JOINT EQUIDISTRIBUTION OF MAXIMAL FLAT CYLINDERS AND HOLONOMIES FOR ANOSOV HOMOGENEOUS SPACES

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ABSTRACT. Let G be a connected semisimple real algebraic group and $P < G$ be a minimal parabolic subgroup with Langlands decomposition $P = MAN$. Let $\Gamma < G$ be a Zariski dense *Anosov* subgroup with respect to P. Since Γ is Anosov, the set of conjugacy classes of primitive elements of Γ is in one-to-one correspondence with the set of (positively oriented) maximal flat cylinders in $\Gamma \backslash G/M$. We describe the joint equidistribution of maximal flat cylinders and their holonomies as their circumferences tend to infinity. This result can be viewed as the Anosov analogue of the joint equidistribution result of closed geodesics and holonomies in rank one by Margulis–Mohammadi–Oh [MMO14].

CONTENTS

1. INTRODUCTION

1.1. **Background and setup.** Let G be a connected semisimple real algebraic group. Let $P < G$ be a minimal parabolic subgroup with Langlands decomposition $P = MAN$ where N is the unipotent radical of P, $A = \exp \mathfrak{a}$ is a maximal real split torus and M is a maximal compact subgroup of P commuting with A. Let $\Gamma < \mathrm{G}$ be a torsion-free discrete subgroup. In this paper, we study the equidistribution of nontrivial closed A-orbits in $\Gamma \backslash G/M$, or equivalently, nontrivial closed AM-orbits in $\Gamma \backslash G$, and their holonomies when Γ is a torsion-free Zariski dense Anosov subgroup with respect to P.

Let $\mathcal{F} := G/P$ denote the Furstenberg boundary and $\mathcal{F}^{(2)}$ denote the unique open G-orbit in $\mathcal{F} \times \mathcal{F}$. A Zariski dense discrete subgroup $\Gamma < G$ is *Anosov* with respect to P if Γ is a finitely generated Gromov hyperbolic group and admits a Γ-equivariant continuous embedding from the Gromov boundary $\partial\Gamma$ of Γ to $\mathcal{F}$ such that if $x, y \in \mathcal{F}$ are the images of two distinct points in $\partial\Gamma$, then $(x, y) \in \mathcal{F}^{(2)}$. The notion of Anosov subgroups (with respect to any parabolic subgroup of G) was first introduced by Labourie [Lab06] for surface groups and later generalized by Guichard–Wienhard [GW12] to Gromov hyperbolic groups (cf. [KLP17, GGKW17, Wie18]). Throughout the paper, all Anosov subgroups are Anosov with respect to P. Anosov subgroups are regarded as natural higher rank generalizations of Zariski dense convex-cocompact subgroups since the two notions coincide when $\mathrm{rank}(G) = 1$.

Let $\Gamma < G$ be a torsion-free Zariski dense Anosov subgroup. Let $\mathfrak{a}^+ \subset \mathfrak{a}$ be the positive Weyl chamber associated to N and let $A^+ = \exp \mathfrak{a}^+$. Let $\psi_\Gamma : \mathfrak{a}^+ \to \{-\infty\} \cup [0, \infty)$ denote the growth indicator function of Γ introduced by Quint [Qui02a] (Definition 3.8). The function ψ_Γ is a higher rank generalization of the critical exponent in rank one. We fix a *tangent form* $\psi \in \mathfrak{a}^*$, that is,

$$\psi : \mathfrak{a} \to \mathbb{R} \text{ linear}, \psi \geq \psi_\Gamma, \psi(\mathsf{v}) = \psi_\Gamma(\mathsf{v}) = 1 \text{ for some } \mathsf{v} \in \mathrm{int}\,\mathfrak{a}^+, \quad (1.1)$$

where $\mathrm{int}\,\mathfrak{a}^+$ denotes the interior of $\mathfrak{a}^+$. The space of such tangent forms is homeomorphic to $\mathbb{R}^{\mathrm{rank}(G)-1}$. Up to multiplicative constant, there exists a unique (Γ, ψ)-conformal measure on $\mathcal{F}$ and it is necessarily supported on the limit set Λ of Γ as shown by Lee–Oh [LO20b, Theorem 1.3], [LO22, Theorem 1.2]. Hence, up to multiplicative constant, there exists a unique Bowen–Margulis–Sullivan (BMS) measure

$$\mathsf{m}_\psi$$

on $\Gamma\backslash G/M$ associated to ψ (Definition 3.19). When $\mathrm{rank}(G) \geq 2$, m_ψ is an infinite measure (see [Sam15, Theorem 3.5], see also [LO20b, Corollary 4.9]). Using the Hopf parametrization $\mathcal{F}^{(2)} \times \mathfrak{a} \cong G/M$ (Definition 3.16), the support of m_ψ is

$$\Omega := \mathrm{supp}\,\mathsf{m}_\psi = \Gamma\backslash(\Lambda^{(2)} \times \mathfrak{a}) \subset \Gamma\backslash(\mathcal{F}^{(2)} \times \mathfrak{a}) \cong \Gamma\backslash G/M,$$

where $\Lambda^{(2)} := (\Lambda \times \Lambda) \cap \mathcal{F}^{(2)}$ (Definition 3.19).

For any closed AM-orbit $C = \Gamma g AM \subset \Gamma\backslash G/M$, its group of periods $g^{-1}\Gamma g \cap AM$ is either trivial or isomorphic to $\mathbb{Z}$, and in the latter case, C is a *maximal flat cylinder*;

$$C \cong (g^{-1}\Gamma g \cap AM)\backslash AM/M \cong \mathbb{R}^{\mathrm{rank}(G)-1} \times \mathbb{S}^1.$$

We emphasize that the closed AM-orbits being cylinders is a feature of Anosov subgroups. For instance, if $\Delta < G$ is a torsion-free lattice, then there exists a closed AM-orbit in $\Delta\backslash G$ whose group of periods is isomorphic to

$\mathbb{Z}^{\mathrm{rank}(G)}$ and hence the closed AM-orbit is a compact torus [PR72, Theorem 2.8].

When $g^{-1}\Gamma g \cap (\mathrm{int}\, A^+)M \neq \emptyset$, where $\mathrm{int}\, A^+$ denotes the interior of A^+, we call $\Gamma g A M \subset \Gamma\backslash G/M$ a *positively oriented* maximal flat cylinder. Let $[\Gamma]$ denote the set of conjugacy classes in Γ and let

$$[\Gamma_{\mathrm{prim}}] := \{[\gamma] \in [\Gamma] : \gamma \in \Gamma \text{ primitive}\}.$$

The set of all positively oriented maximal flat cylinders is in one-to-one correspondence with $[\Gamma_{\mathrm{prim}}]$. We also note that positively oriented maximal flat cylinders are precisely the maximal flat cylinders contained in Ω (Lemma 4.14). In the rest of the introduction, all maximal flat cylinders are positively oriented unless stated otherwise. Denote by $\mathcal{C}_\Gamma$ the set of all maximal flat cylinders:

$$\mathcal{C}_\Gamma = \{C \subset \Omega : C \text{ is a maximal flat cylinder}\}.$$
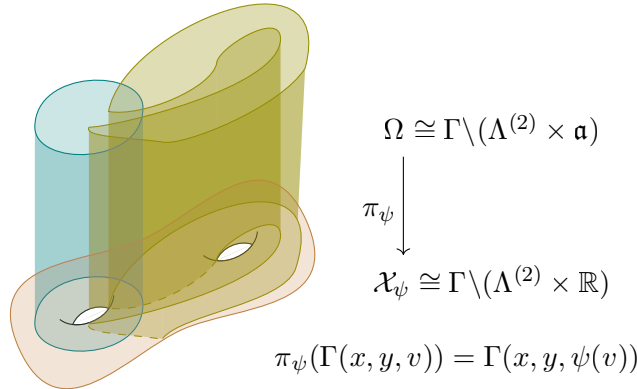

$$\Omega \cong \Gamma\backslash(\Lambda^{(2)} \times \mathfrak{a})$$

$$\pi_\psi \Big\downarrow$$

$$\mathcal{X}_\psi \cong \Gamma\backslash(\Lambda^{(2)} \times \mathbb{R})$$

$$\pi_\psi(\Gamma(x,y,v)) = \Gamma(x,y,\psi(v))$$

FIGURE 1. Maximal flat cylinders in the vector bundle $\pi_\psi : \Omega \to \mathcal{X}_\psi$. In this picture, $\mathcal{X}_\psi$ is depicted as the unit tangent bundle of the closed surface shown.

Since Γ is Anosov and torsion-free, every nontrivial element of $\gamma \in \Gamma$ is *loxodromic*, that is, conjugate to some $\exp(\lambda(\gamma))m \in (\mathrm{int}\, A^+)M$. The element

$$\lambda(\gamma) \in \mathrm{int}\, \mathfrak{a}^+$$

is unique and called the *Jordan projection* of γ and $m \in M$ is in a unique conjugacy class

$$h_\gamma = [m] \in [M]$$

called the *holonomy* of γ. If $C \in \mathcal{C}_\Gamma$ corresponds to $[\gamma] \in [\Gamma_{\mathrm{prim}}]$, then we define the ψ-*circumference* and *holonomy* of C as

$$\ell_\psi(C) := \psi(\lambda(\gamma)) \in (0,\infty) \quad \text{and} \quad h_C := h_\gamma \in [M],$$

respectively (see Theorem 4.3 for the positivity of ψ-circumferences).
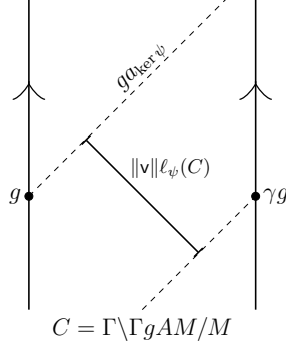

FIGURE 2. A closed A-orbit in $\Gamma\backslash G/M$ laid flat. The left and right boundaries are identified so that g and γg are identified. The ψ-circumference of C can be seen in the width of the slice produced by the $\ker\psi$ hyperplane.

We give a geometric description of ψ-circumferences. As stated before, every maximal flat cylinder is contained in Ω. Another feature of Anosov subgroups is that Ω is homeomorphic to a trivial $\ker\psi$-vector bundle over a compact metric space $\mathcal{X}_\psi$ equipped with a finite measure $m_{\mathcal{X}_\psi}$ on $\mathcal{X}_\psi$ such that

$$d\mathsf{m}_\psi\big|_\Omega = dm_{\mathcal{X}_\psi}\,du, \tag{1.2}$$

for some Lebesgue measure du on $\ker\psi$ (see Subsection 4.2 for details). The ψ-circumference of any maximal flat cylinder is the length of its intersection curve with $\mathcal{X}_\psi$ (Fig. 1). More precisely, $\mathcal{X}_\psi$ is equipped with a *translation flow* and the intersection curve of a maximal flat cylinder with $\mathcal{X}_\psi$ is a periodic orbit of the translation flow (Lemma 4.18). See Fig. 2 for another interpretation.

1.2. **Statement of the main results.** The main result of our paper is the joint equidistribution of maximal flat cylinders and their holonomies as their ψ-circumferences tend to infinity. For $T > 0$, let $\mathcal{C}_\psi(T)$ denote the set of all maximal flat cylinders with ψ-circumference at most T:

$$\mathcal{C}_\psi(T) := \{C \in \mathcal{C}_\Gamma : \ell_\psi(C) \le T\}.$$

We note that $\mathcal{C}_\psi(T)$ is always a finite set (see Lemma 6.1). To formulate the joint equidistribution statement, denote by V_C the volume measure on a maximal flat cylinder C induced by the Haar measure on AM. Let $\mathrm{C_c}(\Gamma\backslash G/M)$ denote the set of continuous compactly supported functions on $\Gamma\backslash G/M$ and let $\mathrm{Cl}(M)$ denote the set of continuous real-valued class functions on M. For $T > 0$, we define Radon measures μ_T and η_T on the product of $\Gamma\backslash G/M$ and

$[M]$ as follows. For $T > 0$, $f \in C_c(\Gamma \backslash G / M)$ and $\varphi \in \mathrm{Cl}(M)$, let

$$\mu_T(f \otimes \varphi) := \sum_{C \in \mathcal{C}_\psi(T)} V_C(f)\varphi(h_C);$$

$$\eta_T(f \otimes \varphi) := \sum_{C \in \mathcal{C}_\psi(T)} \frac{V_C(f)}{\ell_\psi(C)}\varphi(h_C).$$

Let $M_\Gamma < M$ denote the closed subgroup generated by all holonomies of elements in Γ. Then M_Γ is a normal subgroup of M containing the identity component of M [GR07, Corollary 1.10] so in particular, M_Γ has finite index in M. We note that when M is not connected, M_Γ is not necessarily equal to M (see Subsection 1.7 for example). We now state our main theorem describing the joint equidistribution of maximal flat cylinders and their holonomies.

Theorem 1.3 (Joint equidistribution). *Let $\Gamma < G$ be a Zariski dense Anosov subgroup. For any tangent form $\psi \in \mathfrak{a}^*$ and for any $f \in C_c(\Gamma \backslash G / M)$ and $\varphi \in \mathrm{Cl}(M)$, we have*

$$\lim_{T \to \infty} \frac{\mu_T(f \otimes \varphi)}{e^T} = \frac{1}{|m_{\mathcal{X}_\psi}|}\mathsf{m}_\psi(f) \int_{M_\Gamma} \varphi \, dm_\Gamma;$$

$$\lim_{T \to \infty} \frac{\eta_T(f \otimes \varphi)}{e^T/T} = \frac{1}{|m_{\mathcal{X}_\psi}|}\mathsf{m}_\psi(f) \int_{M_\Gamma} \varphi \, dm_\Gamma,$$

where m_Γ denotes the Haar probability measure on M_Γ.

We note that the right-hand side of Theorem 1.3 is independent of the normalization of m_ψ by (1.2). We also obtain the following equidistribution of holonomies from Theorem 1.3. Since m_ψ is an infinite measure when $\mathrm{rank}(G) \geq 2$, this is not an immediate consequence of Theorem 1.3.

Corollary 1.4 (Equidistribution of holonomies). *Using the same notation as in Theorem 1.3, we have*

$$\sum_{C \in \mathcal{C}_\psi(T)} \varphi(h_C) \sim \frac{e^T}{T} \int_{M_\Gamma} \varphi \, dm_\Gamma \quad \text{as } T \to \infty.$$

Remark 1.5. As a consequence of Theorem 1.3, the set of holonomies of Γ is dense in M_Γ, that is,

$$M_\Gamma = \overline{\{m \in M : \exists C \in \mathcal{C}_\Gamma \text{ such that } h_C = [m]\}}.$$

Remark 1.6. Let $\mathcal{L}$ denote the limit cone of Γ, that is, the smallest closed cone containing the Jordan projections of all $\gamma \in \Gamma$. Theorem 1.3 and Corollary 1.4 can be adapted for linear forms $\psi \in \mathfrak{a}^*$ which are positive on $\mathcal{L} \setminus \{0\}$ by using the fact that $\delta_\psi \psi$ is tangent to ψ_Γ for some $\delta_\psi > 0$ called the *ψ-critical exponent* of Γ (see Theorem 4.3). In this context, our theorem implies the ψ-critical exponent is equal to the *ψ-topological entropy*, that is,

$$\delta_\psi = \lim_{t \to +\infty} \frac{1}{t} \log \#\{\gamma \in \Gamma : \psi(\mu(\gamma)) \leq t\}$$

$$= \lim_{t \to +\infty} \frac{1}{t} \log \#\{[\gamma] \in [\Gamma_{\mathrm{prim}}] : \psi(\lambda(\gamma)) \leq t\}, \quad (1.7)$$

where $\mu : G \to \mathfrak{a}^+$ denotes the Cartan projection (Definition 3.8). Moreover, Corollary 1.4 immediately implies

$$\#\mathcal{C}_\psi(T) = \#\{[\gamma] \in [\Gamma_{\mathrm{prim}}] : \psi(\lambda(\gamma)) \leq t\} \sim \frac{e^{\delta_\psi T}}{\delta_\psi T}, \quad (1.8)$$

where for $f_1, f_2 : (0, \infty) \to \mathbb{R}$, we write

$$f_1 \sim f_2 \iff \lim_{T \to \infty} \frac{f_1(T)}{f_2(T)} = 1.$$

The above (1.7) and (1.8) were proved by Sambarino [Sam14b, Theorem 7.8], [Sam14a, Corollary 4.4] when Γ is the fundamental group of a closed connected negatively curved Riemannian manifold and in view of [Car21, Appendix A], Sambarino's work extends to Anosov subgroups. We also mention [BCKM22, Corollary 11.1] which is a counting result analogous to (1.8) for *cusped* Anosov representations.

1.3. **Joint equidistribution with respect to norm-like functions.** Our proofs also allow us to prove in Section 7 similar results when $\mathcal{C}_\Gamma$ is ordered according to a norm-like function. Let $\mathsf{N} : \mathfrak{a}^+ \to \mathbb{R}$ be a *norm-like* function, that is, N is twice continuously differentiable except possibly at the origin, convex, homogeneous of degree 1 and positive on $\mathcal{L} \setminus \{0\}$. For example, L^p norms are norm-like for $1 \leq p < \infty$. The function N determines an ordering on $\mathcal{C}_\Gamma$:

$$\mathcal{C}_{\mathsf{N}}(T) := \{C \in \mathcal{C}_\Gamma : \mathsf{N}(\lambda(\gamma_C)) \leq T\}.$$

We define the N-critical exponent as

$$\delta_{\mathsf{N}} := \max_{\mathsf{N}(w)=1} \psi_\Gamma(w) > 0. \quad (1.9)$$

For simplicity, we only state here the equidistribution of holonomies with respect to N.

Corollary 1.10 (Equidistribution of holonomies with respect to N)**.** *There exists a constant $0 < c_{\mathsf{N}} \leq 1$ such that for any $\varphi \in \mathrm{Cl}(M)$, we have*

$$\sum_{C \in \mathcal{C}_{\mathsf{N}}(T)} \varphi(h_C) \sim c_{\mathsf{N}} \frac{e^{\delta_{\mathsf{N}} T}}{\delta_{\mathsf{N}} T} \int_{M_\Gamma} \varphi \, dm_\Gamma \quad \text{as } T \to \infty.$$

Moreover, $c_{\mathsf{N}} = 1$ if and only if the Hessian of N at v is identically zero where v is the unique vector achieving the maximum in (1.9).

1.4. **Comparison with rank one case.** When $\mathrm{rank}(G) = 1$, Anosov subgroups coincide with Zariski dense convex cocompact subgroups. Denote the critical exponent of Γ by δ and the set of closed geodesics in $\Gamma \backslash G/M$ of length at most T by

$$\mathcal{G}_\Gamma(T) := \{\text{primitive closed geodesics of length at most } T\}.$$

Then using the dictionary in Table 1 and noting that M is connected when G is rank one and center-free, Corollary 1.4 in those cases says that for all $\varphi \in \mathrm{Cl}(M)$, we have

$$\sum_{C \in \mathcal{G}_\Gamma(T)} \varphi(h_C) \sim \frac{e^{\delta T}}{\delta T} \int_M \varphi \, dm \quad \text{as } T \to \infty.$$
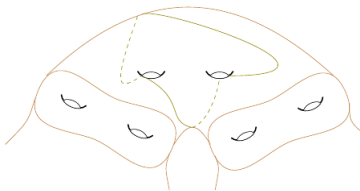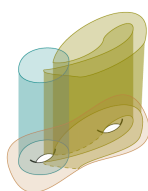
This is a special case of [MMO14, Theorem 1.4].

rank one	higher rank
convex cocompact groups	Anosov groups
primitive closed geodesic C	pos. oriented maximal flat cylinder C
$\delta \times$ (length of C)	ψ-circumference of C
holonomy of C	holonomy of C
finite BMS-measure	infinite BMS-measure associated to ψ

TABLE 1. A dictionary between the rank one setting and the higher rank setting.

For rank one groups, the asymptotic for $\mathcal{G}_\Gamma(T)$ was proved by Margulis [Mar04] [1] when Γ is a uniform lattice, Gangolli–Warner [GW80] when Γ is a nonuniform lattice, and Roblin [Rob03] when Γ is geometrically finite, and the equidistribution of closed geodesics was independently studied by Margulis [Mar04] and Bowen [Bow72a, Bow72b] when Γ is a uniform lattice and Roblin [Rob03] when Γ is geometrically finite. Equidistribution of holonomies was proved by Parry–Pollicott [PP86] when Γ is a uniform lattice in $\mathrm{SO}(n,1)^\circ$. In general rank one groups, equidistribution of holonomies was proved by Sarnak–Wakayama [SW99] when Γ is a lattice and joint equidistribution was proved by Margulis–Mohammadi–Oh [MMO14] when Γ is a Zariski dense geometrically finite subgroup. A joint equidistribution result was obtained by Oh–Pan [OP19] for abelian covers of convex cocompact rank
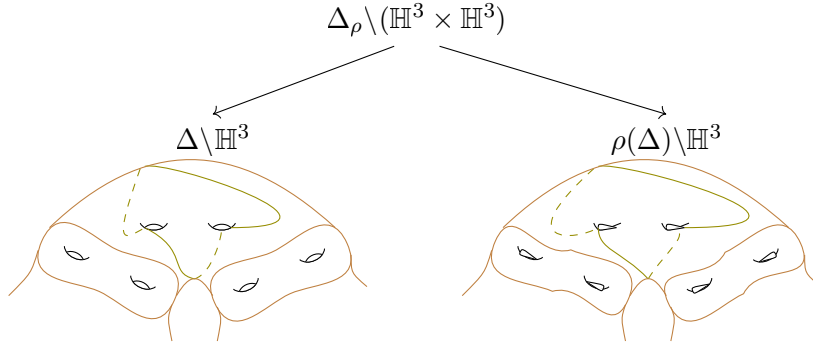
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1. [Mar04] contains Margulis' previously unpublished 1970 thesis. See [Par05] for a review.

one groups; in this case, the BMS measure is infinite unlike in [MMO14]. A key technique in [MMO14] is to use mixing of the frame flow [Win15], and this technique goes back to Margulis' work on the counting and equidistribution of closed geodesics in negatively curved compact manifolds in his 1970 thesis. This mixing is exponential for Γ convex cocompact by Chow–Sarkar [CS22], and for Γ geometrically finite with parabolic elements by Li–Pan–Sarkar [LPS23]. In these cases, the distribution of holonomies can be determined up to an exponential error term (cf. [MMO14, Theorem 1.2]).

1.5. **Some related results in higher rank.** For cocompact lattices of higher rank Lie groups, we mention the thesis of Spatzier [Spa83] where he describes the exponential growth rate of the total volume of maximal flat periodic tori as their regular systole tends to infinity and the recent work by Dang–Li [DL22] on the counting and equidistribution of these maximal flat periodic tori.

1.6. **Example: Self-joinings of convex-cocompact subgroups.** For $G = \mathrm{PSL}_2\,\mathbb{C} \times \mathrm{PSL}_2\,\mathbb{C}$, all Anosov subgroups arise as *convex-cocompact self-joinings*, which have interesting applications to rigidity as shown by Kim–Oh [KO22, KO23a, KO23b]. Let $\Delta < \mathrm{PSL}_2\,\mathbb{C}$ be a Zariski dense convex cocompact subgroup and $\rho : \Delta \to \mathrm{PSL}_2\,\mathbb{C}$ be a convex cocompact discrete faithful representation. Then the *self-joining* Δ_ρ of Δ by ρ is defined as the diagonal embedding of Δ in G via ρ, that is,

$$\Delta_\rho := \{(\gamma, \rho(\gamma)) \in \mathrm{PSL}_2\,\mathbb{C} \times \mathrm{PSL}_2\,\mathbb{C} : \gamma \in \Delta\} < \mathrm{PSL}_2\,\mathbb{C} \times \mathrm{PSL}_2\,\mathbb{C}.$$

Note that Δ_ρ is Zariski dense in $\mathrm{PSL}_2\,\mathbb{C} \times \mathrm{PSL}_2\,\mathbb{C}$ if and only if ρ does not extend to an automorphism of $\mathrm{PSL}_2\,\mathbb{C}$ and is hence conjugation (cf. [KO22, Lemma 4.1]).

For primitive $[\gamma] \in [\Delta]$, let C_γ denote the primitive closed geodesic in $\Delta \backslash \mathbb{H}^3$ corresponding to $[\gamma] \in [\Delta]$ and $\ell(C_\gamma)$ denote the length of C_γ. In this setting, $M_{\Delta_\rho} = M \cong \mathbb{S}^1 \times \mathbb{S}^1$ and we have the following result analogous to [MMO14, Theorem 1.3].

Corollary 1.11 (Holonomy rigidity for convex cocompact groups). *Let $\Delta <$ $\mathrm{PSL}_2\,\mathbb{C}$ be a Zariski dense convex cocompact subgroup with critical exponent δ_Δ. Let Δ_ρ be the self-joining of Δ by a convex-cocompact discrete faithful representation $\rho : \Delta \to \mathrm{PSL}_2\,\mathbb{C}$. Identify holonomies with pairs of angles. If ρ does not extend to $\mathrm{PSL}_2\,\mathbb{C}$, then for any $0 < \theta_1 < \theta_2 < 2\pi$ and any $0 < \theta_3 < \theta_4 < 2\pi$, we have*

$$\#\{C_\gamma : \ell(C_\gamma) \leq T \ \text{and} \ h_{(\gamma,\rho(\gamma))} \in (\theta_1,\theta_2) \times (\theta_3,\theta_4)\}$$
$$\sim \frac{(\theta_2 - \theta_1)(\theta_4 - \theta_3)e^{\delta_\Delta T}}{(2\pi)^2 \delta_\Delta T} \qquad \text{as } T \to \infty.$$

We note that when ρ does extend to $\mathrm{PSL}_2\,\mathbb{C}$, the set of holonomies is contained in the diagonal of $\mathbb{S}^1 \times \mathbb{S}^1$ and hence, is not even dense in $\mathbb{S}^1 \times \mathbb{S}^1$. The representation ρ does not extend if and only if the set of holonomies $\{h_{(\gamma,\rho(\gamma))} \in \mathbb{S}^1 \times \mathbb{S}^1 : \ell(C_\gamma) \leq T\}$ is equidistributed in $\mathbb{S}^1 \times \mathbb{S}^1$ as $T \to \infty$. Hence, Corollary 1.11 can be viewed as a holonomy rigidity statement for convex cocompact subgroups.

1.7. Example: Hitchin representations of surface groups.

Let $\rho : \Gamma_0 \to \mathrm{PSL}_d\,\mathbb{R}$ be a *Hitchin representation*, that is, Γ_0 is the fundamental group of a closed orientable surface of genus at least 2 and ρ can be continuously deformed to $\rho_d \circ \rho_0$, where $\rho_0 : \Gamma_0 \to \mathrm{PSL}_2\,\mathbb{R}$ is some discrete faithful representation and $\rho_d : \mathrm{PSL}_2\,\mathbb{R} \to \mathrm{PSL}_d\,\mathbb{R}$ denotes the irreducible representation which is unique up to conjugation. Suppose $\Gamma = \rho(\Gamma_0) < \mathrm{PSL}_d\,\mathbb{R}$ is Zariski dense.

For $G = \mathrm{PSL}_d\,\mathbb{R}$, $\mathfrak{a} \cong \{(t_1,\dots,t_d) \in \mathbb{R}^d : \sum_i t_i = 0\}$. We choose $\mathfrak{a}^+ = \{(t_1,\dots,t_d) \in \mathbb{R}^d : t_1 \geq \cdots \geq t_d, \sum_i t_i = 0\}$. For each $i \in \{1, 2, \dots, d-1\}$, let $\alpha_i \in \mathfrak{a}^*$ denote the simple root given by

$$\alpha_i(t_1,\dots,t_d) = t_i - t_{i+1}.$$

By Potrie–Sambarino [PS17, Theorem B], α_i is a tangent form. By Labourie [Lab06, Theorem 1.5],

$$M_\Gamma = \{e\}.$$

Denote the eigenvalues of $\gamma \in \Gamma$ by $\lambda_1(\gamma) > \cdots > \lambda_d(\gamma) > 0$ and denote the maximal flat cylinder corresponding to $[\gamma]$ by $C(\gamma)$.

