Lecture Notes in Functional Analysis

Lectures by Andrei Khrennikov

Note that these are rapidly taken and then even more swiftly typed notes, and as such errors might well occur. Be sure to check any oddities against the course literature [KF20]. Last updated January 22, 2016.

Throughout this document, □ signifies end proof, ▲ signifies end of example, and ⧫ signifies end of solution.

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Notes by Jakob Streipel.

1 Lecture I[1]

1.1 Metric Spaces

Example 1.1.1. Let $X = \mathbb{R}$, the real line. Then $\rho(x, y) = |x - y|$ is a distance measure (the ordinary one), with the following properties:

(i) $\rho(x, y) \geq 0$,

(ii) $\rho(x, y) = 0$ if and only if $x = y$,

(iii) $\rho(x, y) = \rho(y, x)$,

(iv) The triangle inequality; for all x, y, z we have $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ (think of the sides of a triangle). ▲

Note that by setting $z = 0$ and taking y to be negative in the triangle inequality we get the perhaps more common form $|x + y| \leq |x| + |y|$.

Definition 1.1.2 (Metric, and metric space)**.** Let X be any set and let $\rho(x, y)$ be any function on $X \times X$, then ρ is called a ***metric*** and (X, ρ) a ***metric space*** if ρ satisfies (i)–(iv) above.

Examples 1.1.3. (i) $X = \mathbb{R}^n$, $\rho_1(x, y) = \sum_{j=1}^{n} |x_j - y_j|$.

(ii) $X = \mathbb{R}^n$, $\rho_\infty(x, y) = \max_{1 \leq j \leq n} |x_j - y_j|$.

(iii) $X = \mathbb{R}^n$, $\rho_2(x, y) = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}$.

(iv) $X = C[a, b] = \{x : [a, b] \to \mathbb{R} \mid x \text{ is a continuous function}\}$, $\rho_\infty(x, y) = \max_{a \leq t \leq b} |x(t) - y(t)|$, (which exists since $[a, b]$ is closed and bounded). ▲

Definitions 1.1.4 (Open ball, closed ball, sphere)**.** Given a metric space (X, ρ), with $a \in X$ and r a positive real number:

(i) Open ball: $B_r(a) = \{x \in X \mid \rho(x, a) < r\}$,

(ii) Closed ball: $\bar{B}_r(a) = \{x \in X \mid \rho(x, a) \leq r\}$,

(iii) Sphere: $S_r(a) = \{x \in X \mid \rho(x, a) = r\}$.

Exercises 1.1.5. Let $X = \mathbb{R}^2$ (because it's easy to visualise), and let a be a fixed point in X and r be a fixed positive real number.

Draw the open ball, closed ball, and sphere around a with radius r given the metric ρ_1, ρ_∞, and ρ_2, respectively.

Solutions. For ρ_2, the open ball of radius r around a is simply the circle centred on a with radius r, since ρ_2 is simply the Cartesian distance.

For ρ_1 the distance between a and x is the sum of the difference of their components. This sum must be less than r. The open ball then becomes a square with diagonals $2r$, the diagonals being parallel to the axes.

[1]Date: November 2, 2015.

For ρ_∞ we take, for a given x, the biggest of the differences of the components of x and a. This will give us a square with sides $2r$, with sides parallel to the axes. ♦

Exercise 1.1.6. Let $X = C[a, b]$, with ρ_∞ as the metric. Draw the ball $B_r(z)$.

Solution. The ball of radius r around the function z will be any function defined in the interval $[a, b]$ such that it is contained within an r-corridor around z. ♦

1.2 Continuous Functions in Metric Spaces

Recall first the definition from single variable calculus: a function $f : \mathbb{R} \to \mathbb{R}$ is called *continuous* at $x_0 \in \mathbb{R}$ (also written $f \in C(x_0)$, if for all $\varepsilon > 0$ there exists a δ such that, when $|x - x_0| < \delta$, we have that $|f(x) - f(x_0)| < \varepsilon$.

We copy this definition into the general settings of metric spaces.

Definition 1.2.1 (Continuity)**.** Given two metric space (X_1, ρ_1) and (X_2, ρ_2), a function $f : X_1 \to X_2$ is called ***continuous*** at the point $x_0 \in X_1$ ($f \in C(x_0)$) if, for all $\varepsilon > 0$ there exists a δ such that, when $\rho_1(x, x_0) < \delta$, we also have $\rho_2(f(x), f(x_0)) < \varepsilon$.

Example 1.2.2. Let us use $X = C[a, b]$ and ρ_∞ as the metric. Further let $f : X \to \mathbb{R}$, given by a fixed point $c \in [a, b]$, such that $f(x) = x(c)$. Then

$$|f(x) - f(x_0)| = |x(c) - x_0(c)| \leq \max_{a \leq t \leq b} |x(t) - x_0(t)| = \rho_\infty(x, x_0),$$

whence $\varepsilon = \delta$ suffices to show continuity. ▲

Exercise 1.2.3. Let $X = \ell_1 = \big\{\boldsymbol{x} = (x_1, x_2, \ldots, x_n, \ldots) \bigm| \sum_{j=1}^{\infty} |x_j| < \infty\big\}$, the space of all absolutely summable series, with $\rho_1(\boldsymbol{x}, \boldsymbol{y}) = \sum_{j=1}^{\infty} |x_j - y_j|$ as a metric.

Further let $T_+ : \ell_1 \to \ell_1$ be defined as

$$T_+(x_1, x_2, \ldots, x_n, \ldots) = (0, x_1, x_2, \ldots, x_n, \ldots),$$

and $T_- : \ell_1 \to \ell_1$ be defined as $T_-(x_1, x_2, \ldots, x_n, \ldots) = (x_2, x_3, \ldots, x_n, \ldots)$.

Is T_+ continuous? How about T_-?

Solution. We simply plug the known information into the definition of continuity, starting with T_+:

$$\rho_1(T_+(\boldsymbol{x}), T_+(\boldsymbol{x}_0)) = 0 + \sum_{j=1}^{\infty} \left|x_j - x_{0_j}\right| = \rho_1(\boldsymbol{x}, \boldsymbol{x}_0),$$

whence $\varepsilon = \delta$ suffices to show continuity. For T_-, on the other hand, we have

$$\rho_1(T_-(\boldsymbol{x}), T_-(\boldsymbol{x}_0)) = \sum_{j=2}^{\infty} \left|x_j - x_{0_j}\right| = -|x_1 - x_{0_1}| + \rho_1(\boldsymbol{x}, \boldsymbol{x}_0),$$

whence T_- is continuous *only* if $|x_1 - x_{0_1}| = 0$, and thereby not continuous in general. ♦

Example 1.2.4. The set

$$X = \ell_\infty = \big\{\boldsymbol{x} = (x_1, x_2, \ldots, x_n, \ldots) \,\big|\, \sup_{1 \le j < \infty} |x_j| < \infty\big\}$$

using $\rho_\infty(\boldsymbol{x}, \boldsymbol{y}) = \sup\limits_{1 \le j < \infty} |x_j - y_j|$ is a metric space. ▲

Example 1.2.5. The set

$$X = \ell_2 = \big\{\boldsymbol{x} = (x_1, x_2, \ldots, x_n, \ldots) \,|\, \sum_{j=1}^{\infty} x_j^2 < \infty\big\}$$

using

$$\rho_2(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{\sum_{j=1}^{\infty} (x_j - y_j)^2}$$

is a metric space. ▲

Exercise 1.2.6. Order ℓ_1, ℓ_2, and ℓ_∞ with respect to inclusion.

Solution. We think of it as follows: for ℓ_1, the sum of the *absolute value* of all of the elements in the sequence is finite. In ℓ_2, the sum of the *square* of all of the elements in the sequence is finite. Finally, in ℓ_∞, the *biggest* (kind of, at least the smallest upper bound) element in the sequence is finite.

It is then easy to see that ℓ_∞ is bigger than both ℓ_1 and ℓ_2 (consider, for example, $(1, 1, \ldots, 1, \ldots) \in \ell_\infty$).

Moreover, if $\sum\limits_{j=1}^{\infty} |x_j| \le \infty$, then we will also have $\sum\limits_{j=1}^{\infty} |x_j|^2 \le \infty$, (take, for example, the harmonic series), giving $\ell_1 \subseteq \ell_2$, but not the other way around.

Thus $\ell_1 \subset \ell_2 \subset \ell_\infty$. ♦

1.3 Complete Metric Spaces

Recall from single variable calculus the definitions of a Cauchy sequence and convergence of a series:

- $a_1, a_2, \ldots, a_n, \ldots \in \mathbb{R}$ is a *Cauchy sequence* is $|a_n - a_m| \to 0$ when $n, m \to \infty$,
- The sequence $\{a_n\}$ is called *convergent* with the *limit* a ($a_n \to a$ as $n \to \infty$) if $|a_n - a| \to 0$ as $n \to \infty$.

Note that, in $\mathbb{R}$, the above are equivalent.

Formulated in the terms of general metric spaces (X, ρ):

- The sequence $\{x_n\}$ is a ***Cauchy sequence*** if $\rho(x_n, x_m) \to 0$ as $n, m \to \infty$,
- The sequence $Setx_n$ ***converges*** to the ***limit*** x ($x_n \to x$) if, for all $\varepsilon > 0$, there exists an N_ε such that for all $n \ge N_varepsilon$ we have $\rho(x_n, x) \le \varepsilon$.

Definition 1.3.1 (Complete metric space)**.** A metric space (X, ρ) is called ***complete*** if each Cauchy sequence has its limit in X. Otherwise (X, ρ) is ***incomplete***.

Examples 1.3.2. $\mathbb{R}$ is complete using the absolute values. $\mathbb{R}^n$ is complete using all three of ρ_1, ρ_2, and ρ_∞. ▲

Example 1.3.3. Take $X = \mathbb{R}^n$ and ρ_∞ as the metric, and let

$$\boldsymbol{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)})$$

be a Cauchy sequence in X. We have that

$$\rho_\infty(\boldsymbol{x}^{(k)}, \boldsymbol{x}^{(m)}) = \max_{1\le j\le n}\left|x_j^{(k)} - x_j^{(m)}\right| \to 0$$

as $k, m \to \infty$. Further we take $x_j = \lim\limits_{k\to\infty} x_j^{(k)}$ and let $\varepsilon > 0$. Then there exists an N_ε such that if $n, m \ge N_\varepsilon$, we have

$$\lim_{m\to\infty}\left(\max_{1\le j\le n}\left|x_j^{(k)} - x_j^{(m)}\right|\right) \le \varepsilon,$$

then $\max\limits_{1\le j\le n}\left|x_j^{(k)} - x_j\right| \le \varepsilon$.

In other words, the limit of the n-tuple sequence is the n-tuple of the limits of the coordinates. ▲

Exercise 1.3.4. Show that ℓ_1, ℓ_2, and ℓ_∞ are complete.

Solution. We show it for ℓ_1. We take a sequence $\{\boldsymbol{x}^{(k)}\}$ of sequences

$$(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}, \ldots).$$

Since $\boldsymbol{x}^{(k)} \in \ell_1$, then $\sum\limits_{j=1}^{\infty}\left|x_j^{(k)}\right| < \infty$ for any fixed k, whence certainly $\rho(x_n^{(k)}, x_m^{(k)})$ must approach 0 as n and m increase (otherwise the series must necessarily converge). Therefore $\{\boldsymbol{x}^{(k)}\}$ is a Cauchy sequence.

Since $\{\boldsymbol{x}^{(k)}\}$ is a Cauchy sequence of real numbers, it must have some limit $\{\boldsymbol{x}\}$. Moreover, this limit is in ℓ_1 since

$$\sum_{j=1}^{\infty} x_j = \sum_{j=1}^{\infty}\lim_{k\to\infty} x_j^{(k)} = \lim_{k\to\infty}\sum_{j=1}^{\infty} x_j^{(k)} < \infty,$$

where switching the order of the limit processes is allowed since $\boldsymbol{x}^{(k)} \in \ell_1$ is absolutely convergent. ⧫

Definition 1.3.5 (Contration)**.** Let (X, ρ) be a metric space. Then the map $A : X \to X$ is called a ***contraction*** if there exists $0 < \alpha < 1$ such that $\rho(Ax, Ay) \le \alpha\rho(x, y)$ for all $x, y \in X$.

Remark 1.3.6. Any contraction is continuous. This is clear by the definition (let $\rho(x, y)$ be sufficiently small, then $\rho(Ax, Ay) \le \alpha\rho(x, y)$ is even smaller).

Definition 1.3.7 (Fixed point)**.** Let $A : X \to X$ be a mapping, then $x \in X$ is called a ***fixed point*** of A if $Ax = x$.

Theorem 1.3.8 ((Banach's) Fixed point theorem)**.** *Let (X, ρ) be a complete metric space and let A be a contraction. Then there exists a unique fixed point $x \in X$ such that $Ax = x$.*

Proof. The following is a constructive proof using the method of iteration.

Take any $x_0 \in X$ and let $x_1 = Ax_0$, $x_2 = Ax_1 = A^2x_0$, et cetera. In general, $x_n = Ax_{n-1} = A^n x_0$.

We claim that $\{x_n\}$ is a Cauchy sequence. Using $n \leq n'$, we show this as follows:

$$\begin{aligned}\rho(x_n, x_{n'}) = \rho(Ax_{n-1}, Ax_{n'-1}) \leq \alpha\rho(x_{n-1}, x_{n'-1}) \leq \\ \leq \ldots \leq \alpha^n \rho(x_0, x_{n'-n}) \leq \\ \leq \alpha^n \left(\rho(x_0, x_1) + \rho(x_1, x_2) + \ldots + \rho(x_{n'-n-2}, x_{n'-n-1})\right) \leq \\ \leq \alpha^n \left(\rho(x_0, x_1) + \alpha\rho(x_0, x_1) + \ldots + \alpha^{n'-n-1}\rho(x_0, x_1)\right) = \\ = \alpha^n \rho(x_0, x_1)\left(1 + \alpha + \alpha^2 + \ldots + \alpha^{n'-n-1}\right),\end{aligned}$$

where in the first inequality we use that A is a contraction and in the third line we use the triangle inequality. The last bracket of the last line is just a geometric sum, however, so

$$\rho(x_n, x_{n'}) \leq \alpha^n \frac{1 - \alpha^{n'-n}}{1 - \alpha} \rho(x_0, x_1) < \frac{\alpha^n}{1 - \alpha} \rho(x_0, x_1).$$

Since $0 < \alpha < 1$, α^n will approach 0 as n approaches infinity, whence $\rho(x_n, x_{n'})$ will approach 0 as n and n' become sufficiently big, meaning that $\{x_n\}$ is a Cauchy sequence, and therefore we have some $X \ni x = \lim_{n\to\infty} x_n$.

Then, since A is continuous (by Remark 1.3.6) we have

$$Ax = A\left(\lim_{n\to\infty} x_n\right) = \lim_{x\to\infty} Ax_n = \lim_{n\to\infty} = x,$$

from which we see that $Ax = x$, so x is a fixed point of A.

It still remains to show that x is the unique fixed point of A. To do this we simply assume that we have two fixed points x and y, such that $Ax = x$ and $Ay = y$. Then:

$$\rho(x, y) = \rho(Ax, Ay) \leq \alpha\rho(x, y).$$

Since $\rho(x, y) \geq 0$ and $0 < \alpha < 1$, for the above to hold we must have $\rho(x, y) = 0$, meaning that $x = y$ (since ρ is a metric). ☐

2 Lecture II[2] & III[3]

2.1 More On Contraction Mappings

The Fixed Point Theorem has the following generalisation which will prove very useful.

Theorem 2.1.1. *Let A be a continuous mapping of a complete metric space (X, ρ) into itself, and suppose that A^n is a contraction mapping (with n being a positive integer greater than 1). Then A has a unique fixed point.*

[2]Date: November 5, 2015. Kind of; it never took place.
[3]Date: November 9, 2015.

Proof. Pick any point $x_0 \in X$ and let

$$x = \lim_{k\to\infty} A^{kn} x_0,$$

(which exists and is unique due to the fixed point theorem since A^n is a contraction). Then, since A is a continuous mapping, we have

$$Ax = A \lim_{k\to\infty} A^{kn} x_0 = \lim_{k\to\infty} AA^{kn} x_0.$$

Since A^n is a contraction mapping, we have

$$\rho(A^{kn}Ax_0, A^{kn}x_0) \leq \alpha\rho(A^{(k-1)n}Ax_0, A^{(k-1)n}x_0) \leq \ldots \leq \alpha^k \rho(Ax_0, x_0),$$

where $0 < \alpha < 1$.

Since the right-hand side of the inequality is positive, and α^k tends to 0 as k tends to infinity, it follows that

$$\rho(Ax, x) = \lim_{k\to\infty} \rho(A^{kn}Ax_0, A^{kn}x_0) = 0,$$

whence $Ax = x$ and x is a fixed point of A. To show that this fixed point is unique we show that if any x is a fixed point of A, it must also be a fixed point of A^n.

Suppose that we have $Ax = x$. Then clearly we also have that $A^n x = A^{n-1}(Ax) = A^{n-1}x = \ldots = Ax = x$, whence x is a fixed point of A^n as well. However since A^n is a contraction is has exactly one fixed point, so A must as well. □

2.2 Applications of the Fixed Point Theorem

This method of successive approximations can be used to prove both the existence and uniqueness of solutions to integral equations. We will study two such equations: *Fredholm equations*, and *Volterra equations*. However to do this we first require the following result.

