Chunyang Ding

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Precalc 06

What is a derivative?

Suppose your teacher were to make a deal with your class. Each time anyone answers a question correctly during class discussion, they earn a “run –of- the-mill” dollar. Every quarter, students would be able to trade these “dollars” for homework passes that would allow you to turn in late homework for full credit! However, your teacher is your math teacher, and he doesn’t want to give you an easy price. He decides to price each ticket at 40 bucks, but decrease the price of each ticket as you purchase more. At the same time, your teacher doesn’t want to allow for too many passes, so that the price would go up after a certain point. He ends up coming up with the following equation to determine the price per ticket given that you bought x tickets: $f(x) = \frac{1}{3}x^3 - 2x^2 - 5x + 40$. Being a smart but lazy student, you decide to find out the number of tickets you should buy so that you get the cheapest price per ticket. This way, you could slack off a bit, and still get a good grade. How would you find this?

It is in dilemmas such as these that you would want to find the relative highest or lowest points in a function. Even though we have already memorized how to find the vertex of a quadratic, how exactly would you find the relative low point on this graph, a cubic? Technically you could just plug in points or just have your calculator draw it, but is there an easier, and more general, way of finding these points?

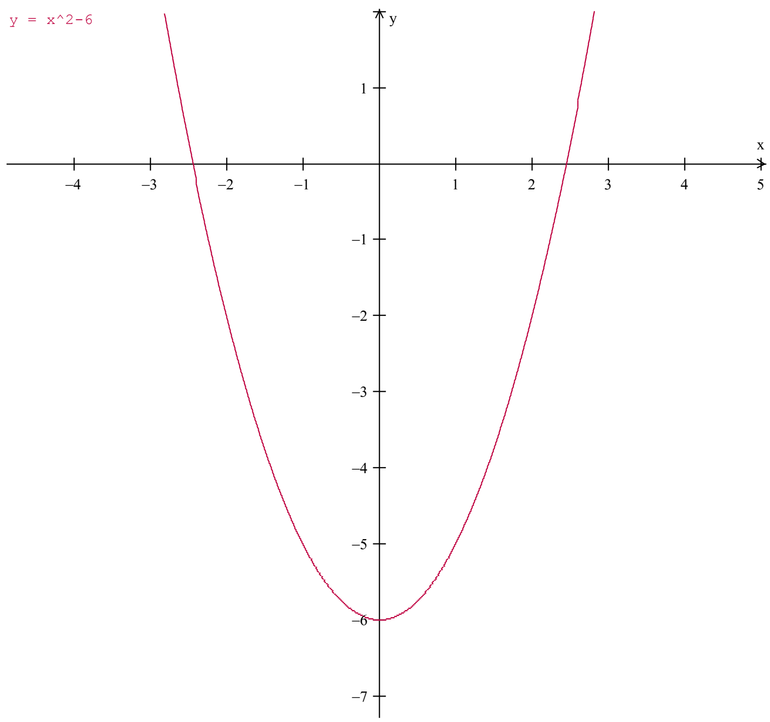
At other times, we are also presented with the dilemma of having to know speed. Even though the average speed formula generally works, as we go further into branches of science such as physics, we find that we not only need to know the speed of an object over a short distance, but we need to know the objects speed at a specific point. Not only that, but the object may be moving at a weird type of polynomial motion. Usually we would just graph the function and the find the tangent line, but is there an easier way?

In the 17^{th} century, people were still having the same problem. Back then, the groundwork for modern physics were being laid out, but giants in the field, such as Newton, found that they were not able to go on with their work. They needed to have more mathematics to solve these problems! However, in 1675, Newton and Leibniz both independently discovered calculus, which involved the math of derivatives and integrals. Now, calculus, integrals, and derivatives sound like big, scary math terms, but in truth, they are just basic mathematical concepts that can help make your life easier while solving physics problems.

The term that we will be focusing on is the derivative. The American Heritage dictionary defines derivative as “resulting from or using derivation.” Not so useful, so let’s take a look at the definition of the word derivation. The definition is: “The act or process of deriving”. Still not enough, so let’s take a final look at the word derive. That definition is: “To obtain or retrieve from a source.” Aha! That’s what we want! A derivative is something that allows us to find the source of a function!

What a derivative equation is in math is to give you an equation for all the instantaneous slopes for each and every point of another graph. That sounds quite confusing, so let’s break it down. We already know the definition of the word “slope”. It’s just how fast that point is increasing or decreasing over a range of times. Now, if we use a simple formula, such as $y = 3x + 2$, we can obviously tell that the slope for each and every point is going to be 3. That is very nice, but what about polynomials of higher degrees? How would we find the slopes for them?

Let’s first put those questions on hold and see if we can answer the first question raised in our discussion: finding the relative high and low points of a function. Those points are called the local minimums and maximums, as all values of f(x) are either higher or lower than they are. In other words, they represent a minimum or maximum compared to the points around them. In a simple quadratic function, we know that there is always at least one maximum or minimum, but do we know anything special about them? Let's take a look at the graph of a simple quadratic formula: $y = x^2 - 6$.