Corollary 1.12. *Let $\Gamma < \mathrm{PSL}_d\,\mathbb{R}$ be a Zariski dense image of a Hitchin representation. Let $i \in \{1,\dots,d-1\}$. Then for any compactly supported continuous function $f : \Gamma \backslash \mathrm{PSL}_d\,\mathbb{R}/M \to \mathbb{R}$, as $T \to \infty$, we have*

$$e^{-T} \sum_{\substack{[\gamma] \in [\Gamma] \\ \lambda_i(\gamma)-\lambda_{i+1}(\gamma) \leq T}} \frac{1}{\lambda_i(\gamma) - \lambda_{i+1}(\gamma)} V_{C(\gamma)}(f) \to \frac{1}{|m_{\mathcal{X}_{\alpha_i}}|} \mathsf{m}_{\alpha_i}(f).$$

1.8. **Outline of the proof of Theorem 1.3.** Our proof follows a similar line of proof as in [MMO14]. However, in our higher rank setting, care is needed to overcome the technical obstructions coming from the higher dimensional nature of A, the BMS-measure being infinite and the A-action not being strong mixing. More precisely, let $g_0 \in G$ and $\varepsilon > 0$. The ε-*flow box centered at g_0* is defined as

$$\mathfrak{B}(g_0, \varepsilon) := g_0(N_\varepsilon^+ N \cap N_\varepsilon N^+ AM)M_\varepsilon A_\varepsilon,$$

where N^+ is the horospherical subgroup opposite to N and a subgroup with ε in the subscript denotes the ε-neighborhood of identity in the subgroup.

Let Θ be a conjugation-invariant Borel subset of M_Γ. It suffices to understand the asymptotic behavior of

$$\mu_T(\tilde{\mathcal{B}}(g_0, \varepsilon) \otimes \Theta),$$

where $\tilde{\mathcal{B}}(g_0, \varepsilon)$ denotes the image of $\mathfrak{B}(g_0, \varepsilon)$ under the projection $G \to \Gamma \backslash G / M$. Let $\mathcal{L}$ denote the *limit cone* of Γ (Definition 3.2) and let

$$\mathcal{L}_T^+ := \{\exp(w) : w \in \mathcal{L}, \psi(w) \leq T\}.$$

Lemma 6.3 relates $\mu_T(\tilde{\mathcal{B}}(g_0, \varepsilon) \otimes \Theta)$ to the number of elements of Γ in the set

$$\mathcal{W}_T(g_0, \varepsilon, \Theta) := \{gamg^{-1} : g \in \mathcal{B}(g_0, \varepsilon), am \in \mathcal{L}_T^+ \Theta\}.$$

We are then led to consider the set

$$\mathcal{V}_T(g_0, \varepsilon, \Theta) := \mathcal{B}(g_0, \varepsilon)\mathcal{L}_T^+ \Theta \mathcal{B}(g_0, \varepsilon)^{-1},$$

which can be thought of as a thickening of $\mathcal{W}_T(g_0, \varepsilon, \Theta)$. We can relate the asymptotic behavior of $\#(\Gamma \cap \mathcal{W}_T(g_0, \varepsilon, \Theta))$ to the asymptotic behavior of $\#(\Gamma \cap \mathcal{V}_T(g_0, \varepsilon, \Theta))$ by using Lemma 2.7 which is an *effective closing lemma for regular directions*. Lemma 2.7 says that if $\gamma \in \Gamma$ corresponds to an AM-orbit in $\Gamma \backslash G$ that almost closes up along some $\exp(w)m \in AM$ where $w \in \mathfrak{a}$ is a sufficiently large vector in a regular direction, then there exists a nearby closed AM-orbit with period approximately $\exp(w)m$. We can apply Lemma 2.7 because Γ being Anosov implies that its limit cone $\mathcal{L}$ is contained in the interior of $\mathfrak{a}^+$.

When $\mathrm{rank}(G) = 1$ and the BMS measure is finite, Margulis–Mohammadi–Oh [MMO14] showed that the asymptotic behavior of $\#(\Gamma \cap \mathcal{V}_T(g_0, \varepsilon, \Theta))$ can be obtained by using strong mixing of the A-action on $\Gamma \backslash G$. When G has higher and hence, m_ψ is infinite, the A-action is not strongly mixing on $\Gamma \backslash G$. In place of strong mixing, we have *local mixing* of the one-parameter diagonal flow $\exp(t\mathsf{v})$ due to Chow–Sarkar [CS23] and Edwards–Lee–Oh [ELO22b], where v is as in (1.1). In fact, their local mixing theorem applies to the more general one parameter family $\exp(t\mathsf{v} + \sqrt{t}u)$ with $u \in \ker \psi$. Our proof needs this more refined version along with an accompanying uniformity statement. Let $C_c(\Gamma \backslash G)$ denote the set of continuous compactly supported functions on $\Gamma \backslash G$ and ρ denote the half sum of the positive roots with respect to $\mathfrak{a}^+$.

Theorem 1.13 (Local mixing, [CS23, Theorem 1.3], [ELO22b, Theorem 3.4]). *There exists $\kappa_{\mathsf{v}} > 0$ such that for any $u \in \ker \psi$ and for any $\phi_1, \phi_2 \in C_c(\Gamma \backslash G)$, we have*

$$\lim_{t \to +\infty} t^{\frac{\mathrm{rank}(G)-1}{2}} e^{(2\rho - \psi)(t\mathsf{v} + \sqrt{t}u)} \int_{\Gamma \backslash G} \phi_1(x \exp(t\mathsf{v} + \sqrt{t}u)) \phi_2(x) \, dx$$
$$= \frac{\kappa_{\mathsf{v}} e^{-I(u)}}{|m_{\mathcal{X}_\psi}|} \sum_{Z \in \mathfrak{Z}_\Gamma} m_\psi^{\mathrm{BR}}\big|_{ZN^+}(\phi_1) \cdot m_\psi^{\mathrm{BR}_\star}\big|_{ZN}(\phi_2),$$

where dx denotes the right G-invariant measure on $\Gamma \backslash G$, $I : \ker \psi \to \mathbb{R}$ is defined by $I(\cdot) = \langle \cdot, \cdot \rangle_* - \frac{\langle \cdot, \mathsf{v} \rangle_*^2}{\langle \mathsf{v}, \mathsf{v} \rangle_*}$ for some inner product $\langle \cdot, \cdot \rangle_*$ on $\mathfrak{a}$, m_ψ^{BR} and $m_\psi^{\mathrm{BR}_\star}$ denote the Burger–Roblin measures associated to ψ and $\mathfrak{Z}_\Gamma$ denotes the finite set of A-ergodic components of m_ψ.

Moreover, there exists constants η_{v} and T_{v} such that for all $\phi_1, \phi_2 \in C_c(\Gamma \backslash G)$, there exists a constant $D_{\mathsf{v}}(\phi_1, \phi_2)$ depending continuously on ϕ_1 and ϕ_2 such that for all $(t, u) \in (T_{\mathsf{v}}, \infty) \times \ker \psi$ such that $t\mathsf{v} + \sqrt{t}u \in \mathcal{L}$, we have

$$\left| t^{\frac{\mathrm{rank}(G)-1}{2}} e^{(2\rho - \psi)(t\mathsf{v} + \sqrt{t}u)} \int_{\Gamma \backslash G} \phi_1(x \exp(t\mathsf{v} + \sqrt{t}u)) \phi_2(x) \, dx \right|$$
$$\leq D_{\mathsf{v}}(\phi_1, \phi_2) e^{-\eta_{\mathsf{v}} I(u)}. \quad (1.14)$$

Due to the higher dimensional nature of $\mathcal{L}_T^+$, we need to show that the error term $E(\phi_1, \phi_2, t, u)$ in Theorem 1.13 for some compact family of functions ϕ_1 and ϕ_2 does not contribute to the asymptotic after integrating on $\mathcal{L}_T^+$. We do this by using (1.14) to bound $|E(\phi_1, \phi_2, t, u)|$ and show that

$$\int_{a_{t\mathsf{v} + \sqrt{t}u} \in \mathcal{L}_T^+} e^t E(\phi_1, \phi_2, t, u) \, dt \, du = o(e^T),$$

where we write $o(f) = g$ if $f, g : \mathbb{R} \to \mathbb{R}$ such that $\lim_{T \to \infty} \frac{f(T)}{g(T)} = 1$. After some technical arguments and applying Theorem 1.13, we obtain Proposition 5.12 which is an asymptotic for the number of elements from Γ in product subsets $S_T := \Xi_1 \mathcal{L}_T^+ \Theta \Xi_2$ of N^+AMN. From Proposition 5.12, we deduce the asymptotic

$$\#\Gamma \cap \mathcal{V}_T(g_0, \varepsilon, \Theta) = \frac{[M : M_\Gamma]}{|m_{\mathcal{X}_\psi}|} e^T \left(\frac{\mathsf{m}_\psi(\tilde{\mathcal{B}}(g_0, \varepsilon) \otimes \Theta)}{b_{\mathsf{r}}(\varepsilon)} (1 + O(\varepsilon)) + o_T(1) \right),$$

where $[M : M_\Gamma]$ denotes the index of M_Γ in M, $b_{\mathsf{r}}(\varepsilon)$ denotes the volume of the Euclidean ε-ball of dimension r and m_ψ denotes here the BMS measure associated to ψ on $\Gamma \backslash G$ (Proposition 5.24). This asymptotic serves as input to obtain the asymptotic for μ_T. Using the asymptotic for μ_T, we deduce the asymptotic for η_T and Corollary 1.4 follows from this by using a careful choice of function f in Theorem 1.3.

Organization. Section 2 begins by fixing notation related to the Lie group G that we will use throughout the paper. In Subsection 2.1, we define ε-flow boxes. In Subsection 2.2, we present and prove an effective closing lemma for regular directions (Lemma 2.7).

In Section 3, we recall geometric objects and geometric measures associated to general Zariski dense discrete subgroups $\Gamma < G$. In particular, we recall the limit set and limit cone in Subsection 3.1, we discuss the holonomy group of Γ in Subsection 3.2 and we recall the constructions of the BMS and BR measures in Subsection 3.5. We also recall the product structures of the BR measures in Subsection 3.6.

In Section 4, we specialize to Anosov subgroups. In Subsection 4.1, we recall ergodic decompositions of the BMS and BR measures and in Subsection 4.2 we recall the vector bundle structure of the support of the BMS measure. In Subsection 4.3, we recall the local mixing theorems we will need. Maximal flat cylinders are introduced and discussed in Subsection 4.4.

In Subsection 5.1, we use local mixing to prove Proposition 5.12 which is an asymptotic for the number of elements in Γ in product subsets $S_T := \Xi_1 \mathcal{L}_T^+ \Theta \Xi_2$ of $N^+ A M N$. In Subsection 5.2, we use Proposition 5.12 to deduce an asymptotic for $\#\Gamma \cap \mathcal{V}_T(g_0, \varepsilon, \Theta)$ (Proposition 5.24).

In Section 6, we show how the asymptotic for $\#\Gamma \cap \mathcal{V}_T(g_0, \varepsilon, \Theta)$ can be used to deduce an asymptotic for $\mu_T(\tilde{\mathcal{B}}(g_0, \varepsilon) \otimes \Theta)$ (Proposition 6.11) and prove the main joint equidistribution result Theorem 6.12 and its corollaries. In Section 7, we show how to modify our arguments to prove joint equidistribution with respect to norm-like functions (Theorem 7.3).

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2. Effective closing lemma for regular directions

Let G be a connected semisimple real algebraic group with identity element $e \in G$. Fix a Cartan involution of the Lie algebra $\mathfrak{g}$ of G and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the associated eigenspace decomposition corresponding to the eigenvalues $+1$ and -1 respectively. Let $K < G$ be the maximal compact subgroup whose Lie algebra is $\mathfrak{k}$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra and choose a closed positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$. Let $A = \exp \mathfrak{a}$, $A^+ = \exp \mathfrak{a}^+$, and denote $a_w = \exp(w)$ for all $w \in \mathfrak{a}$. Define $M := C_K(A)$ and the contracting and expanding *horospherical subgroups* by

$$N := N^- := \left\{ n \in G : \lim_{t \to \infty} a_{-tw} n a_{tw} = e \text{ for all } w \in \text{int } \mathfrak{a}^+ \right\};$$

$$N^+ := \left\{ h \in G : \lim_{t \to \infty} a_{tw} h a_{-tw} = e \text{ for all } w \in \text{int } \mathfrak{a}^+ \right\},$$

respectively, and their Lie algebras $\mathfrak{n} := \mathfrak{n}^- := \log N$ and $\mathfrak{n}^+ := \log N^+$. Let

$$P := P^- := MAN \text{ and } P^+ := MAN^+.$$

Then P and P^+ are opposite minimal parabolic subgroups. Let

$$\mathcal{F} := G/P \cong K/M$$

denote the *Furstenburg boundary* of G, where the isomorphism $G/P \cong K/M$ is given by the Iwasawa decomposition $G \cong K \times A \times N$.

Let $W := N_K(A)/M$ denote the Weyl group. Let $w_0 \in K$ be a representative of the element in W such that $\mathrm{Ad}_{w_0}(\mathfrak{a}^+) = -\mathfrak{a}^+$.

Definition 2.1 (Opposition involution)**.** The map $\mathrm{i} : \mathfrak{a}^+ \to \mathfrak{a}^+$ defined by $\mathrm{i}(v) := -\mathrm{Ad}_{w_0}(v)$ is called the *opposition involution*.

For all $g \in G$, let

$$g^+ := gP \in \mathcal{F}; \qquad\qquad g^- := gw_0P \in \mathcal{F}. \qquad\qquad (2.2)$$

Fix a left G-invariant and right K-invariant Riemannian metric d on G and denote the corresponding inner product and norm on any of its tangent spaces by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. The Riemannian metric on G also induces an inner product and norm on $\mathfrak{a}$, which are invariant under W and we denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. Using the inner product on $\mathfrak{a}$, we identify $\mathfrak{a}$ with $\mathbb{R}^{\mathrm{rank}(G)}$ and equip it with the Lebesgue measure which induces a Haar measure on A. Fix Ad_M-invariant metrics $\| \cdot \|$ on $\mathfrak{n}^\pm$. For $\varepsilon > 0$, we denote several ε-neighborhoods by

$$
\begin{aligned}
G_\varepsilon &:= \{g \in G : d_G(e, g) < \varepsilon\}; & G_\varepsilon(g_0) &:= g_0 G_\varepsilon; \\
N_\varepsilon &:= \{n_x := \exp x : x \in \mathfrak{n}, \|x\| < \varepsilon\}; & A_\varepsilon &:= A \cap G_\varepsilon; \\
N_\varepsilon^+ &:= \{h_x := \exp x : x \in \mathfrak{n}^+, \|x\| < \varepsilon\}; & M_\varepsilon &:= M \cap G_\varepsilon.
\end{aligned}
$$

Throughout the paper, if f_1, f_2 are functions of $\varepsilon > 0$ and $T > 0$, then we use big-O notation to write

$$f_1 = O(f_2) \iff \limsup_{\substack{\varepsilon \to 0 \\ T \to \infty}} \left| \frac{f_1(\varepsilon, T)}{f_2(\varepsilon, T)} \right| < \infty. \qquad\qquad (2.3)$$

2.1. **Flow boxes.** In this subsection, we define *flow boxes* following [Mar04] and [MMO14]. Flow boxes will be used in our effective closing lemma for regular directions (Lemma 2.7) to describe almost closed AM-orbits in $\Gamma \backslash G$.

To motivate the definition of flow boxes, we first recall that the product maps $N \times N^+ \times A \times M \to G$ and $N^+ \times N \times A \times M \to G$ are diffeomorphisms onto Zariski open neighborhoods of $e \in G$. Consequently, we have the following lemma.

Lemma 2.4. *Let $\varepsilon > 0$ be sufficiently small. For any $0 < \varepsilon_1, \varepsilon_2 < \varepsilon$, $h \in N_{\varepsilon_1}^+$ and $n \in N_{\varepsilon_2}$, we have $hn = n_1 h_1 am$ for some unique $h_1 \in N^+$, $n_1 \in N$, $a \in A$ and $m \in M$. Moreover, $h_1 \in N_{O(\varepsilon_1)}^+$, $n_1 \in N_{O(\varepsilon_2)}$, $a \in A_{O(\varepsilon)}$, and $m \in M_{O(\varepsilon)}$. The corresponding statement if we swap the roles of N and N^+ holds as well.*

Definition 2.5 (ε-flow box at g_0). Given $g_0 \in G$ and $\varepsilon > 0$, the ε-*flow box at g_0* is defined by

$$\mathfrak{B}(g_0, \varepsilon) := g_0(N_\varepsilon^+ N \cap N_\varepsilon N^+ AM)M_\varepsilon A_\varepsilon = g_0\mathcal{B}(e, \varepsilon).$$

The set of all ε-flow boxes forms a basis for the topology on G. The next lemma records elementary properties of the flow boxes.

Lemma 2.6. *Let $g_0 \in G$ and $\varepsilon > 0$.*

(1) *For any $g \in \mathfrak{B}(g_0, \varepsilon)$, $\{w \in \mathfrak{a} : g\exp(w) \in \mathfrak{B}(g_0, \varepsilon)\}$ is a Euclidean ball of radius ε.*

(2) *$\mathfrak{B}(g_0, \varepsilon)e^\pm = g_0 N_\varepsilon^\pm e^\pm$ (where $e^\pm$ are defined by (2.2)).*

(3) *For sufficiently small $\varepsilon > 0$, the following holds. For any $h \in N_\varepsilon^+$ and $n \in N_\varepsilon$, $hN \cap nN^+AM$ consists of a single element $g \in \mathfrak{B}(\varepsilon)$. Moreover, by Lemma 2.4,*

$$\mathfrak{B}(g_0, \varepsilon) = g_0(N_\varepsilon^+ N_{O(\varepsilon)} \cap N_\varepsilon N_{O(\varepsilon)}^+ A_{O(\varepsilon)} M_{O(\varepsilon)})M_\varepsilon A_\varepsilon.$$

2.2. **Effective closing lemma for regular directions.** Let $\mathfrak{a}^*$ denote the space of real linear forms on $\mathfrak{a}$. Let $\Phi \subset \mathfrak{a}^*$ denote the restricted root system of $\mathfrak{a}$ and let $\Phi^+ \subset \Phi$ denote the set of positive roots corresponding to $\mathfrak{a}^+$. A direction $w \in \mathfrak{a}$ is called *regular* if

$$\forall \alpha \in \Phi^+, \ \alpha(w) \neq 0.$$

Lemma 2.7 is an effective closing lemma for regular directions, which is an adaptation of [MMO14, Lemma 3.1] which we will be able to use in our Anosov setting later. The proof is similar to that of [MMO14, Lemma 3.1], but we include it for completeness. For $g_1, g_2 \in G$, we write

$$g_1 \sim_{O(\varepsilon)} g_2 \iff d(g_1, g_2) = O(\varepsilon)$$

and we use the same notation for subgroups of G and their lie algebras, relying on context.

Lemma 2.7 (Effective closing lemma for regular directions). *There exists $T_0 > 0$, depending only on G, for which the following holds. Let $\varepsilon > 0$ be sufficiently small and $g_0 \in G$. Suppose there exists $g_1, g_2 \in \mathfrak{B}(g_0, \varepsilon)$ and $\gamma \in G$ such that*

$$g_1\tilde{a}_\gamma\tilde{m}_\gamma = \gamma g_2 \tag{2.8}$$

for some $\tilde{m}_\gamma \in M$ and $\tilde{a}_\gamma \in A$ with

$$T := \min_{\alpha \in \Phi^+} \alpha(\log \tilde{a}_\gamma) \geq T_0. \tag{2.9}$$

Then there exists $g \in \mathfrak{B}(g_0, \varepsilon + O(\varepsilon e^{-T}))$, $a_\gamma \in A$ and $m_\gamma \in M$ such that

$$\gamma = ga_\gamma m_\gamma g^{-1}.$$

Moreover, $a_\gamma \sim_{O(\varepsilon)} \tilde{a}_\gamma$ and $m_\gamma \sim_{O(\varepsilon)} \tilde{m}_\gamma$.

Proof. Let $\varepsilon > 0$ be sufficiently small so that each instance that Lemma 2.4 and Lemma 2.6(3) are used in the proof is valid. By Lemma 2.6(3), we have

$$g_1 = g_0 h_1 n_1 a_1 m_1 \quad \text{and} \quad g_2 = g_0 n_2 h_2 a_2 m_2$$

for some $h_1 \in N_\varepsilon^+$, $n_1 \in N_{O(\varepsilon)}$, $a_1 \in A_\varepsilon$, $m_1 \in M_\varepsilon$, $n_2 \in N_\varepsilon$, $h_2 \in N_{O(\varepsilon)}$, $a_3 \in A_{O(\varepsilon)}$ and $m_3 \in M_{O(\varepsilon)}$. Lemma 2.6(3) also tells us that $h_1 N \cap n_2 N^+ A M = \{g_3\}$, where $g_3 \in \mathfrak{B}(\varepsilon)$ and

$$g_3 = h_1 n_3 = n_2 h_3 a_3 m_3$$

for some $n_3 \in N_{O(\varepsilon)}$, $h_3 \in N_{O(\varepsilon)}^+$, $a_3 \in A_{O(\varepsilon)}$ and $m_3 \in M_{O(\varepsilon)}$. Set $g_4 = g_0 g_3 \in \mathfrak{B}(g_0, \varepsilon)$. Then

$$g_1 = g_4 n_4 a_1 m_1 \quad \text{and} \quad g_2 = g_4 h_4 a_4 m_4,$$

where $n_4 = n_3^{-1} n_1 \in N_{O(\varepsilon)}$, $h_4 = (a_3 m_3)^{-1} h_3^{-1} h_2 (a_3 m_3) \in N_{O(\varepsilon)}^+$, $a_4 = a_3^{-1} a_2 \in A_{O(\varepsilon)}$ and $m_4 = m_3^{-1} m_2 \in M_{O(\varepsilon)}$.

The hypothesis (2.8) becomes $g_4 n_4 a_1 m_1 \tilde{a}_\gamma \tilde{m}_\gamma = \gamma g_4 h_4 a_4 m_4$. Then

$$g_4^{-1} \gamma g_4 = n_4 a_\gamma' m_\gamma' (h_4)^{-1} = a_\gamma' m_\gamma' n_5 (h_4)^{-1},$$

where $a_\gamma' = \tilde{a}_\gamma a_1 a_4^{-1} \sim_{O(\varepsilon)} \tilde{a}_\gamma$, $m_\gamma' = m_1 \tilde{m}_\gamma m_4^{-1} \sim_{O(\varepsilon)} \tilde{m}_\gamma$ and $n_5 = (a_\gamma' m_\gamma')^{-1} n_4 (a_\gamma' m_\gamma') \in N_{O(\varepsilon e^{-T})}$ since $\min_{\alpha \in \Phi^+} \alpha(\log \tilde{a}_\gamma) = T$.

Using Lemma 2.4, we write $n_5 (h_4)^{-1} = a_5 m_5 h_5 n_6$ for some $h_5 \in N_{O(\varepsilon)}^+$, $n_6 \in N_{O(\varepsilon e^{-T})}$, $a_5 \in A_{O(\varepsilon)}$ and $m_5 \in M_{O(\varepsilon)}$. Then

$$g_4^{-1} \gamma g_4 = a_\gamma' m_\gamma' a_5 m_5 h_5 n_6 = h_6 a_\gamma'' m_\gamma'' n_6,$$

where $a_\gamma'' = a_\gamma' a_5 \sim_{O(\varepsilon)} a_\gamma' \sim_{O(\varepsilon)} \tilde{a}_\gamma$, $m_\gamma'' = m_\gamma' m_5 \sim_{O(\varepsilon)} \tilde{m}_\gamma$ and $h_6 = (a_\gamma'' m_\gamma'') n_5 (a_\gamma'' m_\gamma'')^{-1} \in N_{O(\varepsilon e^{-T})}^+$ since $T = \min_{\alpha \in \Phi^+} \alpha(\log \tilde{a}_\gamma)$.

To complete the proof, it suffices to show

$$h_6 a_\gamma'' m_\gamma'' n_6 \in (h_x n_y) a_\gamma'' A_{O(\varepsilon)} m_\gamma'' M_{O(\varepsilon)} (h_x n_y)^{-1}$$

for some $h_x \in N_{O(\varepsilon e^{-T})}^+$ and $n_y \in N_{O(\varepsilon e^{-T})}$, as we then take $g = g_4 h_x n_y$.

For each $h_x \in N_\varepsilon^+$, there is a unique element $h_{\beta(x)} \in N_{O(\varepsilon)}^+$ such that

$$(n_6) h_x \in h_{\beta(x)} N_{O(\varepsilon e^{-T})} A_\varepsilon M_\varepsilon. \tag{2.10}$$

Moreover, since the product map $N^+ \times N \times A \times M \to G$ is a diffeomorphism onto its image, $\beta : \mathfrak{n}^+ \to \mathfrak{n}^+$ is a smooth function. Consider the map

$$f : N_\varepsilon^+ \to N_{\varepsilon + O(\varepsilon e^{-T})}$$

given by

$$h_x \mapsto h_x (a_\gamma'' m_\gamma'') (h_{\beta(x)})^{-1} (a_\gamma'' m_\gamma'')^{-1} = h_x h_{\mathrm{Ad}_{a_\gamma'' m_\gamma''}(\beta(x))}.$$

Assuming T_0 is sufficiently large, $a_\gamma'' \sim_{O(\varepsilon)} \tilde{a}_\gamma$ and hypothesis (2.9) implies that $\|Df - I\|_{op} < 1$ pointwise on N_ε^+. Then f is injective, and therefore a

diffeomorphism onto its image. Since f is a diffeomorphism onto its image, there exists $h_x \in N^+_{O(\varepsilon e^{-T})}$ such that

$$h_x(a''_\gamma m''_\gamma)(h_{\beta(x)})^{-1}(a''_\gamma m''_\gamma)^{-1} = h_6.$$

Recalling (2.10), we write $(h_{\beta(x)})^{-1}n_6 = a_6 m_6 n_7 h_x^{-1}$ for some $a_6 \in A_\varepsilon$, $m \in M_\varepsilon$ and $n_7 \in N_{O(\varepsilon e^{-T})}$. Then

$$h_6 a''_\gamma m''_\gamma n_6 = h_x(a''_\gamma m''_\gamma)(h_{\beta(x)})^{-1}n_6 = h_x(a'''_\gamma m'''_\gamma)n_7 h_x^{-1},$$

where $a'''_\gamma = a''_\gamma a_6$ and $m'''_\gamma = m''_\gamma m_6$. By similar reasoning to the above, there exists $n_y \in N_{O(\varepsilon e^{-T})}$ such that $n_7 = (a'''_\gamma m'''_\gamma)^{-1}n_y(a'''_\gamma m'''_\gamma)n_y^{-1}$ and this completes the proof. ■

3. Geometric measures

Henceforth, let $\Gamma < G$ be a Zariski dense discrete subgroup.

3.1. **Limit set and limit cone.** Recall that $\mathcal{F} := G/P \cong K/M$. Let $m_\mathcal{F}$ denote the unique K-invariant probability measure on $\mathcal{F}$.

