J Econ Inequal
DOI 10.1007/s10888-016-9346-2

The dynamics of capital accumulation in the US: simulations after Piketty

Philippe De Donder[1] · **John E. Roemer**[2]

Received: 26 June 2015 / Accepted: 22 December 2016

Abstract We develop a dynamic model where a competitive firm uses labor and capital, with market clearing rates of return. Individuals are heterogeneous in skills, with an endowment in capital/wealth increasing in skill. Individuals aspire to a socially determined consumption level, with a constant marginal propensity to consume out of income above this level. We also study three variants of the model: one with a higher rate of return for large capitals than for smaller ones, one with a capital levy financing a lump sum transfer, and one with social mobility. We calibrate the model to the US economy and obtain that a steady state exists in all variants, and we obtain convergence to the steady state from the 2012 US wealth distribution. The reduction in the level of the aspirational consumption level is the only way to create wealth for the bottom half of the distribution.

Keywords Piketty · Dynamics of wealth accumulation · Convergence to steady state · Spirit of capitalism · Differential rates of return to capital · Intergenerational mobility · Capital levy · US calibration

Electronic supplementary material The online version of this article (doi:10.1007/s10888-016-9346-2) contains supplementary material, which is available to authorized users.

✉ Philippe De Donder
philippe.dedonder@tse-fr.eu

John E. Roemer
john.roemer@yale.edu

1 Toulouse School of Economics, CNRS, University of Toulouse Capitole, 21 allée de Brienne, Toulouse, 31015 Cedex 6, France

2 Yale University, PO Box 208301, New Haven, CT, 06520, USA

Published online: 13 January 2017

1 Introduction

The recent book by Piketty (2014) has rekindled an interest in understanding why wealth is so concentrated at the top of the distribution, an evident phenomenon in several major economies, and especially in the US.

Benhabib and Bisin (2016) survey the early literature developed to understand the mechanisms at play in generating thick-tailed wealth distributions. All these early attempts were mechanical and lacked economic micro-foundations. Beginning in the 1990s, economists have started to build micro-founded models designed to understand the determinants of the properties of wealth distributions. Quadrini and Ríos-Rull (1997) survey this literature, where the models reviewed are heterogeneous agent versions of standard neoclassical growth models. Two types of model have been developed, *dynastic* and *life-cycle*. In their words (p 23), "The dynastic model includes the infinitely lived agent abstraction and assumes that people care for their descendants as if they were themselves, and the life cycle model includes overlapping generations of finitely lived agents who do not care about their descendants. Thus, the main motive for saving (...) differs in these two types of models: in dynastic models, people save to improve their descendants' consumption, while in life cycle models, people save to improve their own consumption during retirement." In both cases, preferences are represented by the discounted sum of a per period utility function.

Both types of model focus on the steady state (where variables grow at a constant rate – perhaps zero – over time) and model uninsurable idiosyncratic shocks to labor earnings. The seminal papers have adopted the dynastic approach and are due to Bewley (1983) and Aiyagari (1994). In these models, where agents save for precautionary reasons to smooth consumption, the key determinant of the wealth distribution is the volatility of individual earnings, not permanent differences in earnings across households (Constantinides and Duffie 1996). The literature has also added uninsurable labor shocks to life cycle models, and uninsurable lifetime uncertainty (Hugget 1996). In all cases, models underestimate the fraction of wealth accumulated at the top of the distribution. More recently, the literature has introduced stochastic returns to capital as well as labor. Benhabib et al. (2011) obtain a stationary wealth distribution which is Pareto in its right tail, with this tail populated by dynasties which have realized a long streak of high rates of return on capital, so that capital income risk, rather than stochastic labor income, drives the properties of this right tail.

In this paper, we develop in Section 2 a deterministic dynastic model with infinitely lived agents who differ in skills. We mainly depart from the literature surveyed above in how we model individual preferences. Agents do not maximize a discounted infinite sum of consumption over time, but rather a single-period utility function of consumption and savings, repeated in each period. Moreover, we assume there is a socially expected standard of living for people in this society. It is not subsistence consumption, but rather the consumption level to which ordinary people aspire, which is generated by advertising and the media (in the US, this level would define a successful middle-class life).[1] The marginal propensity to consume out of income is assumed to be unity below this standard consumption level, and a constant lower than one above this level.

One way to interpret these preferences is that agents desire to consume and to accumulate wealth for its own sake. They do so because 'money is life's report card,' as the caption of a New Yorker cartoon said. In this view, accumulation for its own sake is the motivation for most members of the wealthy class, for success in the game of life is judged by one's

[1] See Hubbard et al. (1995) for a life cycle model with subsistence consumption.

wealth. As noted by Cooper (1979), "Persons in the wealth category we are now discussing have more current income than they can expend. Beyond a certain point, the real value of greater wealth is power, control, and security (p. 208)". Max Weber (1905) has argued that private accumulation of wealth as an end in itself, rather than for consumption purpose – a behavior he dubbed the *spirit of capitalism* – has been the main driver of Industrial Revolution in Europe. Piketty-and-Zucman (2015, p 1346) study a similar (although not equivalent) "Wealth-in-the-Utility" "Function", where the utility function is defined over consumption and (the increase in) wealth, and justify the latter by referring to wealth as "a signal of their ability or virtue".[2] Our main objective is then to assess the consequences of assuming an aspirational consumption level on the dynamics of capital accumulation in a model calibrated to reflect broadly the US economy.

We assume that the initial distribution of wealth is monotone increasing in an individual's skill. A firm, using a CES production function whose inputs are efficiency units of labor and capital, maximizes profits. Consumer-workers offer inelastically their entire endowment of skilled labor to the firm; they demand the consumption good and supply capital to the firm in order to maximize preferences described above. The interest rate and real wage equilibrate the markets for labor and capital. There are proportional taxes on capital and labor income, the revenues from which are returned as a demogrant to each worker. Skills are assumed to increase at an exogenous rate, as does the aspirational standard of living. The main fundamental changing over time is the distribution of capital/wealth.

The literature surveyed above concentrates on establishing the properties of the steady state of the economy (where real rates of return, capital output ratio and wealth shares remain constant over time). This approach is incomplete, as recognized by Benhabib et al. (2015) who write "This comparison implicitly assumes that the wealth distribution for the U.S. is close to stationary. This might in general not be the case if the wealth distribution is hit frequently enough by aggregate shocks like wars, major business cycle events (e.g., a depression), changes in tax schemes, social insurance institutions, and so on; see Saez and Piketty (2003). We leave the study of the transition of the distribution of wealth for future work" (p 16). Our second objective is then to go beyond the study of the steady state equilibrium and to also investigate whether and how the model converges to this steady state when we start, at period zero, with the wealth distribution observed in the US in 2012 by Saez and Zucman (2016).

Our third objective is to assess how the introduction of three relevant departures from our basic "vanilla" model affects both the steady state and the convergence to this state from the 2012 US wealth distribution. Introducing heterogenous rates of returns is often mentioned as a promising way to move the models closer to explaining actual wealth data, and especially its concentration at the top (see for instance Quadrini and Ríos-Rull (1997), who mention that "the portfolio of wealthy households typically includes assets that yield higher returns than the assets of poorer households" (p 29), and Benhabib and Bisin (2016)).

There is also increasing empirical support for this form of heterogeneity, even though Piketty and Zucman (2015) stress the poor quality of the data. Piketty (2014, chapter 2) estimates these rates of return using university endowments, which are public information in the United States because non-profit institutions must report these data. He obtains returns (over the 1980–2010 period) ranging from 6.2 % for endowments less than \$100 million

[2] Our formulation generates a consumption pattern reflecting Saez and Zucman (2016)'s evidence of substantial saving rate differentials across wealth levels, so that, in Benhabib et al. (2015)'s words, "the rich can get richer through savings, while the poor may not save enough to escape a poverty trap" (p 3).

to 10.2 % for endowments much larger than \$1 billion. Saez and Zucman (2016) find the same pattern for the universe of U.S. foundations, and Piketty and Zucman (2015, Table 15.1) find it as well using Forbes global wealth rankings. Saez and Zucman (2016, Online appendix, Tables B29, B30, and B31) show mildly increasing pre-tax returns in wealth over the period 1980-2012. Administrative data from Scandinavian countries also show pre-tax returns increasing in wealth. Fagereng et al. (2015, 2016) find returns significantly increasing in wealth only for high wealth classes, above the top 10 %, in Norway. Bach et al. (2015) find higher returns on large wealth portfolios for Sweden. In Section 3, we construct a highly simplified model with one rate of return for capitals in the top 1 % of the wealth distribution and a smaller rate of return for capitals in the bottom 99 %. We continue to assume that the average rate of return clears the capital market.

