mathbf{m}_{\psi}(x) = \mathbf{c}_{\mathbf{w}} \mathbf{m}_{\psi}(f_1) \mathbf{m}_{\psi}(f_2) + \mathcal{E}_{f_1, f_2}(t, \mathbf{w}),$$

where $\mathbf{c}_{\mathbf{w}} = c_R^r \int_{\mathbb{R}} \mathcal{G}(c_R(s\mathbf{u} + \mathbf{w})) ds$ and

$$(17.1) \quad |\mathcal{E}_{f_1, f_2}(t, \mathbf{w})| \leq C \|f_1\|_{k, \beta} \|f_2\|_{k, \beta} t^{-\frac{1}{2}} \exp(-a \cdot \min\{\|\mathbf{w}\|^2, \|\mathbf{w}\|\}).$$

Remark 17.2 (More general transverse displacements). Let $\ell : [1, \infty) \rightarrow (0, \infty)$ be any function. Since the constants in Theorem 17.1(2) are independent of $\mathbf{w}$, we may apply that theorem with $\mathbf{w}$ replaced by $\ell(t)\mathbf{w}/\sqrt{t}$. This gives, for all sufficiently large t ,

$$t^{\frac{r-1}{2}} \int_{\Gamma_{\rho} \backslash G} f_1(x a_{t\mathbf{u} + \ell(t)\mathbf{w}}) f_2(x) d\mathbf{m}_{\psi}(x) = \mathbf{c}_{\ell(t)\mathbf{w}/\sqrt{t}} \mathbf{m}_{\psi}(f_1) \mathbf{m}_{\psi}(f_2) + \mathcal{E}_{f_1, f_2} \left(t, \frac{\ell(t)}{\sqrt{t}} \mathbf{w} \right),$$

with the remainder bounded by (17.1) evaluated at $\ell(t)\mathbf{w}/\sqrt{t}$. In particular, if $\ell(t)^2/t \rightarrow q \in [0, \infty)$, then

$$\lim_{t \rightarrow \infty} t^{\frac{r-1}{2}} \int_{\Gamma_\rho \backslash G} f_1(x a_{t\mathbf{u} + \ell(t)\mathbf{w}}) f_2(x) d\mathbf{m}_\psi(x) = \mathbf{c}_{\sqrt{q}\mathbf{w}} \mathbf{m}_\psi(f_1) \mathbf{m}_\psi(f_2),$$

by continuity of $v \mapsto \mathbf{c}_v$. If $\ell(t)^2/t \rightarrow \infty$, the limit is zero for every $\mathbf{w} \neq 0$. This is analogous to the more general transverse local mixing statement in [17, Theorem 1.3].

We first prepare the estimates needed for the proof of Theorem 17.1. After introducing the pullback functions and fixing the parameters, we establish return-time tail bounds and a rough transverse correlation bound in Section 17.3. The transfer-operator decay in Section 17.4, together with the Fourier and stable-projection estimates in Section 17.5, controls frequencies away from zero and reduces the remaining contribution to the expanding map. We then obtain the limiting Taylor coefficients in Section 17.6 and the Gaussian sum estimates in Section 17.7. The proof of Theorem 17.1 is given in Section 17.8.

17.1. Pullbacks of compactly supported functions. For the remainder of this section we fix

$$\text{a compact subset } \mathcal{K} \subset \Gamma_\rho \backslash G, \ L \in \mathbb{Z}_{\geq 0}, \text{ and } f_1, f_2 \in C_{\mathcal{K}}^\infty$$

as in Theorem 17.1. Unless otherwise indicated, all implicit constants depend only on $\mathcal{K}, L$.

In the following, we abuse the notation and write $\Phi : \tilde{\Lambda}_{\mathfrak{a}} \rightarrow \Omega$ for the composition of $\tilde{\Phi} : \tilde{\Lambda}_{\mathfrak{a}} \rightarrow \tilde{\Omega}$ and the projection $\tilde{\Omega} \rightarrow \Omega$. We pull f_1 and f_2 back to the suspension $\tilde{\Lambda}_{\mathfrak{a}}$ and replace the second pullback by a function smooth in the $\mathfrak{a}$ -variable. More precisely, define $F_i : \tilde{\Lambda}_{\mathfrak{a}} \rightarrow \mathbb{C}$, $i = 1, 2$, by

$$(17.2) \quad F_i(x, x_-, s, v) := f_i(\Phi(x, x_-, s, v)) \mathbf{1}_{[0, R(x))}(s).$$

Fix a function $\zeta_0 \in \mathcal{C}^\infty(\mathbb{R}, [0, 1])$ such that $\zeta_0(s) = 0$ for $s \leq -\inf_{x \in \Lambda_+} \frac{R(x)}{10}$ and $\zeta_0(s) = 1$ for $s \geq 0$. Define $F_2^\sharp : \tilde{\Lambda}_{\mathfrak{a}} \rightarrow \mathbb{C}$ by

$$(17.3) \quad F_2^\sharp(x, x_-, s, v) := \zeta_0(s) f_2(\Phi(x, x_-, s, v)) (1 - \zeta_0(s - R(x))).$$

Then $F_2^\sharp$ is smooth in the $\mathfrak{a}$ -variable, is nonzero only if $-\inf_{x \in \Lambda_+} \frac{R(x)}{10} \leq s \leq R(x)$, and satisfies

$$(17.4) \quad F_2 = F_2^\sharp + F_2^\sharp \circ \zeta \quad \text{on } \Lambda_{\mathfrak{a}},$$

where ζ is defined in (16.6), since $\Phi \circ \zeta = \Phi$ by (16.9) and (12.5), and the cutoff factors on $\Lambda_{\mathfrak{a}}$ are $1 - \zeta_0(s - R(x))$ and $\zeta_0(s - R(x))$, respectively.

By Theorem 16.7, there is $C_1 = C_1(\mathcal{K}) > 0$ such that

$$(17.5) \quad \text{supp } F_1 \cup \text{supp } F_2 \cup \text{supp } F_2^\sharp \subset \{(x, x_-, z) : \|z\| \leq C_1 R(x) + C_1\}.$$

Note that by (16.9), the map $(x, x_-) \mapsto \Phi(x, x_-, 0)$ extends smoothly to a neighborhood of the compact set $\overline{\Lambda_+} \times \Lambda_-$ in $\mathcal{F}^{(2)}$. Using Lemma 12.10 and $\|\text{Ad}(a_{-z})\| \ll e^{C_2 \|z\|}$, we obtain $C_2 > 0$ and $\eta \in (0, 1]$ such that

$$(17.6) \quad d_{\text{Riem}}(\Phi(x, x_-, z), \Phi(y, y_-, z)) \ll e^{C_2 \|z\|} (d_{\tilde{\psi}}(x, y)^\eta + d_{\tilde{\psi}}(x_-, y_-)^\eta)$$

for all $(x, x_-), (y, y_-) \in \Lambda_+ \times \Lambda_-$ and $z \in \mathfrak{a}$.

We also introduce the stable projections used below. Recall the conditional measures $\nu_{+,x}$ from Proposition 16.2. For a Borel function $H : \Lambda_+ \times \Lambda_- \rightarrow \mathbb{C}$ and $k \geq 0$, define its k -step stable projection by

$$(17.7) \quad (\pi_k H)(x) := \int_{\Lambda_-} H(\widehat{T}^k(x, x_-)) d\nu_{+,x}(x_-), \quad x \in \Lambda_+,$$

whenever the integral converges absolutely.

17.2. Parameters and the local spectral expansion. Recall $\lambda \in (0, 1)$, $c_0 > 0$, and $\varepsilon_0 > 0$ from Lemma 16.1, Proposition 12.12, and (15.2), respectively; in particular, $\int_{\Lambda_+} e^{\varepsilon_0 R} d\nu_+ < \infty$. With $C_1, \eta, C_2 > 0$ as in (17.5) and (17.6), choose $\beta, \varepsilon_1 \in (0, 1)$ such that

$$(17.8) \quad 8\beta C_1 C_2 < \varepsilon_0, \quad \varepsilon_1 < \frac{1}{4} \left(\frac{r-1}{2} + L + \frac{3}{2} \right)^{-1},$$

and set

$$(17.9) \quad c_1 := \frac{1}{2} \min \{ \beta\eta, c_0, \varepsilon_0/8 \}, \quad \tau := \lambda^{-\beta\eta} > 1, \quad J := 2L + 5.$$

We apply the results of Sections 15 and 16 with $c = c_1$ in the estimates below.

For $t > 1$, set

$$(17.10) \quad k_t := \left\lceil \left(\frac{r-1}{2} + L + 2J \right) \frac{\log t}{\beta\eta |\log \lambda|} \right\rceil.$$

Then $\tau^{-k_t} := \lambda^{\beta\eta k_t} \leq t^{-((r-1)/2 + L + 2J)}$.

Finally, choose $b_0 > 0$ such that Theorem 15.19 and Proposition 15.20 hold on a neighborhood of $[-b_0, b_0]^r$, and for every $z \in (-b_0, b_0)^r$,

$$(17.11) \quad \left(\frac{(r-1)/2 + L + 2J}{\beta\eta |\log \lambda|} \right) |\log |\kappa_z|| < \frac{1 - \varepsilon_1}{10}, \quad |\kappa_z| \leq \widehat{\mathcal{G}}(z/\sqrt{2}).$$

These choices are possible because $\kappa_0 = 1$ and

$$\log |\kappa_z| = -\frac{1}{2} z^t S z + o(\|z\|^2)$$

by (15.30) and Proposition 15.20. Let $\mathcal{O} := (-b_0, b_0)^r$. By analyticity and Proposition 15.20, $\frac{d}{ds} \log |\kappa_{s\xi}| = -s\xi^t S \xi + O(s^2 \|\xi\|^3)$. Since S is positive definite, decreasing b_0 if necessary makes this derivative nonpositive for $\xi \in \mathcal{O}$ and $0 \leq s \leq 1$. Thus $s \mapsto \log |\kappa_{s\xi}|$ is nonincreasing, and hence

$$(17.12) \quad |\kappa_{s\xi}| \geq |\kappa_\xi| \quad (\xi \in \mathcal{O}, 0 \leq s \leq 1).$$

We use the symmetrized tensor product $\otimes$ and write $*$ for full tensor contraction in the fixed orthonormal coordinates. Apply [23, Lemma A.1] to $\tilde{\kappa}$ with its parameters $K = 2L + 3$ and $J = 3$; the latter is distinct from our $J = 2L + 5$. We now record the expansion of κ_ξ^n that produces the Gaussian terms after Fourier inversion. After decreasing b_0 further if necessary, the following expansion and remainder estimate hold uniformly for $\xi \in [-b_0, b_0]^r$ and all sufficiently large n :

$$(17.13) \quad \kappa_\xi^n = e^{inc_R^2 \xi_1} \widehat{\mathcal{G}}(\sqrt{n} \xi) \sum_{k=0}^{2L+4} \sum_{j=3k}^{2L+4+2k} n^k A_{j,k} * \xi^{\otimes j} + O \left(\widehat{\mathcal{G}}(\sqrt{n/2} \xi) n^{-(L+2)} (1 + \|\sqrt{n} \xi\|^{L_0}) \varphi(\xi) \right),$$

where $A_{j,k}$ are fixed symmetric tensors, $A_{0,0} = 1$, $A_{j,0} = 0$ for $j > 0$, $L_0 \in \mathbb{N}$, and φ is bounded on $\mathcal{O}$ with $\varphi(\xi) \rightarrow 0$ as $\xi \rightarrow 0$.

All these choices are independent of t . For each $\xi \in \mathfrak{a}$, the operator $\mathcal{L}_\xi$ extends to a contraction on $L^1(\nu_+)$, given almost everywhere by the same branch formula. Indeed, $|\mathcal{L}_\xi F| \leq \mathcal{L}_0|F|$ and (15.33) give $\|\mathcal{L}_\xi F\|_{L^1(\nu_+)} \leq \|F\|_{L^1(\nu_+)}$.

17.3. Return decomposition and tail estimates. We express the correlations as finite sums indexed by n , the number of iterations of the return map T . The return-time tail bound in Lemma 17.3 will allow us to discard terms with $n \leq t^{1-\varepsilon_1}$ in the return decomposition. The rough correlation bound in Lemma 17.4 provides the transverse decay needed for the integrable remainder in Theorem 17.1(2).

Let ℓ be either the zero function or $t \mapsto \sqrt{t}$ on $[0, \infty)$. For $t \geq 0$ and $\mathbf{w} \in \ker \psi$, set

$$C_t^\ell(f_1, f_2; \mathbf{w}) := \int_{\Gamma_\rho \setminus G} f_1(x a_{t\mathbf{u} + \ell(t)\mathbf{w}}) f_2(x) d\mathbf{m}_\psi(x).$$

Put $t_+ := 2t/\inf R + 2$. If $x \in \Lambda_+$, $0 \leq s < R(x)$, and $R_n(x) \leq s + t$ for some $n \geq 1$, then $(n-1)\inf R \leq R_n(x) - R(x) < t$. Hence $n < t/\inf R + 1 = t_+/2$.

By Theorem 16.7, (16.11), and (16.7),

$$(17.14) \quad C_t^\ell(f_1, f_2; \mathbf{w}) = \sum_{0 \leq n < t_+} \int_{\Lambda_a} (F_1 \circ \zeta^n)(x, x_-, s + t, v + \ell(t)\mathbf{w}) F_2(x, x_-, s, v) d\mathbf{M}_\psi.$$

We first estimate the total measure of N -step branches $\gamma\Lambda_+$ for which $R_\gamma > t$. By Lemma 12.6, there exists $C_3 > 0$ such that, for every $N \geq 1$, $\gamma \in \mathcal{H}^N$, and $x \in \gamma\Lambda_+$,

$$(17.15) \quad R_\gamma - C_3 \leq R_N(x) \leq R_\gamma.$$

Lemma 17.3. *There exists $C_4 > 0$ such that for every $N \geq 1$ and $t > 0$,*

$$\sum_{\gamma \in \mathcal{H}^N, R_\gamma > t} \nu_+(\gamma\Lambda_+) \leq C_4 N e^{-\varepsilon_0 t/N}.$$

Proof. By (17.15), the definition of R_N , and T -invariance of ν_+ ,

$$\begin{aligned} \sum_{\gamma \in \mathcal{H}^N, R_\gamma > t} \nu_+(\gamma\Lambda_+) &\leq \nu_+(\{R_N > t - C_3\}) \\ &\leq \sum_{j=0}^{N-1} \nu_+(T^{-j}\{R > (t - C_3)/N\}) = N\nu_+(\{R > (t - C_3)/N\}) \\ &\leq N e^{-\varepsilon_0(t-C_3)/N} \int_{\Lambda_+} e^{\varepsilon_0 R} d\nu_+. \end{aligned}$$

The integral is finite by the choice of ε_0 and hence the claim follows. $\square$

The next bound gives transverse decay but does not yet give decay in t . In the proof of Theorem 17.1(2), we combine it with a sufficiently high-order approximation to obtain the required integrable error.