Definition 3.1 (Limit set)**.** The *limit set* $\Lambda \subset \mathcal{F}$ of Γ is defined by

$$\Lambda := \{\xi \in \mathcal{F} : \exists \{\gamma_n\}_{n \in \mathbb{N}} \subset \Gamma, (\gamma_n)_* m_\mathcal{F} \xrightarrow{n \to \infty} \delta_\xi\},$$

where δ_ξ denotes the Dirac measure at ξ. The limit set is the unique minimal nonempty closed Γ-invariant subset of $\mathcal{F}$ [Ben97] and Λ is Zariski dense in $\mathcal{F}$ [Ben97, Section 3.6].

An element $\gamma \in G$ is called *loxodromic* if

$$\gamma = g \exp \lambda(\gamma) m g^{-1}$$

for some $g \in G$, $\lambda(\gamma) \in \operatorname{int} \mathfrak{a}^+$ and $m \in M$. In that case,

$$\lambda(\gamma) \in \operatorname{int} \mathfrak{a}^+$$

is unique and called the *Jordan projection* of γ and m belongs to a unique conjugacy class

$$h_\gamma = [m] \in [M]$$

called the *holonomy* of γ. In addition, g^+ and g^- defined by (2.2) are the unique attracting and repelling fixed points of γ, respectively. Moreover, if $\gamma \in \Gamma$, then $g^+, g^- \in \Lambda$.

Definition 3.2 (Limit cone)**.** The *limit cone* $\mathcal{L} \subset \mathfrak{a}^+$ of Γ is the smallest closed cone containing all Jordan projections of loxodromic elements in Γ [Ben97, Sections 4 and 4.3]. We denote the interior of $\mathcal{L}$ by $\operatorname{int} \mathcal{L}$.

Remark 3.3. Recall the opposition involution i (Definition 2.1). We have $\lambda(g^{-1}) = \mathsf{i}(\lambda(g))$ and hence, $\mathcal{L} = \mathsf{i}(\mathcal{L})$.

Theorem 3.4 (Ben97)**.** *The limit cone $\mathcal{L}$ is convex and $\operatorname{int} \mathcal{L} \neq \emptyset$.*

3.2. **The holonomy group.** Let M° denote the identity component of M. Then the identity component of P is $P^\circ = M^\circ AN$.

Definition 3.5 (Holonomy group of Γ). The *holonomy group* of Γ is the closed subgroup $M_\Gamma < M$ generated by all of the holonomies in Γ, that is,

$$M_\Gamma := \overline{\langle \{m : \exists \gamma \in \Gamma \text{ such that } h_\gamma = [m]\} \rangle}.$$

By [GR07, Corollary 1.10], M_Γ is a normal subgroup of M containing M° so in particular, M_Γ has finite index in M.

By [GR07], we have another characterization of M_Γ. Let

$$\mathcal{F}^\circ := G/P^\circ.$$

Fix a Γ-minimal subset $\Lambda_0 \subset \mathcal{F}^\circ$. Then

$$M_\Gamma = \{m \in M : \Lambda_0 m = \Lambda_0\} < M. \tag{3.6}$$

See also [BQ14, Proposition 4.9] for another characterization of M_Γ.

3.3. **Busemann function.** The *Iwasawa cocycle* $\sigma : G \times \mathcal{F} \to \mathfrak{a}$ is the map which assigns to each $(g, kM) \in G \times \mathcal{F}$ the unique element $\sigma(g, kM) \in \mathfrak{a}$ such that $gk \in K a_{\sigma(g,\xi)} N$. It satisfies the cocycle relation $\sigma(g_1 g_2, \xi) = \sigma(g_1, g_2\xi) + \sigma(g_2, \xi)$ for all $g_1, g_2 \in G$ and $\xi \in \mathcal{F}$.

Definition 3.7 (Busemann function). The $\mathfrak{a}$-*valued Busemann function* $\beta : \mathcal{F} \times G \times G \to \mathfrak{a}$ is defined by

$$\beta_\xi(g_1, g_2) := \sigma(g_1^{-1}, \xi) - \sigma(g_2^{-1}, \xi)$$

for all $g_1, g_2 \in G$ and $\xi \in \mathcal{F}$. The properties of the Iwasawa cocycle imply that the Busemann function satisfies

(1) $\beta_\xi(e, g) = -\sigma(g^{-1}, \xi)$;

(2) $\beta_{g\xi}(gg_1, gg_2) = \beta_\xi(g_1, g_2)$;

(3) $\beta_\xi(g_1, g_2) = \beta_\xi(g_1, g) + \beta_\xi(g, g_2)$

for all $g, g_1, g_2 \in G$ and $\xi \in \mathcal{F}$.

3.4. **Conformal measures.** In this subsection, we recall facts about Γ-conformal measures. Patterson [Pat76] and Sullivan [Sul79] were the first to introduce Γ-conformal measures in rank one groups. Their work has been extended to higher rank groups by Albuquerque [Alb99] and Quint [Qui02b]. Generalizing the notion of critical exponent in rank one, Quint introduced the growth indicator function of Γ [Qui02a].

For $g \in G$, let $\mu(g)$ denote the *Cartan projection* of g, that is, $\mu(g) \in \mathfrak{a}^+$ is the unique element in $\mathfrak{a}^+$ such that $g \in K \exp(\mu(g))K$. We note that $\mu(g^{-1}) = \mathsf{i}(\mu(g))$ for all $g \in G$.

Definition 3.8 (Growth indicator function). The *growth indicator function* $\psi_\Gamma : \mathfrak{a}^+ \to \mathbb{R} \cup \{-\infty\}$ of Γ is the degree 1 homogeneous function defined by

$$\psi_\Gamma(w) := \|w\| \inf_{\text{open cones } \mathfrak{C} \ni w} \tau_\mathfrak{C} \qquad \text{for all } w \in \mathfrak{a}^+,$$

where $\tau_\mathfrak{C}$ is the abscissa of convergence of $t \mapsto \sum_{\gamma \in \Gamma, \mu(\gamma) \in \mathfrak{C}} e^{-t\|\mu(\gamma)\|}$.

We note that $\psi_\Gamma \circ \mathsf{i} = \psi_\Gamma$.

Theorem 3.9 (Qui02a, Theorem 4.2.2). *The growth indicator function is concave, upper semicontinuous, and satisfies $\psi_\Gamma|_{\mathfrak{a}^+ \setminus \mathcal{L}} = -\infty$, $\psi_\Gamma|_{\mathcal{L}} \geq 0$, and $\psi_\Gamma|_{\mathrm{int}\,\mathcal{L}} > 0$.*

Definition 3.10 (Conformal measures). Given a closed subgroup $\Delta < G$, a Borel probability measure ν on $\mathcal{F}$ is called a Δ-*conformal measure* if, there exists $\psi \in \mathfrak{a}^*$ such that for any $\gamma \in \Delta$ and $\xi \in \mathcal{F}$,

$$\frac{d\gamma_* \nu}{d\nu}(\xi) = e^{\psi(\beta_\xi(e, \gamma))},$$

where $\gamma_* \nu(Q) := \nu(\gamma^{-1} Q)$ for any Borel subset $Q \subset \mathcal{F}$. In that case, we call ν a (Δ, ψ)-*conformal measure.*

The measure $m_\mathcal{F}$ is a conformal measure. More precisely, let $\rho \in \mathfrak{a}^*$ denote the half sum of the positive roots of $\mathfrak{a}^+$,

$$\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha. \tag{3.11}$$

We have the following lemma.

Lemma 3.12 (Qui06, Proposition 3.3). *The measure $m_\mathcal{F}$ is a $(G, 2\rho)$-conformal measure.*

The following theorem on the existence of Γ-conformal measures is due to Quint. A linear form $\psi \in \mathfrak{a}^*$ is said to be a *tangent form* if

$$\psi \geq \psi_\Gamma \text{ and } \psi(\mathsf{v}) = \psi_\Gamma(\mathsf{v}) \text{ for some } \mathsf{v} \in \mathrm{int}\,\mathfrak{a}^+ \cap \mathcal{L}. \tag{3.13}$$

Note that if $\psi \in \mathfrak{a}^*$ is a tangent form, then $\psi \circ \mathsf{i}$ is also a tangent form.

Theorem 3.14 (Qui02b, Theorems 8.1 and 8.4). *Let $\psi \in \mathfrak{a}^*$.*

  (1) If there exists a (Γ, ψ)-conformal measure, then $\psi \geq \psi_\Gamma$.

  (2) If ψ is a tangent form, then there exists a (Γ, ψ)-conformal measure ν_ψ supported on Λ.

3.5. **Geometric measures.** In this subsection, we recall the definitions of the Bowen–Margulis–Sullivan and Burger–Roblin measures.

We first recall the Hopf parametrization of G/M which will be used to define these measures. There is a unique open G-orbit in $\mathcal{F} \times \mathcal{F}$ given by

$$\mathcal{F}^{(2)} := G \cdot (e^+, e^-) = (\mathcal{F} \times \mathcal{F}) \setminus \bigcup_{wM \notin W \setminus \{w_0 M\}} G \cdot (e^+, we^+), \tag{3.15}$$

where the Weyl group W and the element $w_0 \in K$ were defined in Section 2. If $(x, y) \in \mathcal{F}^{(2)}$, then we say that x and y are in *general position*.

Define a left G-action on $\mathcal{F}^{(2)} \times \mathfrak{a}$ by

$$g \cdot (x, y, v) := (gx, gy, v + \beta_x(g^{-1}, e)) = (gx, gy, v + \sigma(g, x))$$

for all $g \in G$ and $(x, y, v) \in \mathcal{F}^{(2)} \times \mathfrak{a}$. Note that $\mathrm{Stab}_G(e^+, e^-, 0) = M$.

Definition 3.16 (Hopf parametrization). The *Hopf parametrization* is a left G-equivariant diffeomorphism $G/M \to \mathcal{F}^{(2)} \times \mathfrak{a}$ defined by

$$gM \mapsto (g^+, g^-, \beta_{g^+}(e,g)) = (g^+, g^-, \sigma(g, e^+)).$$

For the remainder of Section 3, we fix a

$$\text{tangent form } \psi \in \mathfrak{a}^*. \tag{3.17}$$

Recall that $\psi \circ \mathsf{i}$ is also a tangent form. Using Theorem 3.14, we fix

$$(\Gamma, \psi)\text{-and } (\Gamma, \psi \circ \mathsf{i})\text{-conformal measures } \nu_\psi \text{ and } \nu_{\psi \circ \mathsf{i}}, \tag{3.18}$$

respectively. Let

$$\Lambda^{(2)} := (\Lambda \times \Lambda) \cap \mathcal{F}^{(2)}$$

and let dw denote the Lebesgue measure on $\mathfrak{a}$.

Definition 3.19 (Bowen–Margulis–Sullivan measure). Using the Hopf parametrization, the $(\nu_\psi, \nu_{\psi \circ \mathsf{i}})$-*Bowen–Margulis–Sullivan (BMS) measure* $\mathsf{m}_\psi := m^{\mathrm{BMS}}_{\nu_\psi, \nu_{\psi \circ \mathsf{i}}}$ is defined on $G/M \cong \mathcal{F}^{(2)} \times \mathfrak{a}$ by

$$dm^{\mathrm{BMS}}_{\nu_\psi, \nu_{\psi \circ \mathsf{i}}}(gM) := e^{\psi\left(\beta_{g^+}(e,g)\right) + (\psi \circ \mathsf{i})\left(\beta_{g^-}(e,g)\right)} \, d\nu_\psi(g^+) \, d\nu_{\psi \circ \mathsf{i}}(g^-) \, dw. \tag{3.20}$$

It is clear from (3.20) that m_ψ has support

$$\operatorname{supp} \mathsf{m}_\psi = \Lambda^{(2)} \times \mathfrak{a}. \tag{3.21}$$

The BMS measure is left Γ-invariant and right A-invariant so it descends to a measure on $\Gamma \backslash G/M$ and by lifting using the Haar probability measure on M we also obtain a measure on $\Gamma \backslash G$. Abusing notation, we call also this measure BMS measure and denote it by m_ψ as well, relying on context for the domain. The supports of the BMS measures on these domains are given by

$$\tilde{\Omega} := \{\Gamma g : g^\pm \in \Lambda\} \subset \Gamma \backslash G; \qquad \Omega := \tilde{\Omega}/M \subset \Gamma \backslash G/M. \tag{3.22}$$

Definition 3.23 (Burger–Roblin measures). Following [ELO20, Section 3], the $(m_{\mathcal{F}}, \nu_\psi)$- and $(\nu_{\psi \circ \mathsf{i}}, m_{\mathcal{F}})$-*Burger–Roblin (BR) measures*

$$m^{\mathrm{BR}}_\psi := m^{\mathrm{BR}}_{m_{\mathcal{F}}, \nu_{\psi \circ \mathsf{i}}} \text{ and } m^{\mathrm{BR}\star}_\psi := m^{\mathrm{BR}_*}_{\nu_\psi, m_{\mathcal{F}}},$$

respectively, are defined on G/M in a similar fashion as m_ψ by

$$dm^{\mathrm{BR}}_{m_{\mathcal{F}}, \nu_{\psi \circ \mathsf{i}}}(gM) := e^{2\rho\left(\beta_{g^+}(e,g)\right) + (\psi \circ \mathsf{i})\left(\beta_{g^-}(e,g)\right)} \, dm_{\mathcal{F}}(g^+) \, d\nu_{\psi \circ \mathsf{i}}(g^-) \, dw; \tag{3.24}$$

$$dm^{\mathrm{BR}_*}_{\nu_\psi, m_{\mathcal{F}}}(gM) := e^{\psi\left(\beta_{g^+}(e,g)\right) + 2\rho\left(\beta_{g^-}(e,g)\right)} \, d\nu_\psi(g^+) \, dm_{\mathcal{F}}(g^-) \, dw. \tag{3.25}$$

Then m^{BR}_ψ is a left Γ-invariant and right N^+-invariant measure and $m^{\mathrm{BR}\star}_\psi$ is a left Γ-invariant and right N-invariant measure and they induce measures on $\Gamma \backslash G/M$, $\Gamma \backslash G$ and G that we use the same name and notation for. The supports of the BR measures on $\Gamma \backslash G$ are given by

$$\mathcal{E} := \operatorname{supp} m^{\mathrm{BR}}_\psi = \{\Gamma g : g^- \in \Lambda\}; \quad \mathcal{E}_* := \operatorname{supp} m^{\mathrm{BR}\star}_\psi = \{\Gamma g : g^+ \in \Lambda\}.$$

Definition 3.26. The *right Haar measure dg* on G that we use can be written as

$$dg = e^{2\rho\left(\beta_{g^+}(e,g)+\beta_{g^-}(e,g)\right)} \, dm_{\mathcal{F}}(g^+) \, dm_{\mathcal{F}}(g^-) \, dw \, dm,$$

where dm is the probability Haar measure on M [ELO20, Section 3].

3.6. **Product structure of the BR measures.** In this subsection, we recall a product structure of the BR measures (see [ELO20, Section 4.3] for details) and define some related measures that will be useful in Section 5. We retain the notations in (3.17) and (3.18).

Let dm denote the probability Haar measure on M. We define measures $\tilde{\nu}_\psi$ and $\nu_{\psi\mathsf{oi}}$ on $K \cong \mathcal{F} \times M$ using the M-invariant lifts of the conformal measures by

$$d\nu_\psi^K(k) := d\nu_\psi(k^+) \, dm; \qquad d\nu_{\psi\mathsf{oi}}^K(k) := d\nu_{\psi\mathsf{oi}}(k^-) \, dm. \qquad (3.27)$$

The following lemma gives the decompositions of m_ψ^{BR} and $m_\psi^{\mathrm{BR}\star}$ in KAN-coordinates.

Lemma 3.28 (ELO20, Lemma 4.9)**.** *For all $k_1, k_2 \in K$, $w_1, w_2 \in \mathfrak{a}$, $h \in N^+$ and $n \in N$, we have*

$$dm_\psi^{\mathrm{BR}}(k_1 \exp(w_1)h) = e^{-\psi(w_1)} \, d\nu_{\psi\mathsf{oi}}^K(k_1) \, dw_1 \, dh;$$
$$dm_\psi^{\mathrm{BR}\star}(k_2 \exp(w_2)n) = e^{\psi(w_2)} \, d\nu_\psi^K(k_2) \, dw_2 \, dn,$$

where dh and dn are some Haar measures on N^+ and N, respectively.

Remark 3.29. Recalling that the product maps $M \times A \times N \times N^+ \to MANN^+$, $A \times N^+ \times M \times N \to AN^+MN$ and $N^+ \times M \times A \times N \to G$ are homeomorphisms onto Zariski open subsets of G, using the notation in Lemma 3.28, we have the decomposition

$$dg = dm \, dw \, dn \, dh = dw \, dh \, dm \, dn = e^{2\rho(w)} \, dh \, dw \, dm \, dn. \qquad (3.30)$$

For $g \in G$, we define the measures ν_ψ^g and $\nu_{\psi\mathsf{oi}}^g$ on $\mathcal{F}$ by

$$d\nu_\psi^g(\xi) := e^{\psi(\beta_{g\xi}(e,g))} \, d\nu_\psi(g\xi); \quad d\nu_{\psi\mathsf{oi}}^g(\xi) := e^{(\psi\mathsf{oi})(\beta_{g\xi}(e,g))} \, d\nu_{\psi\mathsf{oi}}(g\xi). \qquad (3.31)$$

Note that ν_ψ^g and $\nu_{\psi\mathsf{oi}}^g$ are $(g^{-1}\Gamma g, \psi)$- and $(g^{-1}\Gamma g, \psi \circ \mathsf{i})$-conformal measures, respectively.

In addition, identifying N^+ with $N^+ e^+ \subset \mathcal{F}$, for all $g \in G$, we define the measures $\tilde{\nu}_\psi^g$ and $\tilde{\nu}_\psi^g|_B$ on N^+ by

$$d\tilde{\nu}_\psi^g(h) := e^{\psi(\beta_{h^+}(e,h))} \, d\nu_\psi^g(h^+); \qquad d\tilde{\nu}_\psi^g|_B(h) := \mathbb{1}_B(h) d\tilde{\nu}_\psi^g(h), \qquad (3.32)$$

where B is a measurable subset of G.

Identifying NM with $Ne^- \times M \subset \mathcal{F} \times M$, for all $g \in G$, we also define the measures $\tilde{\nu}_{\psi\mathsf{oi}}^g$ and $\tilde{\nu}_{\psi\mathsf{oi}}^g|_B$ on NM by

$$d\tilde{\nu}_{\psi\mathsf{oi}}^g(nm) := e^{(\psi\mathsf{oi})(\beta_{n^-}(e,n))} \, d\nu_{\psi\mathsf{oi}}^g(n^-) \, dm; \qquad (3.33)$$
$$d\tilde{\nu}_{\psi\mathsf{oi}}^g|_B(nm) := \mathbb{1}_B(nm) d\tilde{\nu}_{\psi\mathsf{oi}}^g(nm). \qquad (3.34)$$

4. Anosov subgroups and maximal flat cylinders

The Anosov property (with respect to any parabolic subgroup of G) was first introduced by Labourie [Lab06] for surface groups and later generalized by Guichard–Wienhard [GW12] for Gromov hyperbolic groups (cf. [KLP17, GGKW17, Wie18]). For Zariski dense Anosov subgroups with respect to P, [GW12, Corollary 4.16] gives an equivalent characterization of being Anosov which we take as the definition of Anosov we use throughout the paper.

Definition 4.1 (Anosov subgroup). Let $\Gamma < G$ be a Zariski dense discrete subgroup. We say that Γ is *Anosov* if it is a finitely generated Gromov hyperbolic group and it admits a continuous Γ-equivariant continuous embedding from the Gromov boundary of Γ to $\mathcal{F}$ such that for any two distinct points in the Gromov boundary, their images in $\mathcal{F}$ are in general position.

For the rest of the paper, let Γ be a torsion-free Zariski dense Anosov subgroup. It follows from Definition 4.1 that

$$\Lambda^{(2)} := (\Lambda \times \Lambda) \cap \mathcal{F}^{(2)} = \{(x,y) \in \Lambda \times \Lambda : x \neq y\}. \tag{4.2}$$

The following theorem was proved by [Qui03, Proposition 3.2 and Theorem 4.7] when Γ is a Schottky subgroup. In general, Theorem 4.3 follows from [GW12, Lemma 3.1], [Sam14a, Corollaries 3.12, 3.13, and 4.9] and [Sam15, Theorem 4.20] in light of [BCLS15] using the Plücker representation (see also [PS17, Propositions 4.6 and 4.11]).

Theorem 4.3. *The following holds.*

(1) *Every nontrivial element in Γ is loxodromic.*

(2) *The limit cone of Γ is contained in $\operatorname{int} \mathfrak{a}^+ \cup \{0\}$.*

(3) *On $\operatorname{int} \mathcal{L}$, ψ_Γ is analytic and strictly concave except along rays emanating from the origin.*

(4) *If $\psi \in \mathfrak{a}^*$ is tangent to ψ_Γ at $v \in \operatorname{int} \mathfrak{a}^+ \cap \mathcal{L}$, then $v \in \operatorname{int} \mathcal{L}$ and ψ is positive on $\mathcal{L} \setminus \{0\}$.*

(5) *If $\psi \in \mathfrak{a}^*$ is positive on $\mathcal{L} \setminus \{0\}$, then $\delta_\psi \psi$ is tangent to ψ_Γ at some $v \in \operatorname{int} \mathcal{L}$, where $\delta_\psi > 0$ is the ψ-critical exponent, that is, the abscissa of convergence of the ψ-Poincaré series $t \mapsto \sum_{\gamma \in \Gamma} e^{-t\psi(\mu(\gamma))}$.*

Remark 4.4. Except for the requirement that $v \in \operatorname{int} \mathcal{L}$, Theorem 4.3(5) holds for any Zariski dense discrete subgroup as shown by Kim–Minsky–Oh [KMO21, Section 2].

We fix some notation for the rest of the paper. Fix a linear form ψ tangent to ψ_Γ at a normalized direction $v \in \operatorname{int} \mathcal{L}$, that is,

$$\psi \in \mathfrak{a}^*, \ \psi \geq \psi_\Gamma, \ v \in \operatorname{int} \mathcal{L} \text{ and } \psi(v) = \psi_\Gamma(v) = 1.$$

By Theorem 4.3, there is a unique such v for each tangent form ψ. Using Theorem 3.14, we fix

$$(\Gamma, \psi)\text{-and } (\Gamma, \psi \circ i)\text{-conformal measures } \nu_\psi \text{ and } \nu_{\psi \circ i},$$

respectively. We remark that ν_ψ and $\nu_{\psi \circ \mathsf{i}}$ are unique by [LO22, Theorem 1.2], but we do not need to use this uniqueness in this paper. Let $\mathsf{m}_\psi, m_\psi^{\mathrm{BR}}, m_\psi^{\mathrm{BR}\star}$ denote the $(\nu_\psi, \nu_{\psi \circ \mathsf{i}})$-BMS measure, $(m_{\mathcal{F}}, \nu_\psi)$- and $(\nu_{\psi \circ \mathsf{i}}, m_{\mathcal{F}})$-BR measures, respectively, as defined in Subsection 3.5. We will use the related notation from Subsection 3.5.

4.1. Ergodic decompositions of m_ψ, m_ψ^{BR} and $m_\psi^{\mathrm{BR}\star}$.
In this subsection, we recall the A-, N-, and N^+-ergodic decompositions of $\mathsf{m}_\psi, m_\psi^{\mathrm{BR}}$ and $m_\psi^{\mathrm{BR}\star}$, respectively, due to Lee–Oh [LO20a, Theorem 1.1].

Recall that P° denotes the connected component of the identity in P. Denote the set of P°-minimal subsets of $\Gamma \backslash G$ by $\mathfrak{Y}_\Gamma$. Every $Y \in \mathfrak{Y}_\Gamma$ satisfies $Y = (Y \cap \tilde{\Omega})N$. Let $\mathfrak{Z}_\Gamma$ denote the set of all intersections of P°-minimal subsets of $\Gamma \backslash G$ with $\tilde{\Omega}$, that is,

$$\mathfrak{Z}_\Gamma := \{Y \cap \tilde{\Omega} : Y \in \mathfrak{Y}_\Gamma\}. \tag{4.5}$$

Theorem 4.6 (LO20a, Theorem 1.1). *Consider the measures* $\mathsf{m}_\psi, m_\psi^{\mathrm{BR}}$ *and* $m_\psi^{\mathrm{BR}\star}$ *on* $\Gamma \backslash G$. *We have*

(1) *the A-ergodic decomposition*

$$\mathsf{m}_\psi = \sum_{Z \in \mathfrak{Z}_\Gamma} \mathsf{m}_\psi|_Z = \sum_{Z \in \mathfrak{Z}_\Gamma} \mathsf{m}_\psi|_Z;$$

(2) *the N-ergodic decomposition* $m_\psi^{\mathrm{BR}\star} = \sum_{Z \in \mathfrak{Z}_\Gamma} m_\psi^{\mathrm{BR}\star}|_{ZN}$;

(3) *the N^+-ergodic decomposition* $m_\psi^{\mathrm{BR}} = \sum_{Z \in \mathfrak{Z}_\Gamma} m_\psi^{\mathrm{BR}}|_{ZN^+}$.

4.2. The support Ω of m_ψ as a vector bundle.
For this subsection, we refer the reader to [LO20b, Section 4] and [Car21, Appendix A] for more details. The map $\pi_\psi : \Lambda^{(2)} \times \mathfrak{a} \to \Lambda^{(2)} \times \mathbb{R}$ defined by

$$\pi_\psi(x, y, w) = (x, y, \psi(w)) \quad \text{for all } (x, y, w) \in \Lambda^{(2)} \times \mathfrak{a}$$

is a vector bundle with typical fiber $\ker \psi$. Note that Γ acts on $\Lambda^{(2)} \times \mathfrak{a}$ on the left, via the Hopf parametrization. Note also that Γ acts on $\Lambda^{(2)} \times \mathbb{R}$ on the left by

$$\gamma \cdot (x, y, t) = (\gamma x, \gamma y, t + \psi(\beta_x(\gamma^{-1}, e)))$$

for all $\gamma \in \Gamma$, $(x, y, t) \in \Lambda^{(2)} \times \mathbb{R}$.

Theorem 4.7 ([Car21, Proposition A.1], see also [CS23, Theorem 4.15]). *The left Γ-action on $\Lambda^{(2)} \times \mathbb{R}$ is properly discontinuous and cocompact.*

Let

$$\mathcal{X}_\psi := \Gamma \backslash (\Lambda^{(2)} \times \mathbb{R}).$$

By Theorem 4.7, $\mathcal{X}_\psi$ is a compact Hausdorff topological space. The map π_ψ is Γ-equivariant and descends to a map $\pi_\psi : \Omega \to \mathcal{X}_\psi$ which is in fact, a trivial $\ker \psi$-vector bundle. We embed

$$\mathcal{X}_\psi \cong \mathcal{X}_\psi \times \{0\} \subset \mathcal{X}_\psi \times \ker \psi \cong \Omega. \tag{4.8}$$

Define the locally finite Borel measure $m_{\Lambda^{(2)} \times \mathbb{R}}$ on $\Lambda^{(2)} \times \mathbb{R}$ by

$$dm_{\Lambda^{(2)} \times \mathbb{R}}(\xi, \eta, t) := e^{\psi(\beta_\xi(e,g)) + \psi(\mathsf{i}(\beta_\eta(e,g)))} \, d\nu_\psi(\xi) \, d\nu_{\psi \circ \mathsf{i}}(\eta) \, dt,$$

where $g \in G$ is any element with $g^+ = \xi$ and $g^- = \eta$ and dt denotes the Lebesgue measure on $\mathbb{R}$. Note that $m_{\Lambda^{(2)} \times \mathbb{R}}$ is left Γ-invariant, so $m_{\Lambda^{(2)} \times \mathbb{R}}$ descends to a finite measure $m_{\mathcal{X}_\psi}$ on $\mathcal{X}_\psi$. We have

$$d\mathsf{m}_\psi\big|_\Omega = dm_{\mathcal{X}_\psi} \, du, \tag{4.9}$$

where du denotes the Lebesgue measure on $\ker \psi$ which satisfies $dw = dt \, du$, where $w = t\mathsf{v} + u$, $u \in \ker \psi$, dw and dt are the Lebesgue measures on $\mathfrak{a}$ and $\mathbb{R}$, respectively.

