Math Camp

Homework 3

(1) Compute $\lim_{a\to 3} \dfrac{a^2+a-12}{a^2-9}$.

$$\lim_{a\to 3} \frac{a^2+a-12}{a^2-9} = \lim_{a\to 3} \frac{(a+4)(a-3)}{(a+3)(a-3} = \lim_{a\to 3} \frac{(a+4)}{(a+3)} = \frac{7}{6}.$$

(2) Let $f(t) = \dfrac{2t^2-8t+8}{t^2+3t-10}$. Compute $\lim_{t\to 2} f(t)$ and $\lim_{t\to\infty} f(t)$.

$$\lim_{t\to 2} \frac{2t^2-8t+8}{t^2+3t-10} = \lim_{t\to 2} \frac{2(t-2)^2}{(t+5)(t-2)} = \lim_{t\to 2} \frac{2(t-2)}{t+5} = 0$$

$$\lim_{t\to\infty} \frac{2t^2-8t+8}{t^2+3t-10} = \frac{2}{1} = 2.$$

(3) Let $f(x) = \dfrac{e^{-x}+2e^x}{2e^{-x}+e^x}$. Compute $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

As $x \to \infty$, the terms with e^{-x} become negligibly small. So the expression is essentially just $\frac{2e^x}{e^x}$, and the limit is 2.

As $x \to -\infty$, the terms with e^x become negligible (e^{-x} is actually quite large). So the expression is essentially just $\frac{e^{-x}}{2e^{-x}}$, and the limit is $\frac{1}{2}$.

(4) (a) Estimate $\lim_{x\to\infty} \sqrt{x^2+x} - x$ with the aid of a calculator or computer.

(b) Compute the limit by multiplying by $\dfrac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}$ and simplifying the numerator.

The limit is $\frac{1}{2}$. The computation looks like

$$(\sqrt{x^2+x}-x)\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x} = \frac{x^2+x-x^2}{\sqrt{x^2+x}+x} = \frac{x}{\sqrt{x^2+x}+x}.$$

Looking at this last fraction, the term $\sqrt{x^2+x}$ seems like it should be close to $\sqrt{x^2} = x$ when x is very large. Using this approximation, the limit would equal 1/2, which is borne out by numerical evidence.

Note: If we were to use the approximation $\sqrt{x^2+x} \approx x$ at the start of the problem, we would wrongly expect the limit to equal 0. Indeed, $\sqrt{x^2+x}$ is close to x, but the answer to the problem shows that the correct approximation is $\sqrt{x^2+x} \approx x + \frac{1}{2}$. Using this in the fraction, the 1/2 is negligible and goes away in the limit.

(5) Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x+1 & : x \leq 0 \\ 1-x & : 0 < x < 2 \\ 2 & : x = 2 \\ x^2 - 5 & : 2 < x < 3 \\ 1 & : x \geq 3 \end{cases}$$

Where is f continuous? (Your answer could take the form of a union of intervals.)

Clearly f is continuous at all points except possibly $0, 2, 3$. Taking limits from the left and right, we see that

$$lim_{x \to 0} f(x) = f(0) = 1,$$

so f is continuous at 0. At $x = 2$, the left-hand limit equals -1 but $f(2) = 2$, so f is not continuous. At $x = 3$, the left-hand limit equals 4 but the $f(3) = 1$, so again f is not continuous.

The largest domain on which f is continuous is $(-\infty, 2) \cup (2, 3) \cup (3, \infty)$.

(6) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 + 3 & : x \leq 1 \\ ax + b & : 1 < x < 4 \\ 4 - \frac{8}{x} & : x > 4 \end{cases}$$

Find the unique values of a and b that make f continuous on all of $\mathbb{R}$.

This is a matter of "patching" the function so that it is continuous at $x = 1$ and $x = 4$ (note we have two parameters to work with). The limit from the left at $x = 1$ is $1^2 + 3 = 4$, so we need the line $ax + b$ to go through the point $(1, 4)$. At $x = 4$ we need it to go through $(4, 2)$. The line through these two points is given by the equation $y = (-2/3)x + 14/3$.

(7) Use the limit definition of the derivative to compute the derivative of:

(a) $f(x) = x^4$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^4 - x^4}{h} = \lim_{h \to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h}$$

$$= \lim_{h \to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} = \lim_{h \to 0} 4x^3 + 6x^2h + 4xh^2 + h^3 = 4x^3.$$

(b) $g(x) = \dfrac{1}{x^2}$

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \lim_{h \to 0} \frac{1}{h} \frac{x^2 - (x+h)^2}{x^2(x+h)^2} = \lim_{h \to 0} \frac{-2xh - h^2}{hx^2(x+h)^2} = \lim_{h \to 0} \frac{-2x - h}{x^2(x+h)^2} = \frac{-2x}{x^4} = \frac{-2}{x^3}.$$

(8) Use your work in (7) to answer the following:

(a) Find the tangent line to the graph of $y = x^4$ at the point $(2, 16)$.

The line has slope $4(2^3) = 32$, and so is $y = 32x - 48$.

(b) Find the tangent line to the graph of $y = \frac{1}{x^2}$ at the point $(1, 1)$.

The line has slope -2 and so is $y = -2x + 3$.

(9) Suppose you know that $f(x) = x^3 - 12x$ has derivative $f'(x) = 3x^2 - 12$. Find all intervals where f is increasing and all intervals where f is decreasing.

The derivative has the shape of an upward-opening parabola which passes through the x-axis at $x = 2$ and $x = -2$. The derivative is positive, and the function is increasing, for $x \in (-\infty, -2) \cup (2, \infty)$. The derivative is negative, and the function is decreasing, for $x \in (-2, 2)$.