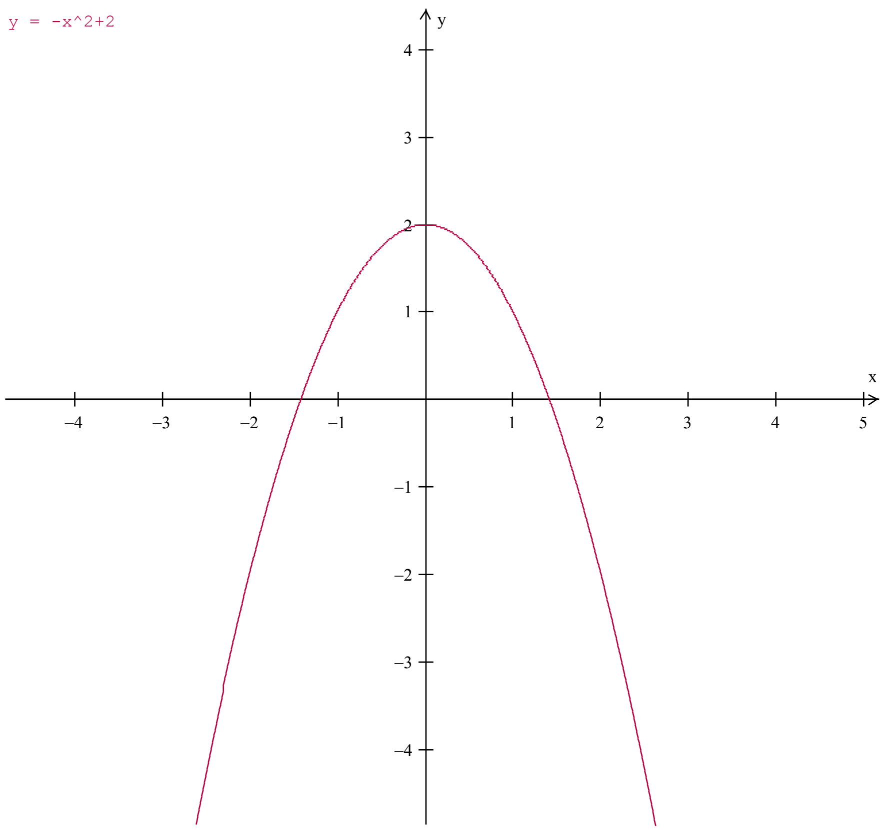


$fig\ 1$

x	$f(x)$
-2	-2
-1	-5
0	-6
1	-5
2	-2

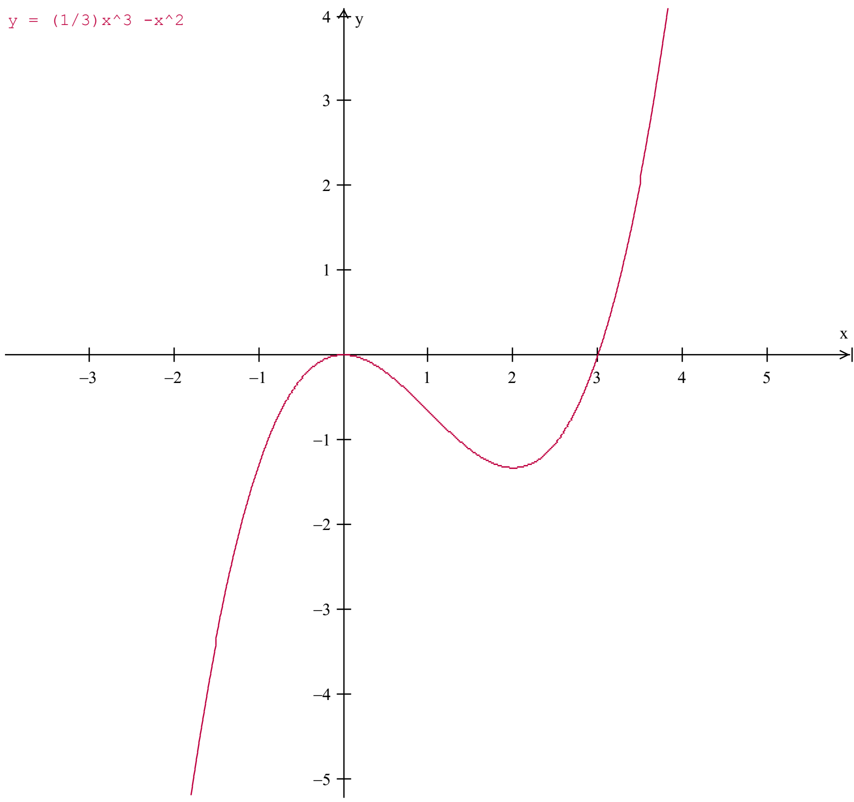
$table\ 1$

We can obviously tell from this graph that the slope is slowly increasing after (0, -6) and was slowly decreasing as it approached (0, -6). On our table, even without finding the actual slope, we know that as x increased from -1 to 0, $f(x)$ went up. As x increased from 0 to -1, $f(x)$ increased as well. However, can we tell what is going on right at (0, -6)? We know that at that point, the function is neither increasing nor decreasing, so therefore that point must have a slope of zero. Because we know that that point is also the local minimum, we may be able to extrapolate that the slope of all local minimums/ maximums are zero. Let's test out some more cases to confirm this hypothesis. First, we'll start with $y = -x^2 = 2$.



x	$f(x)$
-2	-2
-1	1
0	2
1	1
2	-2

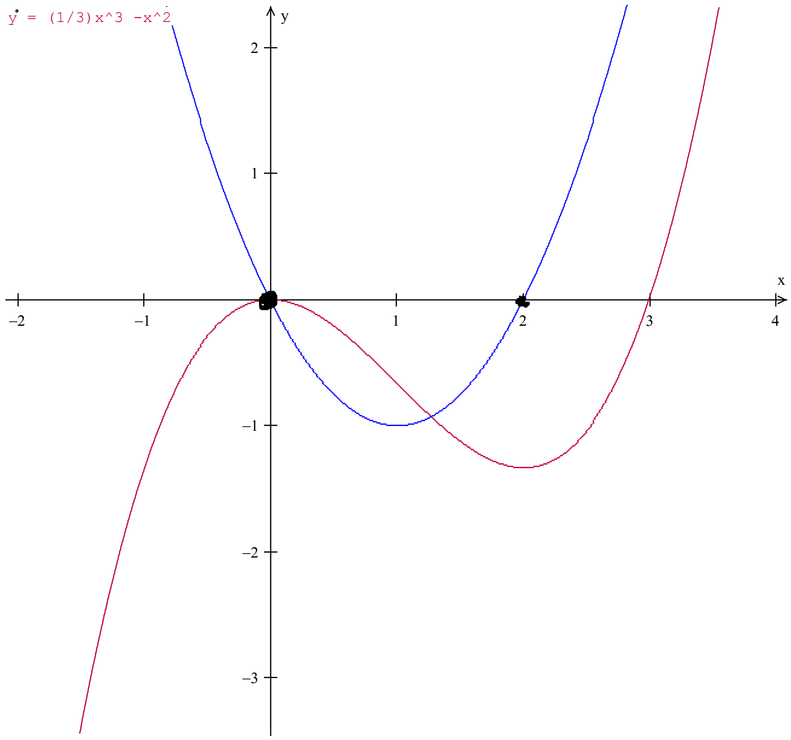
In this graph, we see that the slope of all points are positive before (0, 2) and are all negative after (0, 2). Therefore, we can conclude that the slope of (0, 2) is, again zero. We will also find that (0, 2) is higher than the points that surround it, making it the local maximum. As we have seen that both the local maximum and local minimum of a quadratic have slopes of zero, we may be able to conjecture that all local maximum and local minimum have a slope of zero. Let's go on to find out more about these points. What if a function had both a local minimum and a maximum, such as $y = \frac{1}{3}x^3 - x^2$?



x	$f(x)$
-1	$-\frac{4}{3}$
0	**0**
1	$-\frac{2}{3}$
2	$\boldsymbol{-\frac{4}{3}}$
3	0
4	$\frac{16}{3}$

Again, we see that the two local minimums/maximums are at areas where the slope changes. From the interval of (-∞, 0), or when the domain is between -∞ and 0, the slope is positive; from the interval of (0, 2) the slope is negative, and from the interval of (2, ∞), the slope is positive again. As we see that the slope is always zero when the slope changes signs, and we remember that the definition of a local minimum/maximum is a point where the slope changes signs, we can conclude that all local minimums/maximums have a slope of zero.