The set $\Lambda^{(2)} \times \mathbb{R}$ is also equipped with a natural flow $\Phi_t : \Lambda^{(2)} \times \mathbb{R} \to \Lambda^{(2)} \times \mathbb{R}$ defined by

$$\Phi_t(x, y, s) = (x, y, s + t)$$

for all $(x, y, s) \in \Lambda^{(2)} \times \mathbb{R}$. The flow Φ descends to a flow on $\mathcal{X}_\psi$ which we call the *translation flow* and we also denote by Φ.

4.3. **Local mixing.** We recall the local mixing theorem for the Haar measure on $\Gamma \backslash G$ which will be used in Section 5. Let dx denote the right G-invariant measure on $\Gamma \backslash G$ induced by the Haar measure on G. Given an inner product $\langle \cdot, \cdot \rangle_*$ on $\mathfrak{a}$, let $I : \ker \psi \to \mathbb{R}$ be defined by

$$I(u) = \langle u, u \rangle_* - \frac{\langle u, \mathsf{v} \rangle_*^2}{\langle \mathsf{v}, \mathsf{v} \rangle_*} \text{ for all } u \in \ker \psi. \tag{4.10}$$

Theorem 4.11 ([CS23, Theorem 1.3], [ELO22b, Theorem 3.4])**.** *There exists* $\kappa_{\mathsf{v}} > 0$ *and an inner product* $\langle \cdot, \cdot \rangle_*$ *on* $\mathfrak{a}$ *such that for any* $u \in \ker \psi$ *and* $\phi_1, \phi_2 \in C_c(\Gamma \backslash G)$, *we have*

$$\lim_{t \to +\infty} t^{\frac{\mathrm{rank}(G)-1}{2}} e^{(2\rho - \psi)(t\mathsf{v} + \sqrt{t}u)} \int_{\Gamma \backslash G} \phi_1(x \exp(t\mathsf{v} + \sqrt{t}u))\phi_2(x) \, dx$$

$$= \frac{\kappa_{\mathsf{v}} e^{-I(u)}}{|m_{\mathcal{X}_\psi}|} \sum_{Z \in \mathfrak{Z}_\Gamma} m_\psi^{\mathrm{BR}}\big|_{ZN^+}(\phi_1) \cdot m_\psi^{\mathrm{BR}_\star}\big|_{ZN}(\phi_2),$$

where ρ *is given by* (3.11) *and* $\mathfrak{Z}_\Gamma$ *is given by* (4.5).

Moreover, there exist positive constants η_{v} *and* T_{v} *such that for all* $\phi_1, \phi_2 \in C_c(\Gamma \backslash G)$, *there exists a constant* $D_{\mathsf{v}}(\phi_1, \phi_2)$ *depending continuously on* ϕ_1 *and* ϕ_2 *such that for all* $(t, u) \in (T_{\mathsf{v}}, \infty) \times \ker \psi$ *such that* $t\mathsf{v} + \sqrt{t}u \in \mathcal{L}$, *we have*

$$\left| t^{\frac{\mathrm{rank}(G)-1}{2}} e^{(2\rho - \psi)(t\mathsf{v} + \sqrt{t}u)} \int_{\Gamma \backslash G} \phi_1(x \exp(t\mathsf{v} + \sqrt{t}u))\phi_2(x) \, dx \right|$$

$$\leq D_{\mathsf{v}}(\phi_1, \phi_2) e^{-\eta_{\mathsf{v}} I(u)}.$$

4.4. Maximal flat cylinders.

Definition 4.12 (Maximal flat cylinder)**.** Let $C = \Gamma g A M \subset \Gamma \backslash G / M$ be a closed AM-orbit. We say C is *nontrivial* if its stabilizer $g^{-1}\Gamma g \cap AM$ is nontrivial. We say C is a *maximal flat cylinder* if

$$g^{-1}\Gamma g \cap AM \cong \mathbb{Z}. \tag{4.13}$$

If in addition to (4.13) we have

$$g^{-1}\Gamma g \cap \text{int}(A^+)M \neq \{e\},$$

then we say C is *positively oriented*.

Denote by $\mathcal{C}_\Gamma$ the set of all positively oriented maximal flat cylinders:

$$\mathcal{C}_\Gamma := \{C \subset \Gamma \backslash G / M : C \text{ is a positively oriented maximal flat cylinder}\}.$$

Let Γ_{prim} denote the set of primitive elements of Γ and $[\Gamma_{\text{prim}}]$ denote the set of Γ-conjugacy classes in Γ_{prim}. The following lemma justifies Definition 4.12 and shows why we will consider only positively oriented maximal flat cylinders and not all maximal flat cylinders in subsequent sections.

Lemma 4.14. *Let $C = \Gamma g A M \subset \Gamma \backslash G / M$ be a nontrivial closed AM-orbit in $\Gamma \backslash G / M$. Then the following holds.*

(1) *The stabilizer $g^{-1}\Gamma g \cap AM$ of C is isomorphic to $\mathbb{Z}$. Hence, C is a maximal flat cylinder and homeomorphic to $\mathbb{S}^1 \times \ker \psi$.*

(2) *If C is not positively oriented, then C is disjoint from Ω.*

(3) *If C is positively oriented, then C is contained in Ω.*

(4) *If C is positively oriented, then the semigroup $\Gamma \cap gA^+Mg^{-1}$ is generated by a single element $\gamma_{C,g}$. In particular, the semigroup*

$$(g^{-1}\Gamma g \cap \text{int}(A^+)M)/M \cong \{a \in \text{int}(A^+) : \Gamma gM = \Gamma gaM\}$$

is generated by a single element $\exp(v_C)$ which depends only on C and satisfies $\lambda(\gamma_{C,g}) = v_C$.

(5) *The map*

$$C \mapsto \{\gamma_{C,g} : C = \Gamma g A M\}$$

is a bijection between $\mathcal{C}_\Gamma$ and $[\Gamma_{\text{prim}}]$.

Proof. For (1), suppose for the sake of contradiction that $g^{-1}\Gamma g \cap AM$ contains two elements $g^{-1}\gamma_i g = a_{w_i} m_i$, $i = 1, 2$ that are not generated by a single element in $g^{-1}\Gamma g \cap AM$. Since A and M commute, we have

$$\gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1} = g m_1 m_2 m_1^{-1} m_2^{-1} g^{-1} \in \Gamma.$$

Since Γ is discrete and torsion free, $\gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1}$ must be identity. It follows that Γ contains a subgroup isomorphic to $\mathbb{Z}^2$, which is impossible since Γ is a hyperbolic group.

(Alternatively, we observe that w_1 and w_2 cannot be rational multiples of each other, $\gamma_1^j \gamma_2^k \in g a_{jw_1 + kw_2} M g^{-1}$ for $j, k \in \mathbb{Z}$, and hence $\{\lambda(\gamma_1^j \gamma_2^k) :$

$j, k \in \mathbb{Z}\}$ contains directions arbitrarily close to a wall of $\mathfrak{a}^+$, contradicting $\mathcal{L} \subset \operatorname{int} \mathfrak{a}^+$.)

Hence, $g^{-1}\Gamma g \cap AM \cong \mathbb{Z}$ and

$$C \cong (g^{-1}\Gamma g \cap AM)\backslash AM/M \cong \mathbb{Z}\backslash \mathbb{R}^{\operatorname{rank}(G)} \cong \mathbb{S}^1 \times \mathbb{R}^{\operatorname{rank}(G)-1}.$$

We highlight a more insightful way to see C as a cylinder $\mathbb{S}^1 \times \ker \psi$. Observe that under the vector bundle isomorphism $\Omega \cong \mathcal{X}_\psi \times \ker \psi$, $C = \Gamma g AM$ is identified with

$$\{\Gamma(g^+, g^-, t) : t \in \mathbb{R}\} \times \ker \psi.$$

Since C is closed, so is $\{\Gamma(g^+, g^-, t) : t \in \mathbb{R}\}$. Since the flow Φ is given by an $\mathbb{R}$-action on a compact space (e.g. geodesic flow on the unit tangent bundle of a compact surface rather than the geodesic flow on the surface itself) it follows that the closed Φ-orbit $\{\Gamma(g^+, g^-, t) : t \in \mathbb{R}\}$ is homeomorphic to $\mathbb{S}^1$.

For (2) and (3), we have $\Omega \cong \Gamma\backslash(\Lambda^{(2)} \times \mathfrak{a})$ (3.22) and by the Hopf parametrization (Definition 3.16), $C = \Gamma g AM$ is identified with

$$\{\Gamma(g^+, g^-, v) : v \in \mathfrak{a}\}.$$

Then C is contained in Ω if and only if $g^+ \in \Lambda$. If C is positively oriented, then there exists $\gamma \in g \operatorname{int}(A^+)Mg^{-1}$ and g^+ is the attracting fixed point of γ so $g^+ \in \Lambda$. On the other hand, if C is not positively oriented, then there exists a representative w of a Weyl element in $W \setminus \{M, w_0 M\}$ such that there exists $\gamma \in gw \operatorname{int}(A^+)Mw^{-1}g^{-1}$. Then the attracting fixed point of γ is $(gw)^+ \in \Lambda$. Then $g^+ \notin \Lambda$ otherwise by (4.2), $(g^+, (gw)^+) \in \Lambda^{(2)} \subset \mathcal{F}^{(2)}$, which contradicts the definition of $\mathcal{F}^{(2)}$ (3.15).

Now (4) is immediate from the definitions. For (5), surjectivity of the map $C \mapsto \{\gamma_{C,g} : C = \Gamma g AM\}$ is clear and injectivity follows from the fact that $N_G(A^+M) = AM$. ∎

In view of Lemma 4.14(4),(5), for each $C \in \mathcal{C}_\Gamma$, we fix some

$$\gamma_C \in \{\gamma_{C,g} : C = \Gamma g AM\} \tag{4.15}$$

and define the ψ-*circumference* of C to be

$$\ell_\psi(C) := \psi(\lambda(\gamma_C)) \tag{4.16}$$

which is positive by Theorem 4.3. Let $m_{C,g} \in M$ be the unique element satisfying $\gamma_{C,g} = g \exp(\lambda(\gamma_C))m_{C,g}g^{-1}$. The set

$$h_C := \{m_{C,g} : g \in G\} \in [M]$$

is a conjugacy class called the *holonomy* of C.

Remark 4.17. The notion of ψ-circumference cannot be well-defined for all maximal flat cylinders simultaneously. Indeed, consider a maximal flat cylinder $C = \Gamma g AM$ such that neither generator am nor $(am)^{-1}$ of $g^{-1}\Gamma g \cap AM \cong \mathbb{Z}$ lies in $(\operatorname{int} A^+)M$. Since Theorem 4.3 only guarantees that ψ is positive on $\mathcal{L} \setminus \{0\}$, to define the ψ-circumference, we would need to extract from a and a^{-1} an intrinsic element in $\mathcal{L}$. The natural candidates are the Jordan

projections of one of the generators γ or γ^{-1} of $\Gamma \cap gAMg^{-1}$. However, when G is higher rank, the opposition involution i may be nontrivial and

$$\psi(\lambda(\gamma^{-1})) = \psi(\mathsf{i}(\lambda(\gamma))) \neq \psi(\lambda(\gamma))$$

in general.

The next lemma gives a geometric description of ψ-circumferences.

Lemma 4.18. *The map*

$$C = \Gamma gAM \in \mathcal{C}_\Gamma \to \pi_\psi(C) = \{\Gamma(g^+, g^-, t) : t \in \mathbb{R}\} \subset \mathcal{X}_\psi$$

is a bijection between the set of positively oriented maximal flat cylinders and the set of periodic orbits of the translation flow Φ on $\mathcal{X}_\psi$. Using the embedding of $\mathcal{X}_\psi$ in Ω (4.8), the intersection of $C = \Gamma gAM \in \mathcal{C}_\Gamma$ with $\mathcal{X}_\psi$ is the corresponding periodic orbit $\pi_\psi(C)$ of the translation flow and the ψ-circumference of C is the period of $\pi_\psi(C)$.

Proof. Observe that the Φ-orbit $\{\Gamma(g^+, g^-, t) : t \in \mathbb{R}\}$ is periodic if and only if there exists $\gamma \in \Gamma$ such that x and y are fixed points of some $\gamma \in \Gamma$. By the same reasoning as in the proof of Lemma 4.14(3), x and y must be the attracting and repelling fixed points of γ and this establishes a bijection between the set of periodic orbits of the translation flow and the set of conjugacy classes in Γ. In view of Lemma 4.14(5), this establishes the desired bijection.

For the last assertion, let $C = \Gamma gAM \in \mathcal{C}_\Gamma$. Using the embedding of $\mathcal{X}_\psi$ in Ω (4.8), we have

$$C \cap (\mathcal{X}_\psi \times \{0\}) = \{\Gamma(g^+, g^-, t) : t \in \mathbb{R}\} \times \{0\}$$

and the period of $\{\Gamma(g^+, g^-, t) : t \in \mathbb{R}\}$ is precisely

$$\psi(\beta_{g^+}(\gamma_{C,g}^{-1}, e)) = \psi(\lambda(\gamma_{C,g})) = \ell_\psi(C).$$

$\blacksquare$

For the rest of the paper, we will only consider positively oriented maximal flat cylinders and we will omit the phrase "positively oriented".

5. Counting almost cylindrical maximal flats

The main result of this section is Proposition 5.24. Proposition 5.24 can be thought of as an asymptotic for the number elements in Γ corresponding to maximal flats in $\Gamma \backslash G/M$ which pass through and return to a given flow box in Ω and are almost positively oriented maximal flat cylinders with ψ-circumference at most T (Proposition 5.24). In Section 6, we then relate Proposition 5.24 to the number of maximal flat cylinders of ψ-circumference at most T which pass through $\tilde{\mathcal{B}}(g_0, \varepsilon)$ by applying the effective closing lemma for regular directions (Lemma 2.7), and we prove the main joint equidistribution result Theorem 6.12.

Recall that we have fixed a tangent form ψ tangent to ψ_Γ at a normalized direction $\mathsf{v} \in \operatorname{int} \mathcal{L}$, that is,

$$\psi \in \mathfrak{a}^*, \ \psi \geq \psi_\Gamma, \ \mathsf{v} \in \operatorname{int} \mathcal{L} \text{ and } \psi(\mathsf{v}) = \psi_\Gamma(\mathsf{v}) = 1.$$

We fix a cone

$$\mathfrak{C} \subset \mathcal{L} \text{ with } \mathsf{v} \in \operatorname{int} \mathfrak{C}.$$

For $T > 0$, let

$$\mathfrak{C}_T := \{\exp(w) : w \in \mathfrak{C}, \psi(w) \leq T\}$$

which is a bounded subset of $\mathcal{L}_\Gamma$ since $\mathfrak{C} \subset \mathcal{L}$ and $\psi > 0$ on $\mathcal{L} \setminus \{0\}$.

Recall the definition of flow boxes (Definition 2.5). Let $g_0 \in G$, $T, \varepsilon > 0$, and Θ be a Borel subset of M. We denote

$$\mathcal{V}_T(g_0, \varepsilon, \mathfrak{C}, \Theta) := \mathcal{B}(g_0, \varepsilon)\mathfrak{C}_T \Theta \mathcal{B}(g_0, \varepsilon)^{-1} = g_0 \mathcal{V}_T(e, \varepsilon, \mathfrak{C}, \Theta) g_0^{-1}. \quad (5.1)$$

Each $\gamma \in \Gamma \cap \mathcal{V}_T(g_0, \varepsilon, \mathfrak{C}, \Theta)$ corresponds to a family of AM-orbits in G which pass through the flow box $\mathcal{B}(g_0, \varepsilon)$ and then passes through $\gamma\mathcal{B}(g_0, \varepsilon)$ after translation by an element in $\mathfrak{C}_T\Theta$. In this sense, the elements of $\Gamma \cap \mathcal{V}_T(g_0, \varepsilon, \mathfrak{C}, \Theta)$ correspond to almost cylindrical maximal flats. In this section, we prove an asymptotic (Proposition 5.24) for the number of elements in $\Gamma \cap \mathcal{V}_T(g_0, \varepsilon, \mathfrak{C}, \Theta)$.

5.1. Counting in N^+AMN-coordinates.

In this subsection, we will prove an asymptotic for the number of elements in Γ contained product subsets of N^+AMN of a certain form. Throughout this subsection, we fix bounded Borel sets $\Xi_1 \subset N^+$, $\Xi_2 \subset N$ and $\Theta \subset M$. For $T > 0$ and $\varepsilon > 0$, we denote

$$S_T := S_T(\Xi_1, \Xi_2, \mathfrak{C}, \Theta) := \Xi_1 \mathfrak{C}_T \Theta \Xi_2. \quad (5.2)$$

Let $g_0 \in G$. We prove using local mixing of the Haar measure (Theorem 4.11) an asymptotic for $\#(\Gamma \cap g_0 S_T g_0^{-1})$ (Proposition 5.12) which is the main input in the proof of Proposition 5.24.

Given a bounded Borel subset B of G, define the counting function $F_B : G \times G \to \mathbb{N}$ by

$$F_B(g, h) := \sum_{\gamma \in \Gamma} 1_B(g^{-1}\gamma h) = \#(g^{-1}\Gamma h \cap B) = \#(\Gamma h \cap gB).$$

The function F_B is Γ-invariant in both arguments so it descends to a function on $\Gamma\backslash G \times \Gamma\backslash G$ which we still denote by F_B. Note that

$$F_{S_T}(e, e) = \#(\Gamma \cap S_T).$$

For $F_1, F_2 : \Gamma\backslash G \times \Gamma\backslash G \to \mathbb{R}$, let

$$\langle F_1, F_2 \rangle := \int_{\Gamma\backslash G \times \Gamma\backslash G} F_1(x_1, x_2) F_2(x_1, x_2) \, dx_1 \, dx_2$$

when the integral makes sense, where dx_1, dx_2 are both the Haar measure on $\Gamma\backslash G$.

For $\varepsilon > 0$, we denote

$$S_{T,\varepsilon}^- := \bigcap_{g_1,g_2 \in G_\varepsilon} g_1 S_T g_2; \qquad\qquad S_{T,\varepsilon}^+ := \bigcup_{g_1,g_2 \in G_\varepsilon} g_1 S_T g_2.$$

The sets $S_{T,\varepsilon}^\pm$ can be used to approximate $\#(\Gamma \cap S_T)$ as in the following lemma. For $\varepsilon > 0$ less than the injectivity radius of Γ, we fix a nonnegative function $\psi_\varepsilon \in C^\infty(G)$ with $\operatorname{supp} \psi_\varepsilon \subset G_\varepsilon$ and $\int_G \psi_\varepsilon \, dg = 1$. Let $\Psi_\varepsilon \in C^\infty(\Gamma\backslash G)$ be defined by $\Psi_\varepsilon(\Gamma g) := \sum_{\gamma \in \Gamma} \psi_\varepsilon(\gamma g)$ for all $g \in G$.

Lemma 5.3. *For any $T > 0$ and $\varepsilon > 0$, we have*

$$\langle F_{S_{T,\varepsilon}^-}, \Psi_\varepsilon \otimes \Psi_\varepsilon \rangle \le F_T(e,e) \le \langle F_{S_{T,\varepsilon}^+}, \Psi_\varepsilon \otimes \Psi_\varepsilon \rangle.$$

Our goal is to now estimate $\langle F_{S_{T,\varepsilon}^\pm}, \Psi_\varepsilon \otimes \Psi_\varepsilon \rangle$. We begin with Lemmas 5.4, 5.5, 5.7, and 5.8 which are computations relating to $\langle F_B, \Psi_1 \otimes \Psi_2 \rangle$ for general bounded Borel subset B of G.

A standard folding and unfolding argument gives the following lemma.

Lemma 5.4. *For any bounded Borel subset B of G and for all $\Psi_1, \Psi_2 \in C_c(\Gamma\backslash G)$, we have*

$$\langle F_B, \Psi_1 \otimes \Psi_2 \rangle = \int_B \langle \Psi_1, g\Psi_2 \rangle_{L^2(\Gamma\backslash G)} \, dg,$$

where $g\Psi_2(x) := \Psi_2(xg)$.

In view of the matrix coefficient of $L^2(\Gamma\backslash G)$ in Lemma 5.4, we now express the matrix coefficient in a way that lends itself to using local mixing (Theorem 4.11). For convenience, we denote

$$\mathsf{r} := \operatorname{rank}(G) \quad \text{and} \quad a_w := \exp(w) \text{ for } w \in \mathfrak{a}.$$

Lemma 5.5. *For any bounded Borel subset B of G and for all $\Psi_1, \Psi_2 \in C_c(\Gamma\backslash G)$, we have*

$$\langle F_B, \Psi_1 \otimes \Psi_2 \rangle$$
$$= \frac{\kappa_{\mathsf{v}}}{|m_{\mathcal{X}_\psi}|} \int_{h^{-1}a_{t\mathsf{v}+\sqrt{t}u}mn \in B} e^t e^{-I(u)} \sum_{Z \in \mathfrak{Z}_\Gamma} m_\psi^{\mathrm{BR}_\star}|_{ZN}(h\Psi_1) m_\psi^{\mathrm{BR}}|_{ZN^+}(mn\Psi_2)$$
$$+ e^t E(t,u,h,mn) \, dt \, du \, dh \, dm \, dn,$$

where $h \in N^+, m \in M, n \in N, t \in \mathbb{R}, u \in \ker \psi$ and

$$E(t,u,h,mn) = t^{\frac{\mathsf{r}-1}{2}} e^{2\rho(t\mathsf{v}+\sqrt{t}u)-t} \int_{\Gamma\backslash G} \Psi_1(xh)\Psi_2(xa_{t\mathsf{v}+\sqrt{t}u}mn) \, dx$$

$$- \frac{\kappa_{\mathsf{v}} e^{-I(u)}}{|m_{\mathcal{X}_\psi}|} \sum_{Z \in \mathfrak{Z}_\Gamma} m_\psi^{\mathrm{BR}_\star}|_{ZN}(h\Psi_1) m_\psi^{\mathrm{BR}}|_{ZN^+}(mn\Psi_2) \quad (5.6)$$

is the associated error term in Theorem 4.11 (note that $\psi(t\mathsf{v} + \sqrt{t}u) = t$).

Proof. By Lemma 5.4, we have

$$\langle F_B, \Psi_1 \otimes \Psi_2 \rangle = \int_B \int_{\Gamma \backslash G} \Psi_1(x) \Psi_2(xg) \, dx \, dg.$$

Using the fact that the product map $N^+ \times M \times A \times N \to G$ is a diffeomorphism onto a dense and open subset of G, we write $g = h^{-1} \exp(w) mn \in N^+ MAN$, so $dg = e^{2\rho(w)} \, dh \, dw \, dm \, dn$. Then

$$\int_B \int_{\Gamma \backslash G} \Psi_1(x) \Psi_2(xg) \, dx \, dg$$

$$= \int_{h^{-1} \exp(w) mn \in B} \int_{\Gamma \backslash G} \Psi_1(x) \Psi_2(xh^{-1} a_w mn) e^{2\rho(w)} \, dx \, dh \, dw \, dm \, dn$$

$$= \int_{h^{-1} a_w mn \in B} e^{2\rho(w)} \int_{\Gamma \backslash G} \Psi_1(xh) \Psi_2(xa_w mn) \, dx \, dw \, dh \, dm \, dn.$$

We write $w = t\mathsf{v} + \sqrt{t}u$, where $t \in \mathbb{R}$ and $u \in \ker \psi$. Recall $\mathsf{r} := \operatorname{rank}(G)$. Then $dw = t^{\frac{\mathsf{r}-1}{2}} \, dt \, du$, where dt is the Lebesgue measure on $\mathbb{R}$ and du is the Lebesgue measure on $\ker \psi$ from Subsection 4.2. Then

$$\int_{h^{-1} a_w mn \in B} e^{2\rho(w)} \int_{\Gamma \backslash G} \Psi_1(xh) \Psi_2(xa_w mn) \, dx \, dw \, dh \, dm \, dn$$

$$= \int_{h^{-1} a_{t\mathsf{v} + \sqrt{t}u} mn \in B} t^{\frac{\mathsf{r}-1}{2}} e^{2\rho(t\mathsf{v} + \sqrt{t}u)} \int_{\Gamma \backslash G} \Psi_1(xh) \Psi_2(xa_{t\mathsf{v} + \sqrt{t}u} mn)$$

$$dx \, dt \, du \, dh \, dm \, dn$$

$$= \frac{\kappa_{\mathsf{v}}}{|m_{\mathcal{X}_\psi}|} \int_{h^{-1} a_{t\mathsf{v} + \sqrt{t}u} mn \in B} e^t e^{-I(u)} \sum_{Z \in \mathfrak{Z}_\Gamma} m_\psi^{\mathrm{BR}_\star}\big|_{ZN}(h\Psi_1) m_\psi^{\mathrm{BR}}\big|_{ZN^+}(mn\Psi_2)$$

$$+ e^t E(t, u, h, mn) \, dt \, du \, dh \, dm \, dn.$$

■

We now specialize to the case when $\Psi_1 = \Psi_2 = \Psi_\varepsilon$ in Lemma 5.5. Let $E_\varepsilon(t, u, h, mn)$ denote the associated error term in (5.6) and denote

$$Q_{B, \varepsilon} := \langle F_B, \Psi_\varepsilon \otimes \Psi_\varepsilon \rangle - \int_{h^{-1} a_{t\mathsf{v} + \sqrt{t}u} mn \in B} e^t E_\varepsilon(t, u, h, mn) \, dt \, du \, dh \, dm \, dn.$$

In the next lemma, we use the Iwasawa decompositions to decompose the BR-measures appearing in Lemma 5.5 into a form that will be useful in the proof of Proposition 5.12.

Recall $\mathfrak{Z}_\Gamma$ from (4.5). For $Z \in \mathfrak{Z}_\Gamma$, let $\tilde{Z}$ denote the preimage of Z in G, that is,

$$\tilde{Z} := \{g \in G : \Gamma g \in Z\}.$$

Lemma 5.7. *Let $\varepsilon > 0$ be sufficiently small. Let $Z \in \mathfrak{Z}_\Gamma$, $h \in N^+, m \in M$ and $n \in N$. Then we have*

$$m_\psi^{\mathrm{BR}_\star}\big|_{ZN}(h\Psi_\varepsilon) m_\psi^{\mathrm{BR}}\big|_{ZN^+}(mn\Psi_\varepsilon)$$

$$= \int_{KAN \times KAN^+} \psi_\varepsilon(k_1 a_{w_1} n_1 h) \mathbb{1}_{\tilde{Z}N}(k_1) \psi_\varepsilon(k_2 a_{w_2} h_1 mn) \mathbb{1}_{\tilde{Z}N^+}(k_2)$$
$$\cdot e^{\psi(w_1 - w_2)} \, d\nu_\psi^K(k_1) \, dw_1 \, dn_1 \, d\nu_{\psi \mathrm{oi}}^K(k_2) \, dw_2 \, dh_1,$$

where ν_ψ^K and $\nu_{\psi \mathrm{oi}}^K$ are defined by (3.27).