Lemma 17.4. *There exists $a_0 > 0$ such that, for $\ell(t) = \sqrt{t}$, for all $\mathbf{w} \in \ker \psi$ and sufficiently large t ,*

$$t^{(r-1)/2} |C_t^\ell(f_1, f_2; \mathbf{w})| \ll \|f_1\|_\infty \|f_2\|_\infty t^{(r-1)/2+1} e^{-a_0 \min\{\|\mathbf{w}\|^2, \|\mathbf{w}\|\}}.$$

Proof. Apply Theorem 15.19 on the fixed space $\mathcal{F}_{c_0}(\Lambda_+, \mathbb{C})$. By analyticity and $\nu_+(K^\circ) = 0$, we have $\kappa_{-i\theta} = 1 + O(\|\theta\|^2)$ for $\theta \in \ker \psi$ near 0. The spectral decomposition, with uniformly bounded remainder powers on a smaller complex neighborhood of 0, therefore gives $\delta, C > 0$ such that for every $n \geq 0$, $\theta \in \ker \psi$, and $\|\theta\| \leq \delta$,

$$(17.16) \quad \nu_+(e^{\langle \theta, K_n \rangle}) = \nu_+(\mathcal{L}_{-i\theta}^n \mathbf{1}) \leq C e^{Cn\|\theta\|^2}.$$

Suppose $\mathbf{w} \neq 0$ and set $w := \mathbf{w}/\|\mathbf{w}\|$. For $\ell(t) = \sqrt{t}$, the F_1 -factor in the n th integrand of (17.14) is evaluated at $(\widehat{T}^n(x, x_-), s + t - R_n(x), v')$ for $v' := v + \sqrt{t}\mathbf{w} - K_n^\circ(x)$. On the support of this integrand, (17.5) gives $\|v\| + \|v'\| \leq C_1(2 + R(x) + R(T^n x))$. Taking the inner product with w in $\sqrt{t}\mathbf{w} = K_n^\circ(x) + v' - v$, we obtain

$$\sqrt{t}\|\mathbf{w}\| \leq \langle w, K_n(x) \rangle + D(1 + R(x) + R(T^n x)),$$

where $D > 0$ depends only on $\mathcal{K}$. The volume of the initial (s, v) -support is $O((1 + R(x))^r)$.

Choose $q_0 \in (0, 1)$ with $2q_0 \leq \delta$ and $4Dq_0 \leq \varepsilon_0/2$. For $0 \leq q \leq q_0$, two applications of Cauchy–Schwarz, T -invariance of ν_+ , and (17.16) give

$$\begin{aligned} & \nu_+((1 + R)^r e^{q\langle w, K_n \rangle + qD(R + R \circ T^n)}) \\ & \leq \nu_+(e^{2q\langle w, K_n \rangle})^{1/2} \nu_+((1 + R)^{4r} e^{4qDR})^{1/4} \nu_+(e^{4qDR})^{1/4} \ll e^{Cnq^2}. \end{aligned}$$

Here, the last two factors are uniformly bounded by the exponential moment $\nu_+(e^{\varepsilon_0 R}) < \infty$. Summing over $0 \leq n < t_+$ and absorbing the factor e^{qD} into the implied constant yields

$$|C_t^\ell(f_1, f_2; \mathbf{w})| \ll \|f_1\|_\infty \|f_2\|_\infty t \exp(-q\sqrt{t}\|\mathbf{w}\| + Ctq^2).$$

Taking $q = \min\left\{\frac{\|\mathbf{w}\|}{2C\sqrt{t}}, q_0\right\}$ bounds the exponent above by $-c \min\{\|\mathbf{w}\|^2, q_0\sqrt{t}\|\mathbf{w}\|\}$, with $c > 0$ uniform. For $t \geq q_0^{-2}$, this proves the assertion. The case $\mathbf{w} = 0$ follows from the same support and moment bounds with $q = 0$. $\square$

17.4. Decay of transfer operators. To control the Fourier integral outside $\mathcal{O} = (-b_0, b_0)^r$ fixed in Section 17.2, we need decay estimates with constants independent of the frequency parameter $w \in \mathfrak{a}$. We derive these estimates from the quantitative non-integrability established in Theorem 14.1. In the next subsection (Lemma 17.14), we establish the regularity and norm bounds needed to apply this decay estimate to the functions arising from the stable-projection reduction.

Throughout this subsection, we abbreviate $\mathcal{F}(\Lambda_+, \mathbb{C}) := \mathcal{F}_{c_1}(\Lambda_+, \mathbb{C})$. For each $w \in \mathfrak{a} \setminus \{0\}$ we equip this space with the norm

$$(17.17) \quad \|H\|_w := \max \left\{ \|H\|_\infty, \frac{C_0 |H|_{c_1}}{\|w\| + 1} \right\}, \quad H \in \mathcal{F}(\Lambda_+, \mathbb{C}),$$

where $C_0 > 0$ will be chosen below so that (17.21) holds. The main result of this subsection is the following exponential decay estimate.

Proposition 17.5 (Transfer-operator decay away from zero). *There exist $C, \beta_1, \beta_2 > 0$ such that for every $w \in \mathfrak{a}$ with $\|w\| \geq b_0$, $n \in \mathbb{N}$ and $H \in \mathcal{F}(\Lambda_+, \mathbb{C})$,*

$$\|\mathcal{L}_w^n H\|_\infty \leq C e^{-\beta_1 n \|w\|^{-\beta_2}} \|H\|_w.$$

Absorbing the term $\lambda^{c_1 n} \|H\|_\infty$ in (15.21), we obtain $C_{0,1} > 0$ such that

$$(17.18) \quad |\mathcal{L}_w^n H|_{c_1} \leq C_{0,1}(\|w\| + 1) \|H\|_\infty + C_{0,1} \lambda^{c_1 n} |H|_{c_1}$$

for every $w \in \mathfrak{a}$, $n \in \mathbb{N}$, and $H \in \mathcal{F}(\Lambda_+, \mathbb{C})$. We use the following two conventions throughout this subsection. First, for $t > 0$, we write $\mathcal{L}_w^t := \mathcal{L}_w^{\lfloor t \rfloor}$. Second, we denote again by $\|\cdot\|_w$ the operator norm on $\mathcal{F}(\Lambda_+, \mathbb{C})$ induced by (17.17).

For $z \in \mathbb{C} \setminus \{0\}$, let $\arg z := \{\vartheta \in \mathbb{R} : z = |z|e^{i\vartheta}\}$ and let $\text{Arg}z$ be its unique element in $(-\pi, \pi]$. Write Log for the principal logarithm on $\mathbb{C} \setminus (-\infty, 0]$.

Fix $H \in \mathcal{F}(\Lambda_+, \mathbb{C})$. On a cylinder $\gamma\Lambda_+$, $\gamma \in \langle \mathcal{H} \rangle_+$, where H does not vanish, a branch of $\arg H$ is a function $\varphi_\gamma : \gamma\Lambda_+ \rightarrow \mathbb{R}$ satisfying $\varphi_\gamma(y) \in \arg H(y)$ for every $y \in \gamma\Lambda_+$. Set

$$(17.19) \quad N_0(H) := \min \left\{ n \in \mathbb{N} : |H|_{c_1} \lambda^{nc_1} \text{diam}_{d_{\bar{\psi}}}(\Lambda_+)^{c_1} \leq 1/8 \right\}.$$

Lemma 17.6. *Let $n \geq N_0(H)$ and $\gamma \in \mathcal{H}^n$ satisfy $\inf_{\gamma\Lambda_+} |H| \geq 1/4$, and fix $x_\gamma \in \gamma\Lambda_+$. There is a branch φ_γ of $\arg H$ on $\gamma\Lambda_+$ such that*

- (1) $\varphi_\gamma(x_\gamma) = \text{Arg}(H(x_\gamma))$;
- (2) for every $y, y' \in \gamma\Lambda_+$, $|\varphi_\gamma(y) - \varphi_\gamma(y')| \leq 8|H|_{c_1} d_{\bar{\psi}}(y, y')^{c_1}$.

Proof. Put $a := H(x_\gamma)$. By (15.3) and (17.19),

$$|H(y) - a| \leq |H|_{c_1} \lambda^{nc_1} \text{diam}_{d_{\bar{\psi}}}(\Lambda_+)^{c_1} \leq 1/8 \leq |a|/2 \quad (y \in \gamma\Lambda_+).$$

Thus $\varphi_\gamma(y) := \text{Arg}(a) + \Im(\text{Log}(H(y)/a))$ is a branch with the required normalization. The segment joining $H(y)/a$ and $H(y')/a$ lies in $\{z : |z - 1| \leq 1/2\}$, where $|(\text{Log})'(z)| \leq 2$. Integrating along this segment gives

$$|\varphi_\gamma(y) - \varphi_\gamma(y')| \leq 2|a|^{-1} |H(y) - H(y')| \leq 8|H|_{c_1} d_{\bar{\psi}}(y, y')^{c_1}.$$

□

The following elementary lemma will be used in the proof of Lemma 17.8 to control the angular separation of summands whose weighted sum has modulus close to one.

Lemma 17.7. *Let (x_j) be a countable set of positive numbers x_j and $\sum_j x_j \leq 1$, let $y_j \in \mathbb{R}$, and let $\eta \in (0, 1)$. If $|\sum_j x_j e^{iy_j}| > 1 - \eta$, then for all j, k , $d_{\mathbb{S}^1}(y_j, y_k) < \frac{\pi}{\sqrt{2}} \left(\sqrt{\frac{\eta}{x_j}} + \sqrt{\frac{\eta}{x_k}} \right)$ where $d_{\mathbb{S}^1}(u, v) := \inf_{m \in \mathbb{Z}} |u - v + 2\pi m|$ denotes the angular distance in $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$.*

Proof. If θ is an argument of $\sum_j x_j e^{iy_j}$, then $\sum_j x_j (1 - \cos(y_j - \theta)) = \sum_j x_j - \left| \sum_j x_j e^{iy_j} \right| < \eta$. Use $1 - \cos u \geq 2u^2/\pi^2$ for $|u| \leq \pi$ and the triangle inequality on $\mathbb{S}^1$. □

The following is the main ingredient in the proof of Proposition 17.5.

Lemma 17.8. *There exist $a_1 > 0$, $D_1 > 1$, and $D_2 > 0$ such that, for every $w \in \mathfrak{a}$ with $\|w\| \geq b_0$ and every $H \in \mathcal{F}(\Lambda_+, \mathbb{C})$,*

$$(17.20) \quad \left\| \mathcal{L}_w^{\lfloor D_1 \log(3\|w\|/b_0) \rfloor} H \right\|_w \leq (1 - D_2 \|w\|^{-a_1}) \|H\|_w.$$

Proof. Choose $C_0 > 0$ so that $C_{0,1}C_0 \leq 1/4$, and fix $D_{1,1} > 0$ sufficiently large that $C_{0,1}\lambda^{c_1 \lfloor D_{1,1} \log 3 \rfloor} \leq 1/4$. Now fix $w \in \mathfrak{a}$ with $\|w\| \geq b_0$ and $H \in \mathcal{F}(\Lambda_+, \mathbb{C})$ with $\|H\|_w \leq 1$. By (17.17), $\|H\|_\infty \leq 1$ and $|H|_{c_1} \leq (\|w\| + 1)/C_0$. Thus (17.18) gives, for every $n \geq \lfloor D_{1,1} \log(3\|w\|/b_0) \rfloor$,

$$(17.21) \quad \frac{C_0 |\mathcal{L}_w^n H|_{c_1}}{\|w\| + 1} \leq C_{0,1}C_0 + C_{0,1}\lambda^{c_1 n} \leq \frac{1}{2}.$$

Also, $|\mathcal{L}_w^n H| \leq \mathcal{L}_0^n |H| \leq 1$ by (15.14). It remains to find $a_1, D_2 > 0$, independent of w and H , such that

$$(17.22) \quad \|\mathcal{L}_w^n H\|_\infty \leq 1 - D_2 \|w\|^{-a_1}$$

for $n = \lfloor D_1 \log(3\|w\|/b_0) \rfloor$, with $D_1 > 0$ to be chosen below.

Let $a_{1,2} > 0$ be a constant to be specified in Step 3. We divide the proof into three steps. The constants β' in Step 1 and $D_{1,2}$ in Step 2 are chosen after $a_{1,2}$ is fixed.

Step 1. Denote the constants C, ε_1 , and η in Theorem 14.1 by $C_{\text{NI}}, \varepsilon_{\text{NI}}$, and η_{NI} , respectively, so as to distinguish them from the constants fixed in Section 17.2; retain s_1, s_2, s_3, δ_1 from that theorem. We use that theorem with $\langle \cdot, \cdot \rangle_{\mathfrak{u}}$, decreasing η_{NI} if necessary. Indeed, let $\langle \cdot, \cdot \rangle_K$ denote the Killing-form inner product on $\mathfrak{a}$ used in Theorem 14.1. For each $v \in \mathfrak{a}$ with $\|v\|_{\mathfrak{u}} = 1$, choose v^b so that $\langle v^b, Y \rangle_K = \langle v, Y \rangle_{\mathfrak{u}}$ for all $Y \in \mathfrak{a}$. Apply the theorem to $v^b/\|v^b\|_K$; the quantity $\|v^b\|_K$ has a positive lower bound independent of v with $\|v\|_{\mathfrak{u}} = 1$. Choose $s_4 > 1$ such that $s_1 s_4 c_1 > 1$ and for any $w' \in \mathfrak{a}$ with $\|w'\| \geq b_0$, any $n \in \mathbb{N}$, any $\gamma_1, \gamma_2 \in \mathcal{H}^n$ and $x_1, x_2 \in \Lambda_+$ with $d_{\bar{\psi}}(x_1, x_2) \leq \frac{\varepsilon_{\text{NI}}}{2\|3w'/b_0\|^{s_1 s_4}}$, we have

$$(17.23) \quad \begin{aligned} & |(\langle w', K_n(\gamma_1 x_1) \rangle - \langle w', K_n(\gamma_2 x_1) \rangle) - (\langle w', K_n(\gamma_1 x_2) \rangle - \langle w', K_n(\gamma_2 x_2) \rangle)| \\ & \leq 2\|w'\| \cdot C_K \cdot d_{\bar{\psi}}(x_1, x_2)^{c_1} \leq \frac{b_0}{3} 2^{1-c_1} C_K \varepsilon_{\text{NI}}^{c_1} \cdot \|3w'/b_0\|^{1-s_1 s_4 c_1} < \frac{\pi}{2}, \end{aligned}$$

where $C_K > 0$ is the constant in (12.7), with c_0 replaced by c_1 using the finite $d_{\bar{\psi}}$ -diameter of Λ_+ . We may increase s_4 further as required in Step 3. Once $a_{1,2}$ is fixed in Step 3, choose β' sufficiently large that

$$(17.24) \quad \beta' > \max \left\{ s_1 s_4, \frac{a_{1,2} + 1}{c_1} \right\}.$$

Suppose that there exist $\gamma \in \mathcal{H}_{C_{\text{NI}}}$ and $x \in \gamma\Lambda_+$ such that

$$(17.25) \quad B_{\bar{\psi}} \left(x, \frac{\varepsilon_{\text{NI}}}{2\|3w/b_0\|^{\beta'}} \right) \subset \gamma\Delta_0, \quad |H(x)| < 1 - \frac{1}{2\|3w/b_0\|^{a_{1,2}}}.$$