In the spirit of Piketty (2014), we introduce in Section 4 a capital levy on the top wealth decile, the proceeds from which are redistributed as a lump sum transfer. In the Online Supplementary Material, we add intergenerational mobility to our original model. We take a generation to last for 50 years, and model this by assuming that each individual has a 2 % probability of dying each year, upon which his capital passes down, without taxation, to his single offspring. The offspring's skill level – and hence her labor earnings – are not inherited, but are taken to be determined by the income intergenerational mobility matrix of Chetty et al. (2014).[3]

In all variants of the model, we obtain that a steady state exists. The value of the aspirational consumption level plays an especially important role in our results. Decreasing this level is the only way, in all of our variants, for the bottom half of the distribution to accrue some wealth at the steady state. Lowering the value of this consumption level also has an important downward effect on the wealth shares at the very top of the distribution. The other results obtained with the basic version of the model as well as with each variant (including convergence to the steady state) are summarized at the end of each section, and we recap our main results in Section 5.

2 The basic model

We present the model and solve for its equilibrium date by date in Section 2.1, and we solve for its steady-state (where real rates of return, capital output ratio and wealth shares remain constant over time) in Section 2.2. We then calibrate the model to the modern US economy in Section 2.3 and describe our numerical results in Section 2.4.

2.1 Presentation of the basic model

The economy consists of a continuum of individuals who differ in their skill level s. The distribution of skill levels is represented by the c.d.f. $F(s)$ over $[0, \infty]$, where

$$\bar{s} = \int_0^\infty s dF(s)$$

[3]Kopczuk and Lupton (2007) study empirically bequest motives in the US and obtain that roughly three-fourths of the elderly single population in their sample has a bequest motive, which can best be described as egoistic, namely "a desire to have positive net worth upon death". They model this desire with a per-period utility which is an increasing function of both consumption (if alive) and bequest (if dead) ? i.e., the counterpart of the "wealth in utility function" used in our model and in Piketty and Zucman (2015).

denotes the average skill. We use the subscript t to denote the date at which a variable is measured. The model starts at period $t = 0$, with a distribution of wealth denoted by $S_0(s)$. We assume that wealth $S_0(s)$ is monotone increasing in s, and that skills increase (exogenously) by a factor $(1 + g)$ per period.

There is a single good in the economy. Preferences are non-traditional. We assume there is an aspirational standard of living for people in this society, produced by a consumption level c_0 at date 0. This expected consumption level increases by a factor of $(1 + g)$ per period. We do not call this subsistence consumption – we set it at $100,000 in the simulations. It is the consumption level to which ordinary people aspire, which is generated by advertising and the media (in the US, this level would define a successful middle-class life). A sufficiently wealthy individual at date t chooses her consumption c and investment I to maximize a Stone-Geary utility function as follows:

$$\begin{aligned} \max \quad & (c_t - c_0(1+g)^{t-1})^{\alpha} I_t^{1-\alpha} \\ & \text{subject to} \\ & c_t + I_t \le y_t(s), \\ & c_t \ge c_0(1+g)^{t-1}, \end{aligned} \tag{1}$$

where $y_t(s)$ is the income of individual s at date t. If there is no solution to program (1), because income is insufficient to purchase the consumption level $c_0(1 + g)^{t-1}$, then the individual consumes out of wealth. To be precise,

$$c_t(s) = \begin{cases} y_t(s) + S_{t-1}(s) \text{ if } y_t(s) + S_{t-1}(s) \le (1+g)^{t-1}c_0 \text{ (case 1)}, \\ (1+g)^{t-1}c_0 \text{ if } y_t(s) \le (1+g)^{t-1}c_0 \le y_t(s) + S_{t-1}(s) \text{ (case 2)}, \\ (1+g)^{t-1}c_0 + \alpha\left(y_t(s) - (1+g)^{t-1}c_0\right) \text{ if } y_t(s) > (1+g)^{t-1}c_0 \text{ (case 3)}. \end{cases} \tag{2}$$

In case 1, the individual consumes his income plus his wealth $S_{t-1}(s)$, and those together do not suffice to generate the aspirational consumption of $c_0(1+g)^{t-1}$. In case 2, when her income does not suffice to consume at the aspirational level but her total asset position does, she consumes exactly the aspirational consumption level. In case 3, where her income alone suffices to consume at the aspirational level, she solves program (1) with α the marginal propensity to consume out of income. Thus, investment is given by

$$I_t(s) = \begin{cases} -S_{t-1}(s) \le 0 \text{ if case 1}, \\ y_t(s) - c_0(1+g)^{t-1} \le 0 \text{ if case 2}, \\ y_t(s) - c_t(s) > 0 \text{ if case 3}. \end{cases} \tag{3}$$

The dynamics of wealth are given by

$$S_t(s) = S_{t-1}(s) + I_t(s). \tag{4}$$

We now turn to the production side of the economy. A single firm produces the consumption good, using a CES technology given by

$$y(K, L) = A\left(aK^{\frac{\delta-1}{\delta}} + (1-a)L^{\frac{\delta-1}{\delta}}\right)^{\frac{\delta}{\delta-1}}, \tag{5}$$

where y, K and L are per capita income, capital, and labor in efficiency units. The only technical change in the model is induced by the exogenous increase in labor skills or productivity. The firm faces an interest rate r and a real wage per efficiency unit of labor w, and maximizes profits. We denote by d the annual rate of capital depreciation.

The firm's profit is defined as

$$y(K_t, L_t) - w_t L_t - (r_t + d)K_t,$$

whose differentiation gives the following FOC for the demands for labor

$$w_t = (1-a)y_t^{1/\delta}L_t^{-1/\delta}A^{\frac{\delta-1}{\delta}}, \tag{6}$$

and capital

$$r_t + d = ay_t^{1/\delta}K_t^{-1/\delta}A^{\frac{\delta-1}{\delta}}. \tag{7}$$

The firm replaces depreciated capital from income, so that the investor can cash out his entire capital stock at the end of the period, which explains why the depreciation does not appear in Eq. 4.

The market clearing equations are

$$K_t = \int_0^\infty S_{t-1}(s)dF(s), \tag{8}$$

$$L_t = \int_0^\infty (1+g)^{t-1}sdF(s) = (1+g)^{t-1}\bar{s}. \tag{9}$$

Using Eqs. 5 and 9 in Eqs. 6 and 7, the FOCs with respect to L and K simplify to, respectively,

$$w_t = (1-a)A\left(a\left(\frac{K_t}{(1+g)^{t-1}\bar{s}}\right)^{\frac{\delta-1}{\delta}} + 1 - a\right)^{\frac{1}{\delta-1}}, \tag{10}$$

and

$$K_t = (1+g)^{t-1}\bar{s}\left(\frac{\left(\frac{r_t+d}{aA}\right)^{\delta-1} - a}{1-a}\right)^{\frac{\delta}{1-\delta}}. \tag{11}$$

Finally, we assume an exogenously given income tax rate τ, the revenues from which are returned to citizens as a demogrant. Thus income for an agent of type s in year t is given by

$$y_t(s) = (1-\tau)\left(w_t s(1+g)^{t-1} + r_t S_{t-1}(s)\right) + \tau\left(w_t\bar{s}(1+g)^{t-1} + r_t K_t\right). \tag{12}$$

2.2 The steady-state

We now define the steady state of this economy.

Define first

$$s_1(t) = \sup_s\{s \,|S_t(s) = 0\}$$

as the highest skill level with zero wealth at time t.

Definition 1 A steady-state of the basic model is an equilibrium of the model defined in Section 2.1, where $r_t = r$, $w_t = w$, $s_1(t) = s_1$, for all t, and where the variables $y_t(s)$, $c_t(s)$, $I_t(s)$, K_t, L_t, $S_t(s)$ all grow at rate g.