Proof. For $g_1, g_2 \in G$, write $g_1 = k_1 a_{w_1} n_1 \in KAN$ and $g_2 = k_2 a_{w_2} h_1 \in KAN^+$. Using Lemma 3.28, we have

$$m_\psi^{\mathrm{BR}\star}\big|_{ZN}(h\Psi_\varepsilon) m_\psi^{\mathrm{BR}}\big|_{ZN^+}(mn\Psi_\varepsilon)$$
$$= \int_{G \times G} \psi_\varepsilon(g_1 h) \mathbb{1}_{\tilde{Z}N}(g_1) \psi_\varepsilon(g_2 mn) \mathbb{1}_{\tilde{Z}N^+}(g_2) \, dm_\psi^{\mathrm{BR}\star}(g_1) \, dm_\psi^{\mathrm{BR}}(g_2)$$
$$= \int_{KAN \times KAN^+} \psi_\varepsilon(k_1 a_{w_1} n_1 h) \mathbb{1}_{\tilde{Z}N}(k_1 a_{w_1} n_1) \psi_\varepsilon(k_2 a_{w_2} h_1 mn)$$
$$\cdot \mathbb{1}_{\tilde{Z}N^+}(k_2 a_{w_2} h_1) e^{\psi(w_1 - w_2)} \, d\nu_\psi^K(k_1) \, dw_1 \, dn_1 \, d\nu_{\psi \mathrm{oi}}^K(k_2) \, dw_2 \, dh_1$$
$$= \int_{KAN \times KAN^+} \psi_\varepsilon(k_1 a_{w_1} n_1 h) \mathbb{1}_{\tilde{Z}N}(k_1) \psi_\varepsilon(k_2 a_{w_2} h_1 mn) \mathbb{1}_{\tilde{Z}N^+}(k_2)$$
$$\cdot e^{\psi(w_1 - w_2)} \, d\nu_\psi^K(k_1) \, dw_1 \, dn_1 \, d\nu_{\psi \mathrm{oi}}^K(k_2) \, dw_2 \, dh_1. \qquad \blacksquare$$

To state the next lemma, it will be convenient for us to define some notation. We define $f_B : N^+ \times MN \to \mathbb{R}$ by

$$f_B(h, mn) := \frac{\kappa_{\mathsf{v}}}{|m_{\mathcal{X}_\psi}|} \int_{h^{-1} m a_{t\mathsf{v} + \sqrt{t}u} n \in B} e^t e^{-I(u)} \, dt \, du.$$

Let

$$\tilde{\mathfrak{Z}}_\Gamma := \{\tilde{Z} : Z \in \mathfrak{Z}_\Gamma\}.$$

Define the natural projection maps

$$H_1 : MANN^+ \to M,$$
$$I_1 : MANN^+ \to \mathfrak{a},$$
$$J_1 : MANN^+ \to N^+,$$
$$I_2 : AN^+ MN \to \mathfrak{a},$$
$$J_2 : AN^+ MN \to MN.$$

For each $kM \in K/M \cong \mathcal{F}$, fix a representative $k^* \in K$. In the next lemma, we use these projections and a change of variables to write $Q_{B,\varepsilon}$ using integrals over G_ε.

Lemma 5.8. *For any bounded Borel subset B of G, we have*

$$Q_{B,\varepsilon} = \sum_{\tilde{Z} \in \tilde{\mathfrak{Z}}_\Gamma} \int_{(K/M) \times K} \int_{G_\varepsilon \times G_\varepsilon} f_B(J_1((k_1^*)^{-1} g'), J_2(k_2^{-1} g''))$$
$$\cdot \psi_\varepsilon(g') \mathbb{1}_{\tilde{Z}N}(k_1^* H_1((k_1^*)^{-1} g')) \psi_\varepsilon(g'') \mathbb{1}_{\tilde{Z}N^+}(k_2)$$

$$\cdot \, e^{\psi(I_1((k_1^*)^{-1}g') - I_2(k_2^{-1}g''))} \, dg' \, dg'' \, d\nu_\psi(k_1^+) \, d\nu_{\psi \mathrm{oi}}^K(k_2).$$

Proof. By Lemma 5.5 and Lemma 5.7, we have

$$Q_{B,\varepsilon} = \frac{\kappa_{\mathsf{v}}}{|m_{\mathcal{X}_\psi}|} \sum_{Z \in \mathfrak{Z}_\Gamma} \int_{KAN \times KAN^+} \int_{h^{-1}a_{t\mathsf{v}+\sqrt{t}u}mn \in B} e^t e^{-I(u)} \psi_\varepsilon(k_1 a_{w_1} n_1 h)$$

$$\cdot \, \mathbb{1}_{\tilde{Z}N}(k_1) \psi_\varepsilon(k_2 a_{w_2} h_1 mn) \mathbb{1}_{\tilde{Z}N^+}(k_2) e^{\psi(w_1 - w_2)}$$

$$d\nu_\psi^K(k_1) \, dw_1 \, dn_1 \, d\nu_{\psi \mathrm{oi}}^K(k_2) \, dw_2 \, dh_1 \, dt \, du \, dh \, dm \, dn.$$

Let $m_1 \in M$ such that $k_1 = k_1^* m_1$. Let $g_3 = m_1 a_{w_1} n_1 h$ and $g_4 = a_{w_2} h_1 mn$. Then we have $dg_3 = dm_1 \, dw_1 \, dn_1 \, dh$ and $dg_4 = dw_2 \, dh_2 \, dm \, dn$ as in Subsection 3.5. Using these change of variables, we have

$$Q_{B,\varepsilon} = \sum_{\tilde{Z} \in \tilde{\mathfrak{Z}}_\Gamma} \int_{(K/M) \times K} \int_{G \times G} f_B(J_1(g_3), J_2(g_4)) \psi_\varepsilon(k_1^* g_3) \mathbb{1}_{\tilde{Z}N}(k_1^* H_1(g_3))$$

$$\cdot \, \psi_\varepsilon(k_2 g_4) \mathbb{1}_{\tilde{Z}N^+}(k_2) e^{\psi(I_1(g_3) - I_2(g_4))} \, dg_3 \, dg_4 \, d\nu_\psi(k_1^+) \, d\nu_{\psi \mathrm{oi}}^K(k_2)$$

$$= \sum_{\tilde{Z} \in \tilde{\mathfrak{Z}}_\Gamma} \int_{(K/M) \times K} \int_{G_\varepsilon \times G_\varepsilon} f_B(J_1((k_1^*)^{-1}g'), J_2(k_2^{-1}g''))$$

$$\cdot \, \psi_\varepsilon(g') \mathbb{1}_{\tilde{Z}N}(k_1^* H_1((k_1^*)^{-1}g')) \psi_\varepsilon(g'') \mathbb{1}_{\tilde{Z}N^+}(k_2)$$

$$\cdot \, e^{\psi(I_1((k_1^*)^{-1}g') - I_2(k_2^{-1}g''))} \, dg' \, dg'' \, d\nu_\psi(k_1^+) \, d\nu_{\psi \mathrm{oi}}^K(k_2),$$

where $g' = k_1^* g_3$ and $g'' = k_2 g_4$. ∎

We now specialize to $B = S_{T,\varepsilon}^{\pm}$ and estimate $Q_{S_{T,\varepsilon}^{\pm}, \varepsilon}$. In view of Lemma 5.8, to estimate $Q_{S_{T,\varepsilon}^{\pm}, \varepsilon}$, we need to show that for $k \in K$ that contribute to $Q_{S_{T,\varepsilon}^{\pm}, \varepsilon}$, the images of kG_ε under the projections H_1, I_1, J_1, I_2, J_2 are close to the image of k. This can be done because $\Xi_1 \subset N^+$ and $\Xi_2 \subset N$ are bounded.

Lemma 5.9. *Fix bounded sets $U_1 \subset N^+$ and $U_2 \subset N$. For sufficiently small $\varepsilon > 0$, the following holds. If $g_1 \in G$ with $g_1^{-1} = manh \in MANN^+$ and $g_1^+ \in U_1^+$, then*

$$g_1^{-1} G_\varepsilon \subset m M_{O(\varepsilon)} a A_{O(\varepsilon)} n N_{O(\varepsilon)} h N_{O(\varepsilon)}^+.$$

If $g_2 \in K$ with $g_2^{-1} = ahmn \in AN^+MN$ and $g_2^- \in U_2^-$, then

$$g_2^{-1} G_\varepsilon \subset a A_{O(\varepsilon)} h N_{O(\varepsilon)}^+ m M_{O(\varepsilon)} n N_{O(\varepsilon)}.$$

Proof. We prove the first statement; the proof of the second is similar. The product map $M \times A \times N \times N^+ \to G$ is a diffeomorphism onto an open set containing the identity. In particular,

$$G_\varepsilon \subset M_{O(\varepsilon)} A_{O(\varepsilon)} N_{O(\varepsilon)} N_{O(\varepsilon)}^+.$$

Let $g_\varepsilon \in G_\varepsilon$ with $g_\varepsilon = m_1 a_1 n_1 h_1 \in M_{O(\varepsilon)} A_{O(\varepsilon)} N_{O(\varepsilon)} N_{O(\varepsilon)}^+$. Under the hypotheses, h^{-1} is an element in the bounded set U_1 and

$$
\begin{aligned}
g_1^{-1} g_\varepsilon &= manhm_1 a_1 n_1 h_1 \\
&= mm_1 aa_1 ((m_1 a_1)^{-1} n(m_1 a_1))((m_1 a_1)^{-1} h(m_1 a_1)) n_1 h_1 \\
&= mm_1 m_2 aa_1 a_2 ((m_2 a_2)^{-1} (m_1 a_1)^{-1} n(m_1 a_1)(m_2 a_2)) n_2 h_2 h_1,
\end{aligned}
$$

where $((m_1 a_1)^{-1} h(m_1 a_1)) n_1 = m_2 a_2 n_2 h_2$ and we note that the assumption that h is bounded and $n_1 \in N_{O(\varepsilon)}$ imply that $m_2 \in M_{O(\varepsilon)}$, $a_2 \in A_{O(\varepsilon)}$ and $h_2 \in h N_{O(\varepsilon)}^+$. ∎

We will also need the following lemma which says that bounds the sets $S_{T,\varepsilon}^{\pm}$ by product subsets of $N^+ AMN$ that approximate S_T.

Lemma 5.10. *Assume that $\nu_\psi(\partial \Xi_1^+)$, $\nu_{\psi\text{oi}}(\partial(\Xi_2^{-1})^-)$ and $\int_{\partial\Theta} dm$ are all 0. For all sufficiently small $\varepsilon > 0$, there exists Borel sets $\mathfrak{C}'_{T,\varepsilon} \subset \mathfrak{C}_T$, $\Xi'_{1,\varepsilon} \subset \Xi_1$, $\Xi'_{2,\varepsilon} \subset \Xi_2$ and $\Theta'_\varepsilon \subset \Theta$ such that*

$$
N_{O(\varepsilon)}^+ \Xi'_{1,\varepsilon} \mathfrak{C}'_{T,\varepsilon} M_{O(\varepsilon)} \Theta'_\varepsilon \Xi'_{2,\varepsilon} N_{O(\varepsilon)} \subset S_{T,\varepsilon}^-,
$$

where an $O(\varepsilon)$-neighborhood of $\mathfrak{C}'_{T,\varepsilon}$ contains $\mathfrak{C}_T$ and $\nu_\psi((\Xi'_{1,\varepsilon})^+) \to \nu_\psi(\Xi_1^+)$, $\nu_{\psi\text{oi}}((\Xi'_{2,\varepsilon})^-) \to \nu_{\psi\text{oi}}(\Xi_2^-)$ and $\int_{\Theta'_\varepsilon} dm \to \int_\Theta dm$ as $\varepsilon \to 0$.

Similarly, there exists Borel sets $\mathfrak{C}''_{T,\varepsilon} \supset \mathfrak{C}_T$, $\Xi''_{1,\varepsilon} \supset \Xi_1$, $\Xi''_{2,\varepsilon} \supset \Xi_2$ and $\Theta''_\varepsilon \supset \Theta$ such that

$$
N_{O(\varepsilon)}^+ \Xi''_{1,\varepsilon} \mathfrak{C}''_{T,\varepsilon} M_{O(\varepsilon)} \Theta''_\varepsilon \Xi''_{2,\varepsilon} N_{O(\varepsilon)} \supset S_{T,\varepsilon}^+,
$$

where an $O(\varepsilon)$-neighborhood of $\mathfrak{C}_T$ contains $\mathfrak{C}''_{T,\varepsilon}$ and $\nu_\psi((\Xi''_{1,\varepsilon})^+) \to \nu_\psi(\Xi_1^+)$, $\nu_{\psi\text{oi}}((\Xi''_{2,\varepsilon})^-) \to \nu_{\psi\text{oi}}(\Xi_2^-)$ and $\int_{\Theta''_\varepsilon} dm \to \int_\Theta dm$ as $\varepsilon \to 0$.

Proof. In view of the hypotheses on the boundaries, we may assume Ξ_1, Ξ_2 and Θ are open subsets. Using reasoning similar to that used in Lemma 5.9, we see that for all $g_1, g_2 \in G_\varepsilon$ and for all $g = hamn \in N^+ MAN$ with h and n bounded, we have

$$
g_1 g g_2 \in N_{O(\varepsilon)}^+ h A_{O(\varepsilon)} a M_{O(\varepsilon)} mn N_{O(\varepsilon)}.
$$

Then we can take $\mathfrak{C}'_{T,\varepsilon}$ to be the intersection of $\mathfrak{C}_T$ and the complement of the closed $O(\varepsilon)$-neighborhood of the exterior of $\mathfrak{C}_T$. Similarly for $\Xi'_{1,\varepsilon} \subset \Xi_1$, $\Xi''_{2,\varepsilon} \subset \Xi_2$ and $\Theta'_\varepsilon \subset \Theta$. It is clear that these sets have the desired properties. The proof of the second assertion of the lemma is similar. ∎

We will use the following asymptotic notation. For real-valued functions f_1, f_2 of T, we write

$$
f_1 \sim f_2 \iff \lim_{T \to \infty} \frac{f_1(T)}{f_2(T)} = 1.
$$

For a real-valued function f of ε, we write

$$f = o_\varepsilon(1) \iff \lim_{\varepsilon \to 0} f(\varepsilon) = 0. \tag{5.11}$$

We use local mixing (Theorem 4.11) to prove the following asymptotic for $\#(\Gamma \cap g_0 S_T g_0^{-1})$ for $g_0 \in G$.

Proposition 5.12. *Let $g_0 \in G$. Assume that $\nu_\psi(\partial \Xi_1^+)$, $\nu_{\psi \mathrm{oi}}(\partial(\Xi_2^{-1})^-)$ and $\int_{\partial \Theta} dm$ are all 0. Then*

$$\#(\Gamma \cap g_0 S_T g_0^{-1}) \sim \frac{[M : M_\Gamma]}{|m_{\mathcal{X}_\psi}|} e^T \sum_{\tilde{Z} \in \tilde{\mathfrak{Z}}_\Gamma} \tilde{\nu}_\psi^{g_0}|_{g_0^{-1} \tilde{Z} N}(\Xi_1) \tilde{\nu}_{\psi \mathrm{oi}}^{g_0}|_{g_0^{-1} \tilde{Z} N^+}(\Xi_2^{-1} \Theta^{-1}),$$

$$\tag{5.13}$$

where $\tilde{\nu}_\psi^{g_0}|_{g_0^{-1} \tilde{Z} N}$ and $\tilde{\nu}_{\psi \mathrm{oi}}^{g_0}|_{g_0^{-1} \tilde{Z} N^+}$ are defined by (3.32) and (3.34).

Remark 5.14. We note that the right hand side of (5.13) does not depend on the choice of $\mathfrak{C}$ as long as $\mathsf{v} \in \mathrm{int}\, \mathfrak{C}$. This reflects the fact that the maximum of ψ_Γ on $\{w \in \mathcal{L} : \psi(w) = T\}$ occurs in the v direction.

Proof. It suffices to prove the theorem for $g_0 = e$. For general $g_0 \in G$, we apply the same argument to $g_0 \Gamma g_0^{-1}$, replacing ν_ψ and $\nu_{\psi \mathrm{oi}}$ with $\nu_\psi^{g_0}$ and $\nu_{\psi \mathrm{oi}}^{g_0}$ (3.31) throughout.

By Lemmas 5.4, 5.5, 5.7, and 5.8, we have

$$\langle F_{S_{T,\varepsilon}^\pm}, \Psi_\varepsilon \otimes \Psi_\varepsilon \rangle$$

$$= Q_{S_{T,\varepsilon}^\pm, \varepsilon} + \int_{h^{-1} a_{tv + \sqrt{t}u} mn \in S_{T,\varepsilon}^\pm} e^t E_\varepsilon(t, u, h, mn)\, dt\, du\, dh\, dm\, dn$$

and

$$Q_{S_{T,\varepsilon}^\pm, \varepsilon} = \sum_{\tilde{Z} \in \tilde{\mathfrak{Z}}_\Gamma} \int_{(K/M) \times K} \int_{G_\varepsilon \times G_\varepsilon} f_{S_{T,\varepsilon}^\pm}(J_1((k_1^*)^{-1} g'), J_2(k_2^{-1} g''))$$

$$\cdot \psi_\varepsilon(g') \mathbb{1}_{\tilde{Z} N}(k_1^* H_1((k_1^*)^{-1} g')) \psi_\varepsilon(g'') \mathbb{1}_{\tilde{Z} N^+}(k_2)$$

$$\cdot e^{\psi(I_1((k_1^*)^{-1} g') - I_2(k_2^{-1} g''))}\, dg'\, dg''\, d\nu_\psi(k_1^+)\, d\nu_{\psi \mathrm{oi}}^K(k_2).$$

We now do the proof in 2 steps.

Step 1: We estimate $Q_{S_{T,\varepsilon}^\pm, \varepsilon}$ in terms of S_T.

Recall from Subsection 4.1 that ZN is a P°-minimal subset of $\Gamma \backslash G$. Then using the correspondence between P°-minimal subsets of $\Gamma \backslash G$ and Γ-minimal subsets of G/P°, we have $\tilde{Z} N = \{x \in G : xP^\circ \in \Lambda_1\}$ for some Γ-minimal subset $\Lambda_1 \subset G/P^\circ$. Recall the holonomy group M_Γ of Γ (Definition 3.5). For fixed $k_1 M \in \Lambda$, by (3.6), there exists $m' \in M$ such that $\{m \in M : k_1^* m P^\circ \in \Lambda_1\} = M_\Gamma m'$. Then

$$\mathbb{1}_{\tilde{Z} N}(k_1^* H_1((k_1^*)^{-1} g')) = \mathbb{1}_{M_\Gamma m'}(H_1((k_1^*)^{-1} g')).$$

Since $M^\circ < M_\Gamma$, if ε is sufficiently small, then $\mathbb{1}_{M_\Gamma m'}(H_1((k_1^*)^{-1} g'))$ is constant on G_ε.

It follows from Lemma 5.9 and Lemma 5.10 that

$$\psi(I_1((k_1^*)^{-1}g')) = \psi(I_1((k_1^*)^{-1})) + O(\varepsilon);$$
$$\psi(I_2(k_2^{-1}g'')) = \psi(I_2(k_2^{-1})) + O(\varepsilon)$$

and

$$J_1((k_1^*)^{-1}g')^{-1} \in N_{O(\varepsilon)}^+ J_1((k_1^*)^{-1})^{-1};$$
$$J_2(k_2^{-1}g'') \in M_{O(\varepsilon)} J_2(k_2^{-1}) N_{O(\varepsilon)}.$$

We first prove a lower bound for $Q_{S_{T,\varepsilon}^-,\varepsilon}$. Let $\mathfrak{C}_{T,\varepsilon}' \subset \mathfrak{C}_T$, $\Xi_{1,\varepsilon}' \subset \Xi_1$, $\Xi_{2,\varepsilon}'' \subset \Xi_2$ and $\Theta_\varepsilon' \subset \Theta$ be as in Lemma 5.10. For convenience, let

$$S_{T,\varepsilon}' := \Xi_{1,\varepsilon}' \mathfrak{C}_{T,\varepsilon}' \Theta_\varepsilon' \Xi_{2,\varepsilon}'.$$

Then for all $g', g'' \in G_\varepsilon$, we have

$$f_{S_{T,\varepsilon}^-}(J_1((k_1^*)^{-1}g'), J_2(k_2^{-1}g'')) \ge f_{S_{T,\varepsilon}'}(J_1((k_1^*)^{-1}), J_2(k_2^{-1})).$$

It now follows that

$$Q_{S_{T,\varepsilon}^-,\varepsilon} \ge \sum_{\tilde{Z} \in \tilde{\mathfrak{Z}}_\Gamma} \int_{(K/M) \times K} f_{S_{T,\varepsilon}'}(J_1((k_1^*)^{-1}), J_2(k_2^{-1})) \mathbb{1}_{\tilde{Z}N}(k_1^* H_1((k_1^*)^{-1}))$$
$$\cdot \mathbb{1}_{\tilde{Z}N^+}(k_2)(1 + O(\varepsilon))e^{\psi(I_1((k_1^*)^{-1}) - I_2(k_2^{-1}))} \, d\nu_\psi(k_1^+) \, d\nu_{\psi\mathrm{oi}}^K(k_2). \tag{5.15}$$

Using Lemma 5.10, we observe that

$$\int_{(K/M) \times K} (f_{S_T} - f_{S_T^-})(J_1((k_1^*)^{-1}), J_2(k_2^{-1})) \, d\nu_\psi(k_1^+) \, d\nu_{\psi\mathrm{oi}}^K(k_2)$$
$$= O\left(e^T \int_{(K/M) \times K} \mathbb{1}_{\Xi_1 \setminus \Xi_{1,\varepsilon}^-}(J_1((k_1^*)^{-1})^{-1})\right.$$
$$\left. + \mathbb{1}_{\Theta\Xi_2 \setminus \Theta_\varepsilon \Xi_{2,\varepsilon}^-}(J_2(k_2^{-1})) \, d\nu_\psi(k_1^+) \, d\nu_{\psi\mathrm{oi}}^K(k_2)\right)$$
$$= e^T O\left(\nu_\psi\left(\Xi_1^+ \setminus (\Xi_{1,\varepsilon}^-)^+\right) + \nu_{\psi\mathrm{oi}}\left((\Xi_2^{-1})^- \setminus ((\Xi_{2,\varepsilon}^-)^{-1})^-\right) + m\left(\Theta \setminus \Theta_\varepsilon^-\right)\right)$$
$$= e^T o_\varepsilon(1), \tag{5.16}$$

where o_ε notation is defined by (5.11). Combining (5.15) and (5.16) yields

$$Q_{S_{T,\varepsilon}^-,\varepsilon} \ge e^T o_\varepsilon(1) + \sum_{\tilde{Z} \in \tilde{\mathfrak{Z}}_\Gamma} \int_{(K/M) \times K} (1 + O(\varepsilon)) f_{S_T}(J_1((k_1^*)^{-1}), J_2(k_2^{-1}))$$
$$\cdot \mathbb{1}_{\tilde{Z}N}(k_1^* H_1((k_1^*)^{-1})) \mathbb{1}_{\tilde{Z}N^+}(k_2) e^{\psi(I_1((k_1^*)^{-1}) - I_2(k_2^{-1}))} \, d\nu_\psi(k_1^+) \, d\nu_{\psi\mathrm{oi}}^K(k_2).$$

A similar argument shows that

$$Q_{S_{T,\varepsilon}^+,\varepsilon} \le e^T o_\varepsilon(1) + \sum_{\tilde{Z} \in \tilde{\mathfrak{Z}}_\Gamma} \int_{(K/M) \times K} (1 + O(\varepsilon)) f_{S_T}(J_1((k_1^*)^{-1}), J_2(k_2^{-1}))$$

$$\cdot \mathbb{1}_{\tilde{Z}N}(k_1^* H_1((k_1^*)^{-1})) \mathbb{1}_{\tilde{Z}N^+}(k_2) e^{\psi(I_1((k_1^*)^{-1}) - I_2(k_2^{-1}))} \, d\nu_\psi(k_1^+) \, d\nu^K_{\psi \text{oi}}(k_2)$$

and we conclude that

$$Q_{S^\pm_{T,\varepsilon},\varepsilon} = e^T o_\varepsilon(1) + \sum_{\tilde{Z} \in \tilde{\mathfrak{Z}}_\Gamma} \int_{(K/M) \times K} (1 + O(\varepsilon)) f_{S_T}(J_1((k_1^*)^{-1}), J_2(k_2^{-1}))$$

$$\cdot \mathbb{1}_{\tilde{Z}N}(k_1^* H_1((k_1^*)^{-1})) \mathbb{1}_{\tilde{Z}N^+}(k_2) e^{\psi(I_1((k_1^*)^{-1}) - I_2(k_2^{-1}))} \, d\nu_\psi(k_1^+) \, d\nu^K_{\psi \text{oi}}(k_2).$$

Step 2: We conclude by appropriately decomposing k_1^* and k_2 and applying the asymptotic in Lemma 5.17.