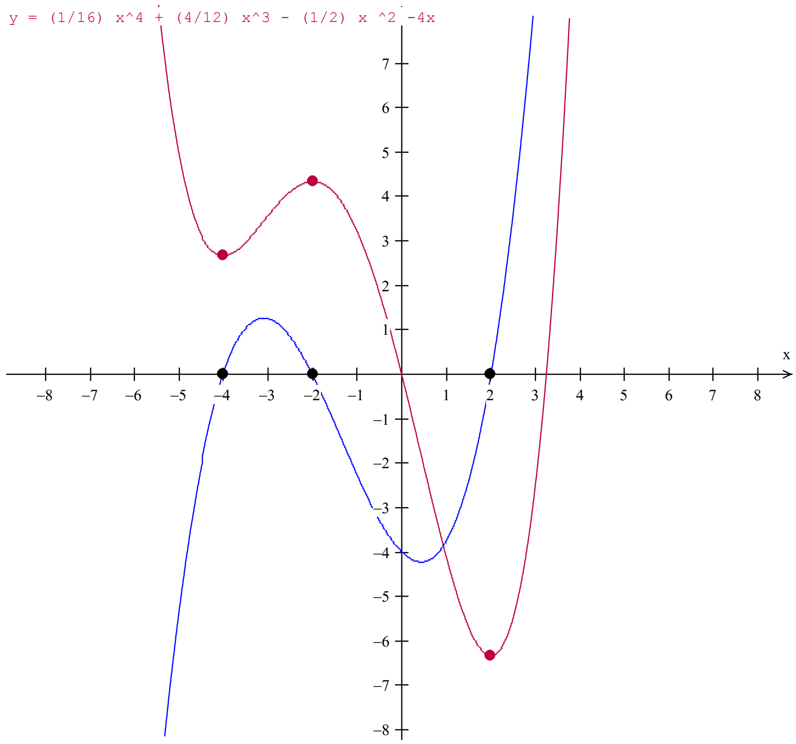
Wow, that is great, but what does that have to do with derivatives? As you should recall, derivatives always give you the slope for any point. Therefore, if we can find the point where the derivative of a point is zero, then we can conclude that that point is also the local minimum or maximum of the original function! For proof, we turn back to our cubic function.



x	$f'(x)$
-1	3
0	0
1	-1
2	0
3	3

Using technology, we found the derivative equation to be the blue line. If we examine that line in comparison with the original function, it is very obvious that it is indeed the derivative. After all, on our table, we see that when $x = 0$, $f'(x) = 0$. Going back to our definition of the derivative, we see that what this really means is that the slope of $f(0)$ is 0, or that that point is a local maximum/minimum. The same thing happens at $x = 2$. It is very obvious that the zeroes of the derivative graph are exactly where the local minimums/maximums are!

Let's see what happens when we try to do the same thing for other graphs, such as a quartic graph. Remember, a quartic graph should have 3 local maximums and minimums. Let's try it out on the graph $y = \frac{1}{16}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 - 4x$.



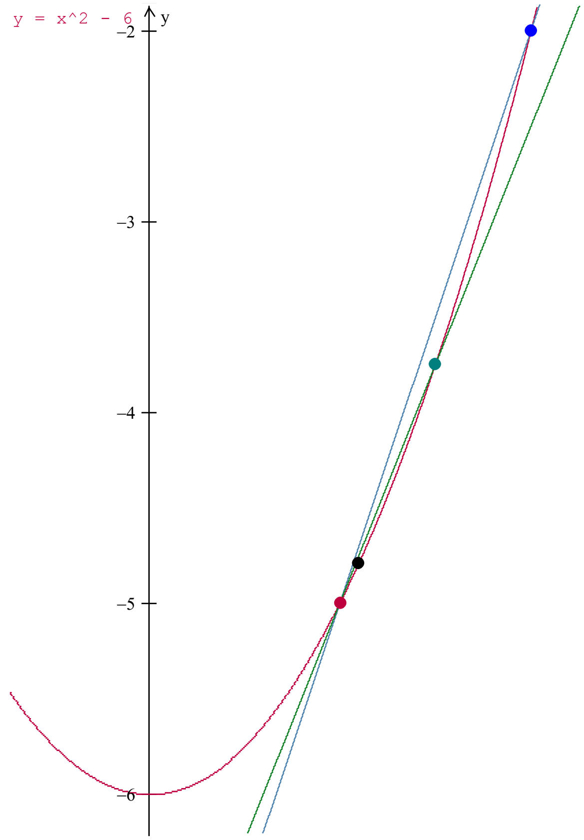
x	$f(x)$	$f'(x)$
-4	2.6667	0
-3	3.5625	1.25
-2	4.3333	0
-1	3.2292	-2.25
0	0	-4
1	-4.104	-3.75
2	-6.333	0

Again, as we can clearly tell, the graph of the derivative function (in blue) has its zeroes at exactly the points of the local minimums/maximum of the original function! By looking at the table, we find that when x is -4, -2, and 2, f'(x) is at exactly 0. Also, at the same time, f(x) is at a local minimum/maximum.

Another thing that you may notice about the derivative graph is that not only did it show zero when there was a local minimum/maximum, but it also showed whether the slope was positive or negative! In our experiment, we found that when x was between the values of $-\infty$ and -4, all values of $f'(x)$ were negative. Between (-4, -2), $f'(x)$ was positive; between (-2, 2), $f'(x)$was again negative, and finally, from (-2, ∞), $f'(x)$ was positive. What does this mean when compared with the $f(x)$ graph? Keeping in mind that this was the derivative of the original function, and that the derivative represents the slope of the original function, we take a look at the original slope. What do you know? It just so happens that while $f'(x)$ was positive, the slope of $f(x)$ is also positive, and when $f'(x)$ is negative, the slope of $f(x)$ is also negative. Even though this isn't anything that is really earthshaking, it does bring us to our next topic: finding the slope of a function by using a derivative.

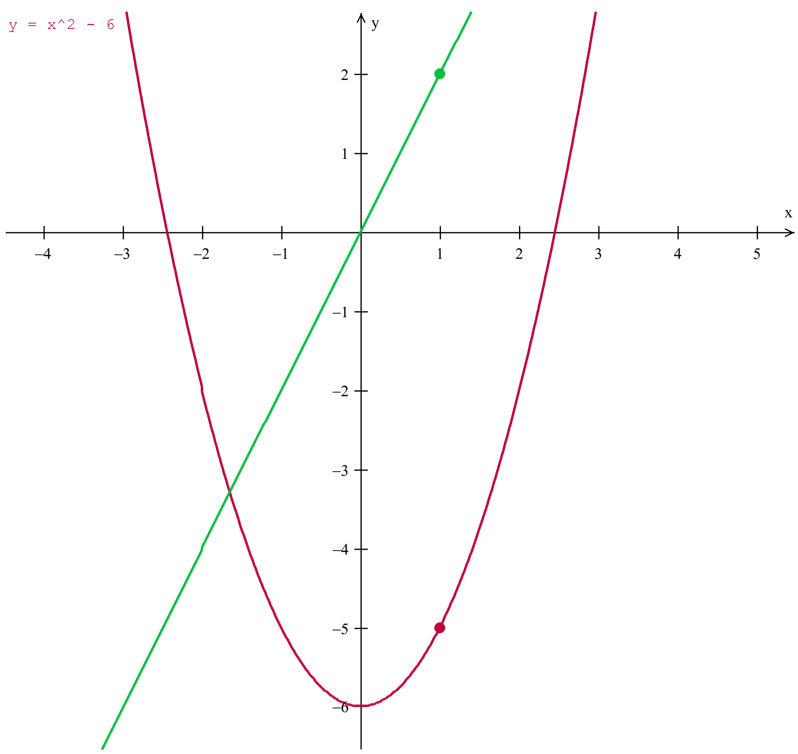
Before we go on to the next topic, let's circle back to our original problem stated in the beginning. Are we any closer to solving it? Yes! In fact, as long as we have a derivative function for $f(x)$, we already know exactly how to find the answer! All you need to do is to factor $f(x)$ and then use the roots! But… we still don't know exactly how to find that derivative function. Even though we are so close to getting some really cheap tickets, let's first understand more about derivatives, so that we can find the derivative formula.