We can then represent

$$\begin{aligned} y_t(s) &= (1+g)^{t-1}y^*(s),\ t \geq 1, \\ S_t(s) &= (1+g)^{t-1}S^*(s),\ t \geq 0, \\ K_t &= (1+g)^{t-1}K^*,\ t \geq 1, \\ L_t &= (1+g)^{t-1}\bar{s},\ t \geq 1. \end{aligned}$$

From the FOC (11) with respect to K and the definition of K^*, we obtain

$$K^* = \bar{s} \left(\frac{\left(\frac{r_t+d}{aA}\right)^{\delta-1} - a}{1-a} \right)^{\frac{\delta}{1-\delta}}. \tag{13}$$

Solving for r, we obtain

$$r = \left(\left(\frac{K^*}{\bar{s}} \right)^{\frac{1-\delta}{\delta}} (1-a) + a \right)^{\frac{1}{\delta-1}} aA - d. \tag{14}$$

Similarly, from the FOC (10) with respect to L , we obtain

$$w = (1-a)A \left(a \left(\frac{K^*}{\bar{s}} \right)^{\frac{\delta-1}{\delta}} + 1 - a \right)^{\frac{1}{\delta-1}}. \tag{15}$$

By definition of s_1,

$$y^*(s) = (1-\tau)ws + \tau(w\bar{s} + rK^*) \leq c_0 \text{ for all } s \leq s_1.$$

We can then define

$$s_1 = \frac{c_0 - \tau(w\bar{s} + rK^*)}{(1-\tau)w}. \tag{16}$$

Observe that there cannot be case 2 agents (see Eq. 2) at the steady state – i.e., agents with positive wealth but negative investments – since their wealth would decrease at every date, contradicting the definition of a steady state. Hence, all those with $s > s_1$ do invest and are such that

$$S_t(s) = S_{t-1}(s) + (1-\alpha)\left(y_t(s) - c_0(1+g)^{t-1} \right).$$

Dividing by $(1+g)^{t-1}$ gives the steady state

$$S^*(s) = \frac{S^*(s)}{1+g} + (1-\alpha)(y^*(s) - c_0), \tag{17}$$

which solves to

$$S^*(s) = \frac{1+g}{g}(1-\alpha)(y^*(s) - c_0).$$

Therefore,

$$S^*(s) = \frac{(1+g)(1-\alpha)}{g} \left((1-\tau) \left(ws + \frac{r}{1+g} S^*(s) \right) + \tau \left(w\bar{s} + rK^* \right) - c_0 \right) \text{ for } s > s_1,$$

which solves to

$$\frac{S^*(s)}{1+g} = \frac{(1-\tau)ws + \tau(w\bar{s} + rK^*) - c_0}{\gamma} \text{ for } s > s_1, \tag{18}$$

where

$$\gamma = \frac{g}{1-\alpha} - r(1-\tau).$$

Using Eq. 8 together with Eq. 18, we obtain:

$$(\gamma - \tau r\,(1 - F(s_1)))\,K^* = (1-\tau)w \int_{s_1}^{\infty} s\,dF(s) + (1 - F(s_1))(\tau w\bar{s} - c_0). \tag{19}$$

Table 1 Steady state allocation

r	6.32 %
w	\$67,750
$K^*/y(K^*, \bar{s})$	4.45
$F(s_1)$	83.8 %

Recall that the lower bound of the integral on the RHS, s_1, depends on K^* (see Eq. 16). Existence of the steady state depends on the existence of a solution to the Eq. 19, an equation in the single unknown K^*. Observe also from Eq. 18 that the distribution of wealth at equilibrium is linear in s for $s > s_1$.

If a steady state equilibrium exists, then we obtain by definition that the wealth of every s grows at rate $(1 + g)$. (This is obviously true for those with wealth zero.) Therefore, the fraction of total wealth owned by any sub-class of the population is constant.

2.3 Calibration

We start with the calibration of the production function. Recall that the production function has the CES form given by Eq. 5. We choose $\delta = 0.85$, and we stress at the end of Section 2.4 that our results are not affected as we increase δ from 0.85 to 1.25. One period is deemed to be one calendar year. The capital income ratio is 4.5 in the U.S. Depreciation is about 10 % of GNP, which suggests a rate of depreciation $d = 0.02$. We assume that $g = 0.02$. The distribution of skills is taken to be lognormal. The unit of skill has no meaning: we take the median skill level to be 0.85 and the mean $\bar{s}$ to be 1. We then have that $L = 1$ at period 1. We develop in Section A of the Online Supplementary Material how we calibrate the production function.

As for preferences, we choose $c_0 = \$100,000$ and $\alpha = 0.6$, based on the fact that the propensity to consume for the wealthy is about 0.6 out of income. There are many estimates of the marginal propensity to consume of the wealthy (see Carroll et al. (2014)), which include 0.6. We chose this value as it generates an aggregate savings rate in our models of about 9 %, conforming roughly with reality.[4]

Finally, the taxation rate τ is set at 0.35 throughout the paper.

2.4 Numerical results: Steady state and convergence

A steady state exists for the parameter values detailed in Section 2.3. The main characteristics of the steady state are described in Table 1.

Observe first that $r(1 - \tau) = 4.11$ %, so that $r(1 - \tau) > g$ is consistent with constant wealth shares in the steady state. The value of the capital output ratio is, at 4.45, very close to the targeted value of 4.5 used to calibrate the parameters of the production function (see Section A of the Online Supplementary Material). Also, Piketty (2014) computes that the share of capital income, net of depreciation, in GNP is 28 % (see Section A of the Online Supplementary Material), from which we infer that $r = 6.22$ %, which is very close

[4] According to US Census data, the savings rate has varied between 5 % and 15 % over the last forty years.

Table 2 Wealth shares at steady state equilibrium and in Saez and Zucman (2016)

Group	Wealth share (%) in	
	steady state	actual
bottom 50 %	0	0
top 10 %	92	77.2
top 5 %	68.7	64.6
top 1 %	25.4	41.8
top 0.5 %	15.4	34.5
top 0.1 %	4.5	22.0
top 0.01 %	0.7	11.2

to our steady state value of 6.32 %.[5] The saving rate of the economy in steady state is then

$$\begin{aligned} g\frac{S^*}{y(K^*,\bar{s})} &= (1+g)g\frac{S^*}{(1+g)y(K^*,\bar{s})} \\ &= (1+g)g\frac{K^*}{y(K^*,\bar{s})} \\ &= 0.091. \end{aligned}$$

As for the wealth distribution, 84 % of individuals have no wealth at the steady state equilibrium, with wealth increasing linearly in skill for the top 16 % of individuals. The steady state distribution of wealth is summarized in Table 2, where we compare it with the actual wealth distribution taken from Saez and Zucman (2016, Appendix Table B1).

The steady state equilibrium nearly reproduces the wealth shares accruing to the bottom 50 % and to the top 5 %, but underestimates both the share going to the "patrimonial middle-class" (from 5th to 9th decile group) and, especially, to the very top (1 % to 0.01 %) of the wealth distribution. Observe also that the underestimation increases as one focuses on the very top of the distribution, since the actual wealth share of the top 1 % is 1.64 times higher in reality than in the computed steady state, with this ratio increasing to 2.24 for the top 0.5 %, 4.89 for the top 0.1 % and 16 for the top 0.01 %. Our review of the literature in the introduction has shown that it is difficult to replicate the thick tail of the wealth distribution, especially in a model without stochastic shocks.

Decreasing the aspirational standard of living c_0 has a very large impact on the steady state results, as reported in Table 3.

As the standard of living c_0 decreases from \$100,000 to \$55,000, the middle class starts accumulating capital, resulting in a more-than-doubling of the capital/output ratio. As a consequence, the equilibrium rate of return of capital decreases while the equilibrium wage increases. The labor share of income in GNP ($w\bar{s}/y$) increases, although in a less spectacular fashion than w. The impact of a lower c_0 on wealth shares is tremendous, with a quarter of capital accruing at steady state to the bottom half of the distribution, one half to the patrimonial middle class, and with the share of the top 1 % decreasing by a factor of five![6]

[5]The share of capital income, including depreciation, in GNP is 37 % in our steady state.

[6]The impact of the elasticity of substitution δ on the results is inconsequential, as the steady state values of w, r, K/y and of the various wealth shares remain practically constant as we vary δ from 0.85 to 1.25 (see

Table 3 Steady state results as a function of c_0

					wealth shares (%)			
c_0	w	r	K/y	$w\bar{s}/y$	0–50	50–90	90–99	top 1 %
\$100,000	\$67,750	6.32 %	4.45	0.629	0	8.01	66.63	25.36
\$85,000	\$75,971	4.82 %	5.28	0.640	0	40.23	47.17	12.6
\$70,000	\$93,281	2.66 %	7.30	0.660	12.57	52.83	28.32	6.27
\$55,000	\$108,034	1.47 %	9.38	0.675	25.06	48.94	21.59	4.41

Going back to the base case value of $c_0 = \$100,000$, the fact that wealth is becoming more concentrated in reality is then either due to a process of convergence to the steady state, or to a departure of reality from the model. Our next step is then to check the convergence to the steady equilibrium, starting from the Saez and Zucman (2016) distribution. More precisely, we assume that the initial distribution $S_0(s)$ is linear by parts over s, with 7 different brackets reproducing Saez and Zucman (2016)'s top brackets. Total capital per worker at the beginning of period 1 is set at 4.5 times \$108,300 (see Section 2.3).