Theorem 2.2.1. *The metric space $C[a, b]$ with the metric*

$$\rho_\infty(f, g) = \max_{a\leq t\leq b} |f(t) - g(t)|$$

is complete.

Proof. To have $f_n \to f$ in ρ_∞ means that, for all $\varepsilon > 0$, there exists some N_ε such that for all $n \geq N_\varepsilon$ we have

$$\max_{a\leq x\leq b} |f(x) - f_n(x)| < \varepsilon$$

and for f_n to be a Cauchy sequence means that, for all $n, m \geq N_\varepsilon$, we have

$$|f_n(x) - f_m(x)| \leq \varepsilon.$$

We fix m and take the limit as n approaches infinity, which yields

$$\lim_{n\to\infty} |f_n(x) - f_m(x)| = |f(x) - f_m(x)| \leq \varepsilon$$

for all x, whence especially

$$\varepsilon \geq \max_{a \leq x \leq b} |f(x) - f_m(x)| = \rho_\infty(f, f_m).$$

That such a limit f must be continuous if f_n are continuous is known from previous courses. □

Example 2.2.2 (Fredholm equation)**.** A ***Fredholm equation*** (of the second kind) is an integral equation of the form

$$f(x) = \lambda \int_a^b K(x, y) f(y)\, dy + \varphi(x), \tag{2.2.1}$$

which involves two given functions K and φ, one unknown function f and an arbitrary parameter λ. The function K is called the ***kernel*** of the equation and the equation is said to be ***homogeneous*** if $\varphi \equiv 0$ (otherwise ***non-homogeneous***).

Suppose now that $K(x, y)$ and $\varphi(x)$ are continuous on the square $a \leq x \leq b$, $a \leq y \leq b$, which implies that $|K(x, y)| \leq M$ is bounded by some $M > 0$ in the square, since it is compact.

Consider the mapping $g = Af$ of the complete metric space $C[a, b]$ into itself given by

$$g(x) = \lambda \int_a^b K(x, y) f(y)\, dy + \varphi(x).$$

It is clear that, if $g_1 = Af_1$ and $g_2 = Af_2$, then

$$\begin{aligned}\rho_\infty(g_1, g_2) &= \max_{a \leq x \leq b} |g_1(x) - g_2(x)| \leq |\lambda| M(b - a) \max_{a \leq x \leq b} |f_1(x) - f_2(x)| \\ &= |\lambda| M(b - a) \rho_\infty(f_1, f_2),\end{aligned}$$

so that if

$$|\lambda| M(b - a) < 1 \quad \Longleftrightarrow |\lambda| < \frac{1}{M(b - a)}, \tag{2.2.2}$$

then A becomes a contraction.

Moreover, by the fixed point theorem, if (2.2.2) holds, the original Fredholm equation (2.2.1) must have a unique solution for the given λ.

More to the point, the successive approximations used to solve the equation are given by

$$f_n(x) = \lambda \int_a^b K(x, y) f_{n-1}(y)\, dy + \varphi(x),$$

for $n \geq 1$, and f_0 can be chosen as *any* continuous function on $[a, b]$. ▲

Exercise 2.2.3. Solve the Fredholm equation on the interval $[0, 1]$ with the kernel $K(x, y) = \sin^2(x - y)$, and $\varphi(x) = \cos x$.

Solution. See Assignment I. ♦

Next we consider the Volterra equation.

Example 2.2.4 (Volterra equation)**.** The ***Volterra equation*** is given by

$$f(x) = \lambda \int_a^x K(x,y) f(y)\, dy + \varphi(x) \tag{2.2.3}$$

which is identical to the Fredholm equation (2.2.1) apart from the upper limit of the integration interval being the variable x rather than the constant b.

We claim that the Volterra equation can be solved for any λ, not just λ sufficiently small, as with the Fredholm equation.

We let A be a mapping on $C[a,b]$ into itself, as before, defined by

$$Af(x) = \lambda \int_a^x K(x,y) f(y)\, dy + \varphi(x),$$

and take any $f_1, f_2 \in C[a,b]$. Then we have

$$\begin{aligned} |Af_1(x) - Af_2(x)| &= \lambda \int_a^x K(x,y)(f_1(y) - f_2(y))\, dy \\ &\le \lambda M(x-a) \max_{a \le x \le b} |f_1(x) - f_2(x)|, \end{aligned}$$

where, as before, $M = \max\limits_{a \le x, y \le b} |K(x,y)|$.

From this it follows that

$$\begin{aligned} \left|A^2 f_1(x) - A^2 f_2(x)\right| &\le \lambda^2 M^2 \max_{a \le x \le b} |f_1(x) - f_2(x)| \int_a^x (x-a)\, dx \\ &= \lambda^2 M^2 \frac{(x-a)^2}{2} \max_{a \le x \le b} |f_1(x) - f_2(x)|, \end{aligned}$$

whence in general

$$\begin{aligned} |A^n f_1(x) - A^n f_2(x)| &\le \lambda^n M^n \frac{(x-a)^n}{n!} \max_{a \le x \le b} |f_1(x) - f_2(x)| \\ &\le \lambda^n M^n \frac{(b-a)^n}{n!} \max_{a \le x \le b} |f_1(x) - f_2(x)|, \end{aligned}$$

since $a \le x \le b$. This implies that

$$\rho(A^n f_1, A^n f_2) \le \lambda^n M^n \frac{(b-a)^n}{n!} \rho(f_1, f_2),$$

in which, regardless of λ, we can always make

$$\lambda^n M^n \frac{(b-a)^n}{n!} < 1$$

by choosing n sufficiently large. Therefore it follows from Theorem 2.1.1 that, since A^n (for n sufficiently large) is a contraction, A must have a unique fixed point f. ▲

This method of iterations is also useful when solving other type of fixed point problems, not only integral solutions.

Example 2.2.5 (Lipschitz functions and Lipschitz continuity)**.** ***Lipschitz functions*** are functions f that satisfy $|f(x) - f(y)| \leq L|x - y|$ for all x and y, where L is the so-called ***Lipschitz constant*** (for f).

It is obvious that these functions are continuous by definition, whence we formulate a sufficient condition for a function in $C^{(1)}[a, b]$ (the space of once continuously differentiable functions) to be Lipschitz.

For such functions we have, by the mean value theorem, that

$$f(x) - f(y) = f'(c)(x - y)$$

for some $c \in [a, b]$. Taking absolute values we therefore get

$$|f(x) - f(y)| = L|x - y|,$$

where $L = \max_{a \leq c \leq b} |f'(c)|$.

We see then that if we interpret f as a map between two complete metric spaces (which it is; from $C[a, b]$ into itself), we have that f almost satisfies the condition of a contraction; it only remains to require that $0 < L < 1$.

It then follows from the fixed point theorem that $f(x) = x$ has a unique solution x, and that it can be found numerically by iterating $x_n = f(x_{n-1})$, with x_0 as any initial value in $[a, b]$. ▲

Exercise 2.2.6. Select a continuous function f on the interval $[a, b]$ such that its Lipschitz constant L is less than 1. Solve, for this f, the equation $f(x) = x$ by the method of iteration.

Solution. See Assignment II. ⧫

2.3 Normed Linear Spaces

Recall first some basics from linear algebra regarding the definition of a space being linear (meaning that we have two operators, addition of vectors and multiplication by scalars, along with an assortment of properties of these).

We leave it as written that $\mathbb{R}^n$, $C[a, b]$, ℓ_1, ℓ_2, and ℓ_∞ are all examples of linear spaces.

Definition 2.3.1 (Norm)**.** A map f, usually denoted $\|\cdot\|$, from a space E to $\mathbb{R}$ is a ***norm*** if it has the following properties:

(*i*) $\|\boldsymbol{x}\| \geq 0$ for all $\boldsymbol{x} \in E$,

(*ii*) $\|\boldsymbol{x}\| = 0$ if and only if $\boldsymbol{x} = 0$,

(*iii*) $\|\lambda\boldsymbol{x}\| = |\lambda|\|\boldsymbol{x}\|$, for all $\boldsymbol{x} \in E$ and real (or complex) constants λ,

(*iv*) Triangle inequality: $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$ for all $\boldsymbol{x}, \boldsymbol{y} \in E$.

A space E equipped with a norm is called a ***normed space***.

Any norm also defines a metric by

$$\rho(x, y) = \|x - y\|,$$

which clearly satisfies the conditions for a metric (see Definition 1.1.2), whereby any normed space is also a metric space. The converse of this is not necessarily true.

Example 2.3.2. All of $\mathbb{R}^n$, $C[a,b]$, ℓ_1, ℓ_2, and ℓ_∞ are normed linear spaces, with norms defined analogously to their metrics. ▲

A very important class of normed spaces are the following. -

Definition 2.3.3 (Banach space)**.** A normed space that is also complete is called a ***Banach space***.

Example 2.3.4. Take $E_1 = C^{(1)}[a,b]$ and $E_2 = C[a,b]$. Then $\|f\|^{(1)} = \|f\|_\infty + \|f'\|_\infty$ is a norm. This can be generalised to $E_1 = C^{(n)}[a,b]$, where we get

$$\|f\|^{(n)} = \|f\|_\infty + \|f'\|_\infty + \ldots + \left\|f^{(n-1)}\right\|_\infty + \left\|f^{(n)}\right\|_\infty = \sum_{j=0}^{n} \left\|f^{(j)}\right\|_\infty. \quad ▲$$

Definition 2.3.5 (Bounded linear operator)**.** A linear operator $A : E_1 \to E_2$ is called ***bounded*** if there exists some $c > 0$ such that $\|Ax\|_2 \leq c\|x\|_1$ for all $x \in E_1$, where $\|\cdot\|_1$ is the norm in E_1 and $\|\cdot\|_2$ is the norm in E_2.

Exercise 2.3.6. Check whether the maps for the Fredholm and Volterra equations are bounded linear mappings. In addition, verify that the differential operator

$$p\Big(\frac{d}{dx}\Big),$$

where

$$p(\xi) = \sum_{j=1}^{n} a_j \xi^j$$

is a polynomial of degree n is a bounded linear operator.

Solution. See Assignment II. ♦

4 Lecture IV[4]

Recall from last lecture: E is a linear normed space, A is a linear bounded operator, then A is bounded if there exists a c such that $\|Ax\| \leq c\|x\|$ for all x. Recall also the definitions of ball and sphere.

4.1 Norm of Bounded Operator

Definition 4.1.1 (Norm of bounded operator)**.** Let A be a linear operator on a linear normed space E, then

$$\|A\| = \sup_{\|x\| \leq 1} \|Ax\|$$

is called the ***norm*** of A.

Remark 4.1.2. Note that $\|A\| \leq c$, whereby we also have $\|A\| = \inf\{c \mid \|Ax\| \leq c\|x\|\}$.

There are different, and often in applications more useful, ways of representing the norm of an operator.

[4]Date: 12 November, 2015.

Theorem 4.1.3. *Let A be a linear bounded operator on a linear normed space E. Then*

(*i*) $\|A\| = \sup_{\|x\|=1} \|Ax\|$,

(*ii*) $\|A\| = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|}$.

Proof. (*i*) Clearly we have $\sup_{\|x\|=1} \|Ax\| \leq \sup_{\|x\|\leq 1} \|Ax\|$, whence it remains to show the opposite inequality.

To do this, take any x such that $\|x\| < 1$. Then, since we have

$$\left\| \frac{x}{\|x\|} \right\| = \|\lambda x\| = |\lambda| \|x\| = \frac{1}{\|x\|} \|x\| = 1$$

we have that $\frac{x}{\|x\|}$ belongs to the unit sphere, whence

$$\|Ax\| = \|x\| \left\| A \frac{x}{\|x\|} \right\| \leq \left\| A \frac{x}{\|x\|} \right\|.$$

By taking supremum we are left with

$$\sup_{\|x\|\leq 1} \|Ax\| \leq \sup_{\|x\|\leq 1} \left\| A \frac{x}{\|x\|} \right\| \leq \sup_{\|y\|=1} \|Ay\|.$$

(*ii*) We have

$$\sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{x\neq 0} \left\| A \frac{x}{\|x\|} \right\| = \sup_{\|y\|=1} \|Ay\|. \qquad \square$$

Why are we concerned with this? Because given Remark 4.1.2 we must have the inequality $\|Ax\| \leq \|A\| \|x\|$. Upper bounds for this are often easy to find, but equalities can prove trickier; this is where the alternate representations above come into use.

Example 4.1.4. Let $E = \ell_\infty = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n, \ldots) \mid \|\boldsymbol{x}\|_\infty = \sup_{1\leq j<\infty} |x_j| < \infty\}$, ans also let $T_- \boldsymbol{x} = (x_2, x_3, \ldots, x_n, \ldots)$ as previously. What is the norm of T_-?

We compute

$$\|T_- \boldsymbol{x}\|_\infty = \sup_{2\leq j<\infty} |x_j|$$

and

$$\|\boldsymbol{x}\|_\infty = \sup_{1\leq j<\infty} |x_j|,$$

whence clearly $\|T_-\|_\infty \leq 1 = c$. Is it equal to $c = 1$? If we can find an $\boldsymbol{x}$ such that $\|\boldsymbol{x}\|_\infty = 1$ whilst $\|T_- \boldsymbol{x}\|_\infty = c$, we will have shown this to be the case, by Theorem 4.1.3 (*i*). One example of such an $\boldsymbol{x}$ is $\boldsymbol{x} = (1, 1, 1, \ldots)$. ▲

Exercise 4.1.5. Find the norm of T_- in ℓ_1 and ℓ_2 as well. Also find the norm of T_+ in ℓ_1, ℓ_2, and ℓ_∞.

Solution. See Assignment III. ⧫

For a more interesting example, consider the following.

Example 4.1.6. Let $E = C[a,b]$, and let $A : E \to \mathbb{R}$ be defined as $A\varphi = \varphi(t_0)$, for some fixed $t_0 \in [a,b]$.

We have then that $\|A\varphi\| = |\varphi(t_0)|$, whereas

$$\|\varphi\|_\infty = \max_{a \le t \le b} |\varphi(t)|$$

whereby obviously $\|A\|$ is bounded by 1, since the norm of the image is the value in a point, and the norm of the function is the maximum of the values in *all* points. Is $\|A\| = 1$? To show this we try to find a φ_0 such that the maximum of φ in $[a,b]$ is 1, and that the value of φ in t_0 is 1 as well. Clearly the trivially simple $\varphi_0(t) = 1$ will do just fine. ▲

Remark 4.1.7. Note that, since sup isn't max, sometimes it might be necessary to study a sequence of x_n such that $\|x_n\| \to 1$ and $\|Ax_n\| \to c$ as $n \to \infty$. See Example 4.2.5.

Theorem 4.1.8 (Properties of operator norm)**.** *Let A and B be bounded linear operators. Then the following hold:*

- (i) $\|A\| \ge 0$,
- (ii) $\|A\| = 0$ *if and only if* $A = 0$,
- (iii) $\|\lambda A\| = |\lambda|\|A\|$,
- (iv) $\|A + B\| \le \|A\| + \|B\|$.

Proof. These are all basic manipulation either of the definition or one of the derived representations from Theorem 4.1.3.

(i) and (ii) are obvious.

(iii) Use Theorem 4.1.3 (ii):

$$\|\lambda A\| = \sup_{x \ne 0} \frac{\|\lambda Ax\|}{\|Ax\|} = |\lambda| \sup_{x \ne 0} \frac{\|Ax\|}{\|Ax\|} = |\lambda|\|A\|.$$

(iv): Use the fact that norms satisfy the triangle inequality:

$$\|A + B\| = \sup_{\|x\| \le 1} \|Ax + Bx\| \le \sup_{\|x\| \le 1} \|Ax\| + \sup_{\|x\| \le 1} \|Bx\| = \|A\| + \|B\|. \qquad \square$$

We note that these properties are exactly those of an ordinary norm, whence the space $\mathscr{L}(E_1, E_2)$ of all bounded operators $A : E_1 \to E_2$ is a linear normed space.

Theorem 4.1.9. *Let E_2 be a Banach space*[5]*, then $\mathscr{L}(E_1, E_2)$ is a Banach space as well.*

Proof. Let (A_n) be a Cauchy sequence in $\mathscr{L}(E_1, E_2)$, meaning that $\|A_n - A_m\| \to 0$ as $m, n \to \infty$. Therefore, for all $x \in E_1$, $\|A_n x - A_m x\| \le \|A_n - A_m\|\|x\| \to 0$ as $m, n \to \infty$.

[5] A complete normed space

Therefore $(A_n x)$ is a Cauchy sequence in E_2. However since E_2 is complete, this Cauchy sequence must have a limit, so there exists some

$$\lim_{n\to\infty} A_n x = Ax.$$

Moreover, since the limit is linear, $A : E_1 \to E_2$ must also be linear.

Consider

$$\|A_n x\| \leq \|A_n x A_1 x\| + \|A_1 x\| \leq \|A_n - A_1\|\|x\| + \|A_1\|\|x\|.$$

Let n go to infinity and we get

$$\lim_{n\to\infty} \|A_n x\| = Ax \leq c\|x\|$$

for some c, whence A is also bounded.