Before we find the slope at each point, which is what the derivative function is all about, let's go back to our textbook definition of average slope. We have speed defined as the rate of change between two points, or $m = \frac{y_2 - y_1}{x_2 - x_1}$. Using this formula, we can easily find the slope between any two points. For example, on the graph $y = x^2 - 6$, the average slope between (1, -5) and (2, -2) is found by $m = \frac{-5 - -2}{1 - 2}$, or $m = 3$.

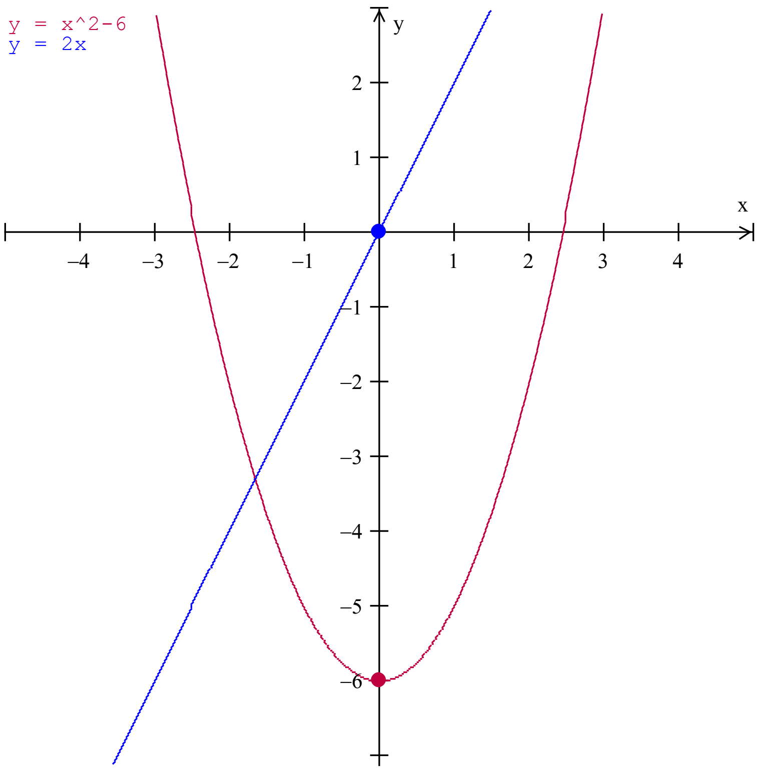


That was pretty easy, but what if we extended that? What if we shortened the distance between the two points to $\frac{1}{2}$? Now the average slope between (1, -5) and (1.5, -3.75) is $\frac{-3.75 - -5}{1.5-1}$, or $m = 2.5$. Now, we have a more accurate slope for the point (1, -5). However, it isn't accurate enough. Let's go even deeper, with the difference being only .1 this time. Now, we are finding the average slope of the points (1, -5) and (1.1, -4.79). The average slope of these two points is $m = \frac{-4.79 - -5}{1.1-1}$, or $m = 2.1$. This is now very close to the actual instantaneous slope, but we must go deeper! How about instead of defining the difference between the two points, we set out a variable for it? Let us set h for the difference between the two points, so that the two points are now (1, -5) and $((1+h), (1+h)^2 - 6,$ or $((1+h), (h^2 + 2x - 5)\ )$. This will make the slope between the two points equal to $\frac{(h^2+2h-5)--5)}{(1+h)-1}$. If you simplify this, you will get $\frac{h^2+2h}{h}$, or $\frac{(h)(h+2)}{h}$. Now, we might be tempted to cancel out the h immediately, but remember that we cannot do so until we mark it that h not equal to zero. Therefore, our final answer of the slope is $h + 2, h \neq 0$. How can we use that to solve for the instantaneous slope? Even though h can never be equal to zero, we can find values for which h is really really really small, perhaps .00001 or even .0000000000000001. This would make the value of h almost be equal to zero, and therefore we can say that the slope of the point (1, -5) is 2.

Hooray! We've done our first derivative! If we check on our graph, shown below, you will see that when x is 1, $f'(x)$ is 2.



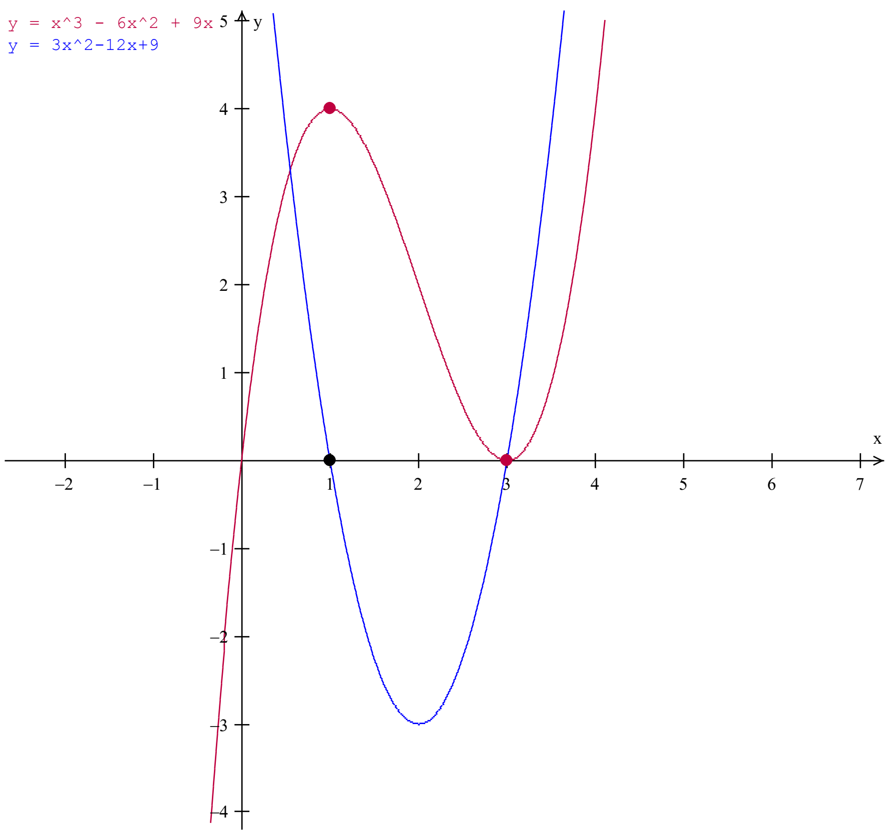
However, how would we find the derivative line as shown above, annotated as $f'(x)$, so that we could find the instantaneous slope for all points of $f(x)$? Let's begin by doing what we already know: the average slope formula. For any point on $f(x)$, the slope between it and another point h units away is $\frac{(x^2+2hx+h^2-6)-(x^2-6)}{(x^2+h)-x^2}$, or simplified, $\frac{(h)2x+h}{h}$. Why would we do a point h units away? Because even though using real numbers is good, sometimes it just isn't accurate enough. If we can find the slope for a point h units away, then we can change h into almost anything we want. Applying what we just found out about the derivative of a point, the previous equation simplifies even more to $2x + h, h \neq 0$. By finally making h infinitely small, we come to the conclusion that $f'(x)$ is equal to 2x.