We run the model for 500 periods, as follows. For each period, we begin with the wealth function $S_{t-1}(s)$ at the beginning of date t. K_t and L_t are determined by Eqs. 8 and 9. Eqs. 10 and 11 determine w_t and r_t. Income y_t is determined by Eq. 12. Consumption and investment are determined by Eq. 2 and 3. $S_t(s)$ is determined by Eq. 4. The next iteration begins.

The model converges to the steady state equilibrium, with a very interesting convergence pattern. The equilibrium interest rate is lower at $t = 1$ ($r_1 = 6.19$ %) than its steady value, and increases for the first 48 periods, to reach a maximum of 6.69 % (see Fig. 1).[7] It then decreases and converges to the steady state value of 6.32 %. A similar pattern of overshooting also occurs for the equilibrium wage rate (see Fig. 2), which starts at $t = 1$ at a higher level (\$68,408) than the steady state, decreases for 48 periods to reach a minimum of \$66,055, and then increases and converges to the steady state level of \$67,750. In both cases, the equilibrium rate of return remains quite close to its steady state value, with a maximum gap (reached in both cases at $t = 48$) of 5.75 % for r and 2.5 % for w.

The capital output ratio behaves very similarly to the equilibrium wage, as can be seen in Fig. 3. It starts above its steady state value, reaches its minimum amount of 4.3 after 48 periods, and then increases to its steady state value. At any point in time its value is within 3.7 % of its steady state value of 4.45.

We now move to the evolution of the wealth shares. The wealth share of the patrimonial middle class (from 5th to 9th decile group) decreases with time, with a rate of decrease higher in the first periods, and a minimum attained after 394 periods (see Fig. 4). Here also, we observe a slight overshooting, since the equilibrium share after 394 periods is, at 7.79 %, lower than its steady state value of 8.01 %.

The other wealth shares we study do not exhibit a pattern of overshooting, with the wealth shares of both the top 1 % and top 0.1 % first increasing (for 29 and 13 periods,

Table B1 in Section B of the Online Supplementary Appendix). This shows that the debate about the value of δ, following Piketty (2014), is somewhat moot, at least for our formulation of the economy.

[7] In all figures illustrating convergence, the horizontal line is set at the steady state level of the variable depicted. Recall also that one period is deemed to be one calendar year.

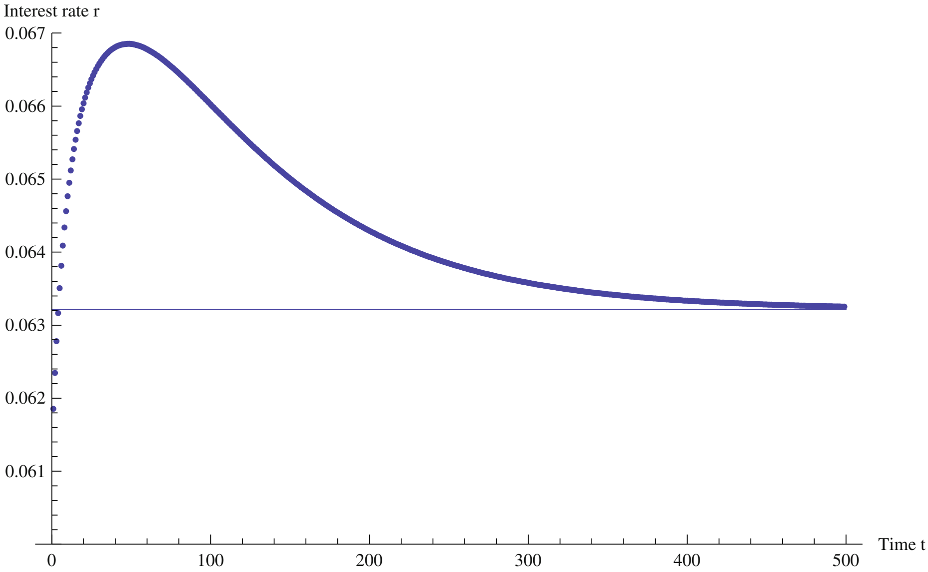

Fig. 1 Basic model, convergence of interest rate

respectively) and then decreasing towards their steady state levels. Observe in Figs. 5 and 6 that the wealth shares of the top of the distribution do not fully converge after 500 periods.

Here are the main conclusions we can draw from our analysis so far. First, decreasing the value of the aspirational consumption level c_0 has a very large and positive (resp., negative) impact on the wealth shares at the bottom (resp., top) of the steady state distribution. Second, we observe convergence to the steady state, with this convergence happening more quickly for r, w and the capital output ratio, and more slowly for wealth shares. Third, during convergence, we may observe overshooting, with a variable crossing its steady state value. Fourth, the evolution of several variables is not monotone with time. For instance, even though the wealth shares of the top 1 % and top 0.1 % are above their steady states values at the beginning of time, they keep on increasing for, respectively, 29 and 13 years before peaking and then starting their downward convergence to their steady state levels. An important message of this section is then that, even when convergence occurs, the non-monotonicity of wealth shares (and other variables such as interest rate, wage or capital output ratio) makes the task of the econometrician extremely difficult, since even the observation of several decades of increasing wealth shares does not mean that we are converging to a higher steady state, but rather the opposite.

We now introduce rates of return which vary with the amount of individual capital invested.

3 Differential rates of return

Large capitals earn significantly higher rates of return than small ones, as shown with data from US university endowments (Piketty 2014), US foundations (Saez and Zucman 2016), Forbes global wealth rankings (Piketty and Zucman 2015), or administrative data from

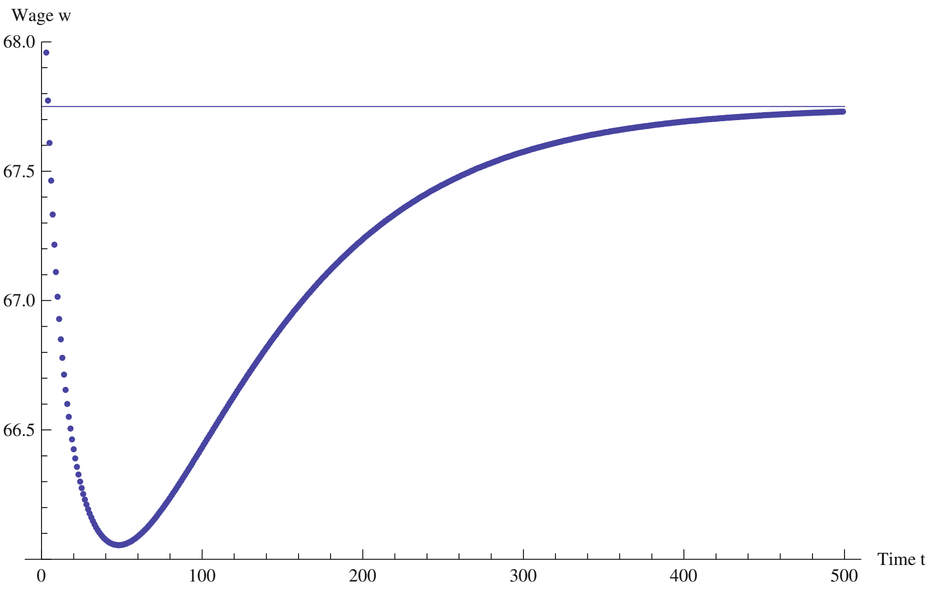

Fig. 2 Basic model, convergence of wage

Scandinavian countries (see Fagereng et al. (2015, 2016) for Norway and Bach et al. (2015) for Sweden).

In this section, we ask whether steady states continue to exist with differential rates of return. To do so, we make a very simple assumption, that the rate of return is some number r_1 for wealths accruing to those individuals in the bottom 99 % of the wealth distribution, and kr_1 for those in the top 1 %, where k is a parameter greater than one.[8] We assume that the average rate of return, r, continues to clear the capital market.