We first prove that (17.25) implies

$$(17.26) \quad \|H\|_{L^1(\nu_+)} \leq 1 - \frac{D_{2,1}}{\|3w/b_0\|^{a_{1,1}}}$$

for some $a_{1,1}, D_{2,1} > 0$ independent of w and H . By (17.17), $|H|_{c_1} \leq (\|w\| + 1)/C_0$. For every $x' \in \Lambda_+$ in the ball in (17.25),

$$(17.27) \quad \begin{aligned} & |H(x')| \leq |H(x)| + |H|_{c_1} d_{\bar{\psi}}(x, x')^{c_1} \\ & \leq 1 - \frac{1}{2\|3w/b_0\|^{a_{1,2}}} + \frac{\|w\| + 1}{C_0} \left(\frac{\varepsilon_{\text{NI}}}{2\|3w/b_0\|^{\beta'}} \right)^{c_1} \leq 1 - \frac{1}{4\|3w/b_0\|^{a_{1,2}}}. \end{aligned}$$

Here x' lies in $\gamma\Lambda_+$ by (12.2), and the last inequality holds for all $\|w\| \geq b_0$ after increasing β' in (17.24). By Corollary 7.9, (5.3), Proposition 11.2, and (15.12),

$$\nu_+ \left(B_{\bar{\psi}} \left(x, \frac{\varepsilon_{\text{NI}}}{2\|3w/b_0\|^{\beta'}} \right) \right) \geq \frac{D_{2,3}}{\|3w/b_0\|^{\beta' a_{1,3}}}$$

for some $a_{1,3} > 1$ and $D_{2,3} \in (0, 1)$. Combining this with (17.27) and $\|H\|_\infty \leq 1$ gives (17.26), with

$$a_{1,1} := a_{1,2} + \beta' a_{1,3}, \quad D_{2,1} := D_{2,3}/4.$$

Step 2. Assuming (17.25), we use the spectral gap of $\mathcal{L}_0$ to turn (17.26) into the uniform supremum-norm bound (17.22). By (17.17), $\|H\|_{c_1} \leq \|H\|_{c_1} \ll \|3w/b_0\|$. Thus Corollary 15.9 and (17.26) give $C > 0$ and $\vartheta \in (0, 1)$, independent of w , H , and n , such that, for every $n \geq 1$,

$$\|\mathcal{L}_w^n H\|_\infty \leq \|\mathcal{L}_0^n H\|_\infty \leq \nu_+(|H|) + \|\mathcal{R}_0^n H\|_\infty \leq 1 - \frac{D_{2,1}}{\|3w/b_0\|^{a_{1,1}}} + C\vartheta^n \|3w/b_0\|.$$

Since $\|3w/b_0\| \geq 3$, we can choose $D_{1,2} > 0$ sufficiently large that, for $n = \lfloor D_{1,2} \log \|3w/b_0\| \rfloor \geq 1$, the last term is strictly less than $D_{2,1}/(2\|3w/b_0\|^{a_{1,1}})$ for every $\|w\| \geq b_0$. Consequently,

$$\|\mathcal{L}_w^{\lfloor D_{1,2} \log \|3w/b_0\| \rfloor} H\|_\infty < 1 - \frac{D_{2,1}}{2\|3w/b_0\|^{a_{1,1}}}.$$

The same bound holds for all further iterates, since $\mathcal{L}_w$ contracts the supremum norm.

Step 3. We now show that (17.25) holds for either H or an iterate $\mathcal{L}_w^{n_0} H$. Step 2 then gives the required supremum-norm bound (17.22). Set $C_h := \inf_{x,y \in \Lambda_+} h(x)/h(y) > 0$, choose $a_{1,6} > 0$ so that $C_h 3^{a_{1,6}} \geq 2$, and put $a_{1,7} := a_{1,2}/2$.

It suffices to find $\gamma \in \mathcal{H}_{C_{\text{NI}}}$ and $x \in \gamma\Lambda_+$ satisfying the ball inclusion in (17.25) such that either H satisfies (17.25) at x , or

$$(17.28) \quad |\mathcal{L}_w^{n_0} H(x)| < 1 - \frac{1}{2\|3w/b_0\|^{a_{1,2}}}, \quad n_0 := \left\lfloor \frac{a_{1,7} - a_{1,6}}{C_{\text{NI}}} \log \|3w/b_0\| \right\rfloor.$$

Choose $a_{1,2}$ sufficiently large that, for all $\|w\| \geq b_0$, we have $n_0 > N_{\text{NI}}$, $n_0 \geq \lfloor D_{1,1} \log \|3w/b_0\| \rfloor$, and

$$(17.29) \quad \frac{\|w\| + 1}{C_0} \lambda^{n_0 c_1} \text{diam}_{d_{\bar{\psi}}}(\Lambda_+)^{c_1} \leq \frac{1}{8}.$$

In the former case, Step 2 applies to H . In the latter, it applies to $\mathcal{L}_w^{n_0} H$, since (17.21) gives $\|\mathcal{L}_w^{n_0} H\|_w \leq 1$.

Applying Theorem 14.1 to the unit vector $\frac{w}{\|w\|} \in \mathfrak{a}$, $b = \|3w/b_0\|^{s_4}$, and n_0 , we obtain $\gamma_1, \gamma_2 \in \mathcal{H}_{C_{\text{NI}}}^{n_0}$ as the two inverse branches, and two points $x_1, x_2 \in \Lambda_+$ satisfying

$$x_2 \in \Lambda_+ \cap \left(B_{\bar{\psi}} \left(x_1, \frac{\varepsilon_{\text{NI}}}{2\|3w/b_0\|^{s_4 s_1}} \right) \setminus B_{\bar{\psi}} \left(x_1, \frac{\delta_1 \varepsilon_{\text{NI}}}{2\|3w/b_0\|^{s_4 s_2}} \right) \right).$$

We also need the ball inclusion in (17.25) at these points. In the proof of Theorem 14.1, x_1 is fixed in the interior of a one-cylinder by (14.16), and $x_2 \rightarrow x_1$ uniformly as $b \rightarrow \infty$ by (14.22). For s_4 sufficiently large, both points lie in a fixed compact subset of that interior. Increasing β' then gives the ball condition at both points. For $\gamma_j x_i$, $i, j \in \{1, 2\}$, the same condition follows directly from (14.2) and $\beta' > s_1 s_4$. These requirements on s_4 are independent of $a_{1,2}$; we now fix s_4 .

Suppose that no pair (γ, x) as above satisfies (17.25) for H . We show that (17.28) holds at x_1 or x_2 . Assume to the contrary that it holds at neither point. Put $F := |H|$. Omitting terms for which $H(\gamma x_1) = 0$, write

$$(\mathcal{L}_w^{n_0} H)(x_1) = \sum_{\gamma \in \mathcal{H}^{n_0}} \frac{e^{-R_{n_0}(\gamma x_1)} F(\gamma x_1) h(\gamma x_1)}{h(x_1)} e^{i\langle w, K_{n_0}(\gamma x_1) \rangle + i \text{Arg} H(\gamma x_1)}.$$

The weights are positive and sum to $\mathcal{L}_0^{n_0} F(x_1) \leq 1$.

Set $\eta := \|3w/b_0\|^{-a_{1,2}}$. Since (17.28) fails at x_1 , $|\mathcal{L}_w^{n_0} H(x_1)| \geq 1 - \eta/2 > 1 - \eta$. Thus Lemma 17.7 applies, and any two terms with weights at least $\|3w/b_0\|^{-a_{1,7}}$ have angular distance at most

$$\sqrt{2\pi} (\|3w/b_0\|^{-a_{1,2}} \|3w/b_0\|^{a_{1,7}})^{1/2} = \frac{\sqrt{2\pi}}{\|3w/b_0\|^{a_{1,8}}}, \quad a_{1,8} := a_{1,2}/4.$$

By this assumption and the ball inclusions at $\gamma_j x_i$, $F(\gamma_j x_i) \geq 1 - \frac{1}{2\|3w/b_0\|^{a_{1,2}}} > \frac{1}{2}$ for all $i, j \in \{1, 2\}$. We claim that

$$(17.30) \quad \frac{e^{-R_{n_0}(\gamma_j x_i)} h(\gamma_j x_i)}{h(x_i)} \geq \frac{2}{\|3w/b_0\|^{a_{1,7}}} \quad (i, j \in \{1, 2\}).$$

Indeed, since $\gamma_j \in \mathcal{H}_{C_{\text{NI}}}^{n_0}$, we have

$$\frac{e^{-R_{n_0}(\gamma_j x_i)} h(\gamma_j x_i)}{h(x_i)} \geq C_h e^{-n_0 C_{\text{NI}}} \geq \frac{C_h}{\|3w/b_0\|^{a_{1,7}-a_{1,6}}}.$$

The choice $C_h 3^{a_{1,6}} \geq 2$ proves (17.30). Including the factor $F(\gamma_j x_i)$, the weights of the selected terms are at least $\|3w/b_0\|^{-a_{1,7}}$. Thus

$$(17.31) \quad d_{\mathbb{S}^1}(\langle w, K_{n_0}(\gamma_1 x_1) \rangle + \text{Arg} H(\gamma_1 x_1), \langle w, K_{n_0}(\gamma_2 x_1) \rangle + \text{Arg} H(\gamma_2 x_1)) \leq \frac{\sqrt{2\pi}}{\|3w/b_0\|^{a_{1,8}}}.$$

The same argument at x_2 gives

$$(17.32) \quad d_{\mathbb{S}^1}(\langle w, K_{n_0}(\gamma_1 x_2) \rangle + \text{Arg} H(\gamma_1 x_2), \langle w, K_{n_0}(\gamma_2 x_2) \rangle + \text{Arg} H(\gamma_2 x_2)) \leq \frac{\sqrt{2\pi}}{\|3w/b_0\|^{a_{1,8}}}.$$

Fix $j \in \{1, 2\}$. By (17.29) and $|H(\gamma_j x_1)| \geq 1/2$, we have $n_0 \geq N_0(H)$ as in (17.19) and $\inf_{\gamma_j \Lambda_+} |H| \geq 3/8$. Thus Lemma 17.6 gives a branch $\varphi_{\gamma_j} : \gamma_j \Lambda_+ \rightarrow \mathbb{R}$ satisfying

$$(17.33) \quad |\varphi_{\gamma_j}(\gamma_j x_1) - \varphi_{\gamma_j}(\gamma_j x_2)| \leq 8|H|_{c_1} d_{\bar{\psi}}(\gamma_j x_1, \gamma_j x_2)^{c_1} \leq \frac{D_{2,7}}{\|3w/b_0\|^{a_{1,9}}},$$

where we use (15.3) and the estimate $|H|_{c_1} \leq (\|w\| + 1)/C_0 \ll \|3w/b_0\|$, absorbing the factor λ^{-c_1} from the integer part in n_0 into $D_{2,7}$, and where $a_{1,9} := -1 - c_1(a_{1,7} - a_{1,6})(\log \lambda)/C_{\text{NI}} > 0$.

Replacing $\text{Arg} H$ by φ_{γ_j} does not change the angular distances in (17.31) and (17.32). The triangle inequality for $d_{\mathbb{S}^1}$ and (17.33) therefore give

$$(17.34) \quad \begin{aligned} & d_{\mathbb{S}^1}(\langle w, K_{n_0}(\gamma_1 x_1) - K_{n_0}(\gamma_2 x_1) \rangle, \langle w, K_{n_0}(\gamma_1 x_2) - K_{n_0}(\gamma_2 x_2) \rangle) \\ & \leq \frac{4\pi}{\|3w/b_0\|^{a_{1,8}}} + \frac{2D_{2,7}}{\|3w/b_0\|^{a_{1,9}}} \leq \frac{D_{2,8}}{\|3w/b_0\|^{a_{1,10}}}. \end{aligned}$$

where $a_{1,10} = \min\{a_{1,8}, a_{1,9}\} = \min\{a_{1,2}/4, -1 - c_1(a_{1,2}/2 - a_{1,6})(\log \lambda)/C_{\text{NI}}\}$.

By (17.23), the phase difference has absolute value less than $\pi/2$, so its distance to $2\pi\mathbb{Z}$ is its absolute value. Thus (17.34) gives

$$|(\langle w, K_{n_0}(\gamma_1 x_1) \rangle - \langle w, K_{n_0}(\gamma_2 x_1) \rangle) - (\langle w, K_{n_0}(\gamma_1 x_2) \rangle - \langle w, K_{n_0}(\gamma_2 x_2) \rangle)| \leq \frac{D_{2,8}}{\|3w/b_0\|^{a_{1,10}}},$$

On the other hand, (14.1) gives

$$|(\langle w, K_{n_0}(\gamma_1 x_1) \rangle - \langle w, K_{n_0}(\gamma_2 x_1) \rangle) - (\langle w, K_{n_0}(\gamma_1 x_2) \rangle - \langle w, K_{n_0}(\gamma_2 x_2) \rangle)|$$

$$\geq \eta_{\text{NI}} \|w\| \left(\frac{\delta_1 \varepsilon_{\text{NI}}}{2 \|3w/b_0\|^{s_4 s_2}} \right)^{s_3}.$$

The lower bound is a fixed positive multiple of $\|3w/b_0\|^{1-s_4 s_2 s_3}$. The constant $D_{2,8}$ can be chosen independently of $a_{1,2}$, whereas $a_{1,10} \rightarrow \infty$ as $a_{1,2} \rightarrow \infty$. Since $\|3w/b_0\| \geq 3$, we may therefore fix $a_{1,2}$ sufficiently large that the upper bound is strictly smaller than the lower bound for every $\|w\| \geq b_0$. Now choose β' and $D_{1,2}$ as in Steps 1 and 2. In either case, Step 2 gives

$$\|\mathcal{L}_w^n H\|_\infty < 1 - \frac{D_{2,1}}{2 \|3w/b_0\|^{a_{1,1}}} \quad (n \geq n_0 + \lfloor D_{1,2} \log \|3w/b_0\| \rfloor).$$

Choose $D_{1,3} \geq D_{1,1} + 1$ with $D_{1,3} \geq D_{1,2} + (a_{1,7} - a_{1,6})/C_{\text{NI}}$. Since $D_{2,1} < 1/4$, this bound and (17.21) prove (17.20), with $a_1 := a_{1,1}$, $D_1 := D_{1,3}$, and $D_2 := \frac{D_{2,1}}{4} \left(\frac{b_0}{3}\right)^{a_1}$. $\square$

Proof of Proposition 17.5. Fix any $w \in \mathfrak{a} \setminus \{0\}$ with $\|w\| \geq b_0$, $n \in \mathbb{N}$, and any $H \in \mathcal{F}(\Lambda_+, \mathbb{C})$ with $\|H\|_w = 1$. Let $a_1 > 0, D_1 > 1, D_2 > 0$ be the constants from Lemma 17.8. Set $n_0 = \lfloor D_1 \log \|3w/b_0\| \rfloor \geq 1$, where the last inequality follows from $D_1 \log 3 \geq 1$ and $\|3w/b_0\| \geq 3$.