x	$f(x)$	$f'(x)$
-2	-2	-4
-1	-5	-2
0	-6	0
1	-5	2
2	-2	4
3	3	6

Now that we know how to find the derivative of a simple, quadratic equation, let's apply that knowledge to a much more complex equation, like $x^3 - 6x^2 + 9x$. Let's start slowly. First, we know that we need the point $(x, x^3 - 6x^2 + 9x)$. Then we find the point $((x + h), \big((x^3 + 3hx^2 + 3h^2x + h^3) - 6(x^2 + 2hx + h^2) + 9(x + h)\big)$, which simplifies to $((x + h), (x^3 + 3hx^2 - 6x^2 + 3h^2x - 12hx + h^3 - 6h^2 + 9h$. If we apply all of that to our average slope equation, we get $\frac{(x^3+3hx^2-6x^2+3h^2x-12hx+9x+h^3-6h^2+9h)-(x^3-6x^2+9x)}{(x+h)-x}$, which simplifies to $\frac{3hx^2+3h^2x-12hx+h^3-6h^2+9h}{h}$, which simplifies even further to $3x^2 - 12x + 3hx + h^2 - 6h + 9, h \neq 0$.

Using the infimestially small h, we find that the derivative equation is $3x^2 - 12x + 9$. We've solved another derivative function! Let's check this function on a graph and table.

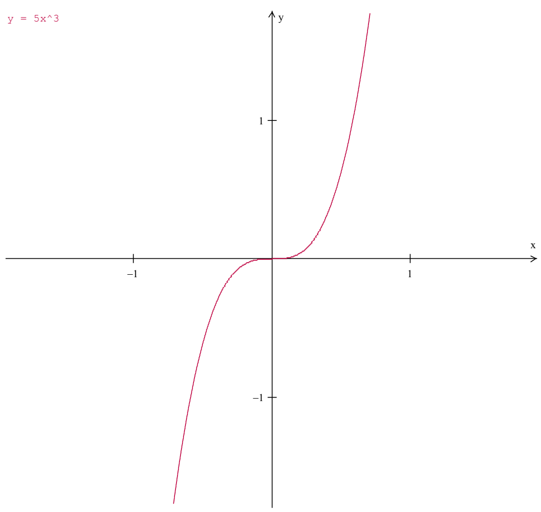


x	$f(x)$	$f'(x)$
0	4	9
1	2	0
2	0	-3
3	4	0

As we could tell from the above graph, we know that the formula that we derived was, in fact, the derivative. Now, what do you notice about the two derivative graphs that you have found? Look back at the first one you derived as well as the second one. Do you notice any patterns? One thing that may pop out at you is the fact that both derivative functions were one degree smaller than their original function. Now, let's think about why that is true. We know that derivative functions are formed by having roots at the local minimum and maximum of the original function. What do we know about the local minimum/maximum of a function then? Of course, we know that the number of local minimum/maximums is always going to be $n - 1$, given that the function was of degree n. Therefore, knowing that the zeroes of the derivative function only occur at $n-1$ points, the function itself must be of the $n-1$ degrees.

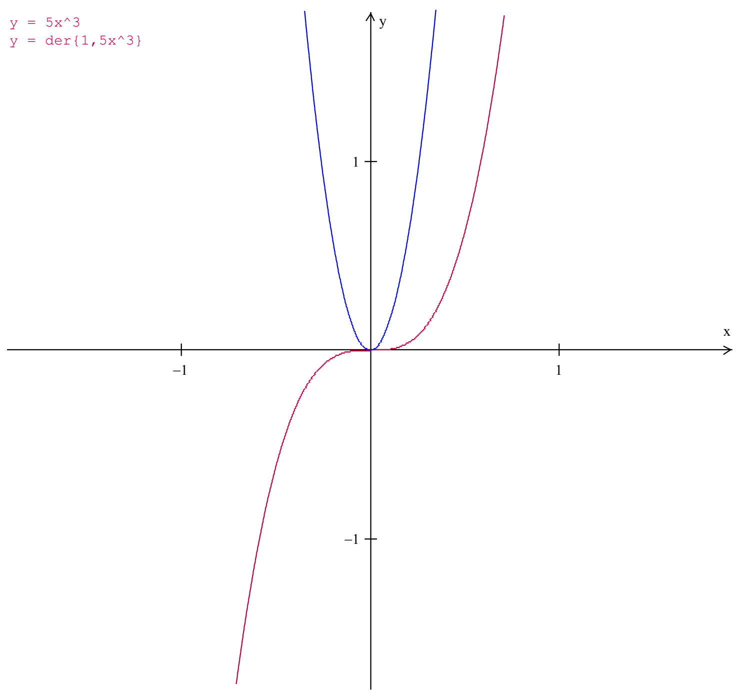
Even though we have now found a way of finding the derivative equation and to find the number of tickets you should buy, wouldn't you agree with me that the way we found it was very complex and convoluted? There must be an easier way, right? So instead of getting to the answer of our tickets right now, let's explore some simple equations to get a better sense of how the derivative is related to the function.

Suppose that we had the function $f(x) = 5x^3$.



x	$f(x)$
-1	-5
0	0
1	5
2	40

What would be the derivative of that function? Before leaping to the actual calculation, let's evaluate what we already know about the derivative of functions. For one, we know that the derivative is always one degree less than the original function, so we know that this derivative function's maximum possible term for x is x^2. Now, let's calculate. First, we take $\frac{5x^3-5(x+h)^3}{x-(x+h)}$, which simplifies to $\frac{5x^3-5(x^3+3hx^2+3h^2x+h^3)}{h}$. When we subtract on the top, we reach $\frac{5*(3hx^2+3h^2x+h^3)}{h}$. Do you notice that the first term, the $5x^3$ term, was cancelled out? Now, after we eliminate the h and place a limit on the value of h, we get $15x^2 + 3h + h^2$. If we make h really really small, we get $15x^2$ for our answer.



x	$f'(x)$
-1	15
0	0
1	15
2	60

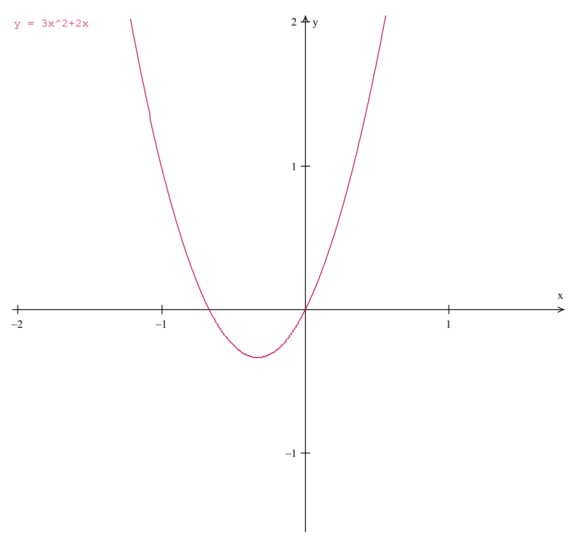
Now, let's compare the two equations: $5x^3$ and $15x^2$. Do we see any relationships between the coefficients? Oh, so the coefficient of the derivative function *just happens* to be 3 times greater than the original function, and the original function *just happened* to have a degree of 3. Could we formulate a theory that when $f(x) = a_0 x^n, f'(x) = (a_0 \cdot n) \cdot x^{n-1}$?