Finally consider the following: for all $\varepsilon > 0$ there is some N_ε such that for all $n, m \geq N_\varepsilon$ we have

$$\|A_n x - A_m x\| \leq \varepsilon$$

(since A_n is a Cauchy sequence). We fix n and let $m \to \infty$:

$$\|A_n x - Ax\| = \|(A_n - A)x\| \leq varepsilon.$$

Now take the supremum and we get $\|A_n - A\| \leq \varepsilon$, whence the limit of the Cauchy sequence exists and the space is therefore complete. □

In other words, it is enough for the image space to be Banach for the space of all linear bounded operators into this space to be Banach as well.

Theorem 4.1.10. *Let E_2 be Banach, let $A_n \in \mathscr{L}(E_1, E_2)$ be bounded, and let $\sum_{n=1}^{\infty} \|A_n\| < \infty$. Then $\sum_{n=1}^{\infty} A_n$ converges in $\mathscr{L}(E_1, E_2)$ and its sum is in $\mathscr{L}(E_1, E_2)$.*

Proof. This is a simple application of the triangle inequality on partial sums. Let $N > M$:

$$\left\|\sum_{n=1}^{N} A_n - \sum_{n=1}^{M} A_n\right\| = \left\|\sum_{n=M+1}^{N} A_n\right\| \leq \sum_{n=M+1}^{N} \|A_n\|,$$

which approaches 0 if we let $M, N \to \infty$. □

4.2 Analytic Functions of Bounded Operators

To explore this we must first discuss compositions of bounded operators.

Theorem 4.2.1 (Composition of bounded operators)**.** *Let E be Banach and let $A, B \in \mathscr{L}(E, E)$. Then $\|AB\| \leq \|A\|\|B\|$.*

Proof. We use the definition of operator norm:

$$\|AB\| = \sup_{\|x\|\leq 1} \|A(Bx)\| = \sup_{normx \leq 1} \|Bx\| \left\| A \frac{Bx}{\|Bx\|} \right\|,$$

where the fraction in the last norm is in the unit sphere, whence

$$\|AB\| \leq \left(\sup_{\|x\|\leq 1} \|Bx\| \right) \left(\sup_{\|x\|\leq 1} \|Ax\| \right) = \|A\|\|B\|.$$ □

Example 4.2.2. Consider the analytic function $e^x = \sum\limits_{n=0}^{\infty} \frac{x_n}{n!}$, for $x \in \mathbb{R}$.

Let $A \in \mathscr{L}(E, E)$, where E is Banach, and consider

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \leq \sum_{n=0}^{\infty} \frac{\|A\|^n}{n!} = e^{\|A\|} < \infty,$$

so e^A is a well-defined bounded operator. ▲

Using similar arguments one may define $\sin A$, $\cos A$ and $\ln(1+A)$. We construct the latter.

Example 4.2.3. We have from calculus that

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

which converges for $|x| < 1$. Thus we also have that $\ln(1+A)$ converges if $\|A\| < 1$. ▲

In general we have that if f is an analytic function, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,$$

which converges for some $|x| \leq R$, then

$$f(A) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} A^n$$

converges for the same $\|A\| \leq R$.

Recall the differential operator from Exercise 2.3.6,

$$p\Big(\frac{d}{dx}\Big) : C^{(m)}[a, b] \to C[a, b].$$

We would prefer it if we had the same domain and codomain, as we have in all of the previous theorems. One obvious candidate for this would be $E = C^{\infty}[a, b]$, but there is no norm on this space (since it'd be the sum of infinitely many derivatives, which needn't converge) so it is impossible to make this a linear normed space (although it is a linear topological space; more on this later).

The reason this would be of use is that it would help us solve certain differential equations, which is one of the main goals of functional analysis.

Example 4.2.4. Suppose that $E = C^{\infty}$. Then, if for all t we have $\varphi(t) \in C^{\infty}$, it follows that $\varphi(t) = \varphi(t, x)$. Let $A = p(\frac{\partial}{\partial x})$. Then

$$\frac{\partial \varphi}{\partial t}(t, x) = p\Big(\frac{\partial}{\partial x}\Big)\varphi(t, x) = \sum_{k=0}^{n} a_k \frac{\partial^k \varphi}{\partial x^k}(t, x).$$ ▲

Example 4.2.5. Let $A = \frac{d}{dx} : C^{(1)}[0, 1] \to C[0, 1]$. Then

$$\|A\varphi\|_{\infty} = \sup_{0 \leq t \leq 1} |\varphi'(t)| \leq \sup_{0 \leq t \leq 1} |\varphi(t)| + \sup_{0 \leq t \leq 1} |\varphi'(t)| = \|\varphi\|^{(1)},$$

so clearly $\|A\| \leq 1$. We would like to show that it is indeed equal to 1, but clearly we can't have $\|\varphi\|^{(1)} = 1$ and

$$\|A\varphi\|_\infty = \|\varphi'\|_\infty = 1$$

simultaneously since

$$\|\varphi\|^{(1)} = \|\varphi\|_\infty + \|\varphi'\|_\infty,$$

meaning that $\|\varphi\|_\infty = 0$.

So instead we look for a sequence φ_δ such that $\|\varphi_\delta\|_\infty \approx 0$ whilst $\|\varphi_\delta'\|_\infty \approx 1$. Consider for example $f_\delta(t) = \sqrt{\delta + t}$. Clearly the maximum of this is $\sqrt{1+\delta}$. Moreover

$$f_\delta'(t) = \frac{1}{2\sqrt{\delta + t}}$$

the maximum of which is $1/(2\delta^{1/2})$.

We want

$$\frac{\sqrt{\delta_1}}{1/(2\delta^{1/2})} = \sqrt{\delta + 1} \cdot 2\delta^{1/2}$$

to be small, which it is if δ is much smaller than 1 (i.e. δ approaches 0).

Therefore we consider the sequence of functions

$$\varphi_\delta(t) = \frac{f_\delta(t)}{\|f_\delta\|^{(1)}}.$$

This gives us

$$\begin{aligned}\|\varphi_\delta\|_\infty &= \frac{1}{\|f_\delta\|^{(1)}}\|f_\delta\|_\infty = \frac{\|f_\delta\|_\infty}{\|f_\delta\|_\infty + \|f_\delta'\|_\infty} \\ &= \frac{\|f_\delta\|_\infty/\|f_\delta'\|_\infty}{1 + \|f_\delta\|_\infty/\|f_\delta'\|_\infty} = \frac{2\sqrt{1+\delta}\cdot\delta^{1/2}}{1 + 2\sqrt{1+\delta}\cdot\delta^{1/2}}\end{aligned}$$

which approaches 0 as δ approaches 0, whilst $\|\varphi_\delta\|^{(1)}$ approaches 1. Therefore

$$\sup_\delta \|A\varphi_\delta\|_\infty = 1.$$ ▲

Exercise 4.2.6. Let $A = \frac{d^2}{dx^2} : C^{(2)}[0,1] \to C[0,1]$. Find $\|A\|$.

Solution. See Assignment III. ♦

5 Lecture V[6]

5.1 More On Bounded Operators

If, in $\mathscr{L}(E_1, E_2)$ from last lecture, $E_1 = E_2 = E$, we will denote this simply $\mathscr{L}(E)$. Since this is the space of all linear bounded operators from E to itself, this is of course a linear space. Moreover, it has a multiplication operation; composition of operators. In other words, for $A, B \in \mathscr{L}(E)$, $AB = A \circ B$.

[6]Date: November 16, 2015.

Moreover, this composition is distributive over addition of operators: $A(B_1 + B_2) = AB_1 + AB_2$.

An algebraic structure with a linear space and a multiplication operation is called an ***algebra***.

Additionally, if E is Banach, then so is $\mathscr{L}(E)$, by Theorem 4.1.10. This multiplication then also satisfies $\|AB\| \leq \|A\|\|B\|$, by Thorem 4.2.1.

We have the following important theorem.

Theorem 5.1.1 (Bounded linear operators are continuous, and vice versa)**.** *Let E_1 and E_2 be linear normed spaces. Then $A : E_1 \to E_2$ being linear and bounded ($A \in \mathscr{L}(E_1, E_2)$) is equivalent to A being linear and continuous.*

Proof. ($\Rightarrow$) Suppose that $A \in \mathscr{L}(E_1, E_2)$. Then by its linearity and, for the final inequality, boundedness, we have

$$\|Ax_1 - Ax_2\| = \|A(x_1 - x_2)\| \leq \|A\|\|x_1 - x_2\|$$

which clearly approaches 0 as $\|x_1 - x_2\|$ approaches 0. Ergo boundedness of a linear operator implies continuity.

($\Leftarrow$) Suppose now that $A : E_1 \to E_2$ is continuous. Further suppose that A is *not* bounded, meaning that

$$\|A\| = \sup_{\|x\| \leq 1} \|Ax\| = \infty.$$

In other words, if we pick x from the unit ball $\bar{B}_1(0) = \{x \mid \|x\| \leq 1\}$, we would like to show that Ax isn't bounded in E_2.

In particular there must exist some sequence $x_n \in \bar{B}_1(0)$ such that $\|Ax_n\| > n$ for all positive integers n.

Dividing this last inequality by n (which is positive and so doesn't change the inequality) we have

$$\left\|A\frac{x_n}{n}\right\| > 1,$$

where

$$\left\|\frac{x_n}{n}\right\| = \frac{\|x_n\|}{n} \leq \frac{1}{n},$$

since x_n come from the unit ball, making their norm between 0 and 1.

We thus have that

$$\left\|A\frac{x_n}{n}\right\| > 1$$

for all n, but also that

$$\frac{x_n}{n}$$

approaches 0 as n approaches infinity, whence we must approach $A0 = 0$, which is a contradiction. Therefore linear operators being continuous implies that they are bounded. □

5.2 Continuous Linear Functionals

We have shown that bounded linear operators and continuous linear operators are equivalent, and in the interest of staying consistent with the literature we will henceforth speak of the latter.

Definition 5.2.1 (Continuous Linear Functionals)**.** Let E be a normed linear space. Then operators $A : E \to \mathbb{R}$ (or, alternatively, $A : E \to \mathbb{C}$) are called ***linear functionals***.

The space of all such linear functionals from E, $\mathscr{L}(E, \mathbb{R})$, is denoted E', which is called the ***dual space*** to E.

Typically we will use $x \in E$ and y for continuous linear functionals.

Note that E' is a normed linear space, and that since $\mathbb{R}$ is Banach, so is E'.

With this we are almost ready to attack one of the main results of functional analysis, however first we require the following definition.

Definition 5.2.2. Let E be a linear space. A function $p : E \to \mathbb{R}$ is called a ***semi-norm*** if, for all $x, y \in E$ and $\lambda \in \mathbb{R}$,

(*i*) $p(x) \geq 0$,

(*ii*) $p(\lambda x) = |\lambda| p(x)$,

(*iii*) $p(x + y) \leq p(x) + p(y)$.

Note the lack of $p(x) = 0 \Leftrightarrow x = 0$, distinguishing the semi-norm from norm.

Example 5.2.3. Consider $E = C(-\infty, \infty)$, the space of all functions continuous on the entire real line. Let $[a, b]$, $-\infty < a < b < \infty$ be some interval on this line, with the semi-norm of a function defined as the maximum of the function on this interval. Then any function that is zero on this interval has semi-norm zero, despite not necessarily being the zero function. ▲

5.3 Hahn–Banach Theorem

Theorem 5.3.1 (Hahn–Banach theorem)**.** *Let L be a linear space and let p be a semi-norm on L. Further let L_0 be a subspace of L, $L_0 \subset L$, and suppose that we have some linear functional $f_0 : L_0 \to \mathbb{R}$ such that $f_0(x) \leq p(x)$ for all $x \in L_0$.*

Then it is possible to extend f_0 onto L such that $f : L \to \mathbb{R}$, with $f|_{L_0} = f_0$, and $f(x) \leq p(x)$ for all $x \in L$.

Proof. We have $L_0 \subset L$, and we assume that $L_0 \neq L$ (otherwise we are done). We take some $z \in L \setminus L_0$, and define using this z

$$L_1 = \{x + tz \mid t \in \mathbb{R}, x \in L_0\}.$$

Using this we attempt to construct a linear functional $f_1 : L_1 \to \mathbb{R}$:

$$f_1(x + tz) = f_0(x) + tf(z)$$

such that

$$f_1(x + tz) \leq p(x + tz).$$

Note that since z is fixed, $f_1(z) = c$ for some fixed constant $c \in \mathbb{R}$.

We thus have

$$f_0(x) + tc \leq p(x + tz). \tag{5.3.1}$$

We consider two scenarios: (*i*) t being positive, and (*ii*) t being negative.

(i) If t is positive, we have

$$f_0\Big(\frac{x}{t}\Big) + c \leq p\Big(\frac{x}{t} + z\Big),$$

whence

$$c \leq -f_0\Big(\frac{x}{t}\Big) + p\Big(\frac{x}{t} + z\Big),$$

where for future reference we'll refer to x/t, $t > 0$, as y'.

(ii) If t is negative, we have instead that

$$\frac{f_0(x)}{t} \geq \frac{p(x+tz}{t},$$

which if we multiply numerator and denominator in the right-hand side by -1 becomes

$$f_0\Big(\frac{x}{t}\Big) + c \geq -p\Big(-\frac{x}{t} - z\Big),$$

giving

$$c \geq -f_0\Big(\frac{x}{t}\Big) - p\Big(-\frac{x}{t} - z\Big),$$

in which we label x/t for $t < 0$ as y''.

Therefore, since y' and y'' are simply scaled versions of x, which came from L_0, we have $y', y'' \in L_0$ as well, which gives us that

$$-f_0(y'') - p(-y'' - z) \leq -f_0(y') + p(y' + z).$$

We solve this for f_0:

$$f_0(y') - f_0(y'') \leq p(y' + z) + p(-y'' - z),$$

in which we rewrite the left-hand side, recalling the triangle inequality for semi-norms:

$$f_0(y' - y'') \leq p(y' - y'') = p(y' + z - z - y'') \leq p(y' + z) + p(-y'' - z),$$

whence the requisite inequality for f's existence holds since there exists some $c \in \mathbb{R}$ so that (5.3.1) holds.

Using this we can thus construct (L_1, f_1) such that the extension f_1 has the properties required. Repeating this on $L_2 = L \setminus L_1$ we can then construct (L_2, f_2), (L_3, f_3), et cetera, up to (L_n, f_n) for any positive integer n.

This also generalises nicely even if we need to do it countably infinitely many times.

How about uncountably infinitely many times? For this we require some set theoretic notions, and so an interlude. . . .

Axiom 5.3.2 (Axiom of choice)**.** Suppose that we have *finitely* many sets $A_1, A_2, \ldots, A_n$. On these we can define a so-called "choice function" f such that $f(A_j) \in A_j$ for all j. Naturally this works also for countably many sets.

Whether this should work even for *uncountably* is not obvious, and in fact is an axiom (read: choice) made in axiomatic set theory; one *decides* whether one accepts it. This is called the ***Axiom of Choice***.

That is, let M be any set. If we accept the axiom of choice, there exists a choice function f such that, for all $A \subset M$, $f(A) \in A$.

An equivalent statement is Zorn's lemma, however to discuss that we first need the following definitions.

Definition 5.3.3 (Partial order, linear order)**.** A set M is called ***partially ordered*** if between *some* pairs $(a, b) \in M \times M$ there can be established a relation $\leq$ such that it satisfies

(i) Reflexivity; $a \leq a$,

(ii) Transitivity; $a \leq b$ and $b \leq c$ implies $a \leq c$,

(iii) Anti-symmetry; $a \leq b$ and $b \leq a$ implies $a = b$.

Such a relation $\leq$ is called a ***partial order relation***.

If the relation $\leq$ holds between *all* pairs $(a, b) \in M \times M$, $\leq$ is called a ***linear order relation***.

Examples 5.3.4. The real numbers $\mathbb{R}$ under the ordinary $\leq$ is a linear order.

The space of continuous functions on an interval, $C[a, b]$, with $\varphi_1 \leq \varphi_2$ if, for all $t \in [a, b]$, $\varphi_1(t) \leq \varphi_2(t)$, is a partial order. ▲

Definition 5.3.5 (Chain)**.** Let M be a partially ordered set. Any linearly ordered subset A of M is called a ***chain***.

Definition 5.3.6 (Supremum)**.** Let M be a partially ordered set, and let $M' \subset M$. Then the smallest $m \in M$ such that $m' \in M'$ we have $m' \leq m$ is called the ***supremum*** with respect to M', denoted $\sup_{M'} = m$.

Lemma 5.3.7 (Zorn's lemma)**.** *Let M be a partially ordered set. If* any *chain in M has a supremum, then M has a maximal ordered element.*

With this we are ready to continue with the proof of the Hahn–Banach theorem.

Proof continued. We have the following partial order on M, with M being pairs of subspaces of L and a function f as defined earlier: Let (L', f') and (L'', f'') be two such a pairs of subsets of L and a functions as described earlier. Then $(L', f') \leq (L'', f'')$ if and only if $L' < L''$ and $f''|_{L'} = f'$.

Consider under this partial order a chain (L_c, f_c), i.e. a linearly ordered subuset of L, and call the collection of all such $M' = \{(L_c, f_c)\}$.

We then take $\tilde{L}$ to be the union of all such L_c, with $\tilde{f}|_{L_c} = f_c$ (which is possible since (L_c, f_c) are all chains. Then $(\tilde{L}, \tilde{f}) \geq (L_c, f_c)$ for all $(L_c, f_c) \in M'$, whence it is the supremum.

Thereby, by Zorn's lemma, there exists a maximal element in M. We claim that the previously described $(\tilde{L}, \tilde{f})$ is this maximal element.

