Math Camp

Homework 10

(1) For what value of a are the vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \\ a \end{pmatrix}$ linearly dependent?

The easiest approach is to take the determinant of the 3×3 matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ 1 & 0 & a \end{pmatrix},$$

which is $\det(A) = (1)(2)(a) - (-1)(a-1) = 3a - 1$. Setting $a = 1/3$ makes the determinant zero, and therefore makes the matrix singular and the columns linearly dependent. With this choice of a, it can be checked that 1/3 the first vector plus 1/3 the second equals the third.

(2) Find the rank of the matrix $\begin{pmatrix} 1 & 2 & 1 & 2 \\ -1 & 3 & 0 & -1 \\ 2 & -1 & 1 & 3 \\ 1 & 7 & 2 & 3 \end{pmatrix}$.

We row reduce the matrix:

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ -1 & 3 & 0 & -1 \\ 2 & -1 & 1 & 3 \\ 1 & 7 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 5 & 1 & 1 \\ 0 & -5 & -1 & -1 \\ 0 & 5 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 5 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix is not yet in reduced row echelon form, but it can already be seen that there will only be 2 nonzero rows in the RREF. So the rank of A is 2.

(3) For the following matrices, compute the eigenvalues, and then find an eigenvector that corresponds to each eigenvalue.

(a) $\begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$

$$p_A(x) = \det(xI - A) = \det \begin{pmatrix} x-1 & 2 \\ -1 & x-4 \end{pmatrix} = (x-1)(x-4) + 2 = x^2 - 5x + 6$$

The roots of $p_A(x)$ are $\lambda_1 = 2$ and $\lambda_2 = 3$. For the eigenvalue 2, an eigenvector v_1 must satisfy

$$\begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} v_1 = \mathbf{0}$$

One is $v_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

For the eigenvalue 3, an eigenvector v_2 must satisfy

$$\begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix} v_2 = \mathbf{0}$$

so one possibility is $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(b) $\begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}$

Here the characteristic polynomial is $(x-1)(x-9)-9 = x^2 - 10x$, so the eigenvalues are 0 and 10. For the eigenvalue 0, an eigenvector v_1 must satisfy

$$\begin{pmatrix} -1 & -3 \\ -3 & -9 \end{pmatrix} v_1 = \mathbf{0}$$

One such vector is $v_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

For the eigenvalue 10, an eigenvector v_1 must satisfy

$$\begin{pmatrix} 9 & -3 \\ -3 & 1 \end{pmatrix} v_2 = \mathbf{0}$$

One is $v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

(c) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & -1 \\ 0 & 0 & 3 \end{pmatrix}$ The eigenvalues are given by the numbers on the diagonal: $\lambda_1 = 1, \lambda_2 = 4$, and $\lambda_3 = 3$. The eigenvector v_i can be found by solving

$$\begin{pmatrix} \lambda_i - 1 & -2 & 0 \\ 0 & \lambda_i - 4 & -1 \\ 0 & 0 & \lambda_i - 3 \end{pmatrix} v_i = \mathbf{0}$$

For $\lambda_1 = 1:$

$$\begin{pmatrix} 0 & -2 & 0 \\ 0 & -3 & -1 \\ 0 & 0 & -2 \end{pmatrix} v_1 = \mathbf{0}$$

One such eigenvector is $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

For $\lambda_2 = 4:$

$$\begin{pmatrix} 3 & -2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} v_1 = \mathbf{0}$$

One such eigenvector is $v_2 = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$.

For $\lambda_3 = 3:$

$$\begin{pmatrix} 2 & -2 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} v_1 = \mathbf{0}$$

One such eigenvector is $v_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.