To see if our hypothesis is true, let's prove the general case. For the general polynomial $f(x) = a_0 x^n$, its derivative is equal to $\frac{a_0 x^n - a_0(x^n + nhx^{n-1} + \cdots nh^{n-1}x + h^n)}{h}$. Now, the true expansion for that formula is much longer, but the truth is, we don't need to worry about the middle part. As we know, in the end we will be making h really small. Therefore, other than h values that cancel out, all other parts with h in front of them are essentially 0! Now, back to the math. By simplifying, we get $\frac{h \cdot a_0(nx^{n-1} + \cdots nh^{n-2}x + h^{n-1})}{h}$. When we divide out the h, we are left with $a_0(nx^{n-1} + \cdots nh^{n-2}x + h^{n-1})$, which we can further simplify to a_0nx^{n-1} by making h really small. Seeing this, we have now proved our original hypothesis.

Before moving on, let's just double check on this rule that we made, seeing if it holds for the previous equation. If we apply the previous rule (let's just call it the law of powers from now on), we are given that $3 \cdot 5x^{3-1}$, or $15x^2$. The answer corresponds to our known derivative, so we know that the law of power does hold for some numbers, at least.

We have now proved a rather basic part of the law of powers, but at the same time, the portion that we proved does not have much use. You will very rarely find a polynomial with only one x term in it. Are there ways to use this rule for more general terms, such as $a_0x^n + a_1x^{n-1} + \cdots$? One way we might be able to prove the more general formula is if we are able to find a commutative property in derivatives. We have learned about the commutative property very long ago, that $if\ a + b = c, and\ e + f = d, then\ (a + b) + (c + d) = e + f$. This property seems very obvious, but could we apply it to derivatives? That is to say, $if\ h(x) = f(x) + g(x), then\ h'(x) = f'(x) + g'(x)$?

Again, let's start with an easy example. Let's make $h(x) = 3x^2 + 2x$.



x	$f(x)$
-1	1
0	0
1	5

We can divide that larger polynomial into two smaller polynomials, $3x^2 and\ 2x$. By applying the method we just proved, we are able to conclude that the derivative of $3x^2$ is $6x$ and the derivative of $2x$ is2. Would this mean that the derivative of $3x^2 + 2x$ be equal to $6x + 2$? Let's try it out. Given the formula for finding the derivative, we find that $h'(x) = \frac{(3x^2+2x)-(3(x^2+2hx+h^2)+2(x+h))}{h}$, and that ultimately simplifies to $\frac{h(3h+6x+2)}{h}$. Eliminating the h, we find that the derivative of $h(x)$is indeed the derivatives of $f(x)\ and\ g(x)$ combined!

Now, let's tackle a more challenging general proof. Given that $h(x) = ax^n + bx^m$, where n is not equal to m, is the derivative equal to $anx^{n-1} + bmx^{m-1}$? Let's test it.

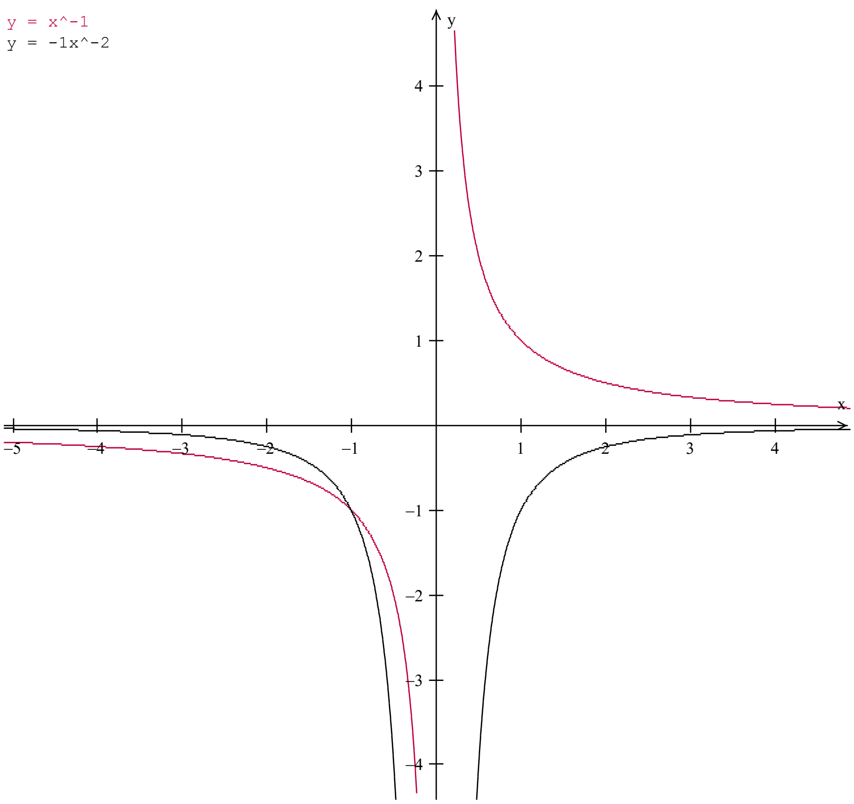
$h'(x) = \frac{ax^n+bx^m-(a(x^n+nhx^{n-1}+\cdots nh^{n-1}x+h^n)+b(x^m+mhx^{m-1}+\cdots mh^{m-1}x+h^m))}{h}$.By reducing and simplifying that gigantic formula, we are left with $\frac{ah(nx^{n-1}+\cdots nh^{n-2}x+h^{n-1})+ bh(nx^{n-1}+\cdots nh^{n-2}x+h^{n-1})}{h}$, or $a(nx^{n-1} + \cdots nh^{n-2}x + h^{n-1}) + b(mx^{m-1} + \cdots mh^{m-2}x + h^{m-1})$. Again, by reducing the h, we are left with $anx^{n-1} + bmx^{m-1}$! We have proved that the commutative property holds for derivatives!

After we have proved that the commutative property is true, it is fairly easy to visualize a much broader general formula for all positive values of n in a polynomial. The way that the general law would work is because you could separate all portions of a larger polynomial into smaller polynomials, find the derivative of that piece using our original proof, and then add it to the overall derivative.

We are finally ready to finish that ticket question! But, before I can let you off the hook, I must explain some more things about the law of powers. So far, we have only worked with n being a positive integer. In fact, our general proof only works for positive integers! What if n wasn't positive? Would our law still hold?

The only way to certify the law of powers is to test a few examples. Let's consider the values of n that seem greatly different each other. We have the $\frac{1}{x}$ equation, which is equivalent to x^{-1}; the $\sqrt{x}$ equation, or $x^{\frac{1}{2}}$, and we also have oddballs like x^{e}, where n is not a rational number. Let's see if we can tackle these three cases!