Write $n = kn_0 + l$ with $0 \leq l < n_0$. Using Lemma 17.8, $1 - u \leq e^{-u}$ for $u \geq 0$, and the fact that $\|\mathcal{L}_w\|_\infty \leq 1$, we obtain

$$\begin{aligned} \|\mathcal{L}_w^n H\|_\infty &\leq \|\mathcal{L}_w^{kn_0} H\|_\infty \leq \|\mathcal{L}_w^{kn_0} H\|_w \\ &\leq \left(1 - \frac{D_2}{\|w\|^{a_1}}\right)^k \leq e^{-kD_2/\|w\|^{a_1}} \leq e^{D_2 b_0^{-a_1}} e^{-\frac{D_2 n}{n_0 \|w\|^{a_1}}} \leq C e^{-\beta_1 n \|w\|^{-(a_1+1)}}, \end{aligned}$$

where we used $n_0 \ll \|w\|$ for $\|w\| \geq b_0$. This proves the proposition with $\beta_2 := a_1 + 1$. $\square$

17.5. Fourier and stable-projection estimates. We now estimate the Fourier transforms of the pullback functions and their stable projections. These estimates justify Fourier inversion, allow us to apply Proposition 17.5, and control the error introduced by averaging over the contracting Λ_- coordinate after applying $\widehat{T}^k$. To streamline the arguments, we introduce

$$\mathcal{S} := \{F_1, \partial_\alpha^\# F_2^\# : \alpha \text{ a multi-index}\}, \quad C_\infty(F_1) := \|f_1\|_\infty, \quad C_\infty(\partial_\alpha^\# F_2^\#) := \|f_2\|_{C^{|\alpha|}}.$$

By (17.2), (17.3), and the product rule, $\|H\|_\infty \ll C_\infty(H)$ for $H \in \mathcal{S}$. Constants in estimates involving derivatives may depend on their orders. Derivatives of $\widehat{F}$ are taken in ξ ; derivatives of F are taken in its last variable z . For $F \in \{F_1, F_2^\#\}$, write $f = f_1$ or f_2 , respectively.

17.5.1. Bounds for Fourier transforms. Since $F_2^\#(x, x_-, \cdot)$ is smooth and compactly supported, integration by parts $|\alpha|$ times gives the following identity.

Lemma 17.9. *Let α be a multi-index and let $(x, x_-) \in \Lambda_+ \times \Lambda_-$. Then, for every $\xi \in \mathfrak{a}$ with $\xi^\alpha \neq 0$,*

$$\widehat{F_2^\#}(x, x_-, \xi) = \frac{i^{|\alpha|}}{\xi^\alpha} \widehat{\partial_\alpha^\# F_2^\#}(x, x_-, \xi).$$

Lemma 17.10 (Supremum norm). *(1) Let $H \in \mathcal{S}$. Then, for all $(x, x_-) \in \Lambda_+ \times \Lambda_-$, all $\xi \in \mathfrak{a}$ and every integer $k \geq 0$,*

$$\begin{aligned} (17.35) \quad |\widehat{H}(x, x_-, \xi)| &\ll C_\infty(H)(R(x) + 1)^r, \\ |(\pi_k \widehat{H}(\cdot, \xi))(x)| &\ll C_\infty(H)(R(T^k x) + 1)^r. \end{aligned}$$

- (2) For all $(x, x_-) \in \Lambda_+ \times \Lambda_-$, all $\xi \in \mathfrak{a} \setminus \{0\}$, $m \in \mathbb{N}$ and integers $k \geq 0$,
- $$(17.36) \quad |\widehat{F_2^\sharp}(x, x_-, \xi)| \ll_m \|\xi\|^{-m} \|f_2\|_{C^m} (R(x) + 1)^r,$$
- $$|(\pi_k \widehat{F_2^\sharp}(\cdot, \xi))(x)| \ll_m \|\xi\|^{-m} \|f_2\|_{C^m} (R(T^k x) + 1)^r.$$
- (3) For $F \in \{F_1, F_2^\sharp\}$, every multi-index α , all $(x, x_-) \in \Lambda_+ \times \Lambda_-$, $\xi \in \mathfrak{a}$ and $k \geq 0$,
- $$|\partial_\alpha^\alpha \widehat{F}(x, x_-, \xi)| \ll \|f\|_\infty (R(x) + 1)^{r+|\alpha|},$$
- $$|(\pi_k \partial_\alpha^\alpha \widehat{F}(\cdot, \xi))(x)| \ll \|f\|_\infty (R(T^k x) + 1)^{r+|\alpha|}.$$

Proof. By (17.5), the support of every $H(x, x_-, \cdot)$, $H \in \mathcal{S}$, lies in the ball of radius $C_1(R(x) + 1)$. Hence

$$|\widehat{H}(x, x_-, \xi)| \leq \int_{\mathfrak{a}} |H(x, x_-, z)| dz \ll C_\infty(H) (R(x) + 1)^r.$$

Integrating this bound against the probability measure in (17.7) proves (1). For (2), choose j with $|\xi_j| \geq \|\xi\|/\sqrt{r}$ and apply Lemma 17.9 with $\alpha = me_j$, followed by (1).

For (3), differentiation under the Fourier integral gives

$$(17.37) \quad \partial_\alpha^\alpha \widehat{F}(x, x_-, \xi) = i^{|\alpha|} \int_{\mathfrak{a}} e^{i\langle \xi, z \rangle} z^\alpha F(x, x_-, z) dz.$$

On the support, $|z^\alpha| \ll_\alpha (R(x) + 1)^{|\alpha|}$ by (17.5). Apply the argument for (1). $\square$

Lemma 17.11 (L^1 -norm). For all $(x, x_-) \in \Lambda_+ \times \Lambda_-$,

$$(17.38) \quad \|\widehat{F_2^\sharp}(x, x_-, \cdot)\|_{L^1(\mathfrak{a})} \ll \|f_2\|_{C^{r+1}} (R(x) + 1)^r.$$

Proof. On $\|\xi\| \leq 1$ we bound the integrand by (17.35), and on $\|\xi\| \geq 1$ by (17.36) with $m = r + 1$, which gives

$$\|\widehat{F_2^\sharp}(x, x_-, \cdot)\|_{L^1(\mathfrak{a})} \ll \|f_2\|_{C^{r+1}} (R(x) + 1)^r \left(1 + \int_{\|\xi\| \geq 1} \|\xi\|^{-(r+1)} d\xi \right).$$

The last integral is finite because $r + 1 > r = \dim \mathfrak{a}$. $\square$

Lemma 17.12 (Hölder norm). Let $x \in \Lambda_+$, let $x_-, x'_- \in \Lambda_-$, and let $\xi \in \mathfrak{a}$.

(1) For every $H \in \mathcal{S}$,

$$|\widehat{H}(x, x_-, \xi) - \widehat{H}(x, x'_-, \xi)| \ll C(H) e^{\beta C_1 C_2 R(x)} (R(x) + 1)^r d_{\bar{\psi}}(x_-, x'_-)^{\beta\eta},$$

where $C(F_1) := \|f_1\|_{0,\beta}$ and $C(\partial_\alpha^\alpha F_2^\sharp) := \|f_2\|_{|\alpha|,\beta}$.

(2) For $F \in \{F_1, F_2^\sharp\}$ and every multi-index α ,

$$|\partial_\alpha^\alpha \widehat{F}(x, x_-, \xi) - \partial_\alpha^\alpha \widehat{F}(x, x'_-, \xi)| \ll \|f\|_{0,\beta} e^{\beta C_1 C_2 R(x)} (R(x) + 1)^{|\alpha|+r} d_{\bar{\psi}}(x_-, x'_-)^{\beta\eta}.$$

Proof. The cutoff factors in (17.2) and (17.3) do not depend on x_- . Thus (17.6) and the product rule give

$$|H(x, x_-, z) - H(x, x'_-, z)| \ll C(H) e^{\beta C_2 \|z\|} d_{\bar{\psi}}(x_-, x'_-)^{\beta\eta}.$$

If this difference is nonzero, (17.5) gives $\|z\| \leq C_1(R(x) + 1)$. Integrating over this ball proves (1). For (2), use (17.37); the extra factor z^α contributes at most $O_\alpha((R(x)+1)^{|\alpha|})$. $\square$

Lemma 17.13 (Stable projection error). Let $k \geq 0$ be an integer and $(x, x_-) \in \Lambda_+ \times \Lambda_-$.

(1) For every $H \in \mathcal{S}$ and $\xi \in \mathfrak{a}$,

$$|\widehat{H}(\widehat{T}^k(x, x_-), \xi) - (\pi_k \widehat{H}(\cdot, \xi))(x)| \ll C(H) e^{\beta C_1 C_2 R(T^k x)} \tau^{-k} (R(T^k x) + 1)^r.$$

(2) For every integer $m > r$ and $\xi \in \mathfrak{a} \setminus \mathcal{O}$,

$$|\widehat{F_2^\sharp}(\widehat{T}^k(x, x_-), \xi) - (\pi_k \widehat{F_2^\sharp}(\cdot, \xi))(x)| \ll_m \|\xi\|^{-m} \|f_2\|_{m, \beta} e^{\beta C_1 C_2 R(T^k x)} \tau^{-k} (R(T^k x) + 1)^r.$$

(3) For $F \in \{F_1, F_2^\sharp\}$, every multi-index α and $\xi \in \mathfrak{a}$,

$$|\partial_{\mathfrak{a}}^\alpha \widehat{F}(\widehat{T}^k(x, x_-), \xi) - (\pi_k \partial_{\mathfrak{a}}^\alpha \widehat{F}(\cdot, \xi))(x)| \ll \|f\|_{0, \beta} e^{\beta C_1 C_2 R(T^k x)} \tau^{-k} (R(T^k x) + 1)^{|\alpha|+r}.$$

Proof. By Lemma 16.1, two points of a stable fiber have images under $\widehat{T}^k$ at stable distance at most $\lambda^k \text{diam}_{d_{\widehat{\psi}}}(\Lambda_-)$. Integrate the bounds of Lemma 17.12 against $\nu_{+, x}$, using (17.7) and $\lambda^{\beta \eta k} = \tau^{-k}$ from (17.9). This proves (1) and (3). For (2), choose j with $|\xi_j| \geq \|\xi\|/\sqrt{r}$ and apply Lemma 17.9 with $\alpha = me_j$, followed by (1) and the definition of $C(H)$. $\square$

We record a direct estimate for the branch formula after composing with inverse branches. For $\xi \in \mathfrak{a}$, a continuous function F , and a multi-index α , we have

$$(17.39) \quad \|\mathcal{L}_\xi(K^\alpha F)\|_\infty + \frac{|\mathcal{L}_\xi(K^\alpha F)|_{c_1}}{1 + \|\xi\|} \ll_\alpha \sum_{\gamma \in \mathcal{H}} e^{-R_\gamma} (1 + R_\gamma)^{|\alpha|} \|F \circ \gamma\|_{c_1}$$

whenever the sum is finite. Here K^α is the coordinate monomial in the fixed basis of $\mathfrak{a}$. To prove this, apply Proposition 12.12, (15.19), and (15.3) to each summand, using $h, h^{-1} \in \mathcal{F}_{c_1}(\Lambda_+)$ and $\|K\| \ll R$ (Lemma 12.6). The resulting series converges in $\mathcal{F}_{c_1}(\Lambda_+, \mathbb{C})$.

Using the stable projection of (17.7), define

$$(17.40) \quad \begin{aligned} H_{1,t}(x, \xi) &:= \pi_{k_t}(\widehat{F}_1(\cdot, \xi))(x), & H_{2,t}(x, \xi) &:= \pi_{k_t}(\widehat{F_2^\sharp}(\cdot, \xi))(x), \\ \widetilde{H}_{j,t}(x, \xi) &:= e^{i\langle \xi, K_{k_t}(x) \rangle} H_{j,t}(x, \xi) & (j = 1, 2). \end{aligned}$$

After cancellation of the k -step phase, the remaining stable average may still grow with $R(x)$. One further application of $\mathcal{L}_\xi$, using (17.39), gives the norm bounds needed both for the large-frequency contribution and for the normalized expectations in Lemma 17.16.

Lemma 17.14. *For every integer $k \geq 0$ and $\xi \in \mathfrak{a}$, set*

$$H_{k, \xi} := \mathcal{L}_\xi^{k+1} \left(e^{-i\langle \xi, K_k \rangle} \pi_k(\widehat{F_2^\sharp}(\cdot, -\xi)) \right).$$

Then $H_{k, \xi} \in \mathcal{F}_{c_1}(\Lambda_+, \mathbb{C})$. Moreover, the following estimates hold with implied constants independent of k .

(1) For every $\xi \in \mathfrak{a}$ with $\|\xi\| \geq b_0$ and $m \in \mathbb{N}$,

$$\|H_{k, \xi}\|_\xi \ll_m \|\xi\|^{-m} \|f_2\|_{m, \beta},$$

(2) For every $\xi \in \mathcal{O}$ and multi-index α ,

$$\|\partial_{\mathfrak{a}}^\alpha H_{k, \xi}\|_{c_1} \ll_\alpha \|f_2\|_{0, \beta}.$$

For $k = k_t$, the input to $\mathcal{L}_\xi^{k+1}$ is $\widetilde{H}_{2,t}(\cdot, -\xi)$ by (17.40).

Proof. For a Borel function H bounded on each stable fiber, (17.7) and (16.5) give

$$\mathcal{L}_0^k(\pi_k H)(x) = \sum_{\gamma \in \mathcal{H}^k} p_\gamma^{(k)}(x) \int_{\Lambda_-} H(x, \gamma^{-1}x_-) d\nu_{+, \gamma x}(x_-) = \pi_0 H(x).$$

The series converges absolutely because its sum of absolute values is at most $\sup_{y_- \in \Lambda_-} |H(x, y_-)|$. The factors $e^{i\langle \xi, K_k \rangle}$ and $e^{-i\langle \xi, K_k \rangle}$ cancel in the k -step branch formula, giving

$$(17.41) \quad \mathcal{L}_\xi^k \left(e^{-i\langle \xi, K_k \rangle} \pi_k(\widehat{F_2^\sharp}(\cdot, -\xi)) \right) = \mathcal{L}_0^k \pi_k(\widehat{F_2^\sharp}(\cdot, -\xi)) = \pi_0(\widehat{F_2^\sharp}(\cdot, -\xi)).$$

Thus both assertions reduce to estimates for $\mathcal{L}_\xi(\pi_0(\widehat{F_2^\sharp}(\cdot, -\xi)))$, independently of k . Fix an inverse branch $\gamma \in \mathcal{H}$. We estimate $(\pi_0 \widehat{F_2^\sharp}(\cdot, -\xi)) \circ \gamma$. For x, y in a common one-cylinder, Proposition 12.12 gives $|R(\gamma x) - R(\gamma y)| \ll d_{\bar{\psi}}(x, y)^{c_1}$. Use this bound for the cutoff factors in (17.3), and (17.6) with (15.3) for the derivatives of f_2 . Since $c_1 \leq \min\{c_0, \beta\eta\}$, the product rule and (17.5) give, for $H = \partial_a^\delta F_2^\sharp$,

$$|\widehat{H}(\gamma x, x_-, \xi) - \widehat{H}(\gamma y, x_-, \xi)| \ll_\delta C(H) e^{\beta C_1 C_2 R_\gamma} (1 + R_\gamma)^r d_{\bar{\psi}}(x, y)^{c_1}.$$

For the conditional measures, (16.3) applied at $\gamma x, \gamma y$ bounds their contribution by

$$C \|\widehat{H}(\gamma y, \cdot, \xi)\|_\infty d_{\bar{\psi}}(\gamma x, \gamma y)^{c_1}.$$

Together with Lemma 17.10(1) and (15.3), this gives

$$(17.42) \quad \|[\pi_0(\widehat{H}(\cdot, \xi))] \circ \gamma\|_{c_1} \ll_\delta C(H) e^{\beta C_1 C_2 R_\gamma} (1 + R_\gamma)^r.$$

The same argument and (17.37) give

$$(17.43) \quad \|[\pi_0(\partial_a^\alpha \widehat{F_2^\sharp}(\cdot, -\xi))] \circ \gamma\|_{c_1} \ll_\alpha \|f_2\|_{0, \beta} e^{\beta C_1 C_2 R_\gamma} (1 + R_\gamma)^{r+|\alpha|}.$$

For (1), choose j with $|\xi_j| \geq \|\xi\|/\sqrt{r}$ and apply Lemma 17.9 with $\delta = m e_j$. By (17.42) and (17.39),

$$\left\| \mathcal{L}_\xi(\pi_0(\widehat{F_2^\sharp}(\cdot, -\xi))) \right\|_\xi \ll_m \|\xi\|^{-m} \|f_2\|_{m, \beta} \sum_{\gamma \in \mathcal{H}} e^{-(1-\beta C_1 C_2) R_\gamma} (1 + R_\gamma)^r \ll_m \|\xi\|^{-m} \|f_2\|_{m, \beta}.$$

The last inequality follows from $\beta C_1 C_2 < \varepsilon_0$ in (17.8) and (15.2), absorbing the polynomial factor into the remaining exponential margin. Together with (17.41), this proves (1).