Considering $k_1 \in K$ such that $(k_1^*)^{-1} = m a_w n h^{-1} \in MANN^+$, we have

$$J_1((k_1^*)^{-1}))^{-1} = h;$$
$$h^+ = k_1 M;$$
$$k_1^* H_1((k_1^*)^{-1}) = h n^{-1} a_{-w};$$
$$I_1((k_1^*)^{-1}) = w = \beta_{h^+}(e, h)$$

and hence,

$$\int_{(J_1((k_1^*)^{-1}))^{-1} \in \Xi_1} \mathbb{1}_{\tilde{Z}N}(k_1^* H_1((k_1^*)^{-1})) e^{\psi(I_1((k_1^*)^{-1}))} \, d\nu_\psi(k_1^+)$$
$$= \int_{h \in \Xi_1} \mathbb{1}_{\tilde{Z}N}(h) e^{\psi(\beta_{h^+}(e,h))} \, d\nu_\psi(h^+) = \tilde{\nu}_\psi|_{\tilde{Z}N}(\Xi_1).$$

Similarly, considering $k_2^{-1} \in K$ such that $k_2^{-1} = a_w h m^{-1} n^{-1} \in AN^+ MN$, we have

$$J_2(k_2^{-1}) = m^{-1} n^{-1} \in MN;$$
$$k_2^- = n^-;$$
$$I_2(k_2^{-1}) = w = -i(\beta_{n^-}(e, n))$$

and

$$\int_{\substack{k_2 \in K \\ J_2(k_2^{-1}) \in \Theta\Xi_2}} \mathbb{1}_{\tilde{Z}N^+}(k_2) e^{-\psi(I_2(k_2^{-1}))} d\nu^K_{\psi \text{oi}}(k_2)$$
$$= \int_{nm \in \Xi_2^{-1}\Theta^{-1}} \mathbb{1}_{\tilde{Z}N^+}(nm) e^{(\psi \text{oi})(\beta_{n^-}(e,n))} \, d\nu_{\psi \text{oi}}(n^-) \, dm$$
$$= \tilde{\nu}_{\psi \text{oi}}|_{\tilde{Z}N^+}(\Xi_2^{-1}\Theta^{-1}).$$

Hence, we obtain

$$\langle F_{S^\pm_{T,\varepsilon}}, \Psi_\varepsilon \otimes \Psi_\varepsilon \rangle$$
$$= (1 + O(\varepsilon)) \frac{\kappa_\mathsf{v}}{|m_{\mathcal{X}_\psi}|} \int_{a_{t\mathsf{v} + \sqrt{t}u} \in \mathfrak{C}_T} e^t e^{-I(u)} \, dt \, du$$
$$\cdot \sum_{\tilde{Z} \in \tilde{\mathfrak{Z}}_\Gamma} \tilde{\nu}_\psi|_{\tilde{Z}N}(\Xi_1) \tilde{\nu}_{\psi \text{oi}}|_{\tilde{Z}N^+}(\Xi_2^{-1}\Theta^{-1})$$

$$+ \int_{h^{-1}a_{t\mathsf{v}+\sqrt{t}u}mn\in S_{T,\varepsilon}^{\pm}} e^t E_\varepsilon(t,u,h,mn)\, dt\, du\, dh\, dm\, dn + e^T o_\varepsilon(1).$$

Using Lemma 5.17 and Lemma 5.3, taking $T \to \infty$ and then $\varepsilon \to 0$, we conclude that

$$\#(\Gamma \cap S_T) \sim e^T \frac{\kappa_\mathsf{v}}{|m_{\mathcal{X}_\psi}|} \int_{\ker \psi} e^{-I(u)}\, du \sum_{\tilde{Z} \in \tilde{\mathfrak{Z}}_\Gamma} \tilde{\nu}_\psi|_{\tilde{Z}N}(\Xi_1) \tilde{\nu}_{\psi \text{oi}}|_{\tilde{Z}N^+}(\Xi_2^{-1}\Theta^{-1})$$

and we note that $\kappa_\mathsf{v} \int_{\ker \psi} e^{-I(u)}\, du = [M : M_\Gamma]$ by Proposition A.1.   ∎

Lemma 5.17. *We have*

$$\lim_{T \to \infty} e^{-T} \int_{a_{t\mathsf{v}+\sqrt{t}u} \in \mathfrak{C}_T} e^t e^{-I(u)}\, dt\, du = \int_{\ker \psi} e^{-I(u)}\, du \qquad (5.18)$$

and for all sufficiently small $\varepsilon > 0$,

$$\lim_{T \to \infty} e^{-T} \int_{h^{-1}a_{t\mathsf{v}+\sqrt{t}u}mn\in S_{T,\varepsilon}^{\pm}} e^t E_\varepsilon(t,u,h,mn)\, dt\, du\, dh\, dm\, dn = 0. \quad (5.19)$$

Proof. First, we show (5.18). For $u \in \ker \psi$ and $T > 0$, let

$$R_T(u) = \{t > 0 : a_{t\mathsf{v}+\sqrt{t}u} \in \mathfrak{C}_T\} = \{0 < t \le T : t\mathsf{v} + \sqrt{t}u \in \mathfrak{C}\}.$$

Then we have

$$e^{-T} \int_{a_{t\mathsf{v}+\sqrt{t}u} \in \mathfrak{C}_T} e^t e^{-I(u)}\, dt\, du = \int_{\ker \psi} e^{-I(u)} e^{-T} \int_{R_T(u)} e^t\, dt\, du.$$

Observe that $e^{-I(u)} e^{-T} \int_{R_T(u)} e^t\, dt \le e^{-I(u)}$ and by definition of $I(u)$ (4.10), $e^{-I(u)} \in L^1(\ker \psi)$. Then by the Lebesgue dominated convergence theorem,

$$\lim_{T \to \infty} e^{-T} \int_{a_{t\mathsf{v}+\sqrt{t}u} \in \mathfrak{C}_T} e^t e^{-I(u)}\, dt\, du$$
$$= \int_{\ker \psi} e^{-I(u)} \lim_{T \to \infty} e^{-T} \int_{R_T(u)} e^t\, dt\, du = \int_{\ker \psi} e^{-I(u)}\, du,$$

where the last equality uses the observation that for fixed u, $t \in R_T(u)$ for all sufficiently large $t \le T$.

Now we show (5.19). Since $\Xi_1 \subset N^+$ and $\Xi_2 \subset N$ are bounded, by Theorem 4.11, there exists positive constants $\eta_\mathsf{v}, D_\mathsf{v}$ and T_v such that

$$|E_\varepsilon(t,u,h,mn)| \le D_\mathsf{v} e^{-\eta_\mathsf{v} I(u)}$$

for all $(t,u) \in (T_\mathsf{v}, \infty) \times \ker \psi_\mathsf{v}$ and h, mn such that $h^{-1}a_{t\mathsf{v}+\sqrt{t}u}mn \in S_{T,\varepsilon}^{\pm}$. Then (5.19) follows by using similar reasoning as before and the fact that for fixed $u \in \ker \psi$, $\lim_{t \to \infty} E_\varepsilon(t,u,h,mn) = 0$.   ∎

5.2. **Counting in $\mathcal{B}(g_0, \varepsilon)\mathfrak{C}_T\Theta\mathcal{B}(g_0, \varepsilon)^{-1}$.** In this subsection, we relate re-late $\mathcal{V}_T(g_0, \varepsilon, \mathfrak{C}, \Theta)$ (5.1) and $g_0 S_T(N_\varepsilon^+, N_\varepsilon^{-1}, \mathfrak{C}, \Theta)g_0^{-1}$ (5.2) and then apply Proposition 5.12 to get an asymptotic for $\#(\Gamma \cap \mathcal{V}_T(g_0, \varepsilon, \mathfrak{C}, \Theta))$ (Proposition 5.24).

For Borel maps $f_1 : \Gamma\backslash G/M \to \mathbb{R}$, $f_2 : M \to \mathbb{R}$ and Borel sets $B_1 \subset \Gamma\backslash G/M$ and $B_2 \subset M$, we define

$$\mathsf{m}_\psi(f_1 \otimes f_2) := \left(\int_X f_1 \, d\mathsf{m}_\psi\right)\left(\int_M f_2 \, dm\right);$$
$$\mathsf{m}_\psi(B_1 \otimes B_2) := \mathsf{m}_\psi(\mathbb{1}_{B_1} \otimes \mathbb{1}_{B_2}).$$

Denote the projection of $\mathcal{B}(g_0, \varepsilon)$ to $\Gamma\backslash G/M$ by

$$\tilde{\mathcal{B}}(g_0, \varepsilon) := \Gamma\mathcal{B}(g_0, \varepsilon)M \subset \Gamma\backslash G/M.$$

Recall $\mathsf{r} = \mathrm{rank}(G)$. Let $b_\mathsf{r}(\varepsilon)$ denote the volume of the Euclidean r-ball of radius ε. We have the following formula for $\mathsf{m}_\psi(\tilde{\mathcal{B}}(g_0, \varepsilon) \otimes \Theta)$.

Lemma 5.20. *For any $g_0 \in G$, $\varepsilon > 0$ and Borel subset Θ of M, we have*

$$\mathsf{m}_\psi(\tilde{\mathcal{B}}(g_0, \varepsilon) \otimes \Theta) = (1 + O(\varepsilon))b_\mathsf{r}(\varepsilon)\tilde{\nu}_\psi^{g_0}(N_\varepsilon^+)\tilde{\nu}_{\psi\mathrm{oi}}^{g_0}(N_\varepsilon\Theta),$$

where $\tilde{\nu}_\psi^{g_0}$ and $\tilde{\nu}_{\psi\mathrm{oi}}^{g_0}$ are defined by (3.32) *and* (3.33).

Proof. We assume that $g_0 = e$. The proof for general $g_0 \in G$ is similar. By Lemma 2.6(2) and (3), we have $\mathcal{B}(e, \varepsilon)e^+ = N_\varepsilon^+e^+$, $\mathcal{B}(e, \varepsilon)e^- = N_\varepsilon e^-$ and the projection of $\mathcal{B}(e, \varepsilon)M \subset G/M \cong \mathcal{F}^{(2)} \times \mathfrak{a}$ into $\mathcal{F}^{(2)}$ is $N_\varepsilon^+e^+ \times N_\varepsilon e^-$. Fix $g \in \mathcal{B}(e, \varepsilon)$ and write $w = \beta_{g^+}(e, g)$, $g = h_1 n_1 a_1 m_1 \in N_\varepsilon^+NA_\varepsilon M_\varepsilon$ and $g = n_2 h_2 a_2 m_2 \in N_\varepsilon N^+ A_{O(\varepsilon)} M_{O(\varepsilon)}$. Using (3.20) and properties of the Busemann function (Definition 3.7), the lemma follows immediately from the following computation:

$$\mathsf{m}_\psi(\mathcal{B}(e, \varepsilon)M)$$
$$= \int_{g \in \mathcal{B}(e, \varepsilon)} e^{\psi\left(\beta_{g^+}(e,g)\right) + (\psi\mathrm{oi})\left(\beta_{g^-}(e,g)\right)} \, d\nu_\psi(g^+) \, d\nu_{\psi\mathrm{oi}}(g^-) \, dw$$
$$= \int_{g \in \mathcal{B}(e, \varepsilon)} e^{\psi\left(\beta_{h_1^+}(e,h_1 a_1)\right) + (\psi\mathrm{oi})\left(\beta_{(n_2)^-}(e,n_2 a_2)\right)} \, d\nu_\psi(h_1^+) \, d\nu_{\psi\mathrm{oi}}(n_2^-) \, dw$$
$$= \int_{g \in \mathcal{B}(e, \varepsilon)} e^{\psi\left(\beta_{h_1^+}(e,h_1) + \log(a_1)\right) + (\psi\mathrm{oi})\left(\beta_{n_2^-}(e,n_2) - \mathrm{i}(\log(a_2))\right)}$$
$$d\nu_\psi(h_1^+) \, d\nu_{\psi\mathrm{oi}}(n_2^-) \, dw$$
$$= \int_{g \in \mathcal{B}(e, \varepsilon)} e^{\psi(\log(a_1) - \log(a_2))} \, d\tilde{\nu}_\psi^{g_0}(h_1) \, d\tilde{\nu}_{\psi\mathrm{oi}}^{g_0}(n_2 M) \, dw$$
$$= (1 + O(\varepsilon))b_\mathsf{r}(\varepsilon)\tilde{\nu}_\psi^{g_0}(N_\varepsilon^+)\tilde{\nu}_{\psi\mathrm{oi}}^{g_0}(N_\varepsilon M). \qquad \blacksquare$$

The next lemma relates $\mathcal{V}_T(e, \varepsilon, \mathfrak{C}, \Theta)$ and $S_T(N_\varepsilon^+, N_\varepsilon^{-1}, \mathfrak{C}, \Theta)$. To state the lemma, we fix a constant $C > 0$ such that

$$a_{-w} N_\varepsilon a_w \subset N_{\varepsilon e^{-C\psi(w)}} \text{ and } a_w N_\varepsilon^+ a_{-w} \subset N_{\varepsilon e^{-C\psi(w)}}^+ \text{ for all } w \in \mathcal{L}. \quad (5.21)$$

This is possible since $\mathcal{L} \subset \operatorname{int} \mathfrak{a}^+$.

Lemma 5.22. *For all $T > T' > 1$ and sufficiently small $\varepsilon > 0$,*

$$S_T(N_\varepsilon^+, N_\varepsilon^{-1}, \mathfrak{C}, \Theta) \subset \mathcal{V}_T(e, \varepsilon, \mathfrak{C}, \Theta)$$

and we have

$$\mathcal{V}_T(e, \varepsilon, \mathfrak{C}, \Theta) \setminus \mathcal{V}_{T'}(e, \varepsilon, \mathfrak{C}, \Theta)$$
$$\subset S_{T+O(\varepsilon)}(N_{\varepsilon+O(\varepsilon e^{-CT'})}^+, (N_{\varepsilon+O(\varepsilon e^{-CT'})})^{-1}, \hat{\mathfrak{C}}_{O(\varepsilon)}, \hat{\Theta}_{O(\varepsilon)})$$
$$\setminus S_{T'-O(\varepsilon)}(N_{\varepsilon+O(\varepsilon e^{-CT'})}^+, (N_{\varepsilon+O(\varepsilon e^{-CT'})})^{-1}, \hat{\mathfrak{C}}_{O(\varepsilon)}, \hat{\Theta}_{O(\varepsilon)}),$$

where $\hat{\Theta}_\varepsilon := \bigcup_{m_1, m_2 \in M_\varepsilon} m_1 \Theta m_2$ and $\hat{\mathfrak{C}}_\varepsilon$ is the smallest closed cone in $\mathfrak{a}$ containing the ε-neighborhood of $\mathfrak{C}_T \setminus \mathfrak{C}_{T'}$ for all $T > T' > 1$.

Proof. Since $N_\varepsilon^\pm \subset \mathcal{B}(e, \varepsilon)$, the first inclusion is clear from the definitions.

For the second inclusion, consider

$$g \in \mathcal{V}_T(e, \varepsilon, \mathfrak{C}, \Theta) \setminus \mathcal{V}_{T'}(e, \varepsilon, \mathfrak{C}, \Theta).$$

There exists $g_1, g_2 \in \mathcal{B}(e, \varepsilon)$, $a \in \mathfrak{C}_T \setminus \mathfrak{C}_{T'}$ and $m \in \Theta$ such that $g = g_1 m a g_2^{-1}$. By Lemma 2.6(3), we have $g_1 = h_1 n_1 m_1 a_1 \in N_\varepsilon^+ N_{O(\varepsilon)} M_\varepsilon A_\varepsilon$ and $g_2 = n_2 h_2 m_2 a_2 \in N_\varepsilon N_{O(\varepsilon)}^+ M_{O(\varepsilon)} A_{O(\varepsilon)}$. Then

$$g = h_1 n_1 m_1 a_1 m a (m_2 a_2)^{-1} h_2^{-1} n_2^{-1}.$$

Let $a' = a a_1 a_2^{-1}$, $m' = m_1 m m_2^{-1}$ and $n_3 = (m'a')^{-1} n_1 m' a'$. Then $a' \in (\hat{\mathfrak{C}}_{O(\varepsilon)})_{T+O(\varepsilon)} \setminus (\hat{\mathfrak{C}}_{O(\varepsilon)})_{T'-O(\varepsilon)}$, $n_3 \in N_{O(\varepsilon e^{-CT'})}$ and $g = h_1 m' a' n_3 h_2^{-1} n_2^{-1}$. Using our choice of constant C in (5.21), by Lemma 2.4, we have

$$n_3 h_2^{-1} = m_3 a_3 h_4 n_4 \in M_{O(\varepsilon)} A_{O(\varepsilon)} N_{O(\varepsilon)}^+ N_{O(\varepsilon e^{-CT'})}.$$

Then

$$g = h_1 m' a' m_3 a_3 h_4 n_4 n_2^{-1} = h_5 m'' a'' n_5,$$

where $a'' = a' a_3$, $m'' = m' m_3$, $h_5 = h_1 (m''a'') h_4 (m''a'')^{-1}$ and $n_5 = n_4 n_2^{-1}$. Note that $a'' \in (\hat{\mathfrak{C}}_{O(\varepsilon)})_{T+O(\varepsilon)} \setminus (\hat{\mathfrak{C}}_{O(\varepsilon)})_{T'-O(\varepsilon)}$, $m'' \in \hat{\Theta}_{O(\varepsilon)}$, $h_5 \in N_{\varepsilon+O(\varepsilon e^{-CT'})}^+$ and $n_5 \in N_{\varepsilon+O(\varepsilon e^{-CT'})}$ which completes the proof. ∎

To state the next proposition, we will use the following asymptotic notation. For a real-valued function f of ε and T, we write

$$f = o_T(1) \iff \lim_{T \to \infty} f(\varepsilon, T) = 0 \quad (5.23)$$

Recall the holonomy group M_Γ of Γ (Definition 3.5). By Lemma 5.22 and Proposition 5.12, we obtain the following asymptotic for the number of elements in $\Gamma \cap \mathcal{V}_T(g_0, \varepsilon, \mathfrak{C}, \Theta)$.

Proposition 5.24. *Let $g_0 \in G$ and $\Theta \subset M_\Gamma$ be a Borel set with $\int_{\partial\Theta} dm = 0$. For all sufficiently small $\varepsilon > 0$, we have*

$$\#(\Gamma \cap \mathcal{V}_T(g_0, \varepsilon, \mathfrak{C}, \Theta))$$

$$= \frac{[M : M_\Gamma]}{|m_{\mathcal{X}_\psi}|} e^T \left(\frac{\mathsf{m}_\psi(\tilde{\mathcal{B}}(g_0, \varepsilon) \otimes \Theta)}{b_{\mathsf{r}}(\varepsilon)} (1 + O(\varepsilon)) + o_T(1) \right).$$

Proof. Again, we give a proof for $g_0 = e$ and the general case is similar. Note that $\partial N_\varepsilon^\pm$ are proper real algebraic subvarieties of $\mathcal{F}$ and hence $\nu_\psi(\partial N_\varepsilon^+) = \nu_{\psi\circ\mathsf{i}}(\partial N_\varepsilon) = 0$ by [KO23c] (this was proved in [ELO22a] for the case $\psi = \psi \circ \mathsf{i}$.)

By Lemma 5.22 and Proposition 5.12, we have

$$\#(\Gamma \cap \mathcal{V}_T(g_0, \varepsilon, \mathfrak{C}, \Theta))$$

$$\geq \frac{[M : M_\Gamma]}{|m_{\mathcal{X}_\psi}|} e^T \left(\sum_{\tilde{Z} \in \tilde{\mathfrak{Z}}_\Gamma} \tilde{\nu}_\psi|_{\tilde{Z}N}(N_\varepsilon^+) \tilde{\nu}_{\psi\circ\mathsf{i}}|_{\tilde{Z}N^+}(N_\varepsilon\Theta^{-1}) + o_T(1) \right). \quad (5.25)$$

We claim that for sufficiently small $\varepsilon > 0$, we have

$$\sum_{\tilde{Z} \in \tilde{\mathfrak{Z}}_\Gamma} \tilde{\nu}_\psi|_{\tilde{Z}N}(N_\varepsilon^+) \tilde{\nu}_{\psi\circ\mathsf{i}}|_{\tilde{Z}N^+}(N_\varepsilon\Theta^{-1}) = \tilde{\nu}_\psi(N_\varepsilon^+) \tilde{\nu}_{\psi\circ\mathsf{i}}(N_\varepsilon\Theta). \quad (5.26)$$

To see why the claim is true, observe that

$$\tilde{\nu}_\psi(N_\varepsilon^+) = \sum_{\tilde{Z} \in \tilde{\mathfrak{Z}}_\Gamma} \tilde{\nu}_\psi|_{\tilde{Z}N}(N_\varepsilon^+) \text{ and } \tilde{\nu}_{\psi\circ\mathsf{i}}(N_\varepsilon\Theta) = \sum_{\tilde{Z} \in \tilde{\mathfrak{Z}}_\Gamma} \tilde{\nu}_{\psi\circ\mathsf{i}}|_{\tilde{Z}N^+}(N_\varepsilon\Theta^{-1}),$$

where in the second equality we use the fact that $\tilde{Z}N$ is right M_Γ-invariant for each $\tilde{Z} \in \tilde{\mathfrak{Z}}_\Gamma$. If $e^+ \notin \Lambda$ or $e^- \notin \Lambda$, then for sufficiently small ε, either $N_\varepsilon^+ \cap \tilde{Z}N = \emptyset$ for all $\tilde{Z} \in \tilde{\mathfrak{Z}}_\Gamma$ or $N_\varepsilon \cap \tilde{Z}N^+ = \emptyset$ for all $\tilde{Z} \in \tilde{\mathfrak{Z}}_\Gamma$ and the claim is clear since both sides of the equation would be 0. Otherwise, $e^+, e^- \in \Lambda$, so for sufficiently small ε, $N_\varepsilon^\pm \cap \Lambda \subset \tilde{Z}$ for a single $\tilde{Z} \in \tilde{\mathfrak{Z}}_\Gamma$ and hence, the claim.

Combining (5.25) and (5.26) and using Lemma 5.20, we obtain

$$\#(\Gamma \cap (\mathcal{V}_T(e, \varepsilon, \mathfrak{C}, \Theta)))$$

$$\geq \frac{[M : M_\Gamma]}{|m_{\mathcal{X}_\psi}|} e^T \left(\frac{\mathsf{m}_\psi(\tilde{\mathcal{B}}(e, \varepsilon) \otimes \Theta)}{b_{\mathsf{r}}(\varepsilon)} (1 + O(\varepsilon)) + o_T(1) \right).$$

It remains to establish the reverse inequality. By Lemma 5.22, we know $\mathcal{V}_T(e, \varepsilon, \mathfrak{C}, \Theta) \setminus \mathcal{V}_{T/2}(e, \varepsilon, \mathfrak{C}, \Theta)$ is contained in

$$S_{T+O(\varepsilon)}(N_{\varepsilon+O(\varepsilon e^{-CT/2})}^+, (N_{\varepsilon+O(\varepsilon e^{-CT/2})}^-)^{-1}, \hat{\mathfrak{C}}_{O(\varepsilon)}, \hat{\Theta}_{O(\varepsilon)})$$

and

$$\mathcal{V}_{T/2}(e, \varepsilon, \mathfrak{C}, \Theta) \subset S_{T/2+O(\varepsilon)}(N_{O(\varepsilon)}^+, (N_{O(\varepsilon)})^{-1}, \hat{\mathfrak{C}}_{O(\varepsilon)}, \hat{\Theta}_{O(\varepsilon)}).$$

Assuming ε is sufficiently small so that $M_{O(\varepsilon)} \subset M^{\circ} \subset M_{\Gamma}$, similar reasoning as above yields

$$\#(\Gamma \cap \mathcal{V}_T(e, \varepsilon, \mathfrak{C}, \Theta)) - \#(\Gamma \cap \mathcal{V}_{T/2}(e, \varepsilon, \mathfrak{C}, \Theta))$$
$$\leq \frac{[M : M_{\Gamma}]}{|m_{\mathcal{X}_{\psi}}|} e^{(T + O(\varepsilon))}$$
$$\cdot \left(\frac{m_{\psi}(\tilde{\mathcal{B}}(e, \varepsilon + O(\varepsilon e^{-CT/2})) \otimes \hat{\Theta}_{O(\varepsilon)})}{b_{r}(\varepsilon + O(\varepsilon e^{-CT/2}))} (1 + O(\varepsilon + \varepsilon e^{-CT/2})) + o_T(1) \right);$$
$$\tag{5.27}$$

$$\#(\Gamma \cap \mathcal{V}_{T/2}(e, \varepsilon, \mathfrak{C}, \Theta))$$
$$\leq \frac{[M : M_{\Gamma}]}{|m_{\mathcal{X}_{\psi}}|} e^{(T/2 + O(\varepsilon))} \left(\frac{m_{\psi}(\tilde{\mathcal{B}}(e, O(\varepsilon)) \otimes \hat{\Theta}_{O(\varepsilon)})}{b_{r}(O(\varepsilon))} (1 + O(\varepsilon)) + o_T(1) \right).$$
$$\tag{5.28}$$

Combining (5.27) and (5.28) and taking T large, we obtain the desired inequality. ∎

6. Joint equidistribution with respect to ψ-circumferences

In this section, we prove the main theorem of this paper, Theorem 1.3, and Corollary 1.4. In fact, the results we prove here are slightly more general as we allow ourselves to consider only maximal flat cylinders C with corresponding Jordan projections $\lambda(\gamma_C)$ lying in a fixed cone $\mathfrak{C} \subset \mathcal{L}$, where γ_C is as in (4.15).

As in Section 5, for this section, we fix

$$\psi \in \mathfrak{a}^* \text{ tangent to } \psi_{\Gamma} \text{ at } \mathsf{v} \in \text{int } \mathcal{L} \text{ with } \psi(\mathsf{v}) = \psi_{\Gamma}(\mathsf{v}) = 1$$

and we fix

$$\text{cone } \mathfrak{C} \subset \mathcal{L} \text{ with } \mathsf{v} \in \text{int } \mathfrak{C}.$$

6.1. Preparatory lemmas. Throughout this subsection, we fix

— g_0 with $\Gamma g_0 M \in \text{supp } m_{\psi}$;

— $\varepsilon > 0$ sufficiently small as in Proposition 5.24;

— a conjugation-invariant Borel set $\Theta \subset M_{\Gamma}$ (Definition 3.5).

For $T > 0$, we set

$$\mathcal{W}_T(g_0, \varepsilon, \mathfrak{C}, \Theta) = \{gamg^{-1} : g \in \mathcal{B}(g_0, \varepsilon), am \in \mathfrak{C}_T \Theta\}.$$

Recall that $\mathcal{C}_{\Gamma}$ is the set of all (positively oriented) maximal flat cylinders in $\Gamma \backslash G / M$. For $C \in \mathcal{C}_{\Gamma}$, choose a lift $\tilde{C} \subset G/M$. Its stabilizer $\Gamma_{\tilde{C}}$ is generated by some element in $[\gamma_C]$, and C can be identified with $\Gamma_{\tilde{C}} \backslash \tilde{C}$. Let

$$I(C) := \{\Gamma_{\tilde{C}} \sigma \in \Gamma_{\tilde{C}} \backslash \Gamma : (\sigma \mathcal{B}(g_0, \varepsilon) M) \cap \tilde{C} \neq \emptyset\}.$$

Note that $\#I(C)$ is independent of the choice of $\tilde{C}$.

For $T > 0$, let $\mathcal{C}_\psi(T)$ denote the set of all maximal flat cylinders with ψ-circumference (4.16) at most T:

$$\mathcal{C}_\psi(T) := \{C \in \mathcal{C}_\Gamma : \ell_\psi(C) \leq T\}.$$

We will also consider the set of maximal flat cylinders C with corresponding Jordan projections $\lambda(\gamma_C)$ lying in the cone $\mathfrak{C}$:

$$\mathcal{C}_\psi(T, \mathfrak{C}) = \{C \in \mathcal{C}_\psi(T) : \lambda(\gamma_C) \in \mathfrak{C}\}.$$

Lemma 6.1.   *(1) The set $\mathcal{C}_\psi(T)$ is finite for any $T > 0$.*

(2) For any $C = \Gamma g A M \in \mathcal{C}_\Gamma$, $I(C)$ is finite.

Proof. By discreteness of Γ, for any $g_0 \in G$ and $\varepsilon > 0$, there are finitely many elements of Γ in $\mathcal{W}_T(g_0, \varepsilon, \mathcal{L}, M)$. On the other hand,

$$
\begin{aligned}
\#\Gamma \cap \mathcal{W}_T(g_0, \varepsilon, \mathcal{L}, M) &= \#\{\gamma = g a_w m g^{-1} : g \in \mathcal{B}(g_0, \varepsilon),\ \psi(w) \leq T\} \\
&= \#\{\tilde{C} : \tilde{C} \cap \mathcal{B}(g_0, \varepsilon) M \neq \emptyset,\ \ell_\psi(\Gamma \backslash \Gamma \tilde{C}) < T\} \\
&= \sum_{C \in \mathcal{C}_\psi(T)} \#I(C).
\end{aligned}
$$

From this, (2) immediately follows, and we see that

$$\#\{C \in \mathcal{C}_\psi(T) : C \cap \tilde{\mathcal{B}}(g_0, \varepsilon) \neq \emptyset\} < \infty.$$

Then (1) follows, using the fact that every maximal flat cylinder intersects the embedded compact set $\mathcal{X}_\psi \subset \Omega$ (Subsection 4.2). $\blacksquare$

Denote by V_C the measure on C induced by the Haar measure on A. Let $\mathrm{Cl}(M)$ denote the set of continuous real-valued class functions on M and $[M]$ denote the set of conjugacy classes in M. For each $T > 0$, we define a Radon measure $\mu_{\mathfrak{C},T}$ on $\Gamma \backslash G / M \times [M]$ as follows. For $T > 0$, $f \in \mathrm{C}_\mathrm{c}(\Gamma \backslash G / M)$ and $\varphi \in \mathrm{Cl}(M)$, let

$$\mu_{\mathfrak{C},T}(f \otimes \varphi) := \sum_{C \in \mathcal{C}_\psi(T, \mathfrak{C})} V_C(f) \varphi(h_C),$$

$$\mu_T(f \otimes \varphi) := \mu_{\mathcal{L},T}(f \otimes \varphi) = \sum_{C \in \mathcal{C}_\psi(T)} V_C(f) \varphi(h_C).$$

We record several appropriately adapted lemmas from [MMO14, Section 5]. Lemma 6.2 and Lemma 6.3 relates the $\mu_{\mathfrak{C},T}$-measure of $\tilde{\mathcal{B}}(g_0, \varepsilon) \otimes [\Theta]$ to the number of elements in $\Gamma \cap \mathcal{W}_T(g_0, \varepsilon, \mathfrak{C}, \Theta)$. The proofs are similar to those in [MMO14], but we provide them here for the sake of completeness.