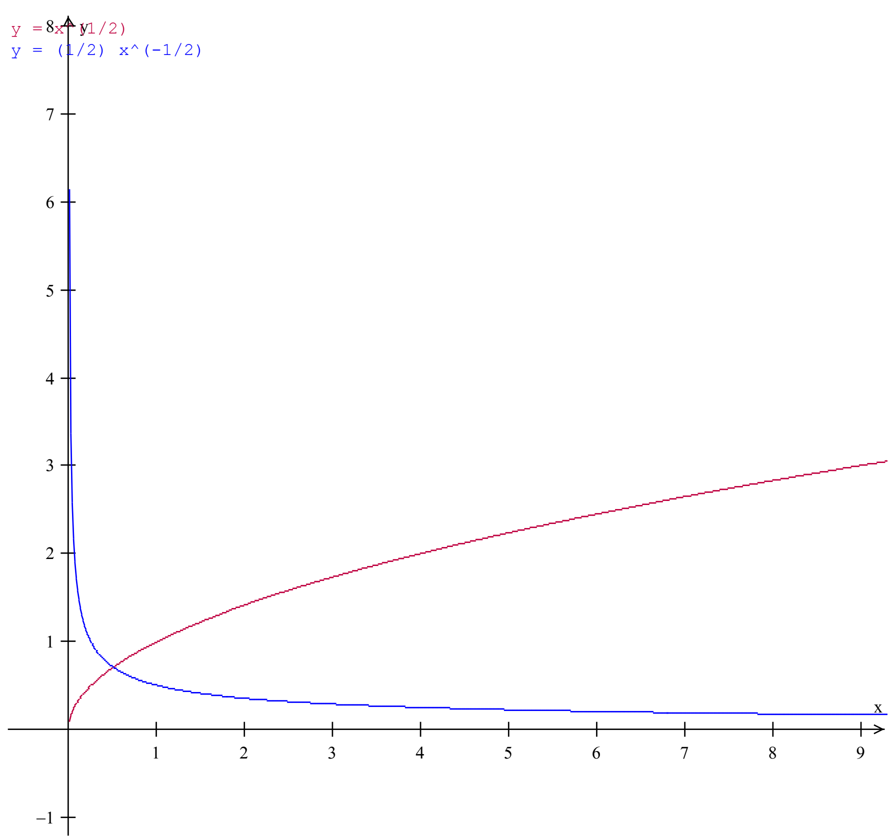
Case 1: What do we do if n is negative? What would the derivative of x^{-1} be? Before we start looking for the formula based on our definition, let's try to find it by our law of powers. Based on our law of powers, we see that $f'(x) = -1x^{-1-1}$, simplified to $-1x^{-2}$. Let's visualize these two graphs.



x	$f(x)$	$f'(x)$
-3	$-\frac{1}{3}$	$-\frac{1}{3}$
-2	$-\frac{1}{2}$	$-\frac{1}{2}$
-1	-1	-1
0	$undef$	$undef$
1	1	-1
2	$\frac{1}{2}$	$-\frac{1}{2}$
3	$\frac{1}{3}$	$-\frac{1}{3}$

Right off the bat, we see that something weird is going on. $f(x)$ never has a local minimum or maximum! But wait, why should we panic about that? The only thing that that tells us is that $f'(x)$ will never equal zero, and that's not too bad. Now, let's begin to find the actual derivative of $f(x)$. Using the points (x, $f(x)$) and $\left(x+h, f(x+h)\right)$, our slope equation turns out to be $\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$ By manipulating the fractions, we are left with $\frac{\frac{x-(x+h)}{(x+h)(x)}}{h}$, and then $-\frac{h}{x(x+h)}$, and by bringing h very close to zero once more, we are left with something close to $-\frac{1}{x^2}$.

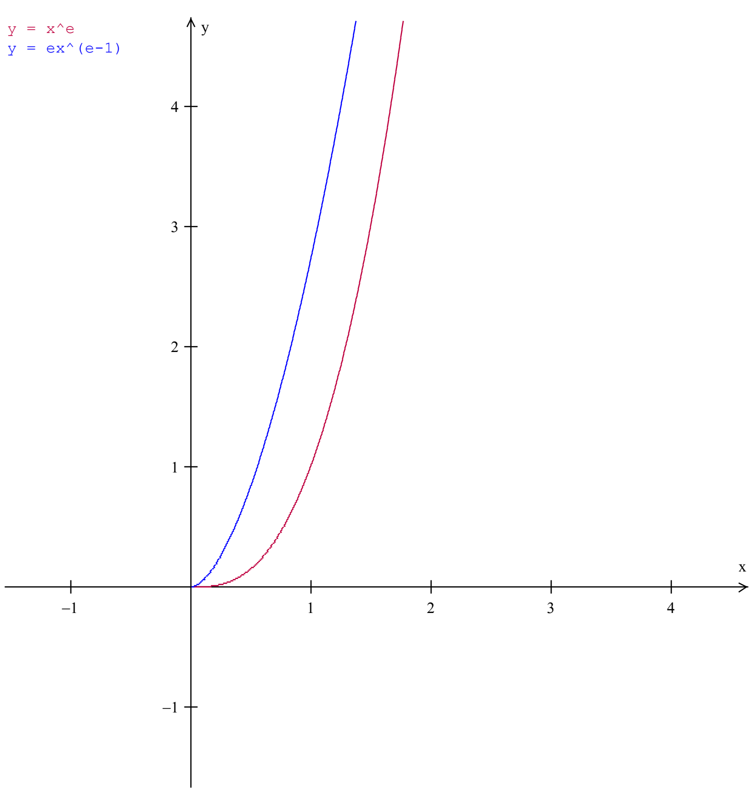
Case 2: What do we do if n is a fraction? What is the derivative of $x^{\frac{1}{2}}$? Let's tackle this problem like we did to the previous one, by first finding what it should be. The derivative of $f(x) = x^{\frac{1}{2}}$ should be $\frac{1}{2}x^{\frac{1}{2}-1}$, or $\frac{1}{2}x^{\frac{-1}{2}}$.



x	$f(x)$	$f'(x)$
0	0	$undef$
1	1	.5
4	2	.25
9	3	.1667
16	4	.125
25	5	.1

This graph shows us, again, how weird graphs can be. Instead of having a giant hole at the point where a local minimum/maximum was supposed to occur, this graph…. Just doesn't have anything at all! Sure, the point (0, 0) is the lowest point, but because the graph's domain, or possible x-values, starts from there, we can't count that as a true local minimum. Again, we can only just shrug this off, and assume that $f'(x)$ won't have a root either. Now, we will begin to tackle this derivative! First, let's set the points that we will be using as (x, $f(x)$), and $(x+h, f(x+h))$. Our slope formula, though a bit messy, is now $\frac{\sqrt{x+h}-\sqrt{x}}{h}$. In order to get rid of some of the radicals, you would multiply everything by $\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$, so that you get $\frac{x+h-x}{(\sqrt{x+h}+\sqrt{x})\cdot h}$, which simplifies down to $\frac{1}{\sqrt{x+h}+\sqrt{x}}$.. Now, we just take h down close to 0, and we are left with $\frac{1}{2\sqrt{x}}$, or $\frac{1}{2}x^{-\frac{1}{2}}$. Done!