For (2), differentiating the Fourier transform at $-\xi$ contributes the factor $(-1)^{|\alpha_2|}$. The product rule gives

$$\partial_a^\alpha \left[\mathcal{L}_\xi(\pi_0(\widehat{F_2^\sharp}(\cdot, -\xi))) \right] = \sum_{\alpha_1 + \alpha_2 = \alpha} \binom{\alpha}{\alpha_1} \mathcal{L}_\xi \left((iK)^{\alpha_1} (-1)^{|\alpha_2|} \pi_0(\partial_a^{\alpha_2} \widehat{F_2^\sharp}(\cdot, -\xi)) \right).$$

Apply (17.39) and (17.43). Since $\mathcal{O}$ is bounded,

$$\left\| \partial_a^\alpha \left[\mathcal{L}_\xi(\pi_0(\widehat{F_2^\sharp}(\cdot, -\xi))) \right] \right\|_{c_1} \ll_\alpha \|f_2\|_{0, \beta} \sum_{\gamma \in \mathcal{H}} e^{-(1-\beta C_1 C_2) R_\gamma} (1 + R_\gamma)^{r+|\alpha|} \ll_\alpha \|f_2\|_{0, \beta},$$

using (15.2) as above. These summable bounds, also at one higher derivative order, justify differentiation in $\mathcal{F}_{c_1}(\Lambda_+, \mathbb{C})$. Equation (17.41) now proves (2). The estimates in (1) and in (2) with $\alpha = 0$ also show that $H_{k, \xi} \in \mathcal{F}_{c_1}(\Lambda_+, \mathbb{C})$ for every $\xi \in \mathfrak{a}$. $\square$

17.5.2. *Reduction to the expanding map.* The following lemma quantifies the error introduced by stable projection, both outside $\mathcal{O}$ and for derivatives of the normalized expectations inside $\mathcal{O}$. We write (N) for the N th derivative with respect to ξ , viewed as a symmetric N -linear form on $\mathfrak{a}$. For $\xi \in \mathfrak{a}$, put

$$(17.44) \quad \Theta_n(\xi, x) := e^{-i\langle \xi, t\mathbf{u} + \ell(t)\mathbf{w} \rangle} e^{i\langle \xi, K_n(x) \rangle}.$$

Lemma 17.15. *Let $k \geq 0$ be an integer.*

(1) *For every $n \in \mathbb{N}$ and $t > 1$,*

$$\begin{aligned} & \int_{\mathfrak{a} \setminus \mathcal{O}} \int_{\Lambda_+ \times \Lambda_-} \Theta_n(\xi, x) \widehat{F}_1(\widehat{T}^n(x, x_-), \xi) \widehat{F}_2^\sharp(x, x_-, -\xi) d\nu^\diamond(x, x_-) d\xi \\ &= \int_{\mathfrak{a} \setminus \mathcal{O}} \int_{\Lambda_+} \Theta_n(\xi, T^k x) (\pi_k \widehat{F}_1(\cdot, \xi)) (T^n x) (\pi_k \widehat{F}_2^\sharp(\cdot, -\xi))(x) d\nu_+(x) d\xi \\ & \quad + O((\|f_1\|_{0,\beta} \|f_2\|_{C^{r+1}} + \|f_1\|_\infty \|f_2\|_{r+1,\beta}) \tau^{-k}). \end{aligned}$$

(2) *For every $n \in \mathbb{N}$, $N \in \mathbb{Z}_{\geq 0}$, and $\xi \in \mathcal{O}$,*

$$\begin{aligned} & \left[\kappa_\xi^{-n} \int_{\Lambda_+ \times \Lambda_-} e^{i\langle \xi, K_n(x) \rangle} \widehat{F}_1(\widehat{T}^n(x, x_-), \xi) \widehat{F}_2^\sharp(x, x_-, -\xi) d\nu^\diamond(x, x_-) \right]^{(N)} \\ &= \left[\kappa_\xi^{-n} \int_{\Lambda_+} e^{i\langle \xi, K_n(T^k x) \rangle} (\pi_k \widehat{F}_1(\cdot, \xi)) (T^n x) (\pi_k \widehat{F}_2^\sharp(\cdot, -\xi))(x) d\nu_+(x) \right]^{(N)} \\ & \quad + O_N(\|f_1\|_{0,\beta} \|f_2\|_{0,\beta} \tau^{-k} n^N |\kappa_\xi^{-n}|), \end{aligned}$$

where the derivatives are taken with respect to ξ .

Proof. We prove (1); the proof of (2) is similar, and we indicate the changes at the end. Fix $n \in \mathbb{N}$, $t > 1$, and $k \geq 0$, and recall the unimodular factor $\Theta_n(\xi, x)$ of (17.44). By the $\widehat{T}$ -invariance of $\nu^\diamond$, the left-hand side equals

$$\int_{\mathfrak{a} \setminus \mathcal{O}} \int_{\Lambda_+ \times \Lambda_-} \Theta_n(\xi, T^k x) \widehat{F}_1(\widehat{T}^{n+k}(x, x_-), \xi) \widehat{F}_2^\sharp(\widehat{T}^k(x, x_-), -\xi) d\nu^\diamond d\xi.$$

Replacing each of the two factors by its stable projection, we obtain the asserted main term together with the two errors

$$\begin{aligned} E_1 &:= \int_{\mathfrak{a} \setminus \mathcal{O}} \int_{\Lambda_+ \times \Lambda_-} \Theta_n(\xi, T^k x) \left(\widehat{F}_1(\widehat{T}^{k+n}(x, x_-), \xi) - (\pi_k \widehat{F}_1(\cdot, \xi))(T^n x) \right) \\ & \quad \cdot \widehat{F}_2^\sharp(\widehat{T}^k(x, x_-), -\xi) d\nu^\diamond d\xi, \\ E_2 &:= \int_{\mathfrak{a} \setminus \mathcal{O}} \int_{\Lambda_+ \times \Lambda_-} \Theta_n(\xi, T^k x) (\pi_k \widehat{F}_1(\cdot, \xi))(T^n x) \\ & \quad \cdot \left(\widehat{F}_2^\sharp(\widehat{T}^k(x, x_-), -\xi) - (\pi_k \widehat{F}_2^\sharp(\cdot, -\xi))(x) \right) d\nu^\diamond d\xi. \end{aligned}$$

It therefore suffices to show that

$$|E_1| \ll \|f_1\|_{0,\beta} \|f_2\|_{C^{r+1}} \tau^{-k}, \quad |E_2| \ll \|f_1\|_\infty \|f_2\|_{r+1,\beta} \tau^{-k}.$$

Since $|\Theta_n| \equiv 1$, Lemma 17.13(1), Fubini's theorem and (17.38) give

$$|E_1| \ll \|f_1\|_{0,\beta} \|f_2\|_{C^{r+1}} \tau^{-k} \int_{\Lambda_+} e^{\beta C_1 C_2 R(T^{k+n}x)} (R(T^{k+n}x) + 1)^r (R(T^k x) + 1)^r d\nu_+(x)$$

$$\begin{aligned} &\ll \|f_1\|_{0,\beta} \|f_2\|_{C^{r+1}} \tau^{-k} \left(\int_{\Lambda_+} e^{2\beta C_1 C_2 R(x)} (R(x) + 1)^{2r} d\nu_+ \int_{\Lambda_+} (R(x) + 1)^{2r} d\nu_+ \right)^{1/2} \\ &\ll \|f_1\|_{0,\beta} \|f_2\|_{C^{r+1}} \tau^{-k}, \end{aligned}$$

where we used Cauchy–Schwarz, T -invariance of ν_+ , the exponential moment fixed in Section 17.2, and $2\beta C_1 C_2 < \varepsilon_0$ from (17.8). The estimate for E_2 follows by the same argument, applying (17.35) to $\pi_k \widehat{F}_1$ and Lemma 17.13(2) to $\widehat{F}_2^\sharp - \pi_k \widehat{F}_2^\sharp$ with $m = r + 1$, which produces $\|f_1\|_\infty \|f_2\|_{r+1,\beta}$ in place of $\|f_1\|_{0,\beta} \|f_2\|_{C^{r+1}}$.

For (2), apply the product rule. Derivatives of the Fourier factors and of their projection errors satisfy Lemma 17.10(3) and Lemma 17.13(3). Derivatives of the cocycle phase insert powers of $K_n \circ T^k$, and, for every $j \in \mathbb{N}$,

$$\|K_n \circ T^k\|^j \ll_j n^{j-1} \sum_{l=0}^{n-1} R(T^{k+l}x)^j$$

by $\|K\| \ll R$ (Lemma 12.6). Hölder's inequality with exponents 4, 4, 2, T -invariance, and the exponential moment of R bound each such integral by $O_j(n^j)$ after removing the factor τ^{-k} ; here $4\beta C_1 C_2 < \varepsilon_0$ by (17.8). Finally, Theorem 15.19 gives analyticity and nonvanishing of κ_ξ on a neighborhood of $\overline{\mathcal{O}}$, and hence $\|D^j(\kappa_\xi^{-n})\| \ll_j n^j |\kappa_\xi|^{-n}$. The total derivative order is N , so the product rule contributes at most n^N . This proves (2). $\square$

17.6. Convergence of the normalized expectations. Near zero frequency, we separate the leading eigenvalue from the Fourier-transformed return correlations. We show that the derivatives of the resulting normalized expectations converge, providing the coefficients for the correlation expansion. Their zeroth-order limit will identify the leading mixing constant.

For $0 \leq N \leq J$, $n \geq 1$, and $\xi \in \mathcal{O}$, set

$$(17.45) \quad \mathcal{A}_{N,n}(\widehat{F}_1, \widehat{F}_2^\sharp; \xi) := \left[\kappa_\xi^{-n} \int_{\Lambda_+ \times \Lambda_-} e^{i\langle \xi, K_n(x) \rangle} \widehat{F}_1(\widehat{T}^n(x, x_-), \xi) \widehat{F}_2^\sharp(x, x_-, -\xi) d\nu^\diamond(x, x_-) \right]^{(N)}.$$

We omit ξ when $\xi = 0$ and write

$$(17.46) \quad \mathcal{A}_N(\widehat{F}_1, \widehat{F}_2^\sharp) := \lim_{n \rightarrow \infty} \mathcal{A}_{N,n}(\widehat{F}_1, \widehat{F}_2^\sharp)$$

for the limits whose existence is established in Lemma 17.16(1). We suppress the two function arguments when no confusion can arise.

Lemma 17.16. (1) For each $N \in \{0, \dots, J\}$ the limit (17.46) exists and

$$(17.47) \quad \left| \mathcal{A}_N(\widehat{F}_1, \widehat{F}_2^\sharp) \right| \ll \|f_1\|_{0,\beta} \|f_2\|_{0,\beta}.$$

Moreover, for every $\bar{L} \in \mathbb{N}$ and $n \in \mathbb{N}$,

$$(17.48) \quad \left| \mathcal{A}_N(\widehat{F}_1, \widehat{F}_2^\sharp) - \mathcal{A}_{N,n}(\widehat{F}_1, \widehat{F}_2^\sharp) \right| \ll_{\bar{L}} n^{-\bar{L}} \|f_1\|_{0,\beta} \|f_2\|_{0,\beta}.$$

(2) For every $\xi \in \mathcal{O}$ and $n \in \mathbb{N}$,

$$|\mathcal{A}_{J,n}(\widehat{F}_1, \widehat{F}_2^\sharp; \xi)| \ll \|f_1\|_{0,\beta} \|f_2\|_{0,\beta} \left(n^{-(\frac{r-1}{2} + L + J)} |\kappa_\xi^{-n}| + n^{2/5} \right).$$

(3) We have

$$\mathcal{A}_0(\widehat{F}_1, \widehat{F}_2^\sharp) = c_R^4 \mathbf{M}_\psi(F_1) \mathbf{M}_\psi(F_2).$$

Proof. Fix $N \in \{0, \dots, J\}$ and abbreviate $\mathcal{A}_{N,n}(\xi) := \mathcal{A}_{N,n}(\widehat{F}_1, \widehat{F}_2^\sharp; \xi)$. For an integer $1 \leq k \leq n-2$, set

$$G_{j,k}(\cdot, \xi) := e^{i\langle \xi, K_k(\cdot) \rangle} \pi_k(\widehat{F}_j(\cdot, \xi)), \quad j = 1, 2,$$

where $\widehat{F}_2 := \widehat{F}_2^\sharp$, so that $G_{j,k_t} = \widetilde{H}_{j,t}$ by (17.40). By Lemma 17.15(2) and the cocycle and transfer-operator duality identities,

$$(17.49) \quad \mathcal{A}_{N,n}(\xi) = \widetilde{\mathcal{A}}_{N,n,k}(\xi) + O\left(\|f_1\|_{0,\beta} \|f_2\|_{0,\beta} n^N \tau^{-k} |\kappa_\xi^{-n}|\right),$$

where

$$\widetilde{\mathcal{A}}_{N,n,k}(\xi) := \left[\kappa_\xi^{-n} \int_{\Lambda_+} G_{1,k}(x, \xi) \mathcal{L}_\xi^{n-k-1} \left(\mathcal{L}_\xi^{k+1} G_{2,k}(\cdot, -\xi) \right) (x) d\nu_+(x) \right]^{(N)}.$$

For every multi-index α with $|\alpha| \leq N$, Lemma 17.10(3), $\|K\| \ll R$ from the proof of Proposition 12.8, and the product rule give

$$|\partial_a^\alpha G_{1,k}(x, \xi)| \ll_N \|f_1\|_\infty (1 + R_k(x))^{|\alpha|} (1 + R(T^k x))^{r+|\alpha|}.$$

Since $\|R_k\|_{L^p(\nu_+)} \leq k\|R\|_{L^p(\nu_+)}$ for $p \geq 1$ by the triangle inequality in $L^p(\nu_+)$ and T -invariance, Cauchy–Schwarz and the exponential moment of R give

$$(17.50) \quad \|\partial_a^\alpha G_{1,k}(\cdot, \xi)\|_{L^1(\nu_+)} \ll_N \|f_1\|_\infty k^{|\alpha|}.$$

We pair this bound with Lemma 17.14(2), which gives

$$\|\partial_a^\alpha (\mathcal{L}_\xi^{k+1} G_{2,k}(\cdot, -\xi))\|_{c_1} \ll_N \|f_2\|_{0,\beta}$$

uniformly in k . These estimates also justify differentiating the expectation above. By Theorem 15.19,

$$(17.51) \quad \kappa_\xi^{-n} \mathcal{L}_\xi^{n-k-1} = \kappa_\xi^{-(k+1)} \mathcal{P}_\xi + \kappa_\xi^{-(k+1)} (\mathcal{R}_\xi / \kappa_\xi)^{n-k-1}.$$

We use this identity on $\mathcal{F}_{c_1}(\Lambda_+, \mathbb{C})$.