That $\tilde{f}$ satisfies the criteria is clear. What about $\tilde{L} = L$? We see that if this is not the case, i.e. that $\tilde{L} \neq L$, we can use the previous construction by taking $z \in L \setminus \tilde{L}$ and constructing $\tilde{L}'$ using $\tilde{L}$ and this z, whence $\tilde{L}$ wasn't maximal, giving us a contradiction. □

6 Lecture VI[7]

Recall from last time the Hahn–Banach theorem which states that, if $L_0 \subset L$ is a linear space and p is a semi-norm on L, with $f_0 : L_0 \to \mathbb{R}$ a linear functional on L_0 such that $f_0(x) \le p(x)$ for all $x \in L_0$, then if we assume the axiom of choice we have that f_0 can be extended to $f : L \to \mathbb{R}$, where f is a linear functional on L such that $f(x) \le p(x)$ for all $x \in L$ and $f|_{L_0} = f_0$.

We now discuss one of the main uses of the theorem.

6.1 A Special Case of the Hahn–Banach Theorem

Let E be a linear normed space. Then, since $p(x) = c\|x\|$, for some constant c, is of course a semi-norm, we have that $f_0(x) \le c\|x\|$ for all $x \in L_0$ implies that $f(x) \le c\|x\|$ for all $x \in E$. Moreover $-f_0(x) \le |f(x)| \le c\|x\|$ for all $x \in E$.

Thus finally $|f_0(x)| \le \|f_0\|_{L_0}\|x\|$, which implies that $|f(x)| \le \|f\|_L\|x\|$, and additionally $\|f_0\|_{L_0} = \|f\|_L$.

Notation 6.1.1. Let E be a linear normed space and let $x \in E$ and $f \in E'$. Then $x \mapsto f(x) = \langle x, f\rangle$ is the so-called ***duality form***, commonly used in physics and distribution theory. I.e. in general $\langle \cdot, \cdot \rangle : E \times E' \to \mathbb{R}$.

This is useful because when E is normed, so is E', and thus so is $(E')'$, et cetera. But in fact often we will have $E = E'$, or, sometimes, $E, E', E'' = E'$, which we call a ***reflexive space***.

6.2 Adjoint Operator

Definition 6.2.1 (Adjoint operator)**.** Let E_1 and E_2 be linear normed spaces, and let $A \in \mathscr{L}(E_1, E_2)$. Then whilst $A : E_1 \to E_2$, its ***adjoint operator*** A^* does $E_2' \to E_1'$. In other words, we have a picture like

$$E_1 \xrightarrow{A} E_2$$
$$E_1' \xleftarrow[A^*]{} E_2'.$$

In the new duality notation we have, by definition, $\langle Ax, f\rangle = \langle x, A^*f\rangle$, where, if we pay attention, we note that $x \in E_1$, $Ax \in E_2$, $f \in E_2'$, and $A^*f \in E_1'$.

So for each $f \in E_2'$ we have some functional on E_1 such that $x \mapsto \langle x, A^*f\rangle = \langle Ax, f\rangle$, so $x : E_1 \to \mathbb{R}$.

We ask a few questions:

1. Is x linear? Yes, since A and f are linear, and $\langle x, A^*f\rangle = f(Ax)$ is a composition of two linear maps.

2. Is it continuous? Yes, again because A and f are continuous and x is simply their composition.

Therefore $f \mapsto A^*f$ is really from $E_2' \to E_1'$.

[7] Date: November 19, 2015.

We estimate its norm:

$$\begin{aligned}\|A^*\| &= \sup_{\|f\|_{E_2'}\le 1} \|A^* f\|_{E_1'} = \sup_{\|f\|_{E_2'}\le 1} \sup_{\|x\|_{E_1}\le 1} |\underbrace{\langle x, A^* f\rangle}_{(A^*f)(x)}| \\ &= \sup_{\|f\|_{E_2'}\le 1} \sup_{\|x\|_{E_1}\le 1} |\langle Ax, f\rangle|,\end{aligned}$$

where Ax is a functional on E_2', whence $\langle Ax, f\rangle \in E_2''$, so

$$\|A^*\| = \sup_{\|x\|_{E_1}\le 1} \sup_{\|f\|_{E_2'}\le 1} |\langle Ax, f\rangle| = \sup_{\|x\|_{E_1}\le 1} \|Ax\|_{E_2} = \|A\|,$$

so we have that $\|A^*\| = \|A\|$.

There is a striking resemblance between adjoint operators and the transpose of matrices, as we demonstrate in the following example.

Example 6.2.2. Consider an operator $A : \mathbb{R}^n \to \mathbb{R}^m$.

First of all we claim that $(\mathbb{R}^n)' = \mathbb{R}^n$. To show this, let $\boldsymbol{e}_j = (0, \dots, 1, \dots, 0)$ be the elements of a base in $\mathbb{R}^n$ (with the 1 of course occurring in the jth coordinate). Then naturally we have

$$\boldsymbol{x} = \sum_{j=1}^{n} x_j \boldsymbol{e}_j,$$

whence for $f \in (\mathbb{R}^n)'$ we must, because of the linearity of the operator, have

$$f(\boldsymbol{x}) = f\Big(\sum_{j=1}^{n} x_j \boldsymbol{e}_j\Big) = \sum_{j=1}^{n} x_j f(\boldsymbol{e}_j),$$

so for any $f \in (\mathbb{R}^n)'$ there exists $f = (\boldsymbol{y}_1, \boldsymbol{y}_2, \dots, \boldsymbol{y}_n)$ such that

$$f(\boldsymbol{x}) = \sum_{j=1}^{n} x_j \boldsymbol{y}_j,$$

whence f is itself an element in $\mathbb{R}^n$, so $(\mathbb{R}^n)' = \mathbb{R}^n$.

On $\mathbb{R}^n$ we can introduce many norms ($\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$, et cetera), however by the following theorem (unfortunately without proof, for it is quite tricky) the choice doesn't matter.

Theorem. *All norms on $\mathbb{R}^n$ are equivalent, meaning that for all $\|\cdot\|$ and $\|\cdot\|'$ there exists some constants c_1 and c_2 such that, for all $\boldsymbol{x}$, $c_1\|\boldsymbol{x}\| \le \|\boldsymbol{x}\|' \le c_2\|\boldsymbol{x}\|$.*

Let us then again consider $\langle Ax, y\rangle = \langle x, A^* y\rangle$. We have

$$A\boldsymbol{x} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \sum_{j=1}^{n} \begin{pmatrix} a_{1j}x_j \\ a_{2j}x_j \\ \vdots \\ a_{mj}x_j \end{pmatrix},$$

since $\langle A\boldsymbol{e}_i, \boldsymbol{e}_j\rangle = a_{ij}$.

Thus

$$\langle A\boldsymbol{x}, \boldsymbol{y}\rangle = \sum_k (Ax)_k y_k = \sum_k \sum_j a_{kj} x_j y_k,$$

wherein we can switch the order of the sums since they are all finite, giving us

$$\langle A\boldsymbol{x}, \boldsymbol{y}\rangle = \sum_j \sum_k a_{kj} y_k x_j.$$

In the first case we have that the inner sum represents $A = (a_{jk})$ operating, and in the second one, after switching order of the sums, we have that $A^T = (a_{kj})$ is operating. So A^* is a matrix, and in particular it is the matrix A^T. ▲

6.3 Hilbert Space

The Hilbert space is a particularly important kind of normed linear space, however to define it we first need the following definition.

Definition 6.3.1 (Scalar product)**.** Let H be a linear space. Then $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ (or $\mathbb{R}$) is a ***scalar product*** if it satisfies the following properties:

(*i*) $\langle \boldsymbol{x}, \boldsymbol{x}\rangle \geq 0$ for all $\boldsymbol{x} \in H$;

(*ii*) $\langle \boldsymbol{x}, \boldsymbol{x}\rangle = 0$ if and only if $\boldsymbol{x} = \boldsymbol{0}$;

(*iii*) It is linear with respect to the first argument:

$$\langle \lambda\boldsymbol{x} + \mu\boldsymbol{y}, \boldsymbol{z}\rangle = \lambda\langle \boldsymbol{x}, \boldsymbol{z}\rangle + \mu\langle \boldsymbol{y}, \boldsymbol{z}\rangle,$$

for all $\lambda, \mu \in \mathbb{C}$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in H$;

(*iv*) It is skew-symmetric: $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \overline{\langle \boldsymbol{y}, \boldsymbol{x}\rangle}$.

Examples 6.3.2. Tale $H = \mathbb{C}^n$, such that $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$. Then $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \sum_{j=1}^{n} x_j \overline{y_j}$ is a scalar product.

If instead we take

$$H = \ell_2 = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \mid \sum_{j=1}^{\infty} |x_j|^2 < \infty\}$$

with $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \sum_{j=1}^{\infty} x_j \overline{y_j}$, we again have a scalar product, but this time on an infinite dimensional space.

Finally consider

$$H = L_2(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \mid \int_{-\infty}^{+\infty} |f(t)|^2 \, dt < \infty\},$$

i.e. the space of complex square differentiable functions, with the scalar product

$$\langle f, g\rangle = \int_{-\infty}^{+\infty} f(t)\overline{g(t)} \, dt.$$ ▲

Lemma 6.3.3. *Let H be a linear space with a scalar product $\langle\cdot,\cdot\rangle$. Then $\|\boldsymbol{x}\| = \sqrt{\langle\boldsymbol{x},\boldsymbol{x}\rangle}$ defines a norm.*

Proof. It is clear that $\|\boldsymbol{x}\| \geq 0$, since the scalar product is nonnegative and taking the square root keeps it nonnegative. Likewise it is clear that $\|\boldsymbol{x}\| = 0$ is equivalent with $\boldsymbol{x} = \boldsymbol{0}$, since the square root of 0 is 0. We also have that

$$\begin{aligned}\|\lambda\boldsymbol{x}\| &= \sqrt{\langle\lambda\boldsymbol{x},\lambda\boldsymbol{x}\rangle} = \sqrt{\lambda\langle\boldsymbol{x},\lambda\boldsymbol{x}\rangle} = \sqrt{\lambda\overline{\langle\lambda\boldsymbol{x},\boldsymbol{x}\rangle}} \\ &= \sqrt{\lambda\bar{\lambda}\,\overline{\langle\boldsymbol{x},\boldsymbol{x}\rangle}} = \sqrt{|\lambda|^2\langle\boldsymbol{x},\boldsymbol{x}\rangle} = |\lambda|\|\boldsymbol{x}\|.\end{aligned}$$

It remains to verify that the triangle holds:

$$\begin{aligned}\|\boldsymbol{x}+\boldsymbol{y}\|^2 &= \langle\boldsymbol{x}+\boldsymbol{y},\boldsymbol{x}+\boldsymbol{y}\rangle = \langle\boldsymbol{x},\boldsymbol{x}+\boldsymbol{y}\rangle + \langle\boldsymbol{y},\boldsymbol{x}+\boldsymbol{y}\rangle \\ &= \overline{\langle\boldsymbol{x}+\boldsymbol{y},\boldsymbol{x}\rangle} + \overline{\langle\boldsymbol{x}+\boldsymbol{y},\boldsymbol{y}\rangle} \\ &= \overline{\langle\boldsymbol{x},\boldsymbol{x}\rangle} + \overline{\langle\boldsymbol{y},\boldsymbol{x}\rangle} + \overline{\langle\boldsymbol{x},\boldsymbol{y}\rangle} + \overline{\langle\boldsymbol{y},\boldsymbol{y}\rangle} \\ &= \langle\boldsymbol{x},\boldsymbol{y}\rangle^2 + \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2.\end{aligned}$$

The square $\langle\boldsymbol{x},\boldsymbol{y}\rangle^2$ is clearly positive, whence if we take it away it becomes smaller, so we have

$$\|\boldsymbol{x}+\boldsymbol{y}\|^2 \leq \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2.$$

Since the square root is a monotone function the inequality remains:

$$\|\boldsymbol{x}+\boldsymbol{y}\| \leq \sqrt{\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2} \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|. \qquad \square$$

Therefore on any space with a scalar product there also exists a norm given by that scalar product.

Definition 6.3.4 (Hilbert space)**.** Let H be a linear Banach space (over $\mathbb{R}$ or $\mathbb{C}$, though we will generally use $\mathbb{C}$) on which there is defined a scalar product $\langle\cdot,\cdot\rangle$ and a norm $\|\boldsymbol{x}\| = \sqrt{\langle\boldsymbol{x},\boldsymbol{x}\rangle}$. Then H is called a ***Hilbert space***.

Theorem 6.3.5. *If H is a Hilbert space, then $H' = H$.*

6.4 Adjoint Operator in a Hilbert Space

Usually we have the picture

$$\begin{aligned}E_1 &\xrightarrow{A} E_2 \\ E_1' &\xleftarrow[A^*]{} E_2'\end{aligned}$$

from before, however the dual space of a Hilbert space is again the Hilbert space in question, whence we now get the much simplified picture

$$H_1 \underset{A^*}{\overset{A}{\rightleftarrows}} H_2.$$

Example 6.4.1. Let us once again consider $H = \mathbb{C}^n$, with basis elements $\boldsymbol{e}_j = (0,\dots,0,1,0,\dots,0)$, and $A = (a_{ij} = (\langle A\boldsymbol{e}_i,\boldsymbol{e}_j\rangle)$. Then, due to the skew-symmetry of the scalar product, $a_{ij} = \langle A\boldsymbol{e}_i,\boldsymbol{e}_j\rangle = \langle\boldsymbol{e}_i,A^*\boldsymbol{e}_j\rangle = \overline{\langle A^*\boldsymbol{e}_j,\boldsymbol{e}_i\rangle} = \overline{a^*_{ji}}$, whereby we have that $a^*_{km} = \overline{a_{mk}}$. ▲

Thus we also have that

$$H_1 \xrightarrow{A^{**}} H_2.$$

6.5 Hermitian Operators

We have from just above the picture

$$H \underset{A^*}{\overset{A}{\rightleftarrows}} H.$$

What if $A = A^*$? Then A is a so-called Hermitian operator.

Definition 6.5.1 (Hermitian operator)**.** Let H be a Hilbert space and let $A : H \to H$ and A^* its adjoint operator. Then if $A = A^*$, A is said to be an ***Hermitian operator***. Therefore $a_{km} = \overline{a_{mk}}$.

Example 6.5.2. Let $H = \mathbb{C}^2$ and take A to be

$$A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}.$$ ▲

In particular we must have $a_{ii} = \overline{a_{ii}}$ the diagonal elements a_{ii} must be real.

6.6 Hermitian Operators in Quantum Mechanics

Quantum mechanics has the following axioms.

(*i*) Given a complex Hilbert space H, unit vectors $\psi \in H$ (i.e. that $\|\psi\| = \sqrt{\langle \psi, \psi \rangle} = 1$) represent states of quantum systems.

(*ii*) Observables (energy, position, (angular) momentum, polarisation, et cetera) are represented by Hermitian operators.

In classical physics, state space and phase space are the same. Knowing the position q and momentum $p = mv$ uniquely determines all observables, for example energy

$$E = \frac{p^2}{2m} + \frac{v}{2}.$$

In quantum mechanics, on the other hand, ψ determines the value of an observable only with some probability. That is to say, quantum mechanics produces averages of observables with respect to states.

(*iii*) The average of an observable A is given by $\langle A \rangle_\psi = \langle A\psi, \psi \rangle$.

For Hermitian operators A, this quantity is real. This follows immediately from this argument:

$$\langle A \rangle_\psi = \langle A\psi, \psi \rangle = \langle \psi, A\psi \rangle = \overline{\langle A\psi, \psi \rangle},$$

wherein the first step is by A being Hermitian, and the second comes from the scalar product by definition being skew-symmetric. Therefore the average of an observable is equal to its own conjugate, whence it must be real.

We know from probability theory that it isn't enough knowing the average μ; we also require knowledge of the standard deviation $\sigma = \sqrt{\sigma^2}$, where σ^2 is the dispersion, defined as

$$\sigma^2 = E[(\xi - \mu)^2],$$

where ξ are experimental measurements.

In quantum mechanics we get

$$\begin{aligned}\sigma_A^2 &= \langle (A - \langle A\rangle_\psi I)^2\rangle_\psi = \langle (A - \langle A\rangle_\psi I)^2\psi, \psi\rangle \\ &= \langle (A - \langle A\rangle_\psi I)(A - \langle A\rangle_\psi I)\psi, \psi\rangle \\ &= \langle (A - \langle A\rangle_\psi I)\psi, (A - \langle A\rangle_\psi I)\psi\rangle = \left\|(A - \langle A\rangle_\psi I)\psi\right\|^2 \geq 0,\end{aligned}$$

wherein I is the unit operator ($I\psi = \psi$), whence we can define quantum standard deviation.

7 Lecture VII[8]

7.1 Schrödinger Inequality

Take two Hermitian operators A and B. Then the Schrödinger inequality states that

$$\sigma_A^2\sigma_B^2 \geq \left|\frac{1}{2}\langle\{A,B\}\rangle_\psi - \langle A\rangle_\psi\langle B\rangle_\psi\right|^2 + \left|\frac{1}{2i}\langle[A,B]\rangle_\psi\right|^2,$$

where $\{A,B\} = AB + BA$ is the so-called anti-commutator of A and B (note that it is 0 if and only if A and B anti-commute, i.e. $AB = -BA$), and $[A,B] = AB - BA$ is the commutator of A and B, which is 0 if and only if $AB = BA$.

