Math Camp

Homework 1

(1) Write the following sets as lists of elements:

(a) $A = \{x \in \mathbb{N} \mid x > 3 \text{ and } x < 9\}$

$A = \{4, 5, 6, 7, 8\}$

(b) $B = \{a \in \mathbb{N} \mid a < 30 \text{ and } a = b^2 + 2 \text{ for some } b \in \mathbb{N}\}$

$B = \{2, 3, 6, 11, 18, 27\}$

(c) $C = \{t \in \mathbb{Q} \mid t^2 = 16\}$

$C = \{-4, 4\}$

(d) $D = \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Z}, y \in \mathbb{Z}, \text{ and } x^2 + y^2 = 5\}$

$D = \{(1, 2), (-1, 2), (1, -2), (-1, -2), (2, 1), (-2, 1), (2, -1), (-2, -1)\}$

(2) Let $A = \{1, 2\}$, $B = \{2, 3, 4, 5\}$, $C = \{3, 4, 5, 6, 7\}$.
Compute $A \cap B$, $A \cap C$, and $A \cup (B \cap C)$.

$A \cap B = \{2\}$
$A \cap C = \emptyset$
$A \cup (B \cap C) = \{1, 2, 3, 4, 5\}$

(3) Find the range of each of the following functions f.

(a) $f : \{0, 1, 2, 3\} \to \mathbb{Z}$ defined by $f(x) = x^2 - 2x$

range of $f = \{0, -1, 6\}$. Notice that $f(0) = 0$ and $f(2) = 0$, but 0 does not appear twice in the range (elements can't be in a set twice).

(b) $f : [-2, 2] \to \mathbb{R}$ defined by $f(x) = 3x + 5$

range of $f = [-1, 11]$.

(c) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 3$

range of $f = [3, \infty)$.

(d) $f : (-\infty, 3) \cup (3, \infty) \to \mathbb{R}$ defined by $f(x) = \dfrac{2x}{x-3}$
(Hint: there is only one real number that is not in the range. One approach is to write $f(x) = a$ and try to solve for x in terms of a by clearing denominators. You'll be able to do this for all $a \in \mathbb{R}$ with one exception.)

range of $f = \{x \in \mathbb{R} \mid x \neq 2\}$. Every real number except 2 can be attained as the output of the function. Setting

$$\frac{2x}{x-3} = a$$

we multiply both sides by $x - 3$ to get $2x = ax - 3a$. Rearranging, this becomes $(2-a)(x) = -3a$ and so $x = \dfrac{-3a}{2-a}$. This gives a valid x for every a except $a = 2$, where there are division by 0 issues.

Notice that because of the -3 in the denominator, the fraction $\dfrac{2x}{x-3}$ can never equal 2, as the numerator is never exactly twice the denominator.

(4) Write an equation, in $y = mx + b$ form, of the line that passes through the points $(2, 4)$ and $(4, 0)$ in $\mathbb{R}^2$.

The slope of the line is

$$m = \frac{0-4}{4-2} = -2.$$

Thus the line is in $y = -2x + b$ form. Using the point $(2, 4)$ we have $4 = -2(2) + b$, from which $b = 8$. So the line is $y = -2x + 8$.

(5) Let $A = [2, 5]$ and $B = [3, 10)$. Write $A \cup B$, $A \cap B$, and $A \cap \overline{B}$ using interval notation.

$A \cup B = [3, 10)$.
$A \cap B = [3, 5]$.
$A \cap \overline{B} = [2, 3)$. Note the complement is taken with respect to $\mathbb{R}$.

(6) Let the functions $f, g, h : \mathbb{R} \to \mathbb{R}$ by defined by $f(x) = x^2$, $g(x) = 2x + 4$, and $h(x) = \frac{1}{2}x - 2$.
(a) Compute $f \circ g$ and $g \circ f$.

$f \circ g = (2x + 4)^2$ and $g \circ f = 2x^2+$.

(b) Show that g and h are inverse functions, i.e., show that $g \circ h = x$ and $h \circ g = x$.

$$g \circ h = 2(\tfrac{1}{2}x - 2) + 4 = x - 4 + 4 = x$$
$$h \circ g = \tfrac{1}{2}(2x + 4) - 2 = x + 2 - 2 = x$$

(7) Are the following functions injective? Surjective? Explain.

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 3$

This function is injective. If $f(x_1) = f(x_2)$, then $2x_1 + 3 = 2x_2 + 3$. Subtracting 3 and then dividing by 2, we have $x_1 = x_2$. Thus the only way x_1 and x_2 can produce the same output is if they are indeed the same number. (Alternatively, the function has positive derivative everywhere, so is increasing everywhere.)

It is also surjective. Let $a \in \mathbb{R}$. Setting $2x + 3 = a$, we can solve and get $x = \frac{a-3}{2}$. So there is x such that $f(x) = a$ for every a.

(b) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 2x + 3$

This function is injective for exactly the same reason as above, but it is no longer surjective. Trial and error quickly shows that only odd numbers are in the range of f. For example, there is no $x \in \mathbb{Z}$ with $f(x) = 2$. This works for $x = -\frac{1}{2}$, but of course this is not in $\mathbb{Z}$.