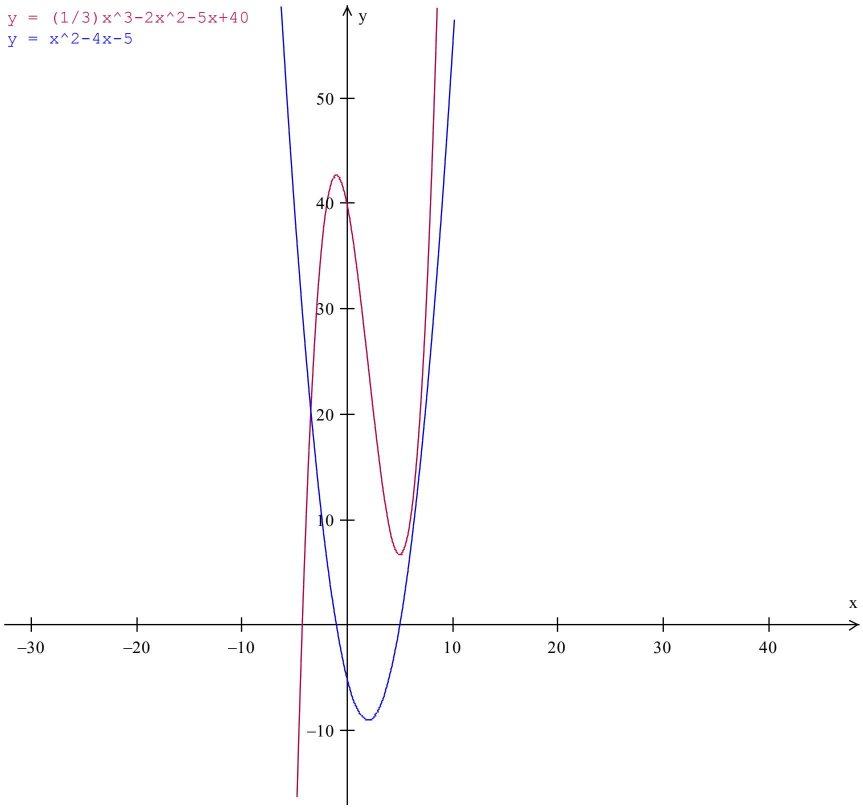
Case 3: What do we do if n is irrational? What is the derivative of $f(x) = x^e$? Using the law of powers, we get our derivative to be ex^{e-1}. This one may be one of the weirdest, so before we do anything else, let's see it on the graph.



x	$f(x)$	$f'(x)$
1	6.58	2.71
2	19.81	8.94
3	43.31	17.95
4	79.43	29.43

Woah. This one is weird. Luckily for us, we can still use our original proof for a polynomial to the nth degree though, as e is just another n. Using the points (x, $f(x)$), and $(x+h, f(x+h))$, we are left with $\frac{x^e+ehx^{e-1}+\cdots eh^{e-1}x+h^e-x^e}{h}$, and we all know that that simplifies to $(ex^{e-1}+\cdots eh^{e-2}x+h^{e-1})$, also known as e^{x-1}. Even though this is the weirdest graph, It has one of the most straightforward answers. However, there is one part of this answer that we did miss. Notice that In the graph, the domain of x is limited between (0, ∞). This is because while positive numbers can be raised to irrational powers, negative numbers can't. Therefore, the true derivative of this function would be $e^{x-1}, x \geq 0$.

Finally, we can solve our problem with the homework passes with ease of mind! After all of those difficult derivatives, this one should be a piece of cake! First, to find the derivative function, we must apply the law of powers that we have touched up on so many times. We find that the derivative function should be $\frac{1}{3} \cdot 3x^{3-2} - 2 \cdot 2x^{2-1} - 5 \cdot 1x^{1-1}$, or $x^2 - 4x - 5$. After factoring that out, we are left with the roots of 5 and -1. Seeing that we can't possibly buy negative tickets, we know that 5 is our only chance. Before saying flat out that 5 is the correct answer, let's plug it back into our original equation to make sure it actually works. We find that the answer is $\frac{1}{3} \cdot 125 - 2 \cdot 25 - 5 \cdot 5 + 40$, or about 13 dollars per ticket, making for a total cost of 65 dollars. Pay up, please!



x	$f(x)$	$f'(x)$
1	33.33	-8
2	24.66	-9
3	16	-8
4	9.33	-5
5	6.66	0
6	10	7
7	21.33	16

As we are now finished with all that needs to be done, we can spend some time investigating into derivatives a little deeper. For example, our original proof on finding the derivative of $a_0x^n + a_1x^{n-1} + \cdots$ was slightly flawed, as it did not compensate completely for fractions, negatives, or other cases, and we instead showed examples of why those other cases worked. However, there is a through proof of the matter. First, for a polynomial of $y = \ln(x)$, what would be the general derivative of it? If we use some special manipulations, we can turn that into $e^y = x$. After deriving that. We get that $e^y \cdot \frac{dy}{dx} = 1$. Because $\frac{dy}{dx}$ is our intended result, all we have to do is move it into $\frac{dy}{dx} = \frac{1}{e^y}$. By substituting y for e^x, the $\ln(\quad)$ and the e cancel each other out, leaving us with $\frac{1}{x}$. Therefore, the derivative of $\ln(x)$ is $\frac{1}{x}$. We will now use that to find what the derivative of $y = a^x$ is. We first take the natural log of both sides, getting $\ln(y) = x \cdot \ln(a)$ and then proceed to take the derivative of that. Since we already know that $\frac{dy}{dx}$ of $\ln(y)$is $\frac{1}{y}$, it is easy to get $\frac{1}{y} \cdot \frac{dy}{dx} = \ln(a)$. By cross multiplying, we get our result, which is

$\frac{dy}{dx} = y\ln(a)$. Now that we have proved the exponential and logarithmic rules of derivatives, we can proceed to tackle the actual general case. If our function was $y = x^n$, then we can manipulate that into $e^{n \cdot \ln(x)}$, as this form will be helpful when we use the logarithmic rule. By taking the derivative of this, we are left with $e^{n \cdot \ln(x)} \cdot \ln e^{n \cdot \ln(x)}$, as this is just using e for a and n ln(x) for x in our exponential law. When we finish this part off, we are left with $x^n \cdot n \cdot \ln(x)$. By simplifying $\ln(x)$into $\frac{1}{x}$, we are left with $x^n \cdot \frac{n}{x}$. $\frac{n}{x}$ is the same as saying nx^{-1}, and by applying laws of exponents (the general kind) , we get nx^{n-1}. Amazing, isn't it! This form can be used for all n, as we did not actually manipulate n itself.

With everything done, we can conclude the law of powers, formally recognized in the mathematical world as the power rule of derivatives, can be used for all n, with infinite terms of x. The only limitation we have found is that while n is a fraction or an irrational number, the domain of x must be positive.

Throughout this paper, I hope that the basic premises of what a derivative is and what it can do has been made clear. Many key topics in basic derivative knowledge were covered very quickly, but it was to the best of my ability to summarize and simplify the understanding of complex ideas such as the power rule as well as the functions of a derivative function. In reading this paper, students of all math backgrounds should be able to understand the clear and precise wording, as well as the easy to read graphs and tables, so that they would be able to understand how a derivative can help them.