If we let $f_A = (A - \langle A\rangle_\psi I)\psi$ and $f_B = (B - \langle B\rangle_\psi I)\psi$, we get

$$\sigma_A^2 = \langle (A - \langle A\rangle_\psi I)\psi, (A - \langle A\rangle_\psi I)\psi\rangle = \langle f_A, f_A\rangle = \|f_A\|^2,$$

and similarly $\sigma_B^2 = \|f_B\|^2$.

Using this, we find that the Schrödinger inequality reduces to a known inequality, abbreviated CBS.

Theorem 7.1.1 (Cauchy–Bunyakovsky–Schwartz inequality)**.** *Let H be a Hilbert space and let $f, g \in H$. Then*

$$|\langle f, g\rangle| \leq \|f\|\|g\|.$$

To prove it we will require the following generalisation of the Pythagorean theorem in general Hilbert spaces.

Lemma 7.1.2. *Let H be a Hilbert space and let $f, g \in H$ such that $\langle f, g\rangle = 0$ (i.e. that f and g are orthogonal). Then $\|f+g\|^2 = \|f\|^2 + \|g\|^2$.*

Proof. We simply apply the definition of norms in Hilbert spaces and then use the linearity of its scalar product (along with its skew-symmetry), and also that $\langle f, g\rangle = 0$:

$$\begin{aligned}\|f+g\|^2 &= \langle f+g, f+g\rangle = \langle f, f+g\rangle + \langle g, f+g\rangle = \overline{\langle f+g, f\rangle} + \overline{\langle f+g, g\rangle} \\ &= \overline{\langle f, f\rangle} + \overline{\langle g, f\rangle} + \overline{\langle f, g\rangle} + \overline{\langle g, g\rangle} = \langle f, f\rangle + \langle g, g\rangle = \|f\|^2 + \|g\|^2. \quad \square\end{aligned}$$

[8]Date: November 23.

Proof of the Cauchy–Bunyakovsky–Schwartz inequality. Let

$$z = f - \frac{\langle f, g \rangle}{\|g\|^2} g$$

and consider

$$\langle z, g \rangle = \left\langle f - \frac{\langle f, g \rangle}{\|g\|^2} g, g \right\rangle = \langle f, g \rangle - \frac{\langle f, g \rangle}{\|g\|^2} \langle g, g \rangle = 0,$$

whence z and g are orthogonal (meaning that the previous lemma applies).

Solving the definition of z for f we have

$$f = z + \frac{\langle f, g \rangle}{\|g\|^2} g,$$

using which we compute $\|f\|^2$, in which we may apply the lemma:

$$\|f\|^2 = \|z\|^2 + \left\| \frac{\langle f, g \rangle}{\|g\|^2} g \right\|^2 = \|z\|^2 \frac{|\langle f, g \rangle|^2}{\|g\|^4} \|g\|^2 = \|z\|^2 + \frac{|\langle f, g \rangle|^2}{\|g\|^2},$$

in which, if we subtract the nonnegative quantity $\|z\|^2$, clearly becomes smaller, so we have

$$\|f\|^2 \geq \frac{|\langle f, g \rangle|^2}{\|g\|^2}.$$

Now we multiply both sides by $\|g\|^2$, which is nonnegative meaning that the inequality remains the same, giving us

$$\|f\|^2 \|g\|^2 \geq |\langle f, g \rangle|^2. \qquad \square$$

As an aside it is perhaps of interest to note that Cauchy's contribution to the inequality was to consider $H = \mathbb{C}^n$, whereas Bunyakovsky, a student of Cauchy's, solved it for $H = L_2(\mathbb{C})$. It was finally Schwartz who solved it for general Hilbert spaces.

Using f_A and f_B in the CBS inequality we thus get $\sigma_A^2 \sigma_B^2 = \|f_A\|^2 \|f_B\|^2 \geq |\langle f_A, f_B \rangle|$, which we recognise as $z\overline{z}$, if we take $z = \langle f_A, f_B \rangle$, which moreover gives us $z\overline{z} = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2$.

If in addition we note that

$$\operatorname{Re} z = \frac{1}{2}(z + \overline{z}) \qquad \text{and} \qquad \operatorname{Im} z = \frac{1}{2i}(z - \overline{z}), \tag{7.1.1}$$

we get that

$$\sigma_A^2 \sigma_B^2 \geq (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 = \left(\frac{1}{2}(z + \overline{z}) \right)^2 + \left(\frac{1}{2i}(z - \overline{z}) \right)^2.$$

If we now consider

$$\begin{aligned} z = \langle f_A, f_B \rangle &= \langle (A - \langle A \rangle_\psi I)\psi, (B - \langle B \rangle_\psi I)\psi \rangle \\ &= \langle (B - \langle B \rangle_\psi I)(A - \langle A \rangle_\psi I)\psi, \psi \rangle, \end{aligned}$$

the last step following from $(B - \langle B\rangle_\psi I)$ being Hermitian.

Expanding the parentheses due to the linearity, we have, by remembering that $\|\psi\|^2 = \langle \psi, \psi\rangle = 1$ by the first axiom of quantum mechanics:

$$\begin{aligned} z &= \langle BA\rangle_\psi - \langle A\rangle_\psi \langle B\rangle_\psi - \langle B\rangle_\psi \langle A\rangle_\psi + \langle A\rangle_\psi \langle B\rangle_\psi \langle \psi, \psi\rangle \\ &= \langle BA\rangle_\psi - \langle A\rangle_\psi \langle B\rangle_\psi. \end{aligned}$$

By almost identical computations we have that

$$z^* = \overline{z} = \langle f_B, f_A\rangle = \langle AB\rangle_\psi - \langle A\rangle_\psi \langle B\rangle_\psi,$$

with which we in (7.1.1) get

$$\operatorname{Re} z = \frac{\langle BA\rangle_\psi - \langle A\rangle_\psi \langle B\rangle_\psi + \langle AB\rangle_\psi - \langle A\rangle_\psi \langle B\rangle_\psi}{2} = \frac{1}{2}\{A, B\} - \langle A\rangle_\psi \langle B\rangle_\psi$$

and

$$\operatorname{Im} z = \frac{\langle BA\rangle_\psi - \langle A\rangle_\psi \langle B\rangle_\psi - \langle AB\rangle_\psi + \langle A\rangle_\psi \langle B\rangle_\psi}{2i} = \frac{1}{2i}[A, B].$$

Plugging this back into the CBS inequality finally gives us

$$\sigma_A^2 \sigma_B^2 \geq (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 = \left| \frac{1}{2}\langle \{A, B\}\rangle_\psi - \langle A\rangle_\psi \langle B\rangle_\psi \right|^2 + \left| \frac{1}{2i}\langle [A, B]\rangle_\psi \right|^2,$$

as sought.

That is to say, from a mathematician's perspective, the famous Schrödinger inequality from quantum mechanics simply reduces to thee CBS inequality, since the operators are Hermitian.

7.2 Robertson Inequality

Another famous inequality from quantum mechanics, which was discovered before Schrödinger's, is that of Robertson. It states that

$$\sigma_A^2 \sigma_B^2 \geq \left| \frac{1}{2i}\langle [A, B]\rangle_\psi \right|^2,$$

which we observe is an immediate consequence of the Schrödinger inequality, since both of the terms in the sum of that inequality are nonnegative.

We will use this instead as a reason to discuss why the physicists insist on the i in the denominator, even though it disappears when taking the absolute values. The reason is that physicists like self-adjoint operators (i.e. operators that are their own adjoint).

Consider $C = [A, B] = AB - BA$. To consider the adjoint of this we first need to know what the adjoint of a composition is.

Lemma 7.2.1. *Let A and B be operators on a Hilbert space H. Then $(AB)^* = B^* A^*$.*

Proof. This is a simple use of the definition of the adjoint and the skew-symmetry of scalar products. Let $f, g \in H$:

$$\begin{aligned} \langle (AB)^* f, g\rangle = \langle f, ABg\rangle = \overline{\langle ABg, f\rangle} &= \overline{\langle Bg, A^* f\rangle} \\ &= \overline{\langle g, B^* A^* f\rangle} = \langle B^* A^* f, g\rangle, \end{aligned}$$

whence by comparing the first and last steps we see that $(AB)^* = B^* A^*$. $\square$

We therefore have $C^* = (AB - BA)^* = (AB)^* - (BA)^* = B^*A^* - A^*B^*$, which, recalling that A and B are taken to be Hermitian (meaning that $A^* = A$ and $B^* = B$) becomes $C^* = BA - AB = -C = -[A, B]$.

That is to say, $[A, B]^* = -[A, B]$, whence $[A, B]$ is *not* self-adjoint.

However if we instead compute

$$\left(\frac{1}{2i}[A, B]\right)^* = \frac{-[A, B]}{-2i} = \frac{[A, B]}{2i},$$

which is self-adjoint, if we use the following lemma.

Lemma 7.2.2. *Let $\lambda \in \mathbb{C}$ and let A be a continuous linear operator on a Hilbert space H. Then $(\lambda A)^* = \overline{\lambda} A^*$.*

Proof. Take $f, g \in H$. Then by the definition of adjoint operators, along with the linearity of of the first component of scalar products and their skew-symmetry, we have

$$\begin{aligned} \langle (\lambda A)^* f, g \rangle &= \langle f, \lambda A g \rangle = \overline{\langle \lambda A g, f \rangle} = \overline{\lambda}\, \overline{\langle A g, f \rangle} \\ &= \overline{\lambda} \langle f, A g \rangle = \overline{\lambda} \langle A^* f, g \rangle = \langle \overline{\lambda} A^* f, g \rangle. \end{aligned}$$

We now compare the first and last step and observe that $(\lambda A)^* = \overline{\lambda} A^*$. □

What does the Robertson inequality mean? If we assume that $[A, B] \neq 0$, i.e. that A and B do not commute, then σ_A^2 and σ_B^2 cannot both be arbitrarily small simultaneously. This then leads us nicely on to a fundamental result of quantum mechanics.

7.3 Heisenberg's Uncertainty Principle

The famous Heisenberg's uncertainty principle is simply the Robertson inequality, taking $H = L_2$ (the space of absolutely square integrable functions) and A and B to be two particular operators.

The first one is $A = q$, the position operator, in quantum mechanics, for reasons unknown, taken to be $q(\varphi)(x) = x\varphi(x)$, i.e. multiplying by the variable.

The second one is $B = p$, where p is the momentum operator, defined as

$$p(\varphi)(x) = \frac{\hbar}{i} \frac{\partial \varphi}{\partial x}(x),$$

where $\hbar$ is the so-called reduced Planck constant.

We plug these into the Robertson inequality and compute:

$$\sigma_q^2 \sigma_p^2 \geq \frac{1}{4} \left| \langle [q, p] \rangle_\psi \right|^2.$$

Here we get

$$[q, p]\varphi(x) = (qp - pq)(\varphi)(x),$$

wherein

$$(qp)(\varphi)(x) = \frac{\hbar}{i} q(\varphi')(x) = \frac{\hbar}{i} x\varphi'(x),$$

and

$$(pq)(\varphi(x) = p(x\varphi(x)) = \frac{\hbar}{i} x\varphi'(x) + \varphi(x)$$

by the product rule for derivatives. Thus

$$[q,p]\varphi(x) = (qp - pq)(\varphi)(x), = \frac{\hbar}{i}(x\varphi'(x) - x\varphi'(x) - \varphi(x)) = -\frac{\hbar}{i}\varphi(x) = \hbar i\varphi(x),$$

whence $[q,p]$ is therefore the operator $\hbar i I$. Of course $\langle I\rangle_\psi = \langle I\psi, \psi\rangle = 1$, so

$$\sigma_q^2\sigma_p^2 \geq \frac{\hbar^2}{4}.$$

This is the fundamental limit of uncertainty in quantum mechanics; if one knows the momentum well, one cannot know the position well, and vice versa. This is also known as Bohr's principle of complementarity.

Exercise 7.3.1. Consider the Hilbert space $H = \mathbb{C}^2$ and let

$$A = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

(which both obviously Hermitian) and let $\boldsymbol{\psi} = 1/\sqrt{2}(\boldsymbol{e}_1 + \boldsymbol{e}_2)$, with $\boldsymbol{e}_1 = (1,0)$ and $\boldsymbol{e}_2 = (0,1)$.

Study for these the Robertson inequality and the Schrödinger inequality.

Solution. See Assignment V. ♦

8 Lecture VIII[9]

8.1 Dense Subsets of Metric Spaces

Definition 8.1.1 (Dense subset)**.** Let (X,ρ) be a metric space, and $A \subset X$ be some subset of X. We say that A is ***dense*** in X if for any $x \in X$ and any $\varepsilon > 0$ there exists an $a \in A$ such that $a \in B_\varepsilon(x)$.

Examples 8.1.2. Let $X = \mathbb{R}$ and $A = \mathbb{Q}$. Then A is dense in X, because for any $x \in \mathbb{R}$ we have the decimal expansion

$$x = \alpha_N\alpha_{N-1}\ldots\alpha_1\alpha_0\boldsymbol{.}\alpha_{-1}\alpha_{-2}\ldots\alpha_{-M}\ldots = \sum_{j=-\infty}^{N}\alpha_j 10^j.$$

Take x_M to be the truncation of this to M decimal places, i.e.

$$x_M = \sum_{j=-M}^{N}\alpha_j 10^j = \alpha_N\alpha_{N-1}\ldots\alpha_1\alpha_0\boldsymbol{.}\alpha_{-1}\alpha_{-2}\ldots\alpha_{-M}.$$

Then clearly x_M is a rational number, and $|x - x_M| \leq 10^{-M} \to 0$ as $M \to \infty$.

Consider now $X = \mathbb{R}^n$ instead, with $\rho(\boldsymbol{x},\boldsymbol{y})$ being any of the distances we are used to ($\|\boldsymbol{x}-\boldsymbol{y}\|_\infty$, $\|\boldsymbol{x}-\boldsymbol{y}\|_1$, or $\|\boldsymbol{x}-\boldsymbol{y}\|_2$). Then $A = \mathbb{Q}^n$ is dense in X, by the same argument as above, componentwise. ▲

Example 8.1.3. Consider instead $X = C[a,b]$, with the distance $\rho_\infty(\varphi,\psi) = \|\varphi - \psi\|_\infty$. We then need the following theorem, stated without proof.

[9] Date: November 26, 2015.

Theorem (Weierstrass approximation theorem). *Any continuous function defined on a closed interval $[a,b]$ can be uniformly approximated by a polynomial.*

This means that for any $\varphi \in C[a,b]$ and for any $\varepsilon > 0$ there exists some polynomial $p \in P[a,b]$,

$$p(t) = c_k t^k,$$

c_k being real numbers, such that $\max_{a \le t \le b} |\varphi(t) - p(t)| < \varepsilon$, in other words $P[a,b]$ is dense in $C[a,b]$. ▲

8.2 Separable Metric Spaces

Definition 8.2.1 (Separable metric space). The metric space (X, ρ) is called ***separable*** if there exists a countable dense subset. Otherwise the space is called ***non-separable***.

Examples 8.2.2. All of the previous examples are separable spaces. In the first case because $\mathbb{Q}$ is countable, in the second case because $\mathbb{Q}^n$ is a finite union of countable sets, which is again countable.

The space of polynomials $P[a,b]$, on the other hand, is not countable, because the coefficients are real. However we can approximate any real polynomial with a rational polynomial from $P_{\mathbb{Q}}[a,b]$, which is countable and also dense in $C[a,b]$.

That is to say, for any $\varphi \in C[a,b]$ and for any $\varepsilon > 0$, by Weierstrass theorem there exists some $p_\varepsilon = \sum\limits_{k=0}^{N} c_k t^k \in P[a,b]$, $c_k \in \mathbb{R}$, such that

$$\max_{a \le t \le b} |\varphi(t) - p_\varepsilon(t)| < \varepsilon,$$

but there also for every coefficient c_k exists some $q_k \in \mathbb{Q}$ such that for every $\delta > 0$ we have $|q_k - c_k| < \delta$, whence

$$\max_{a \le t \le b} |p_\varepsilon(t) - p_\varepsilon^\delta(t)| = \sum_{k=0}^{N} |c_k - q_k| b_k < \delta \sum_{k=0}^{N} b_k,$$

where p_ε^δ is the polynomial with q_k as its coefficients, so we take

$$\delta < \frac{\varepsilon}{\sum_{k=0}^{N} b_k},$$

whence $\|p_\varepsilon - p_\varepsilon^\delta\|_\infty < \varepsilon$, finally giving us $\|\varphi - p_\varepsilon^\delta\|_\infty < 2\varepsilon$, whence $C[a,b]$ is separable since $P_{\mathbb{Q}}[a,b]$ is countable and dense. ▲

Example 8.2.3. Take the space $X = \ell_1 = \{\boldsymbol{x} = (x_j), x_j \in \mathbb{R} \mid \|\boldsymbol{x}\|_1 < \infty\}$ of all absolutely convergent. Since

$$\|\boldsymbol{x}\|_1 = \sum_{j=1}^{\infty} |x_j| < \infty,$$

there must for every $\varepsilon > 0$ exist some N such that

$$\sum_{j=N+1}^{\infty} |x_j| < \varepsilon.$$

(This is simply because of the absolute convergence of the series.)

