Research on Prime Derivatives

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The fields of number theory and of calculus are well defined, but by creating a new number chain pattern, we find interesting conclusions. By rigorously defining a series of numbers, patterns that exist in nature are analyzed through algebra as well as through graphs and figures. We focus on finding loops and divergences in the prime derivative numbers, as well as creating graphs and patterns that could possibly be followed visually. In addition, our research diverges towards parity issues Mersenne Primes. While it is very difficult to draw definitive conclusions from such a tricky analysis, our research explores the fundamental issues within this field.

1. Introduction

There are several very interesting number chains that exist in pure mathematics that are especially interesting to analyze. Perhaps the most famous of them is the Collatz Conjecture, which applies a "simple" algorithm to any integer. Afterwards, a graph can be created of the "flow" between different integers. Through this process, it is hypothesized that every single integer can eventually trace back to the number 1. However, this is a very challenging problem that has not yet been proven.

In this exploration, we investigate a similar number chain problem: The Prime Derivative. This also applies a series of algorithms to integers and traces how they evolve over time. Although it is evident that not every integer returns to the number 1, it is clear that there are certain patterns that could be understood through analysis. We will begin by setting our definitions and slowly building our knowledge from those base cases.

2. Analysis

2.1. Definitions

The Prime Derivative operates on the following principles:

IF:

$$n = 1$$

then:

$$n' = 0$$

IF: n is prime, then

$$n' = 1$$

IF: n is composite of form

$$a \cdot b$$

then:

$$n' = a'b + b'a$$

IF: n is composite of form

$$a \cdot b \cdot c$$

then,

$$n' = a' \cdot (b \cdot c)' = a' \cdot b \cdot c + a \cdot b' \cdot c + a \cdot b \cdot c'$$

As is evident, this creates a system that seems to be very similar to regular derivatives of functions. We can then impose many different test scenarios onto these basic principles to test how prime derivatives work.

2.2. Graphs

One of the first things that we did in investigating the Prime Derivative was to create several graphs. This was the easiest way that we had to visualize what a prime derivative chain looks like, which is similar to the Collatz Conjecture graphs. We see immediately that there are clear differences from the Collatz conjecture graph, namely that there are some points that seem to be "fixed" and several chains that do not lead to 1. Therefore, our next step is to investigate those scenarios.

In Figure 2, we see a small slice of the much larger graph. This graph was generated with number chains from 1 to 1000, but as is evident, there does not seem to

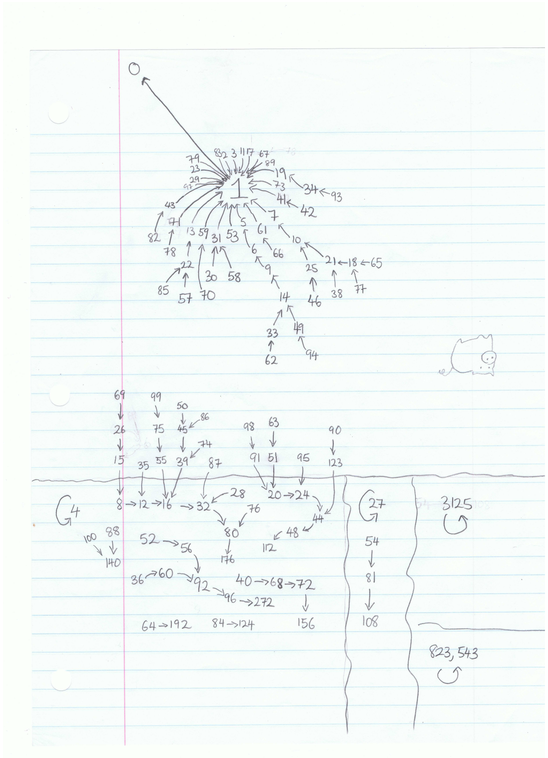

Figure 1: The original, hand-drawn graph for prime derivatives by Benny Cheung

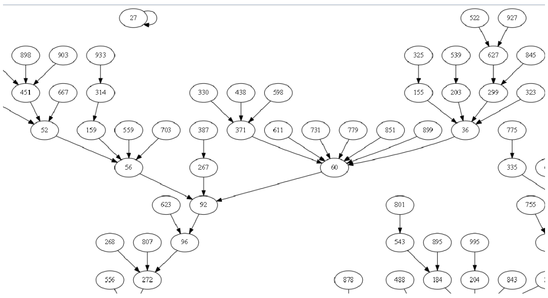

Figure 2: A small, randomly selected, portion of the graph

be any particular reason for links. Even numbers, odd numbers, perfect numbers, squares and cubes all lead haphazardly to other numbers. It isn't well defined how many numbers should lead to 1 number, nor is it even clear if a number would diverge or converge.