Consequently, in the steady state, if one exists, we must have

$$rK^* = r_1 \int_{s_1}^{q99} \frac{S^*(s)dF(s)}{1+g} + kr_1 \int_{q99}^{\infty} \frac{S^*(s)dF(s)}{1+g}, \tag{20}$$

where $q99$ is the 99th centile of the distribution F, which is also the 99th centile of the wealth distribution in our model. The equations derived in Section 2.2 for r, K^*, w and s_1 remain identical. However, Eq. 18 for $S^*(s)$ now bifurcates into

$$\frac{S^*(s)}{1+g} = \frac{(1-\tau)ws + \tau(w\bar{s} + rK^*) - c_0}{\gamma(s)} \text{ for } s > s_1, \tag{21}$$

where

$$\gamma(s) = \begin{cases} \gamma_1 = \frac{g}{1-\alpha} - r_1(1-\tau) \text{ if } s_1 < s < q^{99}, \\ \gamma_2 = \frac{g}{1-\alpha} - kr_1(1-\tau) \text{ if } s \geq q^{99}. \end{cases}$$

[8] This assumption is in line with the empirical literature referred above, where differentiated rates of returns are significant especially for the top of the wealth distribution.

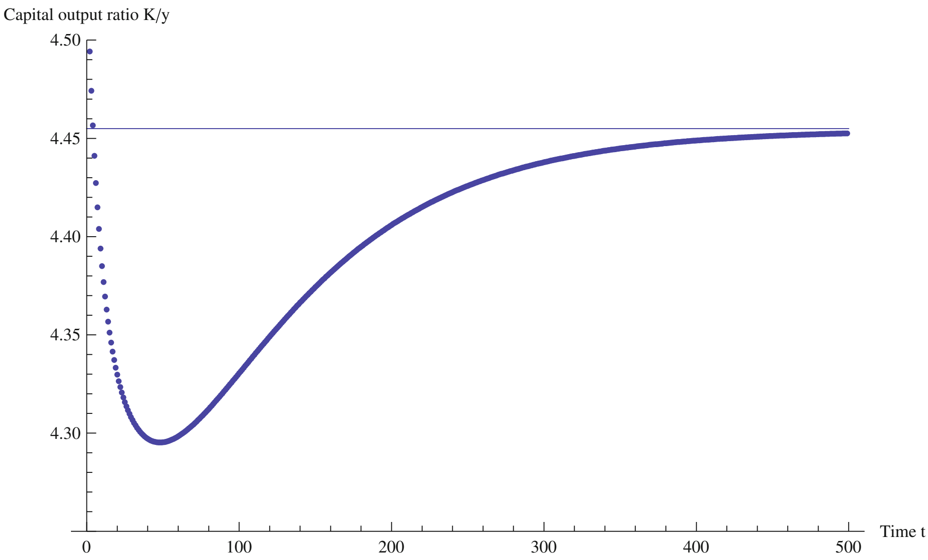

Fig. 3 Basic model, convergence of capital/output ratio

A steady state exists for a value $k > 1$ if we can solve simultaneously the Eqs. 20, 21, and the Eqs. 13 to 16 in Section 2.2 that define K^*, r, w and s_1. Since r, w and s_1 are expressed as functions of K^*, and

$$K^* = \int \frac{(1-\tau)ws + \tau(w\bar{s} + rK^*) - c_0}{\gamma(s)} dF(s) \tag{22}$$

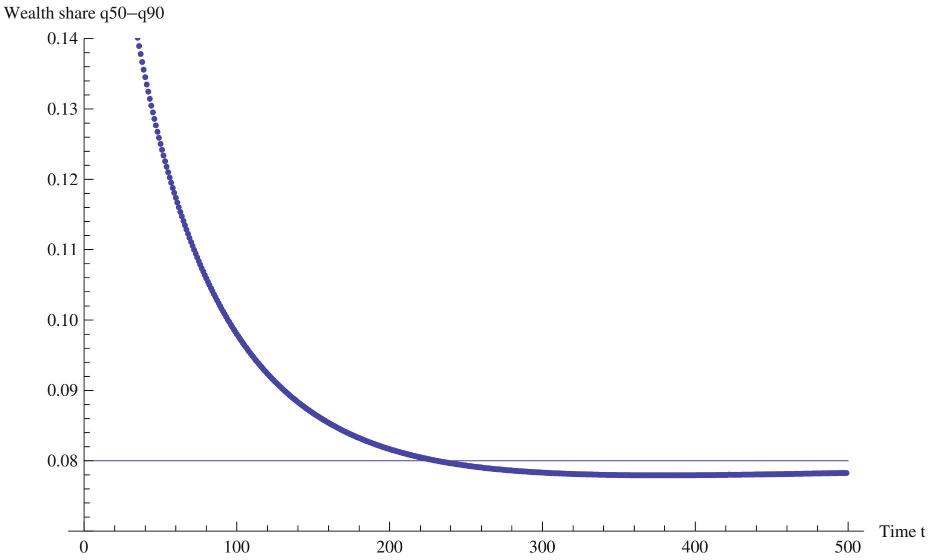

Fig. 4 Basic model, convergence of patrimonial middle class wealth share

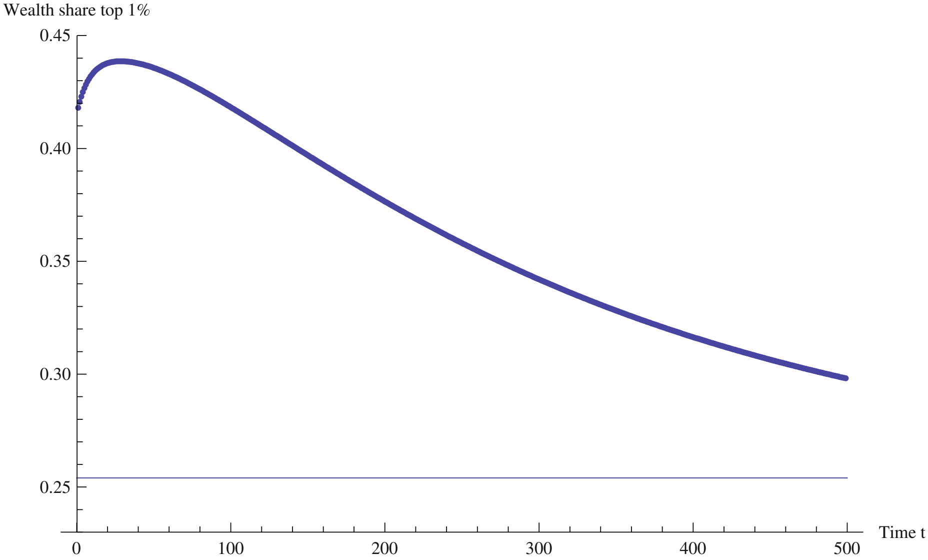

Fig. 5 Basic model, convergence of top 1 % wealth share

by integrating (21), this requires only solving the two simultaneous Eqs. 20 and 22 for K^* and r_1.

We know there is a solution when $k = 1$ (the previous section). What happens as k increases? Computation shows that a steady state exists for all values of $k > 1$. We do not prove this analytically, but demonstrate it, hopefully convincingly, by reporting the solution to Eqs. 20 and 22 for many values of k in Table 4.