Take therefore $\boldsymbol{x}_\varepsilon = (x_1, x_2, \ldots, x_N, 0, \ldots)$. Then clearly $\|\boldsymbol{x} - \boldsymbol{x}_\varepsilon\|_1 < \varepsilon$.

Moreover let $q_j \in \mathbb{Q}$ be rational numbers chosen in such a way that $|q_j - x_j| < \delta$ for some small δ, and construct the sequence $\boldsymbol{q}_\varepsilon^\delta = (q_1, q_2, \ldots, q_N, 0, \ldots)$. The collection of all such $\boldsymbol{q}_\varepsilon^\delta$ is countable since there are finitely many nonzero positions, and each of them have countably many options from $\mathbb{Q}$. In addition we have that $\|\boldsymbol{x}_\varepsilon - \boldsymbol{q}_\varepsilon^\delta\|_1 < N\delta$, whence if we take $\delta < \varepsilon/N$, we have $\|\boldsymbol{x} - \boldsymbol{q}_\varepsilon^\delta\|_1 < 2\varepsilon$, so it is dense in ℓ_1, and therefore since it's countable ℓ_1 is separable. ▲

Example 8.2.4. With an almost identical proof (just replace the norm, but use the same technique) we see that ℓ_2 is separable as well. ▲

On the other hand, not all spaces we are used to are separable.

Example 8.2.5. The space $X = \ell_\infty = \{\boldsymbol{x} = (x_j) | \|\boldsymbol{x}\|_\infty < \infty\}$ is non-separable, meaning that there exists no dense and countable subset. This relies on $\mathbb{R}$ not being countable, indeed not even $[0, 1]$ is countable.

Consider the binary expansions of real numbers on this interval between 0 and 1. They have the form $x = 0.\beta_1\beta_2 \ldots \beta_j \ldots$, where $\beta_j \in 0, 1$ for all j, and the set of all such binary strings is uncountable. We translate these into sequences belonging to ℓ_∞; let $Y = \{\boldsymbol{x} = (\beta_1, \beta_2, \ldots, \beta_j, \ldots)\} \subset \ell_\infty$. That this is a subset of ℓ_∞ is clear since the supremum of terms in such a sequence is 1 (or 0, if it's constantly 0).

Moreover, for any $\boldsymbol{x}, \boldsymbol{y} \in Y$, we clearly have $\|\boldsymbol{x} - \boldsymbol{y}\|_\infty = \max\limits_{1 \le j < \infty} |x_j - y_j| = 1$, if $\boldsymbol{x} \neq \boldsymbol{y}$. Therefore if we take $\varepsilon = 1/2$ we clearly have $B_{1/2}(\boldsymbol{x}) \cap B_{1/2}(\boldsymbol{y}) = \varnothing$.

Suppose now that ℓ_∞ is separable. This means that there must exist some $A = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n, \ldots\}$, where $\boldsymbol{a}_j \in \ell_\infty$ for all j, such that for all $\boldsymbol{z} \in \ell_\infty$ and all $\varepsilon > 0$ there exists some $\boldsymbol{a}_j \in A$ such that $\boldsymbol{a}_j \in B_\varepsilon(\boldsymbol{z})$.

In particular, then, this must be true if we pick $\varepsilon = 1/2$ and take $\boldsymbol{z} \in Y$, our set of binary strings. But we have shown that Y is uncountable, and since A is countable there aren't enough members of A for this to be true, so we have a contradiction. Therefore ℓ_∞ cannot be separable. ▲

Exercise 8.2.6. Let $X = c_0 = \left\{\boldsymbol{x} = (x_1, x_2, \ldots, x_n, \ldots) \;\middle|\; \lim\limits_{j\to\infty} x_j = 0\right\}$ with the norm

$$\|\boldsymbol{x}\|_\infty = \max_{1 \le j < \infty} |x_j|.$$

Investigate whether c_0 is separable or not.

Solution. Since the sequence $\boldsymbol{x} = (x_1, x_2, \ldots, x_n, \ldots)$ decays to 0 as n increases, then by the very definition of the limit there must for every $\varepsilon > 0$ exist some N such that $|x_n| < \varepsilon$ for all $n > N$.

Therefore the maximum of the sequence, that is $\|\boldsymbol{x}\|_\infty$, must be x_k for some $1 \le k \le N$. This serves to tell us that there are only countably many positions k where this maximum can be.

We then for any $\delta > 0$ construct the sequence $\boldsymbol{x}_\varepsilon^\delta = (0, \ldots, 0, q_k, 0, \ldots)$, where q_k is in the kth position of $\boldsymbol{x}_\varepsilon^\delta$, just like the maximum of $\boldsymbol{x}$, and moreover $q_k \in \mathbb{Q}$ is chosen in such a way that $|x_k - q_k| < \delta$.

Since there are countably many positions $1 \le k \le N$ to choose from, and countably many rational numbers in $\mathbb{Q}$, the space of all such sequences $Q =$

$\{\boldsymbol{x}_\varepsilon^\delta = (0, \ldots, 0, q_k, 0, \ldots)\}$ is countable (since the countable union of countable sets is again countable).

It then remains to show that this set Q is dense in c_0. This is clear by construction: we have, if we let q_j denote the jth term in $\boldsymbol{x}_\varepsilon^\delta$, that

$$\|\boldsymbol{x} - \boldsymbol{x}_\varepsilon^\delta\|_\infty = \max_{1 \leq j < \infty} |x_j - q_j| = x_k - q_k < \delta,$$

where δ can be made arbitrarily small. Thus the for any radius δ, there exists some $\boldsymbol{x}_\varepsilon^\delta \in Q$ such that $\boldsymbol{x}_\varepsilon^\delta \in B_\delta(\boldsymbol{x})$, meaning that Q is dense in c_0.

Thus since $Q \subset c_0$ is dense and countable, c_0 is separable. ♦

8.3 Unbounded Linear Operators

By Theorem 5.1.1, this is of course the same as discontinuous linear operators.

The aim here is to explore differential operators, like $d/dx : C^{(1)} \to C$, but instead $d/dx : H \to H$ for some Hilbert space H. What people generally use in applications is $H = L_2[a, b] = \{f : [a, b] \to \mathbb{C} \mid \int_a^b |f(x)|^2 \, dx < \infty\}$. Of course not all of these need be differentiable, so we consider $C^\infty \subset L_2$, and $d/dx : C^\infty \to C^\infty$.

Definition 8.3.1 (Domain of definition)**.** Let H be a Hilbert space and let $\mathcal{D}$ be an uncountable linear subspace (meaning that for all $x, y \in \mathcal{D}$ we have that $\lambda x + \mu y \in \mathcal{D}$) which is dense in H. Moreover let $A : \mathcal{D} \to H$ be a linear operator. Then we call A a linear operator with the ***domain of definition*** $\mathcal{D}$.

Remark 8.3.2. Note that this definition is interesting only for operators which are *not* continuous, because otherwise, since $\mathcal{D}$ is dense in H, and A is continuous, we have for any $x \in H$ some sequence $x_n \to x$, $x_n \in \mathcal{D}$. We then take $y_n = Ax_n$, which since $x_n \to x$ converges to some $y \in H$, since A is continuous. Thus for any $x \in H$ we can extend A to be defined also for this x, so $\mathcal{D}$ can simply be taken to be the whole H.

For this reason, when we in the future consider $A : \mathcal{D} \to H$, $\mathcal{D} \subset H$ being dense, we always assume that A is *not* continuous.

For such operators we naturally cannot have an inequality of the form $\|Ax\| \leq c\|x\|$, where $c = \|A\|$, which we are so used to, since this wouldn't be finite.

Example 8.3.3. Consider the Hilbert space $H = L_2[-\infty, \infty]$, with the operator $\hat{x} : \varphi(x) \to x\varphi(x)$. This is unbounded, because having

$$\int_{-\infty}^{\infty} |\varphi(x)|^2 \, dx < \infty$$

does not necessarily imply that

$$\int_{-\infty}^{\infty} x^2 |\varphi(x)|^2 \, dx < \infty,$$

since taking for example $\varphi(x) = 1/(1 + |x|)$ we have that

$$|\varphi(x)|^2 \sim \frac{1}{|x|^2}$$

as x approaches either infinity, whereas

$$x^2|\varphi(x)|^2 \sim 1$$

as x approaches plus or minus infinity, whence the latter integral doesn't converge. ▲

A question which remains is what $\mathcal{D}$ is. There is a natural choice, as shown below, but it is not always the best choice.

Definition 8.3.4 (Natural domain of definition)**.** Let H be a Hilbert space. For a linear operator $A : \mathcal{D} \to H$, the ***natural domain of definition*** is $\mathcal{D}_A = \{x \in H \mid Ax \in H\}$.

Examples 8.3.5. If, for example, $H = L_2$, and $\hat{x}$ is as above, we have $\mathcal{D}_{\hat{x}} = \{\varphi \in L_2 \mid x\varphi(x) \in L_2\}$. Similarly, if $A = d/dx$, we have the natural domain of definition $\mathcal{D}_A = \{\varphi \in L_2 \mid \exists\ \varphi' \in L_2\}$. ▲

Example 8.3.6 (Hermitian unbounded operators)**.** Recall that for an operator A to be Hermitian we require that $\langle Ax, y\rangle = \langle x, Ay\rangle$, i.e. that $A = A^*$, and consider again the operator $\hat{x}$ on L_2. Is this Hermitian? Yes, since

$$\langle \hat{x}\varphi_1, \varphi_2\rangle = \int_{-\infty}^{\infty} \hat{x}\varphi_1(x)\overline{\varphi_2(x)}\,dx = \int_{-\infty}^{\infty} x\varphi_1(x)\overline{x\varphi_2(x)}\,dx = \langle \varphi_1, \hat{x}\varphi_2\rangle.$$

How about the operator $A = d/dx$? We compute the scalar product

$$\langle A\varphi)1, \varphi_2\rangle = \int_{-\infty}^{\infty} \varphi_1'(x)\overline{\varphi_2(x)}\,dx,$$

which using integration by parts we handily rewrite as

$$\varphi_1(x)\varphi_2(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \varphi_1(x)\overline{\varphi_2'(x)}\,dx = \varphi_1(x)\varphi_2(x)\Big|_{-\infty}^{\infty} - \langle \varphi_1, A\varphi_2\rangle.$$

So on the natural domain of definition $\mathcal{D}_A$ we have two problems, first the evaluation of $\varphi_1(x)\varphi_2(x)$ which we would like to be zero, and secondly the minus sign in front of the scalar product. So we take instead $\mathcal{D} = \{f \in L_2 \mid f' \in L_2, f(x) \to 0, x \to \pm\infty\}$. On this domain of definition we therefore have $\langle A\varphi)1, \varphi_2\rangle = -\langle \varphi_1, A\varphi_2\rangle$.

But since $\bar{i} = -i$, we consider instead the operator $C = iA = id/dx$, whence $\langle C\varphi)1, \varphi_2\rangle = \langle \varphi_1, C\varphi_2\rangle$, so C is Hermitian on the domain of definition $\mathcal{D}$. ▲

Problem 8.3.7. If H is a Hilbert space and $\mathcal{D} \in H$ is dense, and moreover $A : \mathcal{D} \to H$ is continuous, then $A \in \mathrm{L}(H)$, as we have explored in detail before.

If, on the other hand, A is unbounded on $\mathcal{D}$, then it cannot be extended to H.

Having said that, is it possible to construct a linear operator $A : E \to E$, where E is a Banach space, such that A is unbounded?

Solution. Let $B = \{\boldsymbol{e}_j\}$ be a Hamel basis of E (see [KF20, p. 128]), meaning that any $\boldsymbol{x} \in E$ can be written as a linear combination of *finitely* many basis elements

$$\boldsymbol{x} = \sum_{j=1}^{N} c_j \boldsymbol{e}_{n_j}. \tag{8.3.1}$$

We assume that $\|\boldsymbol{e}_j\| = 1$, because if this were not the case we could instead have taken $B' = \{\boldsymbol{e}_j/\|\boldsymbol{e}_j\|\}$ to be our basis.

We now define the map $\varphi : B \to \mathbb{C}$ by

$$\varphi(\boldsymbol{e}_j) = j \tag{8.3.2}$$

for all j, and extend it to the entirety of E and make it linear in the natural way:

$$\varphi(\boldsymbol{x}) = \sum_{j=1}^{N} c_j \varphi(\boldsymbol{e}_{n_j}).$$

(Here the linear combination (8.3.1) using the Hamel basis being made up of finitely many elements is crucial; if the amount of elements were infinite, we would generally not have convergence of the above sum. Since the sum is now finite, this is not a problem.)

That this is linear is fairly easy to show. Let us take some $\boldsymbol{z} = \lambda\boldsymbol{x} + \mu\boldsymbol{y}$, where $\boldsymbol{x} = \sum_{j=1}^{N} c_j \boldsymbol{e}_{n_j}$ and $\boldsymbol{y} = \sum_{j=1}^{N} d_j \boldsymbol{e}_{n_j}$ are any elements in the Banach space and λ and μ are any constants. Then

$$\boldsymbol{z} = \lambda\boldsymbol{x} + \mu\boldsymbol{y} = \lambda \sum_{j=1}^{N} c_j \boldsymbol{e}_{n_j} + \mu \sum_{j=1}^{N} d_j \boldsymbol{e}_{n_j} = \sum_{j=1}^{N} (\lambda c_j + \mu d_j) \boldsymbol{e}_{n_j},$$

whence if we plug this into φ we get

$$\varphi(\boldsymbol{z}) = \sum_{j=1}^{N} (\lambda c_j + \mu d_j) \varphi(\boldsymbol{e}_{n_j}) = \lambda \sum_{j=1}^{N} c_j \varphi(\boldsymbol{e}_{n_j}) + \mu \sum_{j=1}^{N} d_j \varphi(\boldsymbol{e}_{n_j}) = \lambda\varphi(\boldsymbol{x}) + \mu\varphi(\boldsymbol{y}).$$

Moreover that this is unbounded is quite clear by (8.3.2); since $\|\boldsymbol{e}_j\| = 1$, but at the same time $|\varphi(\boldsymbol{e}_j)| = j$, for all j, we cannot have $|\varphi(\boldsymbol{x})| \leq C\|\boldsymbol{x}\|$, for a constant C, for all $\boldsymbol{x} \in E$. Taking $\boldsymbol{x} = \boldsymbol{e}_j$ and letting j tend to infinity makes the left-hand side of the inequality arbitrarily large.

Finally we want to make this a map not from E to $\mathbb{C}$, but an operator from E to E. To do this we simply multiply $\varphi(\boldsymbol{x})$ by any nonzero $\boldsymbol{x}_0 \in E$ (taking $\boldsymbol{x}_0 = 0$ would of course make it the new operator bounded).

So finally we have the linear operator $A : E \to E$ defined by

$$A\boldsymbol{x} = \varphi(\boldsymbol{x})\boldsymbol{x}_0,$$

which is unbounded.

That this new operator A preserves the linearity of φ is clear by the same calculations as above; just multiply by the fixed $\boldsymbol{x}_0$. ♦

Remark 8.3.8. To prove that a Hamel basis always exists we require Zorn's lemma. With it, we can use the same line of reasoning we used to prove the Hahn–Banach theorem, since by Zorn's lemma there must amongst the set of all linearly independent subsets of the linear space E exist a maximal one. That this maximal set generates the whole space follows from precisely the same contradiction proof used in the proof of Hahn–Banach.

8.4 Separable Hilbert Spaces

To discuss these, we require the notion of an (orthonormal) basis in general Hilbert spaces.

Definition 8.4.1 (Orthonormal basis)**.** Let H be a Hilbert space. The set $\{\boldsymbol{e}_j\}$ is an ***orthonormal basis*** in H if $\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = \delta_{ij}$, where $\delta_{ii} = 1$ and $\delta_{ij} = 0$, $i \neq j$, and

$$\boldsymbol{x} = \sum_{j=1}^{\infty} c_j \boldsymbol{e}_j,$$

where $c_j \in \mathbb{C}$, in the sense that we have convergence in H:

$$\left\| \boldsymbol{x} - \sum_{j=1}^{N} c_j \boldsymbol{e}_j \right\|^2 \to 0$$

as $N \to \infty$.

Note that we then have, since $\|\cdot\|$ is continuous and the series converges, that

$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle = \|\boldsymbol{x}\|^2 = \left\langle \sum_{j=1}^{\infty} c_j \boldsymbol{e}_j, \sum_{j=1}^{\infty} c_j \boldsymbol{e}_j \right\rangle = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_j \overline{c_k} \delta_{jk} = \sum_{j=1}^{\infty} |c_j|^2, \qquad (8.4.1)$$

which is known as ***Parseval's equality***.

Since the norm is well-defined for all $\boldsymbol{x} \in H$, we must have that the above sum converges for all $\boldsymbol{x}$. Is the converse true?