Lemma 6.2.

(1) For any $C \in \mathcal{C}_\Gamma$, we have

$$V_C(\tilde{\mathcal{B}}(g_0, \varepsilon)) = b_\mathsf{r}(\varepsilon) \cdot \#I(C).$$

(2) *For any $T > 0$ and , we have*

$$\mu_{\mathfrak{C},T}(\tilde{\mathcal{B}}(g_0, \varepsilon) \otimes [\Theta]) = b_{\mathsf{r}}(\varepsilon) \cdot \sum_{C \in \mathcal{C}_\psi(T, \mathfrak{C})} \#I(C) \cdot 1_{[\Theta]}(h_C).$$

Proof. It is clear from the definition of $\mu_{\mathfrak{C},T}$ that (2) follows from (1). Let $C = \Gamma g A M$, and choose $\tilde{C} = g A M$. Then we have

$$
\begin{aligned}
V_C(\tilde{\mathcal{B}}(g_0, \varepsilon)) &= \int_{\mathfrak{a} \mod \lambda(\gamma_C)} 1_{\tilde{\mathcal{B}}(g_0,\varepsilon)}(\Gamma g \exp(w)M) \ dw \\
&= \sum_{[\sigma] \in \Gamma_{\tilde{C}} \backslash \Gamma} \int_{\mathfrak{a} \mod \lambda(\gamma_C)} 1_{[\sigma]\mathcal{B}(g_0,\varepsilon)}(g \exp(w)M) \ dw \\
&= b_{\mathsf{r}}(\varepsilon) \cdot \#I(C),
\end{aligned}
$$

where dw denotes the Lebesgue measure on $\mathfrak{a}$, the integral is over a fundamental domain in $\mathfrak{a}$ for the $\mathbb{Z}$-action given by translation by $\lambda(\gamma_C)$ and the last equality uses Lemma 2.6(1). ∎

Recall that Γ_{prim} denotes the set of primitive elements of Γ.

Lemma 6.3. *For all sufficiently large T, we have*

$$\mu_{\mathfrak{C},T}(\tilde{\mathcal{B}}(g_0, \varepsilon) \otimes [\Theta]) = b_{\mathsf{r}}(\varepsilon) \cdot \#(\Gamma_{\mathrm{prim}} \cap \mathcal{W}_T(g_0, \varepsilon, \mathfrak{C}, \Theta)).$$

Proof. Let $C = \Gamma g A M \in \mathcal{C}_\psi(T, \mathfrak{C})$ and wihtout loss of generality, assume that $\tilde{C} = g A M$. By Lemma 6.2(1), it suffices to show that

$$\#I(C) \cdot 1_{[\Theta]}(h_C) = \#([\gamma_C] \cap \mathcal{W}_T(g_0, \varepsilon, \mathfrak{C}, \Theta)). \tag{6.4}$$

If $h_C \notin [\Theta]$, then (6.4) is clear. We assume that $h_C \in [\Theta]$. If $\Gamma_{\tilde{C}} \sigma \in I(C)$, then $\sigma^{-1} g \in \mathcal{B}(g_0, \varepsilon) A M$ and

$$\sigma^{-1} \gamma_{C,g} \sigma = \sigma^{-1} g \exp(\lambda(\gamma_C)) m_{C,g} g^{-1} \sigma \in [\gamma_C] \cap \mathcal{W}_T(g_0, \varepsilon, \mathfrak{C}, \Theta).$$

Conversely, if $\sigma \in \Gamma$ such that $\sigma^{-1} \gamma_{C,g} \sigma \in \mathcal{W}_T(g_0, \varepsilon, \mathfrak{C}, \Theta)$, then we have $\gamma_{C,g} = \sigma g' a m (g')^{-1} \sigma^{-1}$ where $g' \in \mathcal{B}(g_0, \varepsilon)$ and $am \in \mathfrak{C}_T \Theta$, but on the other hand, $\gamma_{C,g} = g \exp(\lambda(\gamma_C)) m_{C,g} g^{-1}$, so $\sigma g' \in g A M$, and hence, $\sigma \mathcal{B}(g_0, \varepsilon) \cap g A M \neq \emptyset$, that is, $\Gamma_{\tilde{C}} \sigma \in I(C)$.

$$\sigma g' \in \sigma \mathcal{B}(g_0, \varepsilon) \cap g A M.$$

Noting that the map $\Gamma_{\tilde{C}} \sigma \in \Gamma_{\tilde{C}} \backslash \Gamma \mapsto \sigma^{-1} \gamma_{C,g} \sigma \in [\gamma_C]$ is bijective since the centralizer of $\gamma_{C,g}$ in Γ is $\langle \gamma_{C,g} \rangle = \Gamma_{\tilde{C}}$, we have thus given a bijection between $I(C)$ and $[\gamma_C] \cap \mathcal{W}_T(g_0, \varepsilon, \mathfrak{C}, \Theta)$. ∎

In the next lemma, we use the effective closing lemma for regular directions (Lemma 2.7) to relate $\Gamma \cap \mathcal{W}_T(g_0, \varepsilon, \mathfrak{C}, \Theta)$ with $\Gamma \cap \mathcal{V}_T(g_0, \varepsilon, \mathfrak{C}, \Theta)$.

Lemma 6.5. *There exists $T_1 > 0$, depending only on Γ and ψ, such that for all sufficiently large T, we have*

$$
\begin{aligned}
\Gamma \cap \big(\mathcal{V}_{T-O(\varepsilon)}(g_0, \varepsilon(1 - O(e^{-T})), \check{\mathfrak{C}}_{O(\varepsilon)}, \check{\Theta}_{O(\varepsilon)}) \setminus \mathcal{V}_{T_1}(g_0, \varepsilon, \mathfrak{C}, \Theta)\big) \\
\subset \Gamma \cap \mathcal{W}_T(g_0, \varepsilon, \mathfrak{C}, \Theta),
\end{aligned}
$$

where $\check{\Theta}_\varepsilon := \bigcap_{g_1,g_2 \in G_\varepsilon} g_1 \Theta g_2$ and $\check{\mathfrak{C}}_\varepsilon$ is any cone containing $\mathbb{R}\mathbf{v}$ such that the ε-neighborhood of $\check{\mathfrak{C}}_\varepsilon \setminus (\check{\mathfrak{C}}_\varepsilon)_{T_1}$ is contained in $\mathfrak{C}$.

Proof. Fix $T_1 > 0$ such that if $w \in \mathcal{L}$ and $\psi(w) \geq T_1$, then $\min_{\alpha \in \Phi^+} \alpha(w) \geq T_0$, where T_0 is as in Lemma 2.7. Suppose

$$\gamma \in \Gamma \cap \left(\mathcal{V}_{T-O(\varepsilon)}(g_0, \varepsilon(1 - O(e^{-T})), \check{\mathfrak{C}}_{O(\varepsilon)}, \check{\Theta}_{O(\varepsilon)}) \setminus \mathcal{V}_{T_1}(g_0, \varepsilon, \mathfrak{C}, \Theta)\right).$$

Then $\gamma = g_1 \exp(w) m g_2$, where $g_1, g_2 \in \mathcal{B}(g_0, \varepsilon(1 - O(e^{-T})))$, $w \in \check{\mathfrak{C}}_{O(\varepsilon)}$ with $T_1 < \psi(w) \leq T - O(\varepsilon)$, and $m \in \check{\Theta}_{O(\varepsilon)}$. By Lemma 2.7, we have $\gamma = g \exp(w') m' g^{-1}$ for some $g \in \mathcal{B}(g_0, \varepsilon)$, $w' \sim_{O(\varepsilon)} w$ and $m \sim_{O(\varepsilon)} m'$. It follows that $w \in \mathfrak{C}$, $\psi(w') \leq T$ and $m' \in \Theta$, so $\gamma \in \Gamma \cap \mathcal{W}_T(g_0, \varepsilon, \mathfrak{C}, \Theta)$. $\blacksquare$

Let m_Γ denote the Haar probability measure on M_Γ. The following lemma gives a lower bound for $\#\Gamma_{\mathrm{prim}} \cap \mathcal{W}_T(g_0, \varepsilon, \mathfrak{C}, \Theta)$.

Lemma 6.6. *Suppose that $m_\Gamma(\Theta) > 0$ and $m_\Gamma(\partial\Theta) = 0$. Then for all sufficiently large T, we have*

$$\#\Gamma \cap (\mathcal{W}_T(g_0, \varepsilon, \Theta) \setminus \mathcal{W}_{2T/3}(g_0, \varepsilon, \Theta)) \leq \#\Gamma_{\mathrm{prim}} \cap \mathcal{W}_T(g_0, \varepsilon, \Theta).$$

Proof. Let $\Gamma_{\mathrm{prim}^k} = \{\sigma^k : \sigma \in \Gamma_{\mathrm{prim}}\}$. We observe that

$$\#\Gamma_{\mathrm{prim}} \cap \mathcal{W}_T(g_0, \varepsilon, \mathfrak{C}, \Theta)$$

$$= \#\left(\Gamma \cap \mathcal{W}_T(g_0, \varepsilon, \mathfrak{C}, \Theta)\right) - \#\left(\bigcup_{k \geq 2} \Gamma_{\mathrm{prim}^k} \cap \mathcal{W}_T(g_0, \varepsilon, \mathfrak{C}, \Theta)\right)$$

$$\geq \#\left(\Gamma \cap \mathcal{W}_T(g_0, \varepsilon, \mathfrak{C}, \Theta)\right) - \#\left(\bigcup_{k \geq 2} \Gamma \cap \mathcal{W}_{T/k}(g_0, \varepsilon, \mathfrak{C}, \sqrt[k]{\Theta})\right),$$

where $\sqrt[k]{\Theta} := \{m \in M : m^k \in \Theta\}$. It suffices to show that for all sufficiently large T, we have

$$\#\left(\bigcup_{k \geq 2} \Gamma \cap \mathcal{W}_{T/k}(g_0, \varepsilon, \mathfrak{C}, \sqrt[k]{\Theta})\right) \leq \#\Gamma \cap \mathcal{W}_{2T/3}(g_0, \varepsilon, \mathfrak{C}, \Theta). \tag{6.7}$$

Since $\mathcal{W}_{T/k}(g_0, \varepsilon, \mathfrak{C}, \sqrt[k]{\Theta}) \subset \mathcal{V}_{T/k}(g_0, \varepsilon, \mathfrak{C}, \sqrt[k]{\Theta})$ and $\Gamma \cap \mathcal{W}_{T/k}(g_0, \varepsilon, \mathfrak{C}, \sqrt[k]{\Theta})$ is empty when T/k is sufficiently small, using Proposition 5.24, it follows that

$$\#\left(\bigcup_{k \geq 2} \Gamma \cap \mathcal{W}_{T/k}(g_0, \varepsilon, \mathfrak{C}, \sqrt[k]{\Theta})\right) = O(Te^{T/2}). \tag{6.8}$$

On the other hand, using Lemma 6.5, Proposition 5.24 and the assumption that $m_\Gamma(\Theta) > 0$, we have

$$\#\Gamma \cap \mathcal{W}_{2T/3}(g_0, \varepsilon, \Theta) \geq O(e^{2T/3}). \tag{6.9}$$

The inequality (6.7) now follows from (6.8) and (6.9). $\blacksquare$

Lemmas 6.3, 6.5, and 6.6 imply the following lemma.

Lemma 6.10 (Comparison Lemma). *Suppose that $m_\Gamma(\Theta) > 0$ and $m_\Gamma(\partial\Theta) = 0$. For all sufficiently large T, we have*

$$b_{\mathsf{r}}(\varepsilon) \cdot \#\Gamma \cap \left(\mathcal{V}_{T-O(\varepsilon)}(g_0, \varepsilon(1 - O(e^{-T})), \check{\mathfrak{C}}_{O(\varepsilon)}, \check{\Theta}_{O(\varepsilon)}) \setminus \mathcal{V}_{2T/3}(g_0, \varepsilon, \mathfrak{C}, \Theta)\right)$$
$$\leq \mu_{\mathfrak{C},T}(\tilde{\mathcal{B}}(g_0, \varepsilon) \otimes \Theta) \leq b_{\mathsf{r}}(\varepsilon) \cdot \#\Gamma \cap \mathcal{V}_T(g_0, \varepsilon, \mathfrak{C}, \Theta).$$

Proof. The upper bound follows directly from Lemma 6.3 and the trivial inclusion $\Gamma \cap \mathcal{W}_T(g_0, \varepsilon, \mathfrak{C}, \Theta) \subset \Gamma \cap \mathcal{V}_T(g_0, \varepsilon, \mathfrak{C}, \Theta)$. The lower bound follows by using Lemma 6.3, Lemma 6.6, and Lemma 6.5:

$$\mu_{\mathfrak{C},T}(\tilde{\mathcal{B}}(g_0, \varepsilon) \otimes \Theta)$$
$$= b_{\mathsf{r}}(\varepsilon)\#(\Gamma_{\mathrm{prim}} \cap \mathcal{W}_T(g_0, \varepsilon, \mathfrak{C}, \Theta))$$
$$\geq b_{\mathsf{r}}(\varepsilon)\#\Gamma \cap (\mathcal{W}_T(g_0, \varepsilon, \mathfrak{C}, \Theta) \setminus \mathcal{W}_{2T/3}(g_0, \varepsilon, \mathfrak{C}, \Theta))$$
$$\geq b_{\mathsf{r}}(\varepsilon)\#\Gamma \cap \left(\mathcal{V}_{T-O(\varepsilon)}(g_0, \varepsilon(1 - O(e^{-T})), \check{\mathfrak{C}}_{O(\varepsilon)}, \check{\Theta}_{O(\varepsilon)}) \setminus \mathcal{V}_{2T/3}(g_0, \varepsilon, \mathfrak{C}, \Theta)\right).$$

$\blacksquare$

Combining Proposition 5.24 and Lemma 6.10, we obtain the following asymptotic for $\mu_{\mathfrak{C},T}(\tilde{\mathcal{B}}(g_0, \varepsilon) \otimes \Theta)$.

Proposition 6.11. *For all $g_0 \in \operatorname{supp} \mathsf{m}_\psi$, for all conjugation-invariant Borel subsets $\Theta \subset M_\Gamma$ with $\int_\Theta dm_\Gamma > 0$ and $\int_{\partial\Theta} dm = 0$ and for all sufficiently small $\varepsilon > 0$, we have*

$$\mu_{\mathfrak{C},T}(\tilde{\mathcal{B}}(g_0, \varepsilon) \otimes \Theta) = \frac{[M : M_\Gamma]}{|m_{\mathcal{X}_\psi}|} e^T \left(\mathsf{m}_\psi(\tilde{\mathcal{B}}(g_0, \varepsilon) \otimes \Theta)(1 + O(\varepsilon)) + o_T(1)\right).$$

Proof. Inputting Proposition 5.24 into Lemma 6.10 yields

$$\mu_{\mathfrak{C},T}(\tilde{\mathcal{B}}(g_0, \varepsilon) \otimes \Theta) \leq \frac{[M : M_\Gamma]}{|m_{\mathcal{X}_\psi}|} e^T \left(\mathsf{m}_\psi(\tilde{\mathcal{B}}(g_0, \varepsilon) \otimes \Theta)(1 + O(\varepsilon)) + o_T(1)\right)$$

and

$$\mu_{\mathfrak{C},T}(\tilde{\mathcal{B}}(g_0, \varepsilon) \otimes \Theta)$$
$$\geq \frac{[M : M_\Gamma]}{|m_{\mathcal{X}_\psi}|} e^{T-O(\varepsilon)}$$
$$\cdot \left(\frac{b_{\mathsf{r}}(\varepsilon)\mathsf{m}_\psi(\tilde{\mathcal{B}}(g_0, \varepsilon(1 - O(e^{-T}))) \otimes \check{\Theta}_{O(\varepsilon)})}{b_{\mathsf{r}}(\varepsilon(1 - O(e^{-T})))}(1 + O(\varepsilon)) + o_T(1)\right)$$
$$- \frac{[M : M_\Gamma]}{|m_{\mathcal{X}_\psi}|} e^{2T/3} \left(\mathsf{m}_\psi(\tilde{\mathcal{B}}(g_0, \varepsilon) \otimes \Theta)(1 + O(\varepsilon)) + o_T(1)\right)$$
$$= \frac{[M : M_\Gamma]}{|m_{\mathcal{X}_\psi}|} e^T \left(\mathsf{m}_\psi(\tilde{\mathcal{B}}(g_0, \varepsilon) \otimes \Theta)(1 + O(\varepsilon)) + o_T(1)\right).$$

$\blacksquare$

6.2. **Proofs of the main results.** We are now ready to prove our main joint equidistribution theorem and its corollaries. We note that Theorem 1.3 and Corollary 1.4 are special cases of the following statements when $\mathfrak{C} = \mathcal{L}$.

Theorem 6.12 (Joint equidistribution). *For any $f \in C_c(\Gamma \backslash G/M)$, $\varphi \in \mathrm{Cl}(M)$ and cone $\mathfrak{C} \subset \mathcal{L}$ with $\mathsf{v} \in \mathrm{int}\,\mathfrak{C}$, we have*

$$\lim_{T \to \infty} \frac{\mu_{\mathfrak{C},T}(f \otimes \varphi)}{e^T} = \frac{\mathsf{m}_\psi(f)}{|m_{\mathcal{X}_\psi}|} \int_{M_\Gamma} \varphi \, dm_\Gamma.$$

Proof. The theorem is clear if $\mathrm{supp}\,\varphi \cap M_\Gamma = \emptyset$. We first prove the theorem when $\varphi = \mathbb{1}_\Theta$ for some conjugation-invariant Borel subset $\Theta \subset M_\Gamma$ with $\int_\Theta dm_\Gamma > 0$ and $\int_{\partial\Theta} dm = 0$. By using a partition of unity, we may assume without loss of generality that f is supported on $\tilde{\mathcal{B}}(g_0, \varepsilon)$ for some $g_0 \in \mathrm{supp}\,\mathsf{m}_\psi$ and a $\varepsilon > 0$ as in Proposition 5.24. Then we can approximate f arbitrarily well by a linear combination of characteristic functions on boxes of the form $\tilde{\mathcal{B}}(h, \rho)$, with $h \in \tilde{\mathcal{B}}(g_0, \varepsilon)$ and arbitrarily small $\rho > 0$. Then by applying Proposition 6.11 to each $\mathbb{1}_{\tilde{\mathcal{B}}(h,\rho)} \otimes \mathbb{1}_\Theta$, we obtain

$$(1 - O(\rho)) \frac{[M : M_\Gamma]}{|m_{\mathcal{X}_\psi}|} \mathsf{m}_\psi(f \otimes \mathbb{1}_\Theta) \leq \liminf_T e^{-T} \mu_{\mathfrak{C},T}(f \otimes \mathbb{1}_\Theta)$$

$$\leq \limsup_T e^{-T} \mu_{\mathfrak{C},T}(f \otimes \mathbb{1}_\Theta)$$

$$\leq (1 + O(\rho)) \frac{[M : M_\Gamma]}{|m_{\mathcal{X}_\psi}|} \mathsf{m}_\psi(f \otimes \mathbb{1}_\Theta)$$

and hence

$$\lim_T e^{-T} \mu_{\mathfrak{C},T}(f \otimes \mathbb{1}_\Theta) = \frac{[M : M_\Gamma]}{|m_{\mathcal{X}_\psi}|} \mathsf{m}_\psi(f \otimes \mathbb{1}_\Theta)$$

$$= \frac{\mathsf{m}_\psi(f)}{|m_{\mathcal{X}_\psi}|} \int_{M_\Gamma} \mathbb{1}_\Theta \, dm_\Gamma.$$

After identifying M with the quotient of the Lie algebra of its maximal torus by the Weyl group relative to its maximal torus, a similar approximation argument can be used for general $\varphi \in \mathrm{Cl}(M)$. $\blacksquare$

For each $T > 0$, we define the Radon measure $\eta_{\mathfrak{C},T}$ on $\Gamma \backslash G/M \times [M]$ in a similar fashion as $\mu_{\mathfrak{C},T}$, but normalizing the Haar measure on maximal flat cylinders by their ψ-circumferences. For $T > 0$, $f \in C_c(\Gamma \backslash G/M)$ and $\varphi \in \mathrm{Cl}(M)$, let

$$\eta_{\mathfrak{C},T}(f \otimes \varphi) := \sum_{C \in \mathcal{C}_\psi(T, \mathfrak{C})} \frac{V_C(f)}{\ell_\psi(C)} \varphi(h_C).$$

Corollary 6.13. *For any $f \in C_c(\Gamma \backslash G/M)$, $\varphi \in \mathrm{Cl}(M)$ and cone $\mathfrak{C} \subset \mathcal{L}$ with $\mathsf{v} \in \mathrm{int}\,\mathfrak{C}$, we have*

$$\lim_{T \to \infty} \frac{\eta_{\mathfrak{C},T}(f \otimes \varphi)}{e^T/T} = \frac{\mathsf{m}_\psi(f)}{|m_{\mathcal{X}_\psi}|} \int_{M_\Gamma} \varphi \, dm_\Gamma.$$

Proof. We observe that $T\eta_{\mathfrak{C},T} \geq \mu_{\mathfrak{C},T}$, and for any $\varepsilon > 0$,

$$Te^{-T}\eta_{\mathfrak{C},T}(f \otimes \varphi)$$

$$= Te^{-T}\left(\sum_{\mathcal{C}_\psi((1-\varepsilon)T,\mathfrak{C})} \frac{V_C(f)}{\ell_\psi(C)}\varphi(h_C) + \sum_{\mathcal{C}_\psi(T,\mathfrak{C})\setminus\mathcal{C}_\psi((1-\varepsilon)T,\mathfrak{C})} \frac{V_C(f)}{\ell_\psi(C)}\varphi(h_C)\right)$$

$$\leq Te^{-T}\left(O\left(\sum_{\mathcal{C}_\psi((1-\varepsilon)T,\mathfrak{C})} V_C(f)\varphi(h_C)\right)\right.$$

$$\left. + \sum_{\mathcal{C}_\psi(T,\mathfrak{C})\setminus\mathcal{C}_\psi((1-\varepsilon)T,\mathfrak{C})} \frac{V_C(f)}{(1-\varepsilon)T}\varphi(h_C)\right)$$

$$= Te^{-T}\left(O(\mu_{\mathfrak{C},(1-\varepsilon)T}(f \otimes \varphi))\right.$$

$$\left. + \frac{1}{(1-\varepsilon)T}\left(\mu_{\mathfrak{C},T}(f \otimes \varphi) - \mu_{\mathfrak{C},(1-\varepsilon)T}(f \otimes \varphi)\right)\right)$$

$$= O(Te^{-\varepsilon T}) + \frac{1}{1-\varepsilon}e^{-T}\mu_{\mathfrak{C},T}(f \otimes \varphi).$$

Using Theorem 6.12 and $\varepsilon > 0$ being arbitrary completes the proof. ∎

Corollary 6.14 (Equidistribution of holonomies)**.** *For any $\varphi \in \mathrm{Cl}(M)$ and cone $\mathfrak{C} \subset \mathcal{L}$ with $\mathsf{v} \in \mathrm{int}\,\mathfrak{C}$, we have*

$$\sum_{C \in \mathcal{C}_\psi(T,\mathfrak{C})} \varphi(h_C) \sim \frac{e^T}{T}\int_{M_\Gamma} \varphi\,dm_\Gamma \quad \text{as } T \to \infty.$$

Proof. Recall from Subsection 4.2 that $\Omega \cong \mathcal{X}_\psi \times \ker\psi$, $d\mathsf{m}_\psi\big|_\Omega = dm_{\mathcal{X}_\psi}\,du$ and the Lebesgue measures we use satisfy $dw = dt\,du$, where $w = t\mathsf{v} + u$. Choose $f \in \mathrm{C}_c(\Omega)$ such that $f = \mathbb{1}_{\mathcal{X}_\psi} \otimes b$, where $b \in \mathrm{C}_c(\ker\psi)$ with $\int_{\ker\psi} b(u)\,du = 1$. Then $\mathsf{m}_\psi(f) = |m_{\mathcal{X}_\psi}|$ and for every $C \in \mathcal{C}_\Gamma$,

$$V_C(f) = \int_{\mathfrak{a} \mod \lambda(\gamma_C)} f(\Gamma g a_w M)\,dw$$

$$= \int_{\ker\psi}\int_0^{\ell_\psi(C)} b(u)\,dt\,du = \ell_\psi(C),$$

since $\lambda(\gamma_C) = \ell_\psi(C)\mathsf{v} + u$ for some $u \in \ker\psi$.

Applying Corollary 6.13 to this choice of f, we obtain

$$\lim_{T\to\infty} \frac{1}{e^T/T}\sum_{C \in \mathcal{C}_\psi(T,\mathfrak{C})} \varphi(h_C) = \int_{M_\Gamma} \varphi\,dm_\Gamma.$$

∎

Recall that Lemma 4.18 gives a bijection between the maximal flat cylinders and periodic orbits of the translation flow Φ on $\mathcal{X}_\psi$. Let

$$\mathcal{G}_{\mathcal{X}_\psi}(T) := \{\Phi\text{-periodic orbits of length at most } T\}.$$

For $C \in \mathcal{G}_{\mathcal{X}_\psi}(T)$, denote by $\mathcal{L}_C$ the length measure on C and associate to C the ψ-circumference $\ell_\psi(C)$ and holonomy h_C of the corresponding maximal flat cylinder. Holonomies also jointly equidistribute with the periodic orbits in $\mathcal{X}_\psi$.