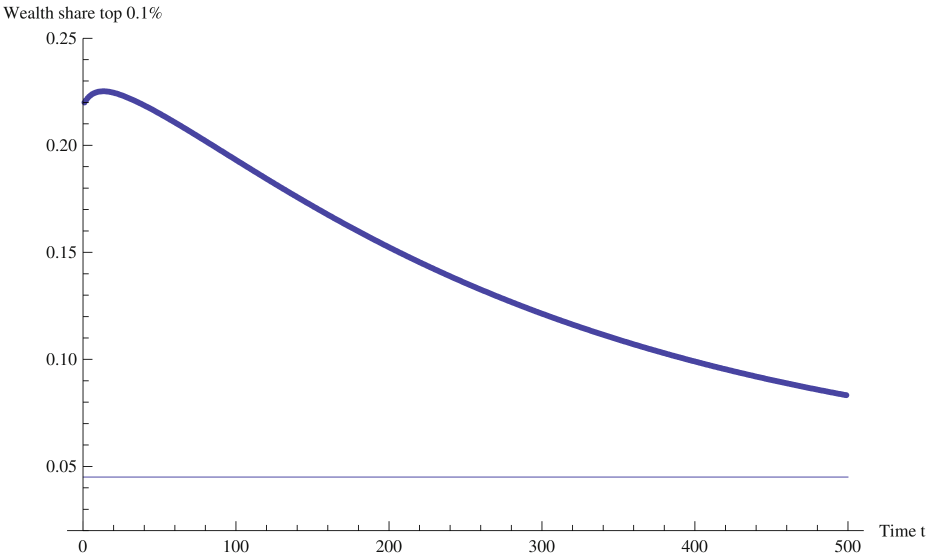

Fig. 6 Basic model, convergence of top 0.1 % wealth share

Table 4 Steady state allocation as a function of k

					Wealth shares (%)			
k	r_1	kr_1	γ_1	γ_2	50th–90th	90th–99th	top 1 %	top .1 %
1	6.32 %	6.32 %	0.0089	0.0089	8.01	66.63	25.36	4.51
1.1	6.10 %	6.71 %	0.0103	0.0064	6.92	57.52	35.56	6.32
1.3	5.43 %	7.06 %	0.0147	0.0041	4.85	40.36	54.79	9.74
1.5	4.77 %	7.16 %	0.019	0.0035	3.76	31.27	64.97	11.55
2	3.58 %	7.23 %	0.027	0.0030	2.67	22.21	75.12	13.35
2.5	2.9 %	7.25 %	0.031	0.0029	2.29	19.06	78.65	13.98
3	2.42 %	7.26 %	0.034	0.0028	2.08	17.28	80.65	14.32
5	1.46 %	7.27 %	0.041	0.0027	1.76	14.65	83.59	14.86
10	0.73 %	7.28 %	0.045	0.0026	1.58	13.12	85.31	15.16

We do not report the values of r, K^*, w and s_1 in Table 4 because they are not affected by the value of k, and are thus the same as in the preceding section. We observe that r_1 decreases with k, and we now show that it converges to 0 as k becomes large. If this were not the case, since the left-hand side of Eq. 20 is constant as a function of k, it would follow that

$$\int_{q99}^{\infty} \frac{S^*(s)dF(s)}{1+g}$$

must tend to zero as k becomes large, which is clearly false from the definitions. Therefore r_1 must tend to zero, and we conjecture that

$$\lim_{k\to\infty} r_2 = \frac{g}{(1-\alpha)(1-\tau)} = 7.69\ \%,$$

which is the value that renders $\gamma_2 = 0$. In other words, γ_2 never becomes negative – for that would indicate the non-existence of a steady state.

Observe from Table 4 that r_1 plunges from 6.3 % to 0.7 % as k increases from 1 to 10, but that kr_1 does not increase very much, moving from 6.3 % to 7.3 %. Even though the absolute level of the return on wealth of the top 1 % does not increase much, the wealth shares of various groups change a lot with k. The wealth shares of the top 1 % and top 0.1 % increase with k, at the expense of the shares of the patrimonial middle class and of the 90th to 99th centiles in the wealth distribution. It is striking that steady state wealth shares are very sensitive to k when k is low: an increase in k from 1 to 1.3 more than doubles the steady state wealth shares of both the top 1 % and the top 0.1 %! In other words, a rate premium of 30 % for the top 1 % results in more than half of total wealth being concentrated among them (up from a quarter with identical rates of return for all).

Since $r_1 \to 0$ as k becomes large, it follows that $\gamma_1 \to g/(1-\alpha)$, and from Eq. 22 that the wealth of the bottom 99 % approaches

$$\int \frac{(1-\tau)ws + \tau(w\bar{s} + rK^*) - c_0}{g/(1-\alpha)} dF(s).$$

Dividing this number by K^* from Table 1, we conclude that, as $k \to \infty$, the wealth share of the top 1 % approaches 86.7 %. In Fig. 7, we report calculation of the wealth share of the top 1 % for steady states up to $k = 20$. The figure bears out our conjecture that the limiting value of the top wealth share is 86.7 %.

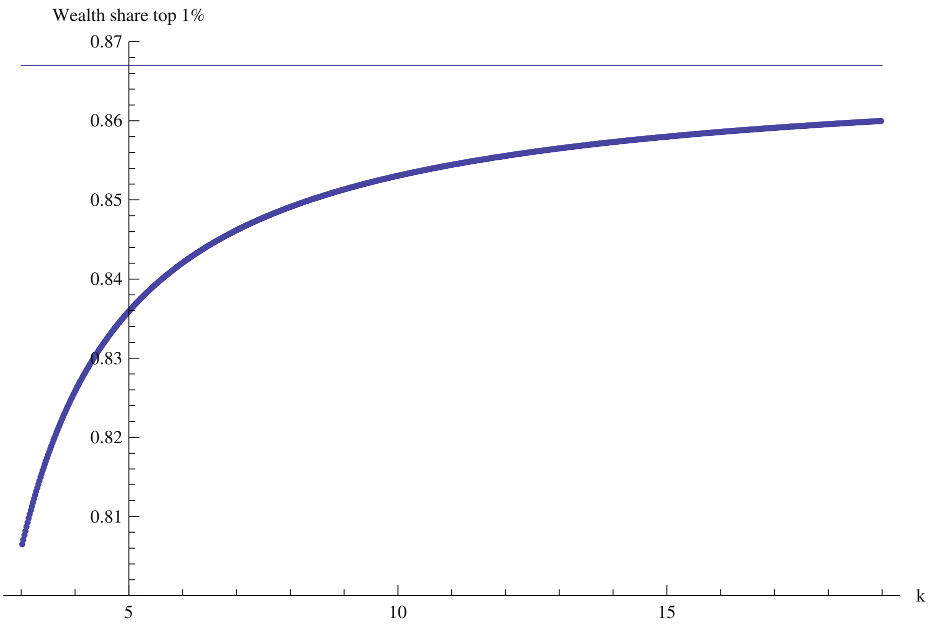

Fig. 7 Top 1 % wealth share as a function of parameter k

We now say a quick word about the convergence pattern to the steady state when one starts with the wealth distribution obtained from Saez and Zucman (2016), as in the preceding section. Fixing a value of $k > 1$, we do observe the same type of convergence as the one reported in Section 2.4, namely a non-monotone convergence to their steady state values for the interest rate, the wage, capital stock, and some wealth shares. The conclusions obtained in this section regarding convergence then carry through to the introduction of differentiated rates of return.

We summarize the results of this section as follows. First, as the rate of return to wealths in the top 1 % becomes an arbitrarily high multiple of the rate of return to wealths in the bottom 99 %, steady states continue to exist, where incomes and wealths grow at the rate g. Second, the values of K^*, r, w and s_1 are independent of how capital income is distributed among owners of capital. Third, the wealth share of the top 1 % increases very fast with k for low values of k, and approaches an asymptotic value less than one when k becomes very large. Fourth, the convergence pattern studied in Section 2.4 carries through to the case of differentiated rates of return on wealth.

We now move to the introduction of capital taxation.

4 Taxing capital

We finally study the effect of capital taxation by examining the steady states generated at different capital levies. Denote by τ_1 a per annum tax on the individual's wealth, collected at date t but upon wealth at date $t - 1$. We choose to levy the tax only on the top decile of the distribution. We amend the vanilla model of Section 2: thus, there is no intergenerational mobility, and there is one rate of return on capital. The revenues from the capital tax will be distributed as a demogrant to the entire population.

At the steady state, it will continue to be true that those who invest positively will be exactly those types for whom $y^*(s) > c_0$. We continue to denote by s_1 the largest type whose savings are zero in the steady state. Thus in the steady state, $S^*(s)$ remains given by Eq. 17 which can be expressed for $s \geq s_1$ as

$$S^*(s) = \frac{(1+g)(1-\alpha)}{g} \left[(1-\tau)\left(ws + \frac{r}{1+g}S^*(s)\right) + \tau\left(w\bar{s} + rK^*\right) - \mathbf{1}_{[q90,\infty]}(s)\tau_1\frac{S^*(s)}{1+g} + \tau_1\Lambda^* - c_0\right], \tag{23}$$

where

$$\mathbf{1}_{[q90,\infty]}(s) = \begin{cases} 1, & \text{if } s \geq q90, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\Lambda^* = \int_{q90}^{\infty} \frac{S^*(s)}{1+g} dF(s)$$

is the steady-state value of capital held by the top decile, and where w, r, and K^* are steady-state values. In the income term in Eq. 23, we collect the capital tax if and only if $s \geq q90$, but everyone receives the demogrant $\tau_1\Lambda^*$. Gathering together terms, we can rewrite (23) as

$$\frac{S^*(s)}{1+g} = \left(\frac{(1-\tau)ws + \tau w\bar{s} + \tau rK^* + \tau_1\Lambda^* - c_0}{\frac{g}{1-\alpha} - r(1-\tau) + \mathbf{1}_{[q90,\infty]}(s)\tau_1}\right) \text{ for } s \geq s_1. \tag{24}$$