That is to say, if we have any sequence (c_j) such that $\sum_{j=1}^{\infty} |c_j|^2 < \infty$, does there exist an $\boldsymbol{x} \in H$ such that

$$\boldsymbol{x} = \sum_{j=1}^{\infty} c_j \boldsymbol{e}_j?$$

The answer is yes, and to show it we finally make use of Hilbert spaces being complete. Set $\boldsymbol{x}_N = \sum_{j=1}^{N} c_j \boldsymbol{e}_j \in H$ be partial sums, and then let $N \leq M$. Then we have by Parseval's equality (8.4.1)

$$\|\boldsymbol{x}_N - \boldsymbol{x}_M\|^2 = \left\| \sum_{j=N+1}^{M} c_j \boldsymbol{e}_j \right\|^2 = \sum_{j=N+1}^{M} |c_j|^2 \to \infty$$

as $N, M \to \infty$ since the series converges. Therefore $\{\boldsymbol{x}_N\}$ is a Cauchy sequence in H, and because of the completeness of H there must exist a within H the limit of the sequence, i.e. some $\boldsymbol{x} \in H$ such that

$$\lim_{N \to \infty} \boldsymbol{x}_N = x,$$

meaning that

$$\boldsymbol{x} = \sum_{j=1}^{\infty} c_j \boldsymbol{e}_j,$$

as claimed.

9 Lecture IV[10]

9.1 Inverse Operators

In what follows we will require the following sufficient condition for the existence of inverse operators.

Theorem 9.1.1. *Let $A \in \mathscr{L}(E)$, where E is a Banach space, and let $\|A\| < 1$. Then there exists an operator $(I - A)^{-1} = \sum_{n=0}^{\infty} A^n$.*

Proof. Note first that

$$(I - A)^{-1} = \sum_{n=0}^{\infty} A^n, \tag{9.1.1}$$

if it exists, follows from the ordinary proof for the sum of a geometric series.

We must now show that the sum in (9.1.1) is in $\mathscr{L}(E)$. To do this, we show that the sequence of partial sums is a Cauchy sequence. Let $N \leq M$, then

$$\Big\|\sum_{n=0}^{N} A^n - \sum_{n=0}^{M} A^n\Big\| = \Big\| \sum_{n=N+1}^{M} A^n \Big\| \leq \sum_{n=N+1}^{M} \|A\|^n,$$

wherein the last step is due to the triangle inequality. Clearly this approaches 0 as N and M approach infinity, since $\|A\| < 1$. Thus

$$C = (I - A)^{-1} = \sum_{n=0}^{\infty} A^n \in \mathscr{L}(E).$$

We must now show that C is indeed the inverse of $I - A$, meaning that we wish to show that $C(I - A) = (I - A)C = I$:

$$(I - A)C = (I - A) \sum_{n=0}^{\infty} A^n = \sum_{n=0}^{\infty} A^n - \sum_{n=0}^{\infty} A^{n+1} = I,$$

and

$$C(I - A) = \Big(\sum_{n=0}^{\infty} A^n\Big)(I - A) = \sum_{n=0}^{\infty} A^n - \sum_{n=0}^{\infty} A^{n+1} = I,$$

whence C is indeed the inverse of $I - A$. □

Theorem 9.1.2. *Let E be a Banach space and let $A, B \in \mathscr{L}(E)$ such that $A^{-1} \in \mathscr{L}(E)$, and further let $\|A^{-1}B\| < 1$. Then there exists an $(A + B)^{-1}$ in $\mathscr{L}(E)$.*

Proof. We factor the inside of $(A + B)^{-1}$, and recall that for any operators C and D we have as a rule that $(CD)^{-1} = D^{-1}C^{-1}$:

$$(A + B)^{-1} = (A(I - A^{-1}B))^{-1} = (I - A^{-1}B)^{-1}A^{-1}.$$

We must thus ask ourselves whether $(I - A^{-1}B)^{-1}$ is in $\mathscr{L}(E)$, but since we have by assumption that $\|A^{-1}B\| < 1$, we have by the previous theorem that this is the case. □

[10]Date: November 30, 2015.

9.2 Spectrum and Resolvent

Definition 9.2.1 (Resolvent)**.** Let $A : \mathcal{D} \to E$ be an operator, possibly unbounded, where E is a normed space and $\mathcal{D}$ is a dense subspace of E. If there exists an operator $R_\lambda = (A - \lambda I)^{-1} \in \mathscr{L}(E)$, then we say that λ belongs to the ***resolvent set*** of A, written $R(A) = \{\lambda \mid \exists\, R_\lambda \in \mathscr{L}(E)\}$.

The motivation for the particular form $A - \lambda I$ naturally comes from linear algebra and the study of eigenvalues and eigenvectors.

Definition 9.2.2 (Spectrum)**.** The ***spectrum*** of A is the complement to the resolvent set, denoted $S(A)$. Thus in the real case we have $S(A) = \mathbb{R} \setminus R(A)$, and in the complex case we have $S(A) = \mathbb{C} \setminus R(A)$.

That is to say, if $\lambda \in S(A)$, then $R_\lambda = (A - \lambda I)^{-1}$ does *not* exist.

Example 9.2.3. Let us for a moment consider eigenvalues and eigenvectors, letting λ_0 being an eigenvalue. Then λ_0 belongs to the spectrum, since $A - \lambda_0 I$ have an inverse, since there exists some nonzero eigenvector $\boldsymbol{\varphi}$ such that $(A - \lambda_0 I)\boldsymbol{\varphi} = \mathbf{0}$, whence $\boldsymbol{\lambda}$ is mapped to $\mathbf{0}$ by $A - \lambda_0 I$, and so is of course $\mathbf{0}$, whereby we do not have injectivity, and so no inverse map can exist. ▲

Definitions 9.2.4 (Kernel and image)**.** Let $C : \mathcal{D} \to E$ be some mapping. Then we denote by $\ker C = \{\varphi \mid C\varphi = 0\}$ the ***kernel*** of C and by $\operatorname{im} C = \{\psi = C\varphi \mid \varphi \in \mathcal{D}\}$ the ***image*** of C.

Example 9.2.5. Consider the finite dimensional case where $C : \mathbb{R}^n \to \mathbb{R}^m$, and let $\ker C = \{\mathbf{0}\}$. This implies that $n = m$ and that $\operatorname{im} C = \mathbb{R}^n$, which in turn implies that there exists some C^{-1}.

In this finite dimensional case $S(A)$ is identical to the set of eigenvalues, since if λ is *not* an eigenvalue, we have that there does not exist any $\boldsymbol{\varphi}$ such that $(A - \lambda I)\boldsymbol{\varphi} = \mathbf{0}$, whence $\ker(A - \lambda I) = \{0\}$, and so $R_\lambda = (A - \lambda I)^{-1}$ exists. ▲

Example 9.2.6. We now consider the infinite dimensional case. Here it can happen that $\ker(A - \lambda I) = \{0\}$, but despite this we have $\operatorname{im}(A - \lambda I) \neq E$, whence there does not exist an $R_\lambda = (A - \lambda I)^{-1}$. Ergo in the infinite dimensional case $S(A)$ is not just the set of eigenvalues. ▲

Let us consider some concrete example.

Example 9.2.7. Let our space be $E = \ell_2$ and take $T_+(x_1, x_2, \ldots, x_n, \ldots) = (0, x_1, x_2, \ldots, x_n, \ldots)$ be our operator.

Clearly we have that $\ker T_+ = \{\mathbf{0}\}$, since every component must be 0 for the shifted ∞-tuple to be $\mathbf{0}$. On the other hand we have that $\operatorname{im} T_+$ is the subspace of all sequences that start with a 0, but otherwise can behave however they like. Thus T_+ is an injection (i.e. one-to-one, which is easy to see), but *not* a surjection. (Note here that the sequences in ell_2 being infinitely long is important; if they're not, then it won't even be injective.)

If we now take $A - I = T_+$, meaning that $A = T_+ + I$, we see that $\lambda = 1 \in S(A)$, but it is not an eigenvalue, because there exists no nonzero $\boldsymbol{x}$ such that $T_+\boldsymbol{x} = \mathbf{0}$. ▲

9.3 Some Elements of Topology

In the discussion that follows we will require some basic topology. Recall that if (X, ρ) is a metric space, then we have the ball $B_r(x_0) = \{x \in X \mid \rho(x_0, x) < r\}$.

Definition 9.3.1 (Open set)**.** Let (X, ρ) be a metric space. Then a set O is called ***open*** is for every $x_0 \in O$ there exists some $r > 0$ such that $B_r(x_0) \subset O$.

Examples 9.3.2. Let $X = \mathbb{R}$, and $O = (a, b)$. Then O is open, but for example $[a, b]$ is not, since any ball around the endpoints will lie partly outside the interval.

Consider not $X = \mathbb{R}^2$. Then any set without boundary is open, whereas if we include the boundary it is not. ▲

Definition 9.3.3 (Closed set)**.** Let (X, ρ) be a metric space. Then a set M is called ***closed*** if it is the complement to some open set.

Example 9.3.4. Let $X = \mathbb{R}$ and $O = (a, b)$ again. Is O closed? No, since its complement $O^c = (-\infty, a] \cup [b, \infty)$ isn't open, due to the endpoints at a and b. ▲

Exercise 9.3.5. Find a metric space in which there exist subsets that are both open and closed at the same time.

Solution. Take X to be any nonempty set and let ρ be a metric such that $\rho(x, x) = 0$, and $\rho(x, y) = 1$ if $x \neq y$, for all $x, y \in X$. That this is in fact a metric is easy to verify; the only part that isn't blindingly obvious is the triangle inequality, which holds true since $\rho(x, y) = 1 \leq 2 = \rho(x, y) + \rho(y, z)$.

Now take any $x \in X$, and pick some radius $r < 1$. Then clearly by the definition of our metric we have that the ball $B_r(x)$ contains only x, whence every single subset of X is open, because around any point x in a set O we can construct balls of radius $r < 1$ such that everything in the ball, which is just x, is in O.

Therefore since every subset is open, there complements must be as well, whence all subsets are closed as well.

(Of course this is also true, but not as interesting, if we take $X = \{x\}$ to be a singleton with any metric ρ. Or indeed (if we allow just one such subset) (X, ρ) to be *any* metric space, because $O = \varnothing$ is both open and closed.) ⧫

With this in hand we are equipped to tackle the next theorem regarding resolvents.

Theorem 9.3.6. *The resolvent set for an operator in a Banach space is always open.*

Proof. We have that $A : \mathcal{D} \to E$, where E is some Banach space. By definition then we have that $\lambda_0 \in R(A)$, meaning that $R_{\lambda_0} = (A - \lambda_0 I)^{-1} \in \mathscr{L}(E)$. Consider now some $\Delta\lambda$ such that $|\Delta\lambda| < \varepsilon$. We would like to show that $\lambda_0 + \Delta\lambda \in R(A)$, for some sufficiently small ε.

We therefore consider the operator

$$R_{\lambda_0 + \Delta\lambda} = (A - (\lambda_0 + \Delta\lambda)I)^{-1} = ((A - \lambda_0 I) - \Delta\lambda I)^{-1},$$

whence by Theorem 9.1.2 we have that this inverse exists if $A - \lambda_0 I$ has an inverse (which it does, since we assumed that $\lambda_0 \in R(A)$), and

$$\|(A - \lambda_0 I)^{-1} \Delta\lambda I\| < 1.$$

Let us therefore investigate that norm. Taking the constant $\Delta\lambda$ outside of the norm and solving for it we have that

$$|\Delta\lambda| = \frac{1}{\|(A-\lambda_0 I)^{-1} I\|} = \frac{1}{\|R_{\lambda_0}\|},$$

so we take this to be our ε, which shows that the resolvent set is open, since $R_{\lambda_0+\Delta\lambda} \in \mathscr{L}(E)$ for all $|\lambda_0 - \Delta\lambda| < \varepsilon$. □

Corollary 9.3.7. *The spectrum of an operator in a Banach set is always closed.*

Proof. This follows from the definition of a set being closed, the definition of the spectrum, and the resolvent set being closed. □

10 Lecture X[11]

10.1 More On Open and Closed Sets

Recall from last lecture that the set O is open if and only if for each $x_0 \in O$ there exists some $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subset O$, and that a set F is closed if $F = X \setminus O$, where O is open.

Theorem 10.1.1. *Let (X,ρ) be a metric space. Then the set F is closed if and only if for each convergent sequence (x_n) where $x_n \in F$, its limit also belongs to F.*

Proof. ($\Rightarrow$) We let F be closed, meaning that we must show that for every sequence (x_n) such that $x_n \to x$, where $x_n \in F$, we must have $x \in F$. To show this we assume that $x \notin F$, meaning that since $F = X \setminus O$, where O is open, we must have $x \in O$. However since O is open and $x \in O$, there exists some $\varepsilon > 0$ such that $B_\varepsilon(x) \subset O$. Therefore, since x is the limit of x_n, there must also exist some sufficiently large N such that, for all $n > N$, we have $x_n \in B_\varepsilon(x) \subset O$, whence $x_n \notin F$, which is a contradiction.

($\Leftarrow$) We now instead assume that for all convergent sequences (x_n), $x_n \in F$, we have $x_n \to x \in F$. Using this we must show that F is closed, that is that $O = X \setminus F$ is open. This means that if we have some $y \notin F$, we cannot have that $x_n \to y$ for any sequence of $x_n \in F$. Thus there must exist some $\varepsilon > 0$ such that $B_\varepsilon(y) \subset O$ since sequences (x_n) can never approach y. □

Definition 10.1.2 (Closure)**.** Let (X,ρ) be a metric space, and let A be a subset of X. Its ***closure*** is the minimal closed set that contains A, denoted $[A] = \min\{F \supset A \mid F \text{ closed}\}$.

A fairly obvious, yet useful, consequence of the definition is that if $A \subset B$, then $[A] \subset [B]$. Moreover, pairing the definition with the previous theorem we see that the closure of a set must also be the union of the set and the set of all limits of sequences in the set.

Exercise 10.1.3. Consider the metric space $(\mathbb{R}, \rho(x,y) = |x-y|)$. What is the closure of the rational numbers, of the irrational numbers, and the natural numbers?

[11]Date: December 3, 2015.

Solution. We know already that sequences of rational numbers have real number limits (see Dedekind sections). Therefore by the previous theorem $[\mathbb{Q}] = \mathbb{R}$. Similarly the closure of the irrational numbers is the set of real numbers, for the same reason.

The closure of the natural numbers is the set of natural numbers, that is to say $[\mathbb{N}] = \mathbb{N}$. This is easy to see since, for any $0 < \varepsilon < 1$ we have that the ball $B_\varepsilon(n)$ contains only n, for all $n \in \mathbb{N}$, whereby the natural numbers are closed, and clearly the closure of a closed set is itself. Of course it is also easy to see that any sequence of natural numbers can have only a natural number as its limit. ♦

Exercise 10.1.4. Consider now $X = C[a, b]$, with $A = C^{(1)}[a, b]$. What is $[A]$?

Solution. Recall that by Weierstrass theorem (page 30) we have that any element in $C[a, b]$ can be uniformly approximated using polynomials from $P[a, b]$. Moreover we clearly have $P[a, b] \subset C^{(1)}[a, b] \subset C[a, b]$, and since polynomials can approximate continuous functions uniformly, it means that $[P[a, b]] = C[a, b]$. Therefore we have that $C[a, b] = [P[a, b]] \subset [C^{(1)}[a, b]] \subset [C[a, b]] = C[a, b]$, whereby we must have that $[C^{(1)}[a, b]] = C[a, b]$ as well. ♦

Problem 10.1.5 (p-adic metric space)**.** The p-adic natural numbers are defined as follows. Let p be a fixed prime. Then for all n in $\mathbb{N}$, we define the p-adic absolute value as $|n|_p = p^{-\alpha}$, where $n = p^\alpha k$, $p \nmid k$. For example, if we take $p = 2$ we have $|2|_2 = 1/2$ since $2 = 2^1$, and $|4|_2 = 1/4$ since $4 = 2^2$. On the other hand, $|5|_2 = 1$ since $5 = 2^0 \cdot 5$. We define p-adic distance analogously to the Euclidean distance: $\rho_p(n_1, n_2) = |n_1 - n_2|_p$.

Consider now the ball $\bar{B}_\varepsilon(x) = \{y \,|\, \rho_p(x, y) \le \varepsilon\}$. Show that this ball is both open and closed. Moreover, show that the same thing is true for the sphere $S_\varepsilon(x) = \{y \mid \rho_p(x, y) = \varepsilon\}$.

Solution. We will spend some time not only to solve the problem, but to explore interesting features of the p-adic metric.

Let us show that the p-adic absolute value not only satisfies the triangle inequality (which of course it does, since it defines a metric) but also the ***strong triangle inequality***, $|x + y|_p \le \max\{|x|_p, |y|_p\}$.

Take $x = p^\alpha m$ and $y = p^\beta n$, where $\alpha, \beta, m, n \in \mathbb{N}$, and neither m nor n are divisible by p. Let us in addition assume without loss of generality that $\alpha \le \beta$. Then we can write $x + y = p^\alpha m + p^\beta n = p^\alpha(m + p^{\beta-\alpha} n)$.

Given this we must therefore have

$$|x + y|_p = |p^\alpha(m + p^{\beta-\alpha} n)|_p \le p^{-\alpha} = |x|_p.$$

Had we on the other hand had $\beta \le \alpha$ we would get $|x + y|_p \le |y|_p$, whence in general we get $|x + y|_p \le \max\{|x|_p, |y|_p\}$. Of course this maximum is smaller than or equal to the sum of the two since they are nonnegative, whence it also satisfies the triangle inequality.

Let us show using this the complement of the original ball $\bar{B}_\varepsilon(x)$ is open. To do so we pick some y outside of the ball, necessarily meaning that $\rho_p(x, y) > \varepsilon$. We now pick some $\delta < \varepsilon$ and form the ball $B_\delta(y)$ and show that the intersection between this ball and $\bar{B}_\varepsilon(x)$ is empty, meaning that the complement of the original ball is open.