2.3. Patterns in Multiples

We continued our analysis through observing patterns in the prime derivatives of multiples of small integers. We quickly realized that the number 4 loops to itself because the sum of its prime factors is equal to the number itself, and that multiples of 4 increase infinitely. We investigated whether numbers lead to 0 or to infinity, and found no evident patterns in multiples of 5, 6, 7, or 9 through 100. This led us to explore different methods of understanding prime derivatives, such as using loops and fixed points.

2.4. Loops

Immediately, it is evident that the prime derivative of some numbers is itself. We first see this with the number 4, and then with 27, 3125 and 823543. If we look at these numbers, it appears that these numbers follow the pattern of being in the form:

$$n = p^p$$

If we try to apply our previous definition to these numbers, we gain some clever insights. For any number in this form, the derivative would be:

$$n' = p' \cdot p^{p-1} + p' \cdot p^{p-1} .... + p' \cdot p^{p-1}$$

$$n' = p' \cdot p \cdot p^{n-1} = p' \cdot p^p = p^p$$

Clearly, any derivative of a prime number raised to the power of a prime number will be itself. We can therefore set a definition that all numbers of that form are essentially fixed points in the universe.

At this point, we will establish another term. For any number n, we can find the ratio of

$$\frac{n'}{n} = \frac{1}{f_1} + \frac{1}{f_2} + ... \frac{1}{f_n}$$

if we assume the form that $n = f_1 \cdot f_2 \cdot ... \cdot f_n$

If the ratio $\frac{n'}{n}$ is equal to 1, we can say that n' will always be equal to n. This is also evident that loops to the same number are valid in these scenarios. This is easily demonstrable as there are n fractions and each f_n is the same number. Therefore,

$$\frac{n'}{n} = \frac{1}{p} + \frac{1}{p} + ... \frac{1}{p}$$

$$\frac{n'}{n} = \frac{p}{p} = 1$$

2.5. Branches Diverging to Infinity

Now, let us consider if $\frac{n'}{n} > 1$. We can know with certainty that $n' > n$, obviously. However, let's consider the number:

$$n = p^p \cdot x, x \in \mathbb{Z}$$

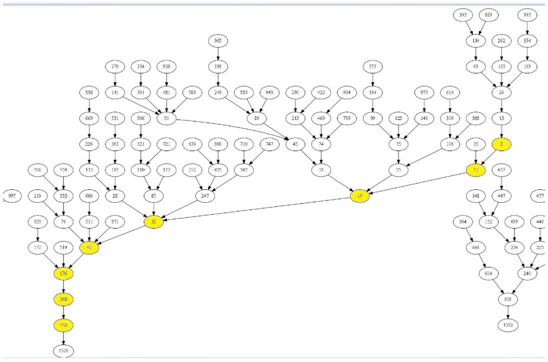

Figure 3: A small portion of the graph for multiples of 4.

We can tell that n' would therefore be:

$$n' = p^{p'} \cdot x + p^p \cdot x'$$

Because we know that $p^{p'} = p^p$, we understand that

$$n' = p^p \cdot x + p^p \cdot x')$$

such that n' is always greater than n. However, this also implies that the derivative of n would simply be

$$n' = p^p \cdot x_1$$

and because we can apply this recursively, our only result is that this derivative diverges to infinity! We can express this as:

$$\lim_{x \to \infty} (p^p \cdot n)^{(x)} \to \infty$$

where x is not a power, but the number of prime derivatives that have been taken. In plain words, for any number of the form $p^p \cdot n$, as you take more prime derivatives, that number diverges to infinity. See Figure 3 for and example of this using numbers with a factor of 4.

2.6. Loops, Part 2

At this point, we will ask: Are numbers of the form p^p the *only* fixed points in the prime derivative scenario? In order to analyze this question, we take the following definition and ideas:

$$n = {p_1}^b \cdot {p_2}^d$$

We realize that the following ratio is true:

$$\frac{n'}{n} = \frac{b}{p_1} + \frac{d}{p_2}$$

Given that formula, we will realize that this ratio will never be equal to 1. If the denominators are prime numbers, as we have defined, the sum of fractions can never be equivalent to one. Therefore, any number that is not of the form p^p cannot be a fixed number.