We continue to have the market-clearing equation for capital

$$K^* = \int_{s_1}^{\infty} \frac{S^*(s)}{1+g} dF(s). \tag{25}$$

The upper bound of types who save zero in steady state is defined by

$$s_1 = \frac{c_0 - \tau\left(w\bar{s} + rK^*\right) - \tau_1\Lambda^*}{(1-\tau)w}.$$

We integrate (24) over the interval $[s_1, \infty)$. The left-hand side becomes K^* by Eq. 25. This new equation contains two unknowns, K^* and Λ^*: note that w, r, and s_1 are all functions of K^* and Λ^*. Secondly, we integrate (24) over the interval $[q90, \infty)$: then the left-hand side of the new equation is Λ^*, and thus we have a second equation in K^* and Λ^*. We now solve these two equations simultaneously for K^* and Λ^*, for various values of the capital tax τ_1. A solution is the steady-state that we seek.[9]

We calculated steady states for values of the capital levy between 0 and 3 %. In this interval, the bottom half of the population continues to accumulate zero wealth: they use the capital levy demogrant to augment consumption. However, the fortunes of the patrimonial middle class, from the 50th to 90th centile, improve dramatically. In Fig. 8, we plot the wealth shares of three quantile groups of the population at the steady state, as a function of the capital levy. With an increase of the wealth tax from zero to 3 %, the wealth share of the

[9] The procedure described is the correct one as long as $s_1 \leq q90$. This turns out to be the case in the region that we examine. If $s_1 > q90$, a slightly different procedure must be used.

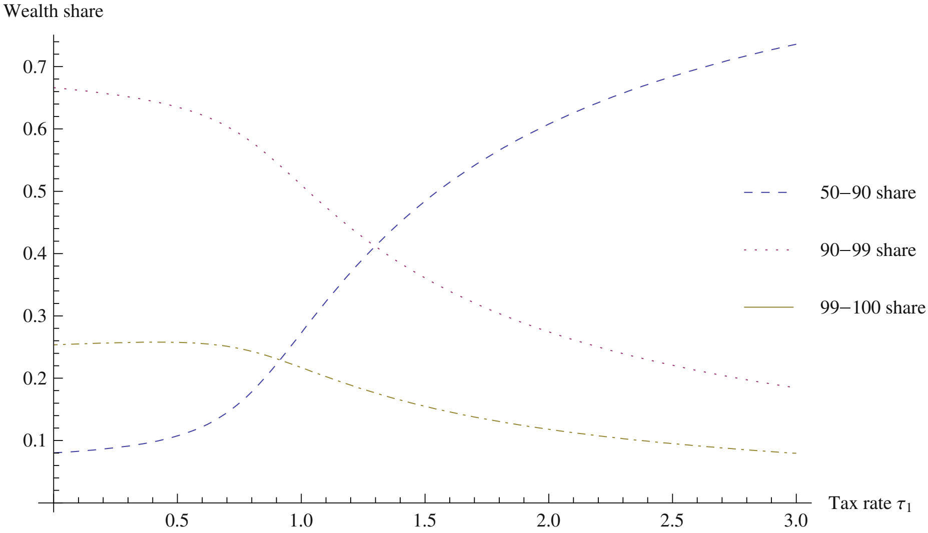

Fig. 8 Quantile wealth shares as function of capital levy

middle class increases from below 10 % to almost 70 %; obviously, this is at the expense of the top decile group.

In Figs. 9, 10 and 11, we plot the interest rate, capital-output ratio and labor's share in national income as a function of the capital tax. It appears that most of the action occurs as the tax increases from zero to 1 %. These variables are all quite stable in the higher part of the range, although the wealth shares continue to change quite dramatically.

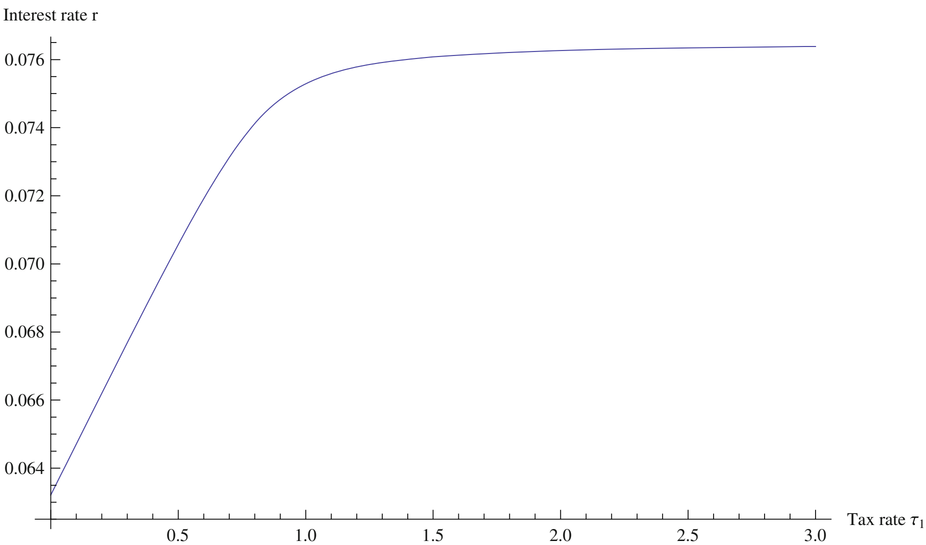

Fig. 9 Interest rate as function of capital levy

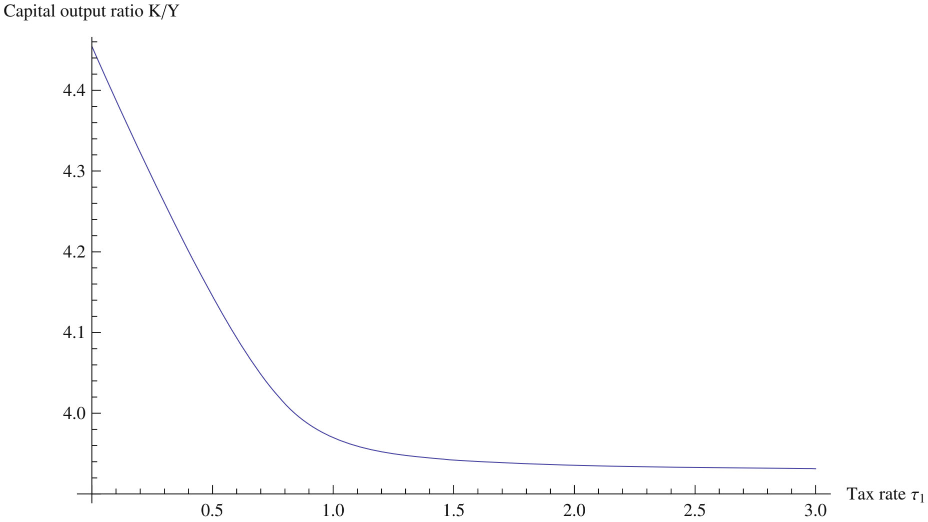

Fig. 10 Capital-output ratio as function of capital levy

The lesson seems to be that quite moderate capital taxation (vastly short of full appropriation of capital) has a dramatic effect on the fortunes of the middle class, but no effect on the wealth of the bottom half, who continue to own nothing. Their consumption, however, increases due to the capital levy demogrant. Evidently, other strategies must be used to create wealth for the bottom half of the income distribution.