Corollary 6.15 (Joint equidistribution in $\mathcal{X}_\psi$). *For any $f \in C_c(\mathcal{X}_\psi)$ and $\varphi \in \mathrm{Cl}(M)$, we have*

$$\lim_{T \to \infty} \frac{1}{e^T} \sum_{C \in \mathcal{G}_{\mathcal{X}_\psi}(T)} \mathcal{L}_C(f)\varphi(h_C) = \frac{m_{\mathcal{X}_\psi}(f)}{|m_{\mathcal{X}_\psi}|} \int_{M_\Gamma} \varphi \, dm;$$

$$\lim_{T \to \infty} \frac{1}{e^T/T} \sum_{C \in \mathcal{G}_{\mathcal{X}_\psi}(T)} \frac{\mathcal{L}_C(f)}{\ell_\psi(C)}\varphi(h_C) = \frac{m_{\mathcal{X}_\psi}(f)}{|m_{\mathcal{X}_\psi}|} \int_{M_\Gamma} \varphi \, dm;$$

Proof. Apply Theorem 6.12 with $\mathfrak{C} = \mathcal{L}$ to the function $f \otimes b$, where $b \in C_c(\ker \psi)$ with $\int_{\ker \psi} b(u) \, du = 1$. ∎

7. Joint equidistribution with respect to norm-like orderings

Let $\mathsf{N} : \mathfrak{a} \to \mathbb{R}$ be twice continuously differentiable except possibly at the origin, convex, homogeneous of degree 1 and positive on $\mathcal{L} \backslash \{0\}$. For example, any L^p-norm on $\mathfrak{a}$ has these properties when $1 \leq p < \infty$. In this section, we give a slight modification of the arguments in the previous sections to prove joint equidistribution with respect to the ordering on $\mathcal{C}_\Gamma$ determined by N:

$$\mathcal{C}_\mathsf{N}(T) := \{C \in \mathcal{C}_\Gamma : \mathsf{N}(\lambda(\gamma_C)) \leq T\}.$$

We define the N-critical exponent as

$$\delta_\mathsf{N} := \max_{\mathsf{N}(w)=1} \psi_\Gamma(w). \tag{7.1}$$

By convexity of N and strict concavity of ψ_Γ, there exists a unique

$$\mathsf{v} \in \mathrm{int}\, \mathcal{L}$$

such that the maximum in (7.1) is achieved at $w = \delta_\mathsf{N}\mathsf{v}$. We note that when N is the Euclidean norm on $\mathfrak{a}$, v is simply the maximal growth direction of ψ_Γ. We also let ψ denote the tangent form such that

$$\psi \text{ is tangent to } \psi_\Gamma \text{ at } \mathsf{v}.$$

We note that $\mathsf{N}(\delta_\mathsf{N}\mathsf{v}) = 1 = \frac{1}{\delta_\mathsf{N}}\psi_\Gamma(\delta_\mathsf{N}\mathsf{v})$ so by convexity of N, we have $\frac{1}{\delta_\mathsf{N}}\psi \leq \mathsf{N}$ and equality holds along the v direction. In particular, the differential of N along the v direction is $\frac{1}{\delta_\mathsf{N}}\psi$. Fix a cone

$$\mathfrak{C} \subset \mathcal{L} \text{ with } \mathsf{v} \in \mathrm{int}\, \mathfrak{C}. \tag{7.2}$$

Define
$$\mathcal{C}_{\mathsf{N}}(T, \mathfrak{C}) := \{C \in \mathcal{C}_{\mathsf{N}}(T) : \lambda(\gamma_C) \in \mathfrak{C}\}.$$
Then we define the measures $\mu_{\mathfrak{C},T}^{\mathsf{N}}$ and $\eta_{\mathfrak{C},T}^{\mathsf{N}}$ on $\Gamma\backslash G/M \times [M]$ as follows. For $T > 0$, $f \in C_c(\Gamma\backslash G/M)$, $\varphi \in \mathrm{Cl}(M)$, and $\mathfrak{C} \subset \mathfrak{a}^+$, let

$$\mu_{\mathfrak{C},T}^{\mathsf{N}}(f \otimes \varphi) := \sum_{C \in \mathcal{C}_{\mathsf{N}}(T,\mathfrak{C})} V_C(f)\varphi(h_C);$$

$$\eta_{\mathfrak{C},T}^{\mathsf{N}}(f \otimes \varphi) := \sum_{C \in \mathcal{C}_{\mathsf{N}}(T,\mathfrak{C})} \frac{V_C(f)}{\ell_\psi(C)}\varphi(h_C).$$

Note that for $\eta_{\mathfrak{C},T}^{\mathsf{N}}$, we still normalize by using the ψ-circumferences, or in other words, the engths of the periodic orbits in $\mathcal{X}_\psi$ because this is useful to deduce equidistribution of holonomies from joint equidistribution. We now state the joint equidistribution theorem with respect to N.

Theorem 7.3 (Joint equidistribution with respect to N)**.** *There exists a constant $c_{\mathsf{N}} > 0$ such that for any $f \in C_c(\Gamma\backslash G/M)$, $\varphi \in \mathrm{Cl}(M)$, and any cone $\mathfrak{C} \subset \mathcal{L}$ with $\mathsf{v} \in \mathrm{int}\,\mathfrak{C}$, we have*

$$\lim_{T\to\infty} \frac{\mu_{\mathfrak{C},T}^{\mathsf{N}}(f \otimes \varphi)}{e^{\delta_{\mathsf{N}}T}} = c_{\mathsf{N}} \frac{\mathsf{m}_\psi(f)}{|m_{\mathcal{X}_\psi}|} \int_{M_\Gamma} \varphi\, dm_\Gamma; \qquad (7.4)$$

$$\lim_{T\to\infty} \frac{\eta_{\mathfrak{C},T}^{\mathsf{N}}T(f \otimes \varphi)}{e^{\delta_{\mathsf{N}}T}/\delta_{\mathsf{N}}T} = c_{\mathsf{N}} \frac{\mathsf{m}_\psi(f)}{|m_{\mathcal{X}_\psi}|} \int_{M_\Gamma} \varphi\, dm_\Gamma. \qquad (7.5)$$

The constant c_{N} is given by the formula

$$c_{\mathsf{N}} := \frac{\kappa_{\mathsf{v}}}{[M : M_\Gamma]} \int_{\ker\psi} e^{-I(u)-\frac{\delta_{\mathsf{N}}^2}{2}u^\top(\mathrm{Hess}(\mathsf{N})(\delta_{\mathsf{N}}\mathsf{v}))u}\, du \leq 1,$$

where $\mathrm{Hess}(\mathsf{N})$ denotes the Hessian of N, I and κ_{v} are as in Theorem 4.11 and du is as in (4.9).

Remark 7.6. If the norm-like function N is in fact a norm induced by an inner product, then $u^\top(\mathrm{Hess}(\mathsf{N})(\delta_{\mathsf{N}}\mathsf{v}))u = \mathsf{N}(u)^2$.

Corollary 7.7 (Equidistribution of holonomies with respect to N)**.** *For any $\varphi \in \mathrm{Cl}(M)$ and cone $\mathfrak{C} \subset \mathcal{L}$ with $\mathsf{v} \in \mathrm{int}\,\mathfrak{C}$, we have*

$$\sum_{C \in \mathcal{C}_{\mathsf{N}}(T,\mathfrak{C})} \varphi(h_C) \sim c_{\mathsf{N}} \frac{e^{\delta_{\mathsf{N}}T}}{\delta_{\mathsf{N}}T} \int_{M_\Gamma} \varphi\, dm_\Gamma \quad as\ T \to \infty.$$

Corollary 7.8. *For any cone $\mathfrak{C} \subset \mathcal{L}$ with $\mathsf{v} \in \mathrm{int}\,\mathfrak{C}$, we have as $T \to \infty$,*

$$\#\{[\gamma] \in [\Gamma_{\mathrm{prim}}] : \mathsf{N}(\lambda(\gamma)) \leq T,\ \lambda(\gamma) \in \mathfrak{C}\} \sim c_{\mathsf{N}} \frac{e^{\delta_{\mathsf{N}}T}}{\delta_{\mathsf{N}}T}.$$

The arguments in Sections 5 and 6 are easily adapted to the current setting with the only points of interest being the replacement of Lemma 5.17 and deducing the asymptotic for $\eta_{\mathfrak{C},T}^{\mathsf{N}}$ from the asymptotic for $\mu_{\mathfrak{C},T}^{\mathsf{N}}$.

Let
$$\mathfrak{C}_T^{\mathsf{N}} := \{\exp(w) : w \in \mathfrak{C}, \mathsf{N}(w) \leq T\}.$$

The following lemma is the appropriate replacement of the first statement in Lemma 5.17.

Lemma 7.9. *We have*
$$\lim_{T \to \infty} e^{-\delta_\mathsf{N} T} \int_{a_{t\mathsf{v}+\sqrt{t}u} \in \mathfrak{C}_T^{\mathsf{N}}} e^t e^{-I(u)} \, dt \, du = \int_{\ker \psi} e^{-I(u) - \frac{\delta_\mathsf{N}^2}{2} u^\top (\mathrm{Hess}(\mathsf{N})(\delta_\mathsf{N}\mathsf{v}))u} \, du.$$

Proof. For $u \in \ker \psi$ and $T > 0$, let
$$R_T(u) = \{t > 0 : a_{t\mathsf{v}+\sqrt{t}u} \in \mathfrak{C}_T^{\mathsf{N}}\},$$
so that we have
$$e^{-\delta_\mathsf{N} T} \int_{a_{t\mathsf{v}+\sqrt{t}u} \in \mathfrak{C}_T^{\mathsf{N}}} e^t e^{-I(u)} \, dt \, du = \int_{\ker \psi} e^{-I(u)} e^{-\delta_\mathsf{N} T} \int_{R_T(u)} e^t \, dt \, du.$$

Using the second order Tylor approximation of N at $\delta_\mathsf{N}\mathsf{v}$ we have

$$\mathsf{N}(t\mathsf{v} + \sqrt{t}u)$$
$$= \frac{t}{\delta_\mathsf{N}} \mathsf{N}\left(\delta_\mathsf{N}\mathsf{v} + \frac{\delta_\mathsf{N}}{\sqrt{t}}u\right)$$
$$= \frac{t}{\delta_\mathsf{N}}\left(1 + \frac{1}{\delta_\mathsf{N}}\psi\left(\frac{\delta_\mathsf{N}}{\sqrt{t}}u\right) + \frac{\delta_\mathsf{N}^2}{2t}u^\top (\mathrm{Hess}(\mathsf{N})(\delta_\mathsf{N}\mathsf{v}))u + O\left(\frac{\|u\|^3}{t^{3/2}}\right)\right)$$
$$= \frac{t}{\delta_\mathsf{N}}\left(1 + \frac{\delta_\mathsf{N}^2}{2t}u^\top (\mathrm{Hess}(\mathsf{N})(\delta_\mathsf{N}\mathsf{v}))u + O\left(\frac{\|u\|^3}{t^{3/2}}\right)\right)$$

and hence for fixed u,
$$R_T(u) = \left[O(1), \delta_\mathsf{N} T - \frac{\delta_\mathsf{N}^2}{2}u^\top (\mathrm{Hess}(\mathsf{N})(\delta_\mathsf{N}\mathsf{v}))u + O\left(\frac{\|u\|^3}{\sqrt{T}}\right)\right].$$

Observe that $e^{-I(u)} e^{-\delta_\mathsf{N} T} \int_{R_T(u)} e^t \, dt \leq e^{-I(u)}$ and by the formula for I from Theorem 4.11, $e^{-I(u)} \in L^1(\ker \psi)$. Then by the Lebesgue dominated convergence theorem,

$$\lim_{T \to \infty} e^{-\delta_\mathsf{N} T} \int_{a_{t\mathsf{v}+\sqrt{t}u} \in \mathfrak{C}_T} e^t e^{-I(u)} \, dt \, du$$
$$= \int_{\ker \psi} e^{-I(u)} \lim_{T \to \infty} e^{-\delta_\mathsf{N} T} \int_{R_T(u)} e^t \, dt \, du$$
$$= \int_{\ker \psi} e^{-I(u)} \lim_{T \to \infty} e^{-\delta_\mathsf{N} T} \left(e^{\delta_\mathsf{N} T - \frac{\delta_\mathsf{N}^2}{2}u^\top (\mathrm{Hess}(\mathsf{N})(\delta_\mathsf{N}\mathsf{v}))u + O\left(\frac{\|u\|^3}{\sqrt{T}}\right)}\right) \, du$$
$$= \int_{\ker \psi} e^{-I(u) - \frac{\delta_\mathsf{N}^2}{2}u^\top (\mathrm{Hess}(\mathsf{N})(\delta_\mathsf{N}\mathsf{v}))u} \, du.$$

∎

Proof of Theorem 7.3. After replacing Lemma 5.17 with Lemma 7.9, the remainder of the proof for the asymptotic (7.4) of $\mu^{\mathsf{N}}_{\mathfrak{C},T}$ is almost identical to the proof of Theorem 6.12.

We present the proof of the asymptotic (7.5) for $\eta^{\mathsf{N}}_{\mathfrak{C},T}$. Some care must be taken since in the definition of $\eta^{\mathsf{N}}_{\mathfrak{C},T}$, the volume measures on maximal flat cylinders C are normalized using their ψ-circumference $\ell_\psi(C)$, rather than $\mathsf{N}(\lambda(\gamma_C))$.

Note that for $C \in \mathcal{C}_{\mathsf{N}}(T, \mathfrak{C})$, we have $\ell_\psi(C) = \psi(\lambda(\gamma_C) \leq \mathsf{N}(\lambda(\gamma_C)) \leq T$ and hence we have

$$T^{-1}\mu^{\mathsf{N}}_{\mathfrak{C},T} \leq \eta^{\mathsf{N}}_{\mathfrak{C},T}. \tag{7.10}$$

Next, we show an upper bound for $\eta^{\mathsf{N}}_{\mathfrak{C},T}$. Fix $\vartheta > 0$ such that $\frac{\psi(w)}{\mathsf{N}(w)} > \vartheta$ for all $w \in \mathcal{L} \setminus \{0\}$. Fix $\varepsilon > 0$ and choose a cone $\mathfrak{C}' \subset \mathfrak{C}$ such that $\mathsf{v} \in \operatorname{int}\mathfrak{C}'$ and $\frac{\psi(w)}{\delta_{\mathsf{N}}\mathsf{N}(w)} > 1 - \varepsilon$ for all $w \in \mathfrak{C}'$. Then we have

$$\eta^{\mathsf{N}}_{\mathfrak{C},T}(f \otimes \varphi)$$

$$\leq O\left(\sum_{\mathcal{C}_{\mathsf{N}}((1-\varepsilon)T,\mathfrak{C})} V_C(f)\varphi(h_C)\right)$$

$$+ \sum_{\mathcal{C}_{\mathsf{N}}(T,\mathfrak{C})\setminus(\mathcal{C}_{\mathsf{N}}(T,\mathfrak{C}')\cup\mathcal{C}_{\mathsf{N}}((1-\varepsilon)T,\mathfrak{C}))} \frac{V_C(f)}{(1-\varepsilon)\vartheta T}\varphi(h_C)$$

$$+ \sum_{\mathcal{C}(T,\mathfrak{C}')-\mathcal{C}((1-\varepsilon)T)} \frac{V_C(f)}{(1-\varepsilon)^2\delta_{\mathsf{N}}T}\varphi(h_C)$$

$$= O\left(\mu^{\mathsf{N}}_{\mathfrak{C},(1-\varepsilon)T}\right) + \frac{1}{(1-\varepsilon)\vartheta T}\left(\left(\mu^{\mathsf{N}}_{\mathfrak{C},T} - \mu^{\mathsf{N}}_{\mathfrak{C}',T}\right) - \left(\mu^{\mathsf{N}}_{\mathfrak{C},(1-\varepsilon)T} - \mu^{\mathsf{N}}_{\mathfrak{C}',(1-\varepsilon)T}\right)\right)$$

$$+ \frac{1}{(1-\varepsilon)^2\delta_{\mathsf{N}}T}\left(\mu^{\mathsf{N}}_{\mathfrak{C}',T} - \mu^{\mathsf{N}}_{\mathfrak{C}',(1-\varepsilon)T}\right). \tag{7.11}$$

Using the asymptotic (7.4) for $\mu^{\mathsf{N}}_{\mathfrak{C},T}$ in (7.10) and (7.11), we obtain

$$c_{\mathsf{N}}\frac{\mathsf{m}_\psi(f)}{|m_{\mathcal{X}_\psi}|}\int_{M_\Gamma}\varphi\,dm \leq \lim_{T\to\infty}\frac{\eta^{\mathsf{N}}_{\mathfrak{C},T}(f \otimes \varphi)}{e^{\delta_{\mathsf{N}}T}/\delta_{\mathsf{N}}T} \leq \frac{1}{(1-\varepsilon)^2}c_{\mathsf{N}}\frac{\mathsf{m}_\psi(f)}{|m_{\mathcal{X}_\psi}|}\int_{M_\Gamma}\varphi\,dm.$$

Since $\varepsilon > 0$ is arbitrary, this completes the proof. ∎

Remark 7.12. We remark that in the above proof of the asymptotic for $\eta^{\mathsf{N}}_{\mathfrak{C},T}$ for a given cone $\mathfrak{C}$ as in (7.2), it was crucial that the asymptotic for $\mu^{\mathsf{N}}_{\mathfrak{C}',T}$ is the same for any smaller $\mathfrak{C}' \subset \mathfrak{C}$ satisfying (7.2).

APPENDIX A. AN IDENTITY BETWEEN κ_{v} AND I

Let $\Gamma < G$ be a Zariski dense Anosov subgroup (Definition 4.1). Let $\psi \in \mathfrak{a}^*$ be a tangent form (3.13) tangent to ψ_Γ at some v in the interior of the limitcone $\mathcal{L}$ of Γ (Definition 3.2). We take v normalized so that

$\psi(\mathsf{v}) = 1$. The purpose of this appendix is to prove the following identity between the constant κ_v and function $I : \ker \psi \to \mathbb{R}$ appearing in the local mixing theorem (Theorem 4.11 or Theorem A.2) and the index $[M : M_\Gamma]$ of the holonomy group M_Γ of Γ (Definition 3.5). This identity was required in Sections 5–7 to determine the constants in the asymptotics we proved.

Proposition A.1. *Using the same notation as in Theorem 4.11 or Theorem A.2, we have*

$$\kappa_\mathsf{v} \int_{\ker \psi} e^{-I(u)} \, du = [M : M_\Gamma],$$

where du denotes the Lebesgue measure on $\ker \psi$ as in (4.9).

We will in fact deduce Proposition A.1 from local mixing with respect to the BMS measure m_ψ and an accompanying uniformity statement.

Theorem A.2 (Local mixing, [CS23, Theorem 1.3]). *There exists $\kappa_\mathsf{v} > 0$ and function $I : \ker \psi \to \mathbb{R}$ is defined by $I(\cdot) = \langle \cdot, \cdot \rangle_* - \frac{\langle \cdot, \mathsf{v} \rangle_*^2}{\langle \mathsf{v}, \mathsf{v} \rangle_*}$ for some inner product $\langle \cdot, \cdot \rangle_*$ on $\mathfrak{a}$ such that for any $u \in \ker \psi$ and for any $\phi_1, \phi_2 \in C_\mathsf{c}(\Gamma \backslash G)$, we have*

$$\lim_{t \to +\infty} t^{\frac{\mathsf{r}-1}{2}} \int_{\Gamma \backslash G} \phi_1(x \exp(t\mathsf{v} + \sqrt{t}u)) \phi_2(x) \, d\mathsf{m}_\psi(x)$$

$$= \frac{\kappa_\mathsf{v} e^{-I(u)}}{|m_{\mathcal{X}_\psi}|} \sum_{Z \in \mathfrak{Z}_\Gamma} \mathsf{m}_\psi \big|_Z(\phi_1) \cdot \mathsf{m}_\psi \big|_Z(\phi_2),$$

where $\mathsf{r} = \mathrm{rank}(G)$, $\mathfrak{Z}_\Gamma$ denotes the finite set of A-ergodic components of m_ψ and $m_{\mathcal{X}_\psi}$ is as in Subsection 4.2.

Moreover, for all $\phi_1, \phi_2 \in C_\mathsf{c}(\Gamma \backslash G)$, there exists a constant $D_\mathsf{v}(\phi_1, \phi_2)$ depending on ϕ_1 and ϕ_2 such that for all $(t, u) \in [0, \infty) \times \ker \psi$, we have

$$\left| t^{\frac{\mathsf{r}-1}{2}} \int_{\Gamma \backslash G} \phi_1(x \exp(t\mathsf{v} + \sqrt{t}u)) \phi_2(x) \, d\mathsf{m}_\psi(x) \right|$$

$$\leq D_\mathsf{v}(\phi_1, \phi_2) e^{-2(\|t\mathsf{v} + \sqrt{t}u\|_* \|\mathsf{v}\|_* - \langle t\mathsf{v} + \sqrt{t}u, \mathsf{v} \rangle_*)},$$

where $\| \cdot \|_*$ denotes the norm induced by $\langle \cdot, \cdot \rangle_*$.

Proof of Proposition A.1. We refer the reader to Subsection 4.2 for the notation used in this proof. Let $S : \mathcal{X}_\psi \to \Omega$ be a continuous section of the trivial $\ker \psi$-vector bundle $\pi_\psi : \Omega \to \mathcal{X}_\psi$. Then we have a continuous function $s : \Lambda^{(2)} \times \mathbb{R} \to \mathfrak{a}$ such that for all $(x, y, r) \in \Lambda^{(2)} \times \mathbb{R}$,

$$S(\Gamma(x, y, r)) = \Gamma(x, y, s(x, y, r)) \text{ and } \psi(s(x, y, r)) = r.$$

Then we have a homeomorphism $\mathcal{X}_\psi \times \ker \psi \cong \Omega$ given by

$$(\Gamma(x, y, r), u') \mapsto \Gamma(x, y, s(x, y, r) + u').$$

Observe that for $T > 0$ and $u \in \ker \psi$, the $\mathfrak{a}$-coordinate of

$$\Gamma(x, y, s(x, y, r) + u') \exp(t\mathsf{v} + \sqrt{t}u) \in \Gamma \backslash (\Lambda^{(2)} \times \mathfrak{a})$$

is given by

$$s(x, y, r) + u' + t\mathsf{v} + \sqrt{t}u = s(x, y, r + t) + (u' + \sqrt{t}u) - \hat{s}(x, y, r, t), \quad \text{(A.3)}$$

where $\hat{s}(x, y, r, t) := s(x, y, r + t) - s(x, y, r) - t\mathsf{v} \in \ker\psi$.

Fix a compactly supported continuous function $b : \ker\psi \to [0, \infty)$ such that $\int_{\ker\psi} b(u')\, du' = 1$, where du' is the Lebesgue measure on $\ker\psi$ satisfying $dm_{\mathcal{X}_\psi}\, du' = dm_\psi\big|_\Omega$. Let $\phi \in C_c(\tilde\Omega)$ be the function given by the M-invariant lift of the function $\mathbb{1}_{X_\psi} \otimes b$ on $\mathcal{X}_\psi \times \ker\psi \cong \Omega$. Note that

$$\mathsf{m}_\psi\big|_Z(\phi) = \frac{1}{\#\mathfrak{Z}_\Gamma}\mathsf{m}_\psi(\phi) = \frac{|m_{\mathcal{X}_\psi}|}{[M : M_\Gamma]}, \quad \text{(A.4)}$$

where the last equality uses $\#\mathfrak{Z}_\Gamma = [M : M_\Gamma]$ [LO20a, Theorem 1.1]. Using Theorem A.2 applied to the functions $\phi_1 = \phi_2 = \phi$ and using (A.4), we have

$$\lim_{t\to+\infty} t^{\frac{\mathsf{r}-1}{2}} \int_{\mathcal{X}_\psi \times \ker\psi} b((u' + \sqrt{t}u) - \hat{s}(x, y, r, t))b(u')\, dm_{\mathcal{X}_\psi}(\Gamma(x, y, r))\, du'$$

$$= \frac{\kappa_\mathsf{v} e^{-I(u)}}{|m_{\mathcal{X}_\psi}|} \sum_{Z \in \mathfrak{Z}_\Gamma} \big(\mathsf{m}_\psi\big|_Z(\phi)\big)^2 = \frac{\kappa_\mathsf{v} e^{-I(u)}|m_{\mathcal{X}_\psi}|}{[M : M_\Gamma]}. \quad \text{(A.5)}$$

Rearranging and integrating (A.5) with respect to $u \in \ker\psi$, we obtain

$$\kappa_\mathsf{v} \int_{\ker\psi} e^{-I(u)}\, du$$

$$= \frac{[M : M_\Gamma]}{|m_{\mathcal{X}_\psi}|} \int_{\ker\psi} \lim_{t\to+\infty} t^{\frac{\mathsf{r}-1}{2}} \int_{\ker\psi} \int_{\mathcal{X}_\psi} b((u' + \sqrt{t}u) - \hat{s}(x, y, r, t))$$

$$b(u')\, dm_{\mathcal{X}_\psi}(\Gamma(x, y, r))\, du'\, du$$
$$\text{(A.6)}$$

Next, we explain how to move the limit in (A.6) outside of the integral. By Theorem A.2, there exists a constant $D := D_\mathsf{v}(\phi, \phi) > 0$ such that for all $(t, u) \in [1, \infty) \times \ker\psi$, we have

$$t^{\frac{\mathsf{r}-1}{2}} \int_{\ker\psi} \int_{\mathcal{X}_\psi} b((u' + \sqrt{t}u) - \hat{s}(x, y, r, t))b(u')\, dm_{\mathcal{X}_\psi}(\Gamma(x, y, r))\, du'\, du$$

$$\leq De^{-2(\|t\mathsf{v}+\sqrt{t}u\|_*\|\mathsf{v}\|_* - \langle t\mathsf{v}+\sqrt{t}u, \mathsf{v}\rangle_*)}. \quad \text{(A.7)}$$

Note that for $t \geq 1$,

$$\|t\mathsf{v} + \sqrt{t}u\|_*\|\mathsf{v}\|_* - \langle t\mathsf{v} + \sqrt{t}u, \mathsf{v}\rangle_* = \frac{\|u\|_*^2\|\mathsf{v}\|_*^2 - \langle u, \mathsf{v}\rangle_*^2}{\|\mathsf{v} + u/\sqrt{t}\|_*\|\mathsf{v}\|_* + \langle \mathsf{v} + u/\sqrt{t}, \mathsf{v}\rangle_*}$$

$$\geq \frac{\|u\|_*^2\|\mathsf{v}\|_*^2 - \langle u, \mathsf{v}\rangle_*^2}{2\|\mathsf{v}\|_*^2 + (\|u\|_*\|\mathsf{v}\|_* + \langle u, \mathsf{v}\rangle_*)}. \quad \text{(A.8)}$$

Define the function $f : \ker\psi \to [0, \infty)$ by

$$f(u) = \frac{\|u\|_*^2\|\mathsf{v}\|_*^2 - \langle u, \mathsf{v}\rangle_*^2}{2\|\mathsf{v}\|_*^2 + \|u\|_*\|\mathsf{v}\|_* + \langle u, \mathsf{v}\rangle_*}.$$

Then combining (A.7) and (A.8), for all $(t, u) \in [1, \infty) \times \ker \psi$, we have

$$t^{\frac{r-1}{2}} \int_{\ker \psi} \int_{\mathcal{X}_\psi} b((u' + \sqrt{t}u) - \hat{s}(x, y, r, t)) b(u') \, dm_{\mathcal{X}_\psi}(\Gamma(x, y, r)) \, du' \, du$$

$$\leq D e^{-2f(u)}. \quad \text{(A.9)}$$

Moreover, we observe that $e^{-2f(u)} \in L^1(\ker \psi)$ since for all $u \in \ker \psi \setminus \{0\}$, we have

$$\lim_{\lambda \to \infty} \frac{f(\lambda u)}{\lambda} = \frac{\|u\|_*^2 \|\mathsf{v}\|_*^2 - \langle u, \mathsf{v}\rangle_*^2}{\|u\|_* \|\mathsf{v}\|_* + \langle u, \mathsf{v}\rangle_*} > 0.$$

Then by (A.9), we can apply the Lebesgue dominated convergence theorem in (A.7), followed by the change of variables $\tilde{u} = \sqrt{t}u$, Fubini's theorem and integrating using $\int_{\ker \psi} b(u') \, du' = 1$ twice to conclude that

$$\kappa_\mathsf{v} \int_{\ker \psi} e^{-I(u)} \, du$$

$$= \frac{[M : M_\Gamma]}{|m_{\mathcal{X}_\psi}|} \lim_{t \to +\infty} t^{\frac{r-1}{2}} \int_{\ker \psi} \int_{\ker \psi} \int_{\mathcal{X}_\psi} b((u' + \sqrt{t}u) - \hat{s}(x, y, r, t))$$

$$b(u') \, dm_{\mathcal{X}_\psi}(\Gamma(x, y, r)) \, du' \, du,$$

$$= \frac{[M : M_\Gamma]}{|m_{\mathcal{X}_\psi}|} \lim_{t \to +\infty} \int_{\ker \psi} \int_{\mathcal{X}_\psi} \int_{\ker \psi} b((u' + \tilde{u}) - \hat{s}(x, y, r, t))$$

$$b(u') \, d\tilde{u} \, dm_{\mathcal{X}_\psi}(\Gamma(x, y, r)) \, du'$$

$$= \frac{[M : M_\Gamma]}{|m_{\mathcal{X}_\psi}|} \lim_{t \to +\infty} \int_{\ker \psi} \int_{\mathcal{X}_\psi} b(u') \, dm_{\mathcal{X}_\psi}(\Gamma(x, y, r)) \, du'$$

$$= [M : M_\Gamma].$$

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