Proof of (1). Fix $\bar{L} \in \mathbb{N}$ and take $k_n := \lceil (\bar{L} + N) \log n / \log \tau \rceil$. For sufficiently large n , $1 \leq k_n + 1 \leq n/2$. Use the same k_n for n and an integer $n' \in [n, 2n]$. The projection term in (17.51) is independent of n , so its contributions to $\widetilde{\mathcal{A}}_{N,n,k_n}(0)$ and $\widetilde{\mathcal{A}}_{N,n',k_n}(0)$ cancel. Since $\kappa_0 = 1$, (15.29) and the product rule give

$$\left\| \partial_a^\alpha \left(\kappa_\xi^{-(k_n+1)} (\mathcal{R}_\xi / \kappa_\xi)^{n-k_n-1} \right) \right\|_{\xi=0} \Big|_{c_1 \rightarrow c_1} \ll_N k_n^{|\alpha|} \vartheta^{n-k_n-1}$$

for some $\vartheta \in (0, 1)$. Combining this with (17.50) and Lemma 17.14(2), we obtain

$$|\widetilde{\mathcal{A}}_{N,n,k_n}(0) - \widetilde{\mathcal{A}}_{N,n',k_n}(0)| \ll \|f_1\|_\infty \|f_2\|_{0,\beta} k_n^N \vartheta^{n/2}.$$

Equation (17.49) and the choice of k_n therefore give

$$\sup_{n \leq n' \leq 2n} |\mathcal{A}_{N,n}(0) - \mathcal{A}_{N,n'}(0)| \ll_{\bar{L}} n^{-\bar{L}} \|f_1\|_{0,\beta} \|f_2\|_{0,\beta}.$$

Summing this estimate over successive dyadic intervals proves the Cauchy property and (17.48) for large n . For each fixed n , (17.45), Lemma 17.10(3), and the exponential moment

of R bound $\mathcal{A}_{N,n}(0)$ by $O_n(\|f_1\|_\infty\|f_2\|_\infty)$. This covers the remaining finitely many n and gives (17.47).

Proof of (2). Take $N = J$ and $k := \left\lceil \frac{(r-1)/2+L+2J}{\log \tau} \log n \right\rceil$. For sufficiently large n , $k+1 \leq n/2$ and $n^J \tau^{-k} \leq n^{-((r-1)/2+L+J)}$. Thus (17.49) gives the first term of the asserted bound. By (17.51), analyticity of κ_ξ and $\mathcal{P}_\xi$, and (15.29),

$$\|\partial_\alpha^\alpha(\kappa_\xi^{-n} \mathcal{L}_\xi^{n-k-1})\|_{c_1 \rightarrow c_1} \ll_J k^{|\alpha|} |\kappa_\xi|^{-(k+1)} (1 + \vartheta^{n-k-1}) \quad (|\alpha| \leq J).$$

Using (17.50) and Lemma 17.14(2), we obtain

$$|\tilde{\mathcal{A}}_{J,n,k}(\xi)| \ll \|f_1\|_\infty \|f_2\|_{0,\beta} k^J |\kappa_\xi|^{-(k+1)} \ll \|f_1\|_{0,\beta} \|f_2\|_{0,\beta} n^{2/5}.$$

Indeed, (17.9) and (17.11) give $|\kappa_\xi|^{-(k+1)} \ll n^{(1-\varepsilon_1)/10}$ uniformly on $\mathcal{O}$, and $k^J \ll (\log n)^J \ll n^{1/10}$. The moment bounds used in (1), together with the nonvanishing of κ_ξ on a neighborhood of $\overline{\mathcal{O}}$, cover the remaining finitely many n .

Proof of (3). Take $N = 0$, $\xi = 0$, and $k = \lceil \log n \rceil$ in (17.49) and (17.51). The projection error and the spectral remainder tend to zero by the estimates in (1). By (15.15), the remaining term is

$$\nu_+(\pi_k \widehat{F}_1(\cdot, 0)) \nu_+(\mathcal{L}_0^{k+1} \pi_k \widehat{F}_2^\sharp(\cdot, 0)) = \nu^\diamond(\widehat{F}_1(\cdot, 0)) \nu^\diamond(\widehat{F}_2^\sharp(\cdot, 0)),$$

where we used (17.7), $\widehat{T}$ -invariance of $\nu^\diamond$, and (15.33). Consequently, Fubini's theorem and (16.8) give $\mathcal{A}_0(\widehat{F}_1, \widehat{F}_2^\sharp) = c_R^4 \widetilde{\mathbf{M}}_\psi(F_1) \widetilde{\mathbf{M}}_\psi(F_2^\sharp)$. Absolute integrability follows from (17.5) and the exponential moment of R . Finally, $\widetilde{\mathbf{M}}_\psi(F_1) = \mathbf{M}_\psi(F_1)$. Integrating (17.4) over $\Lambda_\mathbf{a}$ and translating (s, v) by $(-R(x), -K^\circ(x))$ in its second term gives $\widetilde{\mathbf{M}}_\psi(F_2^\sharp) = \mathbf{M}_\psi(F_2)$. This proves (3). $\square$

17.7. Uniform estimates for Gaussian sums. Fourier inversion of the local spectral expansion produces sums of Gaussian derivatives indexed by the number of returns. The following estimates identify the leading transverse Gaussian and bound the remaining terms uniformly in the transverse displacement. They will be used in the final step of the proof of Theorem 17.1(2).

Lemma 17.17. *There is $a > 0$, depending only on c_R, r, S , with the following properties. For every fixed $\varepsilon \in (0, 1)$, uniformly for $t \geq 2$ and $\mathbf{w} \in \ker \psi$,*

$$(17.52) \quad \sum_{n > t^{1-\varepsilon}} n^{-r/2} \mathcal{G} \left(\frac{t - c_R^2 n}{\sqrt{n}}, \frac{\sqrt{t} \mathbf{w}}{\sqrt{n}} \right) = c_R^{r-2} t^{-(r-1)/2} \int_{\mathbb{R}} ds + O_\varepsilon(t^{-r/2} e^{-a \min\{\|\mathbf{w}\|^2, \|\mathbf{w}\|\}}).$$

Moreover, for integers $p, k \geq 0$ with $p \geq 2k$,

$$(17.53) \quad \sum_{n \geq 1} n^{k-(r+p)/2} \left\| \mathcal{G}^{(p)} \left(\frac{t - c_R^2 n}{\sqrt{n}}, \frac{\sqrt{t} \mathbf{w}}{\sqrt{n}} \right) \right\| \ll_{p,k} t^{-(r-1+p-2k)/2} e^{-a \min\{\|\mathbf{w}\|^2, \|\mathbf{w}\|\}}.$$

Proof. For some $a_1 > 0$ depending only on S , Gaussian decay gives $\|\mathcal{G}^{(p)}(z)\| \ll_p e^{-a_1 \|z\|^2}$ for every $p \geq 0$. Put $W := \|\mathbf{w}\|$ and $P_n := ((t - c_R^2 n)/\sqrt{n}, \sqrt{t} \mathbf{w}/\sqrt{n})$. On $t/(2c_R^2) \leq n \leq 2t/c_R^2$,

$$\|P_n\|^2 \gg (n - t/c_R^2)^2/t + W^2.$$

The Gaussian sum over this range is $O(\sqrt{t})$, proving (17.53) there. For $n < t/(2c_R^2)$, $\|P_n\|^2 \gg t + W^2$; for $n > 2t/c_R^2$, $\|P_n\|^2 \gg n + \sqrt{t}W$ by the arithmetic–geometric mean inequality. Both tails satisfy the same bound.

For (17.52), restrict to $\mathcal{U}_t := \{n : |n - t/c_R^2| \leq t^{5/8}\}$. Splitting the Gaussian exponent in the preceding estimates shows that the omitted sum is $O(e^{-ct^{1/4}} e^{-a \min\{\|\mathbf{w}\|^2, \|\mathbf{w}\|\}})$. For large t , this also permits the cutoff $n > t^{1-\varepsilon}$. For $n \in \mathcal{U}_t$, put $y_n := \frac{t - c_R^2 n}{\sqrt{t}}$, $Q_n := (c_R y_n, c_R \mathbf{w})$ and $\mu_n := \sqrt{\frac{t}{c_R^2 n}}$. Then $P_n = \mu_n Q_n$ and $|\mu_n - 1| \ll |y_n|/\sqrt{t}$. The mean value theorem and the Gaussian bounds for $\mathcal{G}$ and its first derivative give

$$\left| n^{-r/2} \mathcal{G}(P_n) - (t/c_R^2)^{-r/2} \mathcal{G}(Q_n) \right| \ll t^{-(r+1)/2} |y_n| (1 + |y_n| + W) e^{-c(y_n^2 + W^2)}.$$

Summing over the grid of spacing $c_R^2/\sqrt{t}$ gives an error $O(t^{-r/2} e^{-c'W^2})$. For an integrable absolutely continuous function g with integrable derivative,

$$\left| b \sum_{y \in \alpha + b\mathbb{Z}} g(y) - \int_{\mathbb{R}} g(y) dy \right| \leq b \int_{\mathbb{R}} |g'(y)| dy.$$

Apply this with $g(y) = \mathcal{G}(c_R y, c_R \mathbf{w})$ and $b = c_R^2/\sqrt{t}$; the omitted grid points are exponentially small. It follows that

$$\sum_{n \in \mathcal{U}_t} \mathcal{G}(Q_n) = \frac{\sqrt{t}}{c_R^2} \int_{\mathbb{R}} \mathcal{G}(c_R y, c_R \mathbf{w}) dy + O(e^{-cW^2}).$$

Multiply by $t^{-r/2} c_R^r$ to obtain (17.52). The bound (17.53) covers the remaining bounded range of t , after enlarging the implied constant. $\square$

17.8. Proof of the local mixing theorem. We now combine the preceding estimates to prove Theorem 17.1. We follow the proof of [23, Theorem 2.2], using [23, Lemmas A.1, A.3 and A.4] for the Taylor expansions and Gaussian sums. Starting from the return decomposition (17.14), we discard small return counts, control frequencies outside $\mathcal{O}$ using stable projection and transfer-operator decay, and expand the contribution from $\mathcal{O}$ using the normalized expectations. Fourier inversion and summation of the resulting Gaussian terms then give the two assertions of the theorem. We retain the upper cutoff $n < t_+$ until Step IV, including after restricting the Fourier integral.

Step I (Reduction to return sums and Fourier transform).

Split (17.14) as

$$C_t^\ell(f_1, f_2; \mathbf{w}) = R_{t,1}^\ell(f_1, f_2; \mathbf{w}) + C_{t,1}^\ell(f_1, f_2; \mathbf{w}),$$

where $R_{t,1}^\ell$ and $C_{t,1}^\ell$ are the sums over $0 \leq n \leq t^{1-\varepsilon_1}$ and $t^{1-\varepsilon_1} < n < t_+$, respectively.

Claim 1. We have

$$C_t^\ell = C_{t,1}^\ell + O(\|f_1\|_\infty \|f_2\|_\infty t^{-((r-1)/2 + L + 1)}).$$

Proof of Claim 1. Set

$$\Lambda_{\text{bad},t} := \{x \in \Lambda_+ : R_{\lfloor t^{1-\varepsilon_1} \rfloor + 1}(x) > t - \frac{\inf R}{10}\}.$$

Fix an integer $0 \leq n \leq t^{1-\varepsilon_1}$. If the integrand of the summand of $R_{t,1}^\ell$ indexed by n is nonzero at $(x, x_-, s, v) \in \Lambda_a$, then $R_{n+1}(x) > s + t \geq t$. Hence $x \in \Lambda_{\text{bad},t}$. Summing over these indices and using (17.5), we obtain

$$(17.54) \quad |R_{t,1}^\ell| \ll t \|f_1\|_\infty \|f_2\|_\infty \int_{\Lambda_+} (R+1)^r \mathbf{1}_{\Lambda_{\text{bad},t}} d\nu_+.$$

By Lemma 17.3, applied with $N = \lfloor t^{1-\varepsilon_1} \rfloor + 1$ and its parameter t replaced by $t - (\inf R)/10$, we have $\nu_+(\Lambda_{\text{bad},t}) \ll te^{-ct^{\varepsilon_1}}$ for some $c > 0$. The exponential moment of R implies $\int_{\Lambda_+} (R+1)^{2r} d\nu_+ < \infty$. Hence Cauchy–Schwarz gives

$$(17.55) \quad \int_{\Lambda_+} (R+1)^r \mathbf{1}_{\Lambda_{\text{bad},t}} d\nu_+ \ll t^{1/2} e^{-ct^{\varepsilon_1}/2}.$$

Substituting this into (17.54) proves the claim. $\square$

Next, set

$$C_{t,2}^\ell(f_1, f_2; \mathbf{w}) := \sum_{t^{1-\varepsilon_1} < n < t_+} \int_{\tilde{\Lambda}_a} (F_1 \circ \zeta^n)(x, x_-, s+t, v + \ell(t)\mathbf{w}) \cdot F_2^\sharp(x, x_-, s, v) d\tilde{\mathbf{M}}_\psi.$$

Claim 2. We have

$$C_{t,1}^\ell = C_{t,2}^\ell + O(\|f_1\|_\infty \|f_2\|_\infty t^{-((r-1)/2+L+1)}).$$