Take any z in the new ball $B_\delta(y)$. Then

$$\varepsilon < |x-y|_p = |x-z+z-y|_p \leq \max\{|x-z|_p, |z-y|_p\},$$

the first step being true since y is outside $\bar{B}_\varepsilon(x)$, and the last step being the strong triangle inequality. Note now that the second term in the maximum concerns z and y, whence this is less than δ, which was taken to be less than ε, whereby it all reduces to $\varepsilon < |x-z|_p$, whereby the entire ball $B_\delta(y)$ is outside of $\bar{B}_\varepsilon(x)$, and so $\bar{B}_\varepsilon(x)$ is closed.

That the ball $\bar{B}_\varepsilon(x)$ is open follows from the remarkable claim: due to the strong triangle inequality we have the curious property that any (open or closed) ball has every point in it as its centre, which we show thus: take $x, y \in B_\varepsilon(z)$. Then by definition $\rho_p(x,z) < \varepsilon$. Moreover, by the strong triangle inequality, we have $\rho_p(x,y) \leq \max\{\rho_p(x,z), \rho_p(y,z)\} < \varepsilon$ since z is the defined centre of the ball. Therefore x is also in the ball $B_\varepsilon(y)$, which is true for any x and y in the original ball.

Thus every ball of any radius $r \leq \varepsilon$ centred on any point in $\bar{B}_\varepsilon(x)$ is contained in $\bar{B}_\varepsilon(x)$ and so it is open, and all together both closed and open.

Let us now consider the sphere $S_\varepsilon(x) = \{y \mid \rho_p(x,y) = \varepsilon\}$ instead. We take any y on this sphere and, for $\delta < \varepsilon$, form the ball $B_\delta(y)$ of radius δ around y. If we can show that all points of this ball are on the sphere, the sphere is open.

We do this as follows. Let z be an arbitrary point in $B_\delta(y)$. Then consider the distance between z and x, which we would like to be ε:

$$|z-x|_p \leq \max\{|z-y|_p, |y-x|_p\} = \varepsilon$$

since $|z-y|_p < \delta < \varepsilon$ and $|y-x|_p = \varepsilon$. On the other hand we have

$$\varepsilon = |y-x|_p \leq \max\{|z-y|_p, |x-z|_p\},$$

wherein we can guarantee that $|x-z|_p \leq \varepsilon$ is the bigger of the two, $|z-y|_p < \delta$, since the latter can be made sufficiently small by choosing $\delta < \varepsilon$ appropriately. We thereby have $\varepsilon \leq |x-z|_p \leq \varepsilon$, whence $|x-z|_p = \varepsilon$, and so $z \in B_\delta(y)$, making $S_\varepsilon(x)$ open. To see that it is also closed we note that its complement is the union of the ball $B_\varepsilon(x)$ and the same complement we considered for $\bar{B}_\varepsilon(x)$, whereby we are done. ♦

10.2 More On Separability

With the advent of the notion of closure, we can discuss separability in a new light.

Theorem 10.2.1. *The metric space (X, ρ) is separable if and only if there exists a countable set $A = \{a_1, a_2, \ldots, a_n, \ldots\}$ such that $[A] = X$.*

With this we get the following result.

Theorem 10.2.2. *Let H be a Hilbert space. Suppose that there exists a countable orthonormal basis $(\boldsymbol{e}_m)_{n=1}^{\infty}$ in H, meaning that $\langle \boldsymbol{e}_n, \boldsymbol{e}_m \rangle = \delta_{nm}$. Then H is separable.*

Proof. Let us first note that if $H = H_N$ is a finite N-dimensional space, then we are done, because then $H_N = \mathbb{C}^N$. There then exists a dense and countable subset $(\mathbb{Q} \times \mathbb{Q})^N$ such that the distance between

$$\boldsymbol{\varphi} = \sum_{n=1}^{N} c_n \boldsymbol{e}_n \in H_N \quad \text{and} \quad \boldsymbol{\varphi}_\varepsilon = \sum_{n=1}^{N} (q_{1n} + q_{2n} i) \boldsymbol{e}_n$$

is less than ε, for every ε.

For infinite dimensional spaces we use almost the same strategy, with one small modification. We know from previously that Parseval's equality must be satisfied:

$$\|\boldsymbol{\varphi}\|^2 = \sum_{n=1}^{\infty} |c_n|^2.$$

Due to convergence this then means that for every $\boldsymbol{\varphi} \in H$ we have that the partial sums

$$\boldsymbol{\varphi}_N = \sum_{n=1}^{N} c_n \boldsymbol{e}_n$$

become arbitrarily close to $\boldsymbol{\varphi}$, meaning that

$$\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_N\| = \sum_{n=N+1}^{\infty} |c_n|^2 \to 0$$

as $N \to \infty$. This in addition means that for every $\varepsilon > 0$ there exists some N such that $\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_N\| < \varepsilon$. We then have by the above argument in H_N that for every such $\boldsymbol{\varphi}_N$ there exists some vector that is arbitrarily close to it, and since these are countable, we are done. □

The converse is in fact true as well.

Theorem 10.2.3. *Let H be a separable Hilbert space. Then there exists a countable orthonormal basis $(\boldsymbol{e}_n)_{n=1}^{\infty}$ in H.*

Proof. Since H is separable we know that there exists some countable set $A = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n, \ldots\}$ which is dense in H.

Recall from linear algebra that any finite set of vectors can be made orthonormal by, for example the Gram–Schmidt process. We apply this repeatedly to the set A, producing $B = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n, \ldots\}$ where $\langle \boldsymbol{e}_n, \boldsymbol{e}_m \rangle = \delta_{nm}$.

Since A is dense in H, and each $\boldsymbol{e}_j$ by Gram–Schmidt is a linear combination of members from A, the set B is dense in H as well.

Suppose now that there exists some $\boldsymbol{\varphi} \in H$ that cannot be written as

$$\boldsymbol{\varphi} = \sum_{k=1}^{\infty} c_k \boldsymbol{e}_k.$$

This means that $\boldsymbol{\varphi}$ is orthogonal to every element of B, and in turn to every element of A. However this implies that $\boldsymbol{\varphi} = \boldsymbol{0}$, since if $\langle \boldsymbol{a}_j, \boldsymbol{\varphi} \rangle = 0$ for every j, we must have that for every $\varepsilon > 0$ some $\boldsymbol{a}_{k_\varepsilon}$ such that $\|\boldsymbol{a}_{k_\varepsilon} - \boldsymbol{\varphi}\| < \varepsilon$, which by Parsevals equality means that the sum of the squares of the coordinates c_k of $\boldsymbol{\varphi}$ must be arbitrarily small, whence $\boldsymbol{\varphi} = \boldsymbol{0}$. □

Combining the two previous results we get the following.

Theorem 10.2.4. *A Hilbert space has a countable orthonormal basis if and only if it is separable.*

Remark 10.2.5. If we have an uncountable basis $(\boldsymbol{e}_\alpha)$, $\alpha \in A$ not countable, then the sum $\sum\limits_{\alpha \in A} c_\alpha \boldsymbol{e}_\alpha$ has meaning only if only countably many c_α are nonzero.

10.3 Something More About Spectrums

We recall from the end of last lecture that the spectrum $S(A)$ is always a closed subset. So if $\lambda_n \in S(A)$, and there exists some $\lambda = \lim\limits_{n\to\infty} \lambda_n$, then $\lambda \in S(A)$ as well.

Theorem 10.3.1. *Let H be a Hilbert space (or indeed a Banach space), and let $A \in \mathscr{L}(H)$. Then $S(A) \subset B_r(0)$, where $r = \|A\|$.*

Proof. Recall from last lecture that the resolvent set is

$$R(A) = \{\lambda \mid R_\lambda = (A - \lambda I)^{-1} \in \mathscr{L}(H)\},$$

and that the spectrum is its complement. Therefore $S(A)$ is the set of all λ such that $(A - \lambda I)^{-1} \notin \mathscr{L}(H)$. Further recall from Theorem 9.1.1 that a sufficient condition for $(I - C)^{-1}$ existing is that $\|C\| < 1$. We have

$$A - \lambda I = \lambda\Big(\frac{A}{\lambda} - I\Big) = -\lambda\Big(I - \frac{A}{\lambda}\Big),$$

whence $C = A/\lambda$ and therefore we must have $\|A/\lambda\| = 1/|\lambda|\|A\| \geq 1$. Multiplying either side by the nonnegative $|\lambda|$ we thus get $\|A\| \geq |\lambda|$.

This then means that if we pick any λ such that $\|A\| \geq |\lambda|$, then $(A - \lambda I)^{-1}$ does not exist, whence $\lambda \in S(A)$. □

11 Lecture XI[12]

11.1 Something More On Metric Spaces

Recall from previous lectures the notion of separability of metric spaces. We have the following result.

Theorem 11.1.1. *Suppose that (X, ρ) is a separable metric space. Let $Y \subset X$. Then the metric space (Y, ρ) is separable as well.*

Moreover we have the following.

Theorem 11.1.2. *Let (X, ρ) be a complete metric space and let Y be a closed subset of X. Then (Y, ρ) is complete as well.*

Proof. Since (X, ρ) is complete, any Cauchy sequence (y_n) from Y must have a limit, and since Y is closed, this limit must be in Y, and therefore (Y, ρ) is complete. □

[12]Date: December 7, 2015.

Finally let us have a think about Hilbert spaces once more. A consequence of the two above theorems is the following, recalling the definition of Hilbert space.

Corollary 11.1.3. *Let L be a linear subspace of a Hilbert space H, and let L be closed. Then L is a Hilbert space. Moreover, if H is separable, then so is L.*

Moreover, recalling what we have proved previously regarding orthonormal bases of Hilbert spaces, we have this, again following from the previous theorem.

Corollary 11.1.4. *Any closed subspace of a separable Hilbert space has a countable orthonormal basis.*

11.2 Orthogonal Complement of a Closed Linear Subspace

Following the last corollary we will henceforth always work with separable Hilbert spaces.

Definition 11.2.1 (Orthogonal complement)**.** Let H be a separable Hilbert space and let L be a subspace of H. Then we denote by $L^{\perp}$ the ***orthogonal complement*** of L, defined as $L^{\perp} = \{x \in H \mid \langle y, x\rangle = 0 \ \forall y \in L\}$.

Theorem 11.2.2. *For any $L \subset H$, H being a separable Hilbert space, $L^{\perp}$ is a closed linear subspace of H.*

Proof. That $L^{\perp}$ is linear is clear, since the scalar product is linear with respect to its first component.

Next we show that it is closed. Let $x_n \in L^{\perp}$, meaning that $\langle x_n, y\rangle = 0$ for all $y \in L$. Moreover suppose that $x_n \to x$ as $n \to \infty$. Then $0 = \langle x_n, y\rangle \to \langle x, y\rangle = 0$, so the limit x also belongs to $L_{\perp}$. □

Theorem 11.2.3. *Let L be a closed linear subspace of a Hilbert space H. Then any $z \in H$ can be uniquely represented as $z = a + b$ where $a \in L$ and $b \in L^{\perp}$.*

Proof. From Corollary 11.1.4 we have that, say, $(e_n)_{n=1}^{\infty}$ is an orthonormal basis of L and that $(f_n)_{n=1}^{\infty}$ is an orthonormal basis of $L^{\perp}$. We would like to show that (e_n, f_n) is a basis of the whole space H.

To do this, suppose that it is not. Then there must exist some $y \in H$ such that $\langle y, e_n\rangle = 0$ and $\langle y, f_n\rangle = 0$ for all n. The first scalar product implies that $y \in L^{\perp}$, whereas the second one implies that $y \in (L^{\perp})^{\perp} = L$. Therefore y must be orthogonal to itself, whence $y = 0$ is the only option, and so all $x \in H$ can be written uniquely as

$$x = \sum_{n=1}^{\infty} a_n e_n + \sum_{n=1}^{\infty} b_n f_n.$$ □

With the notion of orthogonal complements comes the notion of orthogonal projections (cf. linear algebra).

Definition 11.2.4 (Orthogonal projection)**.** Let L be a closed linear subspace of a Hilbert space H. By the previous theorem we have that any $z \in H$ can be written as $z = a+b$, where $a \in L$ and $b \in L^{\perp}$. We denote by P_L the ***orthogonal projection*** on L; $P_L z = a$.

It is clear that we may write

$$P_L z = \sum_{n=1}^{\infty} \langle z, e_n \rangle e_n,$$

where (e_n) is some basis in L.

Theorem 11.2.5. *Let P be an orthogonal projection on some closed linear subspace L of a Hilbert space H. Then P is* **idempotent***, meaning that $P^2 = P$, and Hermitian, meaning that $P^* = P$.*

Proof. That it is idempotent is clear from the definition. That it is Hermitian is more interesting. We show it in two ways, using $z = a + b$ and $y = c + d$, where these are decompositions in the way described earlier:

$$\begin{aligned}\langle Pz, y \rangle &= \Big\langle \sum_{n=1}^{\infty} \langle z, e_n \rangle e_n, y \Big\rangle = \sum_{n=1}^{\infty} \langle z, e_n \rangle \langle e_n, y \rangle = \sum_{n=1}^{\infty} \langle e_n, y \rangle \langle z, e_n \rangle \\ &= \sum_{n=1}^{\infty} \overline{\langle y, e_n \rangle} \langle z, e_n \rangle = \Big\langle z, \sum_{n=1}^{\infty} \langle y, e_n \rangle e_n \Big\rangle = \langle z, Py \rangle.\end{aligned}$$

Alternatively, we have that $Pz = a$ and $Py = c$. Therefore by the linearity of the scalar product

$$\langle Pz, y \rangle = \langle a, c + d \rangle = \langle a, c \rangle + \langle a, d \rangle,$$

however $\langle a, d \rangle = 0$ since $a \in L$ and $d \in L^{\perp}$. Similarly,

$$\langle P^* z, y \rangle = \langle z, Py \rangle = \langle a + b, c \rangle = \langle a, c \rangle + \langle b, c \rangle,$$

where again $\langle b, c \rangle$ vanishes since they are in L and $L^{\perp}$ respectively. Thus $\langle Pz, y \rangle = \langle P^* z, y \rangle$ whence $P = P^*$. □

The converse is true as well.

Theorem 11.2.6. *Let $P : H \to H$ be a linear operator in a Hilbert space H, such that P is idempotent and Hermitian. Then P is an orthogonal projection.*

Proof. Let us first denote by L the image of P on the whole space H. For any $a \in L$ we of course have $Pa = a$. Hence if we take any $H \ni z = a + b$, this being the same sort of decomposition as above, we want to have $Pz = Pa + Pb = a$, whence we must show that $Pb = 0$.

To do this we first realise that $Pb \in L$, and since $b \in L^{\perp}$, we have $\langle Pb, b \rangle = 0$. Since P is idempotent we also have $\langle P^2 b, b \rangle = 0$, but since P is Hermitian this is the same as $\langle Pb, Pb \rangle = 0$, whence the norm of Pb is 0, which is true if and only if $Pb = 0$. □

Combining the two previous theorems we get the following.

Theorem 11.2.7. *Let H be a Hilbert space and $P : H \to H$. Then P is an orthogonal projector if and only if it is both idempotent and Hermitian.*

11.3 Operator Representation of Quantum States

In quantum mechanics a Schrödinger equation is an equation of the form

$$\frac{\hbar}{i}\frac{\partial \psi}{\partial t} = \mathcal{H}\psi,$$

where $\psi|_{t=0} = \psi_0$. Here ψ_0 is a quantum state, meaning that $\langle \psi_0, \psi_0 \rangle = 1$.

Taking $\psi \mapsto \rho_\psi$ as the (one dimensional) projection on ψ, meaning that $\rho_\psi \varphi = \langle \varphi, \psi \rangle \psi$, we aim to rewrite the Schrödinger equation as an equation of operators.

We first solve the differential equation above as per usual, by first rewriting it as

$$\frac{\partial \psi}{\partial t} = \Big(\frac{i\mathcal{H}}{\hbar}\Big)\psi,$$

giving us

$$\psi(t) = e^{\frac{it}{\hbar}\mathcal{H}}\psi_0,$$

where the exponential is usually denoted U_t, called the ***evolution operator*** and $\mathcal{H}$ is called the ***generator of evolution***, which lets us write it shortly as $\psi(t) = U_t\psi_0$.

Recalling that $\mathcal{H}$ is a quantum observable and that they are always Hermitian, and that exponentials can be written as sums quite neatly, we compute U_t^*:

$$\begin{aligned}\langle U_t\psi_0, \psi_0\rangle &= \Big\langle \sum_{n=0}^{\infty} \frac{\left(\frac{it}{\hbar}\right)^n}{n!}\mathcal{H}^n\psi_0, \psi_0 \Big\rangle = \sum_{n=0}^{\infty} \frac{\left(\frac{it}{\hbar}\right)^n}{n!}\langle \mathcal{H}^n\psi_0, \psi_0\rangle \\ &= \Big\langle \psi_0, \sum_{n=0}^{\infty} \frac{\left(\frac{-it}{\hbar}\right)^n}{n!}\mathcal{H}^n\psi_0 \Big\rangle = \langle \psi_0, e^{\frac{-it}{\hbar}\mathcal{H}}\psi_0\rangle,\end{aligned}$$

where in the last step conjugates have been taken of the terms in the sum. Thus

$$U_t^* = e^{\frac{-it}{\hbar}\mathcal{H}},$$