2.7. Geometric Series

We now attempt to analyze regular sequences using our prime derivatives, in the hope that consistent series can hold some sort of pattern when derived. The most obvious of these patterns would be to use a geometric series, of the form

$$\prod_{n=0}^{a} f^n \cdot x$$

Where a is any integer and f is some prime factor. For our first case, allow x to only be some other prime number.

Therefore, we can find that the derivative of these geometric series is of the form:

$${S_n}' p \cdot n \cdot a^{n-1} + a^n$$

If we find this for the next series, we will notice that it would be

$${S_{n+1}}' p \cdot n + 1 \cdot a^n + a^{n+1}$$

While this is somewhat useful, there is no clear pattern between derivatives of this form. We recognize the addition enclosed in this pattern, and realize that there is no way to derive properly.

2.8. Parity

Exploring the parity of these prime derivatives can prove to be quite interesting. First, let us redefine our numbers using factors. Here, we will use the sample case of having 4 factors.

$$n = f_1 \cdot f_2 \cdot f_3 \cdot f_4$$

$$n' = f_1{}' \cdot (f_2 f_3 f_4) + f_2{}' \cdot (f_1 f_3 f_4)$$
$$+ f_3{}' \cdot (f_1 f_2 f_4) + f_4{}' \cdot (f_1 f_2 f_3)$$

Now, let us assume that each f_n is a prime number. Further, let is first assume that $f_n \neq 2$ for this first case.

Because all prime numbers (other than 2) are odd, and because the product of two odd numbers is also odd, each of the individual addition parts must also be odd. We know from basic addition principles that the sum of an even number of odd numbers will be even, while the sum of an odd number of odd numbers must be odd.

Therefore, we conclude that if there are m factors in n and m is odd, n' must be an odd number. In addition, if m is even, n' must also be even. This is a useful tool for analysis and checking sums.

Next, we will extend our analysis to include the prime factor of 2, but also such that $n\%4 \neq 0$ We establish the second condition because if there was a factor of 4 in n, we know that the derivative of n will be divisible by 4 and repeated derivatives will tend towards infinity and constantly be growing.

Using this condition, we understand it is very similar to our previously established cases. The only difference is that the parity is different by 1. Therefore, we know that if there are m factors in n and m is odd, n' must be an even number. In addition, if m is even, n' must also be odd.

Ultimately, these ideas prove that prime derivatives are *very hard.* In order to even know the parity of a prime derivative, one must know the factorization of that number. This makes algorithms for double derivatives almost impossible. There is no intuitive form for factoring the sum of two numbers, and therefore, there is not a way of understanding the properties of double derivatives.

2.9. Calculus Connections

After making so many calculations here, we realize that the prime derivative is certainly named appropriately. If we model a number of prime factors, the prime derivative would look really similar as if you were conducting the derivative of an algebraic expression. Some examples of this was explored earlier, in what appears to be the equivalence of the chain rule. That is, if

$$n = x^m$$

then

$$n' = m \cdot x' \cdot x^{m-1}$$

This is very similar to the chain rule that is taught in calculus. However, it is difficult to speak to the significance of this idea, other than it makes calculations simpler.

3. Extensions

At this point, we have exhausted all of the "low hanging fruit" that is available in this problem. We have examined the common ratios and eventual patterns within the prime derivative system. The next big problem would be to define the second derivative of any number. However, this can be a much trickier problem than our original problem.

For one thing, any double derivative would require factoring the sum of two or more numbers. There are not clearly defined rules for what possible factors can result from sums of numbers; they are very unpredictable. If it was easy to do this, we would be able to very simply conduct prime factorizations of any number, a problem which is obviously not easy. Many of the most difficult GIMPS searches are using extremely complicated analysis algorithms to conduct such research, most of which are choosing to focus on Mersenne primes. We can analyze a similar class of numbers, and see the eventual end results of these "Mersenne Prime Derivatives.

3.1. Mersenne Prime Derivatives

First, let us define any number to be a Mersenne Number if it is of the form

$$n = 2^a - 1$$

Unfortunately, upon further analysis of these series, there are no clear patterns. Just like geometric, arithmetic, and harmonic sequences, and even like Fibonacci and perfect numbers, there are no patterns for the end behavior of Mersenne Primes. Instead, they seem to "randomly" diverge or converge, with no predictability. Simple experimentation shows that these numbers do not come up with a verifiable pattern.