Proof of Claim 2. Use (17.4). In the term containing $F_2^\sharp \circ \zeta$, translate (s, v) by $(-R(x), -K^\circ(x))$ and use $\widehat{T}$ -invariance of $\nu^\diamond$. Since $F_2^\sharp$ vanishes for $s \leq -\inf R/10$, this gives

$$\begin{aligned} & \int_{\Lambda_a} (F_1 \circ \zeta^n)(x, x_-, s+t, v + \ell(t)\mathbf{w}) (F_2^\sharp \circ \zeta)(x, x_-, s, v) d\mathbf{M}_\psi \\ &= \int_{\{-\inf R/10 \leq s < 0\}} (F_1 \circ \zeta^{n-1})(x, x_-, s+t, v + \ell(t)\mathbf{w}) F_2^\sharp(x, x_-, s, v) d\tilde{\mathbf{M}}_\psi. \end{aligned}$$

Using $F_2 = F_2^\sharp + F_2^\sharp \circ \zeta$ on Λ_a , we sum the preceding identity and reindex the contribution from $F_2^\sharp \circ \zeta$. The upper boundary term vanishes by the support cutoff. Thus $C_{t,1}^\ell = C_{t,2}^\ell + E$, where, with $N_1 := \lfloor t^{1-\varepsilon_1} \rfloor$,

$$E := \int_{\{-\inf R/10 \leq s < 0\}} (F_1 \circ \zeta^{N_1})(x, x_-, s+t, v + \ell(t)\mathbf{w}) F_2^\sharp(x, x_-, s, v) d\tilde{\mathbf{M}}_\psi.$$

Where the integrand defining E is nonzero, $R_{N_1+1}(x) > t + s \geq t - \inf R/10$. Thus (17.5) and (17.55) give the claimed bound. $\square$

For fixed n and $(x, x_-) \in \Lambda_+ \times \Lambda_-$, put

$$g(z) := \int_a F_1(\widehat{T}^n(x, x_-), z + z') F_2^\sharp(x, x_-, z') dz'.$$

Since $F_1(\widehat{T}^n(x, x_-), \cdot)$ is bounded and compactly supported and $F_2^\sharp(x, x_-, \cdot) \in \mathcal{C}_c^\infty(\mathfrak{a})$, we have $g \in \mathcal{C}_c^\infty(\mathfrak{a})$. Since $\widehat{g}(\xi) = \widehat{F}_1(\widehat{T}^n(x, x_-), \xi) \widehat{F}_2^\sharp(x, x_-, -\xi)$, Fourier inversion gives

$$g(tu + \ell(t)\mathbf{w} - K_n(x)) = \frac{1}{(2\pi)^r} \int_a e^{-i\langle \xi, tu + \ell(t)\mathbf{w} - K_n(x) \rangle} \widehat{F}_1(\widehat{T}^n(x, x_-), \xi) \widehat{F}_2^\sharp(x, x_-, -\xi) d\xi.$$

Here and below $d\xi$ is the dual measure fixed above. Define $C_{t,3}^\ell$ by summing the integral on the right without the factor $(2\pi)^{-r}$ over $t^{1-\varepsilon_1} < n < t_+$ and integrating against $d\nu^\diamond(x, x_-)$ on $\Lambda_+ \times \Lambda_-$. By Lemma 17.10(1),(2), the integrals are absolutely convergent. Since $d\tilde{\mathbf{M}}_\psi = c_R^{-2} d\nu^\diamond dz$,

$$(2\pi)^r c_R^2 C_{t,2}^\ell = C_{t,3}^\ell.$$

Step II (Reduction to integration over a compact domain).

Recall $\mathcal{O}$ from Section 17.2 and the oscillatory factor Θ_n from (17.44). We isolate the contribution from frequencies near zero, which will produce the Gaussian term and its higher-order corrections:

$$C_{t,4}^\ell := \sum_{t^{1-\varepsilon_1} < n < t_+} \int_{\mathcal{O}} \int_{\Lambda_+ \times \Lambda_-} \Theta_n(\xi, x) \widehat{F}_1(\widehat{T}^n(x, x_-), \xi) \widehat{F}_2^\sharp(x, x_-, -\xi) d\nu^\diamond d\xi,$$

and set

$$R_{t,2}^\ell := C_{t,3}^\ell - C_{t,4}^\ell.$$

Thus $R_{t,2}^\ell$ is the contribution from frequencies outside $\mathcal{O}$.

Claim 3. There exists $m \in \mathbb{N}$ such that

$$C_{t,3}^\ell = C_{t,4}^\ell + O(\|f_1\|_{0,\beta} \|f_2\|_{m,\beta} t^{-((r-1)/2+L+1)}).$$

Claim 3 will be a consequence of Claim 4.

Let $R_{t,3}^\ell$ denote the expression obtained from $R_{t,2}^\ell$ by composing with $\widehat{T}^{k_t}$ and replacing both factors by their stable projections. By Lemma 17.15(1), this replacement introduces an error of $O(\|f_1\|_{0,\beta} \|f_2\|_{r+1,\beta} \tau^{-k_t})$ for each n in the summation range. Summing over $O(t)$ summands and using (17.10) gives

$$(17.56) \quad R_{t,2}^\ell = R_{t,3}^\ell + O(\|f_1\|_{0,\beta} \|f_2\|_{r+1,\beta} t^{-((r-1)/2+L+1)}).$$

The transfer-operator decay obtained from quantitative non-integrability, combined with the Fourier decay of $F_2^\sharp$, gives the following estimate:

Claim 4. For sufficiently large $m \in \mathbb{N}$,

$$|R_{t,3}^\ell| \ll \|f_1\|_\infty \|f_2\|_{m,\beta} t^{-((r-1)/2+L+1)}.$$

Proof of Claim 4. The cocycle identity $K_n \circ T^{k_t} = K_n - K_{k_t} + K_{k_t} \circ T^n$ and transfer-operator duality give

$$(17.57) \quad R_{t,3}^\ell = \sum_{t^{1-\varepsilon_1} < n < t_+} \int_{\mathfrak{a} \setminus \mathcal{O}} e^{-i\langle \xi, tu + \ell(t)w \rangle} \cdot \int_{\Lambda_+} \widetilde{H}_{1,t}(x, \xi) \mathcal{L}_\xi^{n-k_t-1} (\mathcal{L}_\xi^{k_t+1} \widetilde{H}_{2,t}(\cdot, -\xi))(x) d\nu_+(x) d\xi.$$

Let $m > r$ be an integer, to be chosen below. Set $D_1 := \|f_1\|_\infty \|f_2\|_{m,\beta}$, and let $\beta_1, \beta_2 > 0$ be as in Proposition 17.5. For $\xi \in \mathfrak{a} \setminus \mathcal{O}$, put $B_{t,\xi} := \mathcal{L}_\xi^{k_t+1} \widetilde{H}_{2,t}(\cdot, -\xi)$. Since $k_t = O(\log t)$ and $n > t^{1-\varepsilon_1}$, we have $n - k_t - 1 \geq n/2$ throughout the summation range for sufficiently large t . By Proposition 17.5 and Lemma 17.14(1),

$$\|\mathcal{L}_\xi^{n-k_t-1} B_{t,\xi}\|_\infty \ll e^{-\beta_1 n / (2\|\xi\|^{\beta_2})} \|B_{t,\xi}\|_\xi \ll e^{-\beta_1 n / (2\|\xi\|^{\beta_2})} \|\xi\|^{-m} \|f_2\|_{m,\beta}.$$

Moreover, (17.35), T -invariance of ν_+ , and the exponential moment of R give

$$\int_{\Lambda_+} |\widetilde{H}_{1,t}(x, \xi)| d\nu_+(x) \ll \|f_1\|_\infty \int_{\Lambda_+} (R \circ T^{k_t} + 1)^r d\nu_+ \ll \|f_1\|_\infty.$$

Applying these bounds to (17.57), we obtain

$$|R_{t,3}^\ell| \ll D_1 \sum_{t^{1-\varepsilon_1} < n < t_+} \int_{\mathfrak{a} \setminus \mathcal{O}} e^{-\beta_1 n / (2\|\xi\|^{\beta_2})} \|\xi\|^{-m} d\xi.$$

There are $O(t)$ summands, and $\mathfrak{a} \setminus \mathcal{O} \subset \{\xi : \|\xi\| \geq b_0\}$. Using $n > t^{1-\varepsilon_1}$ and polar coordinates therefore gives

$$|R_{t,3}^\ell| \ll D_1 t \int_{b_0}^\infty e^{-ct^{1-\varepsilon_1}s^{-\beta_2}} s^{r-1-m} ds.$$

With the substitution $s = t^{(1-\varepsilon_1)/\beta_2} y$, the integral on the right becomes

$$t^{-(1-\varepsilon_1)(m-r)/\beta_2} \int_{b_0 t^{-(1-\varepsilon_1)/\beta_2}}^\infty e^{-cy^{-\beta_2}} y^{r-1-m} dy.$$

The last integral is bounded uniformly in t : the exponential ensures integrability near zero, while $m > r$ ensures integrability at infinity. Hence $|R_{t,3}^\ell| \ll D_1 t^{1-(1-\varepsilon_1)(m-r)/\beta_2}$. Choosing m to be any number larger than $\max \left\{ r+1, \left\lceil r + \frac{\beta_2}{1-\varepsilon_1} \left(\frac{r-1}{2} + L + 2 \right) \right\rceil \right\}$ gives the required bound. $\square$

Combining (17.56) and Claim 4 proves Claim 3.

Step III (Expansion of the normalized expectations).

Recall the normalized derivatives $\mathcal{A}_{N,n}$ and their limits $\mathcal{A}_N$ from (17.45) and (17.46). We suppress their two function arguments in this step. Set $D_2 := \|f_1\|_{0,\beta} \|f_2\|_{0,\beta}$ where β is fixed in (17.8). Taylor's formula at $\xi = 0$ gives, for every $n \geq 1$ and $\xi \in \mathcal{O}$,

$$\begin{aligned} \mathcal{A}_{0,n}(\xi) &= \kappa_\xi^{-n} \int_{\Lambda_+ \times \Lambda_-} e^{i\langle \xi, K_n(x) \rangle} \widehat{F}_1(\widehat{T}^n(x, x_-), \xi) \widehat{F}_2^\sharp(x, x_-, -\xi) d\nu^\diamond(x, x_-) \\ &= \sum_{N=0}^{J-1} \frac{\mathcal{A}_{N,n}(0) * \xi^{\otimes N}}{N!} + O \left(\sup_{0 \leq s \leq 1} |\mathcal{A}_{J,n}(s\xi)| \|\xi\|^J \right). \end{aligned}$$

By Lemma 17.16(1),(2) and (17.12), this equals

$$(17.58) \quad \sum_{N=0}^{J-1} \frac{\mathcal{A}_N * \xi^{\otimes N}}{N!} + O \left(D_2 \left(n^{-((r-1)/2+L+J)} |\kappa_\xi|^{-n} + n^{-\bar{L}} + n^{2/5} \|\xi\|^J \right) \right),$$

where $\bar{L} \geq (r-1)/2 + L + 5/2$ is a fixed integer. Multiplying (17.58) by κ_ξ^n and substituting the expansion (17.13), we obtain, using (17.11) and (17.47),

$$\begin{aligned} &\int_{\Lambda_+ \times \Lambda_-} e^{i\langle \xi, K_n(x) \rangle} \widehat{F}_1(\widehat{T}^n(x, x_-), \xi) \widehat{F}_2^\sharp(x, x_-, -\xi) d\nu^\diamond(x, x_-) \\ &= e^{inc_R^2 \xi_1} \widehat{\mathcal{G}}(\sqrt{n} \xi) \sum_{N=0}^{J-1} \sum_{k=0}^{2L+4} \sum_{j=3k}^{2L+4+2k} \frac{n^k}{N!} (\mathcal{A}_N \otimes A_{j,k}) * \xi^{\otimes(N+j)} + E_n(\xi). \end{aligned}$$

where, by the substitution $\xi \mapsto \xi/\sqrt{n}$ and $J = 2L + 5$,

$$\int_{\mathcal{O}} |E_n(\xi)| d\xi \ll D_2 \left(n^{-((r-1)/2+L+J)} + n^{-((r-1)/2+L+5/2)} \right).$$

Summing and using (17.8), we obtain

$$C_{t,4}^\ell = C_{t,5}^\ell + O(D_2 t^{-((r-1)/2+L+1)}),$$

where

$$C_{t,5}^\ell := \sum_{N=0}^{J-1} \frac{1}{N!} \sum_{k=0}^{2L+4} \sum_{j=3k}^{2L+4+2k} \sum_{t^{1-\varepsilon_1} < n < t_+} \mathcal{I}_n^{N,k,j}$$

and

$$\mathcal{I}_n^{N,k,j} := n^k \int_{\mathcal{O}} e^{-i\langle \xi, (t-nc_R^2)\mathbf{u} + \ell(t)\mathbf{w} \rangle} \widehat{\mathcal{G}}(\sqrt{n}\xi) (\mathcal{A}_N \otimes A_{j,k}) * \xi^{\otimes(N+j)} d\xi.$$

Step IV (Fourier inversion).

After the change of variables $\eta = \sqrt{n}\xi$, Gaussian decay and (17.47) allow us to extend the integration domain from $\sqrt{n}\mathcal{O}$ to $\mathfrak{a}$, with error $O(D_2 e^{-cn})$. Fourier inversion then gives

$$\mathcal{I}_n^{N,k,j} = (2\pi)^r i^{N+j} n^{k-(r+N+j)/2} \mathcal{G}^{(N+j)} \left(\frac{t-nc_R^2}{\sqrt{n}}, \frac{\ell(t)\mathbf{w}}{\sqrt{n}} \right) * (\mathcal{A}_N \otimes A_{j,k}) + O(D_2 e^{-cn}).$$

For $n \geq t_+$, we have $c_R^2 n \geq 2t$, and hence $\frac{|t-nc_R^2|}{\sqrt{n}} \geq \frac{c_R^2}{2} \sqrt{n}$. The Gaussian and its derivatives therefore decay exponentially in n on this range. Thus the terms with $n \geq t_+$ contribute $O(D_2 e^{-ct})$, and we may extend the sum to all $n > t^{1-\varepsilon_1}$. Consequently,

$$C_{t,5}^\ell = C_{t,6}^\ell + O(D_2 t^{-((r-1)/2+L+1)}),$$

where

$$(17.59) \quad \begin{aligned} C_{t,6}^\ell &:= (2\pi)^r \sum_{N=0}^{J-1} \sum_{k=0}^{2L+4} \sum_{j=3k}^{2L+4+2k} \frac{i^{N+j}}{N!} \sum_{n>t^{1-\varepsilon_1}} n^{k-(r+N+j)/2} \\ &\quad \cdot \mathcal{G}^{(N+j)} \left(\frac{t-nc_R^2}{\sqrt{n}}, \frac{\ell(t)\mathbf{w}}{\sqrt{n}} \right) * (\mathcal{A}_N \otimes A_{j,k}). \end{aligned}$$

Combining Steps I–IV, for some integer $m \geq r+1$ we have

$$(17.60) \quad \left| t^{(r-1)/2} C_t^\ell - \frac{t^{(r-1)/2}}{(2\pi)^r c_R^2} C_{t,6}^\ell \right| \ll \|f_1\|_{0,\beta} \|f_2\|_{m,\beta} t^{-(L+1)},$$

uniformly in $\mathbf{w} \in \ker \psi$.

Step V (Summing over n).