What is the relative size of the demogrant from taxing capital income, which is $\tau r K^*$, and from the capital levy, which is $\tau_1 \Lambda^*$? We plot these two demogrants in Fig. 12. We see that at a 3 % capital levy, the capital-stock demogrant is only about one-third the value of

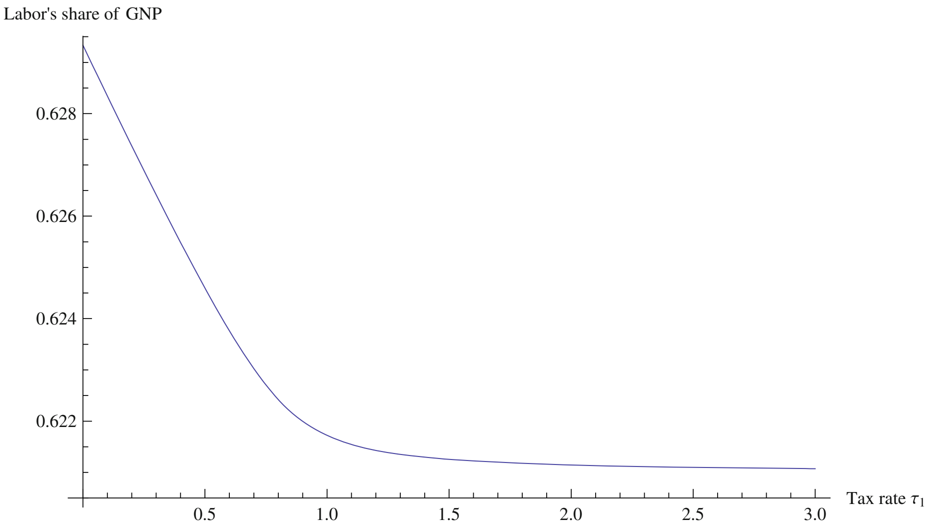

Fig. 11 Labor share as a function of capital levy

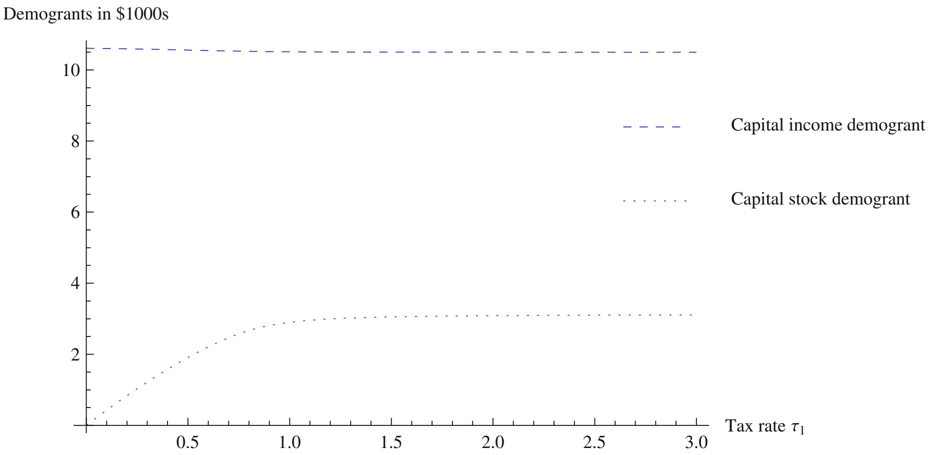

Fig. 12 Demogrants as a function of capital levy

the capital-income demogrant. As Piketty (2014) emphasizes, a principal value to having a capital stock tax, even a small one, is that it would establish statistics on the distribution of the wealth, which would presumably invigorate the social movement for redistribution.

5 Conclusion

We close by drawing attention to some key points. We believe modeling consumers as seeking to reach a socially acceptable and culturally determined level of consumption, which we denoted c_0, and beyond that accumulating wealth for its own sake, in addition to augmenting consumption, is a good approximation to reality in a capitalist society in which high consumption and accumulation of wealth are prized as signals of success. In the vanilla model (Section 2) and its three variants (including the introduction of social mobility in Section C of the Online Supplementary Material), steady states always exist. Although the top quantiles of the distribution can own, in the limit, large fractions of total wealth, these steady-state values are less than unity. When there are differential rates of return to capital, depending on the size of the investment, even if the top 1 % receive a rate of return that is an arbitrarily high factor of that received by the bottom 99 %, the top 1 % does not in the limit own all the capital.

It is striking that the only variant we studied that succeeds in creating wealth for the bottom half of the distribution is a reduction in the level of c_0. If c_0 is halved, from $100,000 to $50,000 per annum, the bottom 90 % of the distribution converge to owning 75 % of the wealth – almost their per capita share. We conjecture that a lower c_0 may correspond to the social compact in Europe. If that is so, we can predict a quite different long-run distribution of wealth in Europe, and a higher capital/output ratio in Europe than in the US.

A capital tax levied on the top decile at a modest rate (below 3 %), and redistributed as a demogrant, has a very dramatic effect on the wealth share owned by the middle class, which increases from 10 % to 70 %. It has no effect on the wealth held by the bottom half, who consume the demogrant in an attempt to reach the aspirational consumption level. Intergenerational mobility (see the Online Supplementary Material) provides some wealth to the bottom half, but it also increases the wealth concentration at the top. This is apparently due to the fact that when wealth travels down the distribution through low-earning children inheriting from wealthy fathers, it is largely consumed rather than saved.

Not only do steady states exist in all our variants, but it appears that convergence to the steady state occurs from an arbitrary initial distribution of wealth following our dynamics. This is a conjecture, stimulated by our simulations. A proof would presumably show that the mapping of the wealth function at date t to date $t + 1$ is a contraction mapping. We illustrate convergence to the steady state from the 2012 wealth distribution of Saez and Zucman (2016). Two features are noteworthy: convergence takes a very long time, and in several variables of interest it is not monotonic. This non-monotonicity illustrates another theme from Piketty (2014): that it is imprudent to attempt to deduce general dynamical laws of capitalism from time series that are short.

Acknowledgments We thank two anonymous referees, Jess Benhabib, Alberto Bisin, as well as audiences in Marseille ("Dynamics of Inequalities and their Perceptions" conference), UQAM and Northwestern University for their comments and suggestions. The usual disclaimer applies.

References

Aiyagari, S.R.: Uninsured idiosyncratic risk and aggregate savings. Q. J. Econ. **109**(3), 659–684 (1994)

Bach, L., Calvet, L., Sodini, P.: Rich pickings? Risk, return, and skill in the portfolios of the wealthy. Unpublished paper (2015)

Benhabib, J., Bisin, A.: Skewed wealth distributions: Theory and empirics. NBER Working Paper 21924 (2016)

Benhabib, J., Bisin, A., Zhu, S.: The distribution of wealth and fiscal policy in economies with finitely-lived agents. Econometrica **79**, 123–157 (2011)

Benhabib, J., Bisin, A., Zhu, S.: Wealth distributions in Bewley models with capital income. J. Econ. Theory **159-A**, 489–515 (2015)

Bewley, T.: A difficulty with the optimum quantity of money. Econometrica **51**, 1485–1504 (1983)

Carroll, C., Slacalek, J., Tokuoka, K.: The distribution of wealth and the MPC: Implications of new European data. Am. Econ. Rev. **104**(5), 107–11 (2014)

Chetty, R., Hendren, N., Kline, P., Saez, E.: Where is the land of opportunity? The geography of intergenerational mobility in the United States. NBER Working Paper 19843 (2014)

Constantinides, G., Duffie, D.: Asset pricing with heterogeneous consumers. J. Polit. Econ. **104**, 219–40 (1996)

Cooper, G.: Voluntary tax? New perspectives on sophisticated tax avoidance, Columbia Law Review, 77 (2), 161–247 (1979)

Fagereng, A., Guiso, L., Malacrino, D., Pistaferri, L.: Wealth return dynamics and heterogeneity. Unpublished paper (2015)

Fagereng, A., Guiso, L., Malacrino, D., Pistaferri, L.: Heterogeneity in returns to wealth, and the measurement of wealth inequality. Am. Econ. Rev. Pap. Proc. **106**(5), 651–655 (2016)

Hubbard, R.G., Skinner, J., Zeldes, S.P.: Precautionary saving and social insurance. J. Polit. Econ. **103**, 360–99 (1995)

Hugget, M.: Wealth distribution in life-cycle economies. J. Monet. Econ. **38**, 469–494 (1996)

Kopczuk, W., Lupton, J.: To leave or not to leave: The distribution of bequest motives. Rev. Econ. Stud. **74**, 207–235 (2007)

Piketty, T.: Capital in the 21st Century. Harvard University Press (2014)

Piketty, T., Zucman, G.: Wealth and inheritance in the long run. Chapter 15 of Handbook of Income Distribution 2B. Elsevier (2015)

Quadrini, V., Ríos-Rull, J.-V.: Understanding the U.S. distribution of wealth, Federal Reserve Bank of Minneapolis Quarterly Review **21**(2), 22–36 (1997)

Saez, E., Piketty, T.: Income inequality in the United States, 1913–1998. Q. J. Econ. **118**(1), 1–39 (2003)

Saez, E., Zucman, G.: Wealth inequality in the United States since 1913: Evidence from capitalized income tax data. Q. J. Econ., forthcoming (2016). Online appendix at http://gabriel-zucman.eu/files/SaezZucman2016QJEAppendix.pdf

Weber, M.: The Protestant Ethic and the Spirit of Capitalism. Charles Scribners Sons, New York (1905)