3.2. Strings of Infinite Divergence

One additional question that we can pose here may not be extremely useful for understanding the entire system, but is an interesting computational problem that is good for visualization. We will examine how many "strings" of numbers that can diverge to infinity, including the maximum "string" number that we can find. We have conducted some simple experiments on this, using the code found in Appendix A. This generates the graph found in Figure 4.

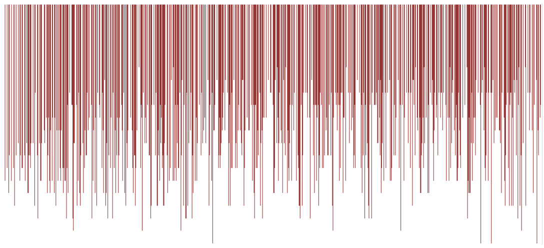

Figure 4: A graph of the number of consecutive numbers that diverge to infinity. The higher the bar, the longer the chain.

Through this analysis, we have seen seemingly random scattered chunks of data throughout the entire spectrum. It does not seem that these strings markedly decrease over time. Instead, it might even be hypothesized that these strings can possibly infinite in length, given that we are operating within the infinitely long list of integers.

3.3. Rings

After all of this analysis, we begin to realize that our "prime derivative" is rather unique. There does not seem to have many clear patterns about numbers in general, but we can see that the product rule holds, that is for any prime derivation,

$$D(ab) = (Da)b + a(Db)$$

We now recognize the existence of algebraic items called rings. A ring is an algebraic structure that has a generalized function of arithmetic and multiplication. Obviously, the integers fall into this category, in fact, they belong to a commutative ring. However, rings can be specialized. For example, a differentiable ring is any ring that obeys the sum rule (commutability) as well as the product rule listead above. When we analyze our prime derivative, we realize that the new function satisfies the product rule. In fact, it is defined to be satisfying that rule. However, it fails for the sum rule. For example,

$$(5+7)' = 12' = 16, 5'+7' = 1+1 = 2, 2 \neq 16$$

This makes the prime derivative as an odd algebraic structure, one that cannot be easily categorized.

4. Conclusions

Through our analysis of this very interesting number chain problem, we have uncovered several connections to the derivative function in algebra as well as prime numbers in number theory. Therefore, the Prime Derivative is actually a fascinating manner of connecting two seemingly separate branches of mathematics in search of something new. On top of that, there seems to be something that looks like fixed points and chaos theory in this system, where we can trace changes that diverge and converge.

Further investigation of this problem would go to the next level of computational analysis. We would like to use better algorithms for generating the graphs found in Section 2.2 This might be able to provide better visuals, and allow us to see patterns that we may have missed in our smaller diagrams. In addition, an extension might be to find the opposite of the prime derivative - the prime integral? If we applied the opposite algorithm to our numbers and attach a condition for isolating a single prime integral for each number, what would that graph look like? The prime derivative certainly has many extensions that will prove to be fascinating in the future.

A. Code for Prime Derivative

Attached is the source code that we used in calculating the prime derivative of any number. This was done through java and printed to a text file. Afterwards, the data was analyzed through Microsoft Excel and Graphviz for generating graphs.

```
public class Research {
  /**
   * @param args
   */
  public static void main(String[]
      args) {
    int largeCount = 10000000;
    int count = 1000;

    int[] num = new int[largeCount];
    int[] der = new int[largeCount];

    for(int i = 0; i < largeCount;
        i++)
    {
      num[i] = i+1;
      if(num[i] == 1) der[i] = 0;
      else if(factor(num[i]) == 1)
          der[i] = 1;
      else der[i] =
          (num[i]/factor(num[i]))
          *der[factor(num[i])-1]
          + factor(num[i])*der[
          (num[i]/factor(num[i]))-1];
    }

    System.out.println("num[i]
        der[i]");
    for(int i = 0; i < count; i++)
    {
      System.out.print(num[i] + " "
          + der[i]);
      //prints sequence
      int j = 0;
      int temp = der[i];
      while(temp-1 >= 0 && j < 20
          && temp < largeCount)
      {
        System.out.print(" " +
            der[temp-1]);
```

```
            temp = der[temp-1];
            j++;
         }
         if(temp > largeCount)
             System.out.print(" MAX");
         System.out.println();
      }
   }
   public static int factor(int n) {
      for(int i = 2; i < n-1; i++)
         if(n$\%$i == 0) return i;
      return 1;
   }
}
```