Case 1: $\ell(t) = 0$. For an integer $p \geq 0$, write $\mathcal{G}^{(p)} := D^p \mathcal{G}$ for the p -th total derivative, with $\mathcal{G}^{(0)} = \mathcal{G}$. For $\sigma \in \mathbb{R}$, define

$$h_{p,\sigma}(z_1, z_2) := z_2^\sigma D^p \mathcal{G} \left(\frac{c_R z_1}{\sqrt{z_2}} \mathbf{u} \right), \quad (z_1, z_2) \in \mathbb{R} \times (0, \infty).$$

Set

$$\mathcal{V} := \left\{ (N, k, j, q) \in \mathbb{Z}_{\geq 0}^4 : \begin{array}{l} N \leq J-1, \ k \leq 2L+4, \\ 3k \leq j \leq 2L+4+2k, \ N+j-2k+q \leq 2L \end{array} \right\}.$$

Apply [23, Lemma A.3] with $Q = 2L + 1$ to each Gaussian sum. The lower cutoff can be changed to any fixed positive multiple of t below t/c_R^2 at an exponentially small cost. By [23, Lemma A.4], the terms with $N + j + q$ odd vanish. Thus

$$(17.61) \quad C_{t,6}^\ell = (2\pi)^r \sum_{(N,k,j,q) \in \mathcal{V}} \frac{i^{N+j} (\sqrt{t}/c_R)^{-((r-1)+N+j-2k+q)}}{N!q! c_R^{q+1}} \cdot \left(\int_{\mathbb{R}} \partial_2^q h_{N+j,k-(N+j+r)/2}(s,1) (-s)^q ds \right) * (\mathcal{A}_N \otimes A_{j,k}) + O(D_2 t^{-((r-1)/2+L+1)}).$$

The remaining omitted terms have $N + j - 2k + q \geq 2L + 2$ and are absorbed into the error. After dividing by $(2\pi)^r c_R^2$ and multiplying by $t^{(r-1)/2}$, define, for $0 \leq m \leq L$,

$$\mathfrak{c}_m(f_1, f_2) := \sum_{\substack{(N,k,j,q) \in \mathcal{V} \\ N+j-2k+q=2m}} \frac{i^{N+j} c_R^{r-4+N+j-2k}}{N!q!} \cdot \left(\int_{\mathbb{R}} \partial_2^q h_{N+j,k-(N+j+r)/2}(s,1) (-s)^q ds \right) * (\mathcal{A}_N \otimes A_{j,k}).$$

With these definitions, (17.61) becomes

$$\frac{t^{(r-1)/2}}{(2\pi)^r c_R^2} C_{t,6}^\ell = \sum_{m=0}^L \mathfrak{c}_m(f_1, f_2) t^{-m} + O(D_2 t^{-(L+1)}).$$

Combining this with (17.60) gives the correlation expansion in Theorem 17.1(1). Uniqueness of the coefficients of an asymptotic expansion makes these definitions compatible as $\mathcal{K}$ and L vary. They therefore define bilinear forms on $\mathcal{C}_c^\infty(\Gamma_\rho \backslash G)^M \times \mathcal{C}_c^\infty(\Gamma_\rho \backslash G)^M$.

Since each defining sum is finite, (17.47) gives

$$|\mathfrak{c}_m(f_1, f_2)| \ll_{\mathcal{K},L} \|f_1\|_{0,\beta} \|f_2\|_{0,\beta} \quad (0 \leq m \leq L)$$

for $f_1, f_2 \in \mathcal{C}_\mathcal{K}^\infty$. This implies the coefficient bounds in Theorem 17.1(1).

By Lemma 17.16(3), $\mathcal{A}_0 = c_R^4 \mathbf{M}_\psi(F_1) \mathbf{M}_\psi(F_2)$. The term $N = k = j = q = 0$ in (17.61) is therefore

$$(2\pi)^r c_R^{r+2} t^{-(r-1)/2} \left(\int_{\mathbb{R}} \mathcal{G}(c_R s, 0) ds \right) \mathbf{M}_\psi(F_1) \mathbf{M}_\psi(F_2).$$

Since $\mathbf{M}_\psi(F_i) = \mathbf{m}_\psi(f_i)$ by Theorem 16.7, division by $(2\pi)^r c_R^2$ proves the stated leading coefficient and completes part (1).

Case 2: $\ell(t) = \sqrt{t}$. For this, carry out Steps I–IV with $L = \lceil (r-1)/2 + 1 \rceil$ and put

$$P_t(\mathbf{w}) := \frac{t^{(r-1)/2}}{(2\pi)^r c_R^2} C_{t,6}^\ell(f_1, f_2; \mathbf{w}).$$

Since $N + j - 2k \geq 1$ unless $N = k = j = 0$, (17.53) bounds every nonleading term by $O(D_2 t^{-1/2} e^{-a_1 \min\{\|\mathbf{w}\|^2, \|\mathbf{w}\|\}})$. Applying Lemma 17.17 to the leading term and using Lemma 17.16(3), we obtain

$$(17.62) \quad P_t(\mathbf{w}) = \mathfrak{c}_\mathbf{w} \mathbf{m}_\psi(f_1) \mathbf{m}_\psi(f_2) + O(D_2 t^{-1/2} e^{-a_1 \min\{\|\mathbf{w}\|^2, \|\mathbf{w}\|\}}),$$

where $a_1 > 0$ is uniform. In particular, $|P_t(\mathbf{w})| \ll D_2 e^{-a_1 \min\{\|\mathbf{w}\|^2, \|\mathbf{w}\|\}}$. Choose $a_2 := \min\{a_0, a_1\}$. By Lemma 17.4, the triangle inequality gives

$$|t^{(r-1)/2} C_t^\ell - P_t(\mathbf{w})| \ll D_2 t^{(r-1)/2+1} e^{-a_2 \min\{\|\mathbf{w}\|^2, \|\mathbf{w}\|\}}.$$

Take the geometric mean of this bound and (17.60). Since $L \geq (r-1)/2 + 1$, we obtain

$$\begin{aligned} |t^{(r-1)/2} C_t^\ell - P_t(\mathbf{w})| &\ll \|f_1\|_{0,\beta} \|f_2\|_{m,\beta} t^{((r-1)/2-L)/2} e^{-(a_2/2) \min\{\|\mathbf{w}\|^2, \|\mathbf{w}\|\}} \\ &\ll \|f_1\|_{m,\beta} \|f_2\|_{m,\beta} t^{-1/2} e^{-(a_2/2) \min\{\|\mathbf{w}\|^2, \|\mathbf{w}\|\}}. \end{aligned}$$

Combining this with (17.62) proves (17.1) with $a = a_2/2$. This completes the proof of Theorem 17.1. $\square$

18. HAAR LOCAL MIXING AND ORBITAL COUNTING

We deduce Haar local mixing and orbital counting from BMS local mixing, using the arguments of Edwards–Lee–Oh [25]. We first specify the measure normalizations.

18.1. Measure normalizations. We use the Hopf coordinates and Lebesgue measure fixed in Section 3.6. Throughout this part, $\mathbf{m}_\psi^{\text{BMS}}$ denotes the BMS measure induced by (3.9), before the normalization (3.10).

Let m_o be the K -invariant probability measure on $\mathcal{F}$, which is (G, ϱ) -conformal, where $\varrho = \sum_{\alpha \in \Phi^+} \dim(\mathfrak{g}_\alpha) \alpha$. Define Haar and Burger–Roblin measures in Hopf coordinates by

$$\begin{aligned} d\tilde{\mathbf{m}}^{\text{Haar}}(g) &= e^{\varrho(\beta_{g^+}(e,g)) + \varrho(\beta_{g^-}(e,g))} dm_o(g^+) dm_o(g^-) d\text{Leb}_{\mathfrak{a}}, \\ d\tilde{\mathbf{m}}_\psi^{\text{BR}}(g) &= e^{\varrho(\beta_{g^+}(e,g)) + \psi(\beta_{g^-}(e,g))} dm_o(g^+) d\nu_\psi(g^-) d\text{Leb}_{\mathfrak{a}}, \\ d\tilde{\mathbf{m}}_\psi^{\text{BR}*}(g) &= e^{\psi(\beta_{g^+}(e,g)) + \varrho(\beta_{g^-}(e,g))} d\nu_\psi(g^+) dm_o(g^-) d\text{Leb}_{\mathfrak{a}}. \end{aligned}$$

We lift these measures from G/M to G using Haar probability measure on M , and denote their descents to $\Gamma_\rho \backslash G$ by the corresponding untilded symbols. For $\psi = \psi_u$, write $m_u^{\text{BR}} := \mathbf{m}_{\psi_u}^{\text{BR}}$ and $m_u^{\text{BR}*} := \mathbf{m}_{\psi_u}^{\text{BR}*}$. Uniqueness of tangent forms gives $\psi_{i(u)} = \psi_u \circ i$. Thus the Haar local mixing limit involves $m_{i(u)}^{\text{BR}}$ and $m_u^{\text{BR}*}$ in the notation of the introduction.

After rescaling the BMS measure to $\mathbf{m}_\psi^{\text{BMS}}$, write the coefficient in Theorem 17.1(2) as $c_\psi e^{-Q_\psi(w)/2}$, where $c_\psi > 0$ and $w \in \ker \psi$.

18.2. Local mixing.

Proof of Theorem 4.1. Fix $u \in \text{int } \mathcal{L}_\rho$. By [14, Theorem 10.1], ψ_u is positive on $\mathcal{L}_\rho \setminus \{0\}$, so Theorem 17.1 applies with $\psi = \psi_u$. Since $u = \psi_\rho(u) \mathbf{u}_{\psi_u}$ by Theorem 15.11, replacing t by $\psi_\rho(u)t$ and w by $w/\sqrt{\psi_\rho(u)}$ gives Theorem 1.1, with $Q_u = \psi_\rho(u)^{-1} Q_{\psi_u}$. Approximation by smooth functions, with error controlled by fixed smooth cutoffs, extends the limit to continuous compactly supported functions. Theorem 1.3 follows from [25, Proposition 6.8]. The convergence is locally uniform in w : on compact sets of transverse parameters, all simple roots of $tu + \sqrt{t}w$ tend to infinity uniformly, so the error terms in that proof are uniform.

For the transverse bound, the upper comparisons in [25, Propositions 5.3, 6.1, and 6.8] bound the Haar correlation by BMS correlations of fixed compactly supported functions at the same A -parameter. These comparisons hold whenever $tu + \sqrt{t}w \in \mathfrak{a}^+$. Dominating the functions by smooth functions and applying Theorem 17.1(2) gives the asserted decay. For functions supported in a fixed compact set, domination by their supremum norms times a fixed smooth majorant gives the stated dependence on $\|f_1\|_\infty \|f_2\|_\infty$. Finally, the time rescaling and the measure normalizations above give $\kappa_u c_u = \psi_\rho(u)^{-(r-1)/2} c_{\psi_u}$. $\square$

Remark 18.1. The uniformity in w in Theorem 1.1(2) controls correlations along other rays: for every $v \in \text{int } \mathcal{L}_\rho \setminus \mathbb{R}_{>0}u$, there exists $b = b(u, v) > 0$ such that, for all $f_1, f_2 \in \mathcal{C}_c^\infty(X_\rho)$,

$$\int_{X_\rho} f_1(xa_{tv})f_2(x) dm_u^{\text{BMS}}(x) = O_{u,v,f_1,f_2}\left(t^{-r/2}e^{-b\sqrt{t}}\right) \quad (t \rightarrow +\infty).$$

Proof of Corollary 1.4. The BMS local mixing asymptotic implies u -balancedness in the sense of [10]. The conclusion follows from [10, Theorem 1.4]. $\square$

18.3. Equidistribution of expanding horospheres. Our local mixing theorem also gives equidistribution of translates of maximal horospheres. Let $\check{N}$ be the maximal horospherical subgroup satisfying $a_{tu}na_{-tu} \rightarrow e$ as $t \rightarrow \infty$ for $n \in \check{N}$ and $u \in \text{int } \mathfrak{a}^+$. For $x = [g] \in \Gamma_\rho \backslash G$, the measure $\mu_{x\check{N},u}^{\text{PS}}$ on $\check{N}$ is defined by $d\mu_{x\check{N},u}^{\text{PS}}(n) = e^{\psi_u(\beta_{gnP}(e, gn))} d\nu_{\psi_u}(gnP)$. We normalize Haar measure dn on $\check{N}$ compatibly with the measures in Theorem 1.3, as in [25, Section 4]. Applying [25, Proposition 6.1] to Theorem 1.1 gives the following.

Corollary 18.2. *For every $u \in \text{int } \mathcal{L}_\rho$, $x \in \Gamma_\rho \backslash G$, $f \in C_c(\Gamma_\rho \backslash G)^M$, and $\phi \in C_c(\check{N})$, we have*

$$\lim_{t \rightarrow \infty} t^{\frac{r-1}{2}} e^{t(\varrho - \psi_u)(u)} \int_{\check{N}} f(xna_{tu})\phi(n) dn = \kappa_u c_u m_{i(u)}^{\text{BR}}(f) \mu_{x\check{N},u}^{\text{PS}}(\phi),$$

where κ_u and c_u are as in Theorem 1.3.

Approximation by smooth functions, using fixed smooth majorants, extends the limit to continuous compactly supported functions.

18.4. Orbital counting in Riemannian balls. Let $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote the inner product and norm on $\mathfrak{a}$ induced by the Killing form. By [63, 13, 14], there is a unique unit vector $u_\rho \in \text{int } \mathcal{L}_\rho$ such that

$$\delta_\rho = \max_{\|v\|=1} \psi_\rho(v) = \psi_\rho(u_\rho).$$

Since i preserves the norm and ψ_ρ , uniqueness gives $i(u_\rho) = u_\rho$. Differentiability of ψ_ρ [13, 14] gives $\psi_{u_\rho}(v) = \delta_\rho \langle v, u_\rho \rangle$. In particular, $\mathbb{R}u_\rho \perp \ker \psi_{u_\rho}$. For $Q \subset K$, write $Q[P] := \{qP : q \in Q\}$.

Corollary 18.3 (Bisector counting in Riemannian balls). *There exists $c_\rho > 0$ such that, for all Borel sets $Q_1, Q_2 \subset K$ satisfying $\nu_{u_\rho}(\partial(Q_1[P])) = \nu_{u_\rho}(\partial(Q_2^{-1}w_0[P])) = 0$,*

$$\lim_{T \rightarrow \infty} e^{-\delta_\rho T} \# \{ \gamma \in \Gamma_\rho \cap Q_1 M A^+ Q_2 : \|\mu(\gamma)\| < T \} = c_\rho \nu_{u_\rho}(Q_1[P]) \nu_{u_\rho}(Q_2^{-1}w_0[P]).$$

Proof. Apply the counting argument of [25, Corollary 9.21 and the proof of Theorem 1.8] with $H = K$, $\mathfrak{b} = \mathfrak{a}$, and the Killing-form norm, using Theorem 1.3 in direction u_ρ . Since $i(u_\rho) = u_\rho$, both angular measures are ν_{u_ρ} . $\square$

Taking $Q_1 = Q_2 = K$ in Corollary 18.3 gives the orbital asymptotic at $o = K$ as stated in Corollary 1.2. Conjugating Γ_ρ gives the result at every $o \in G/K$, since δ_ρ is invariant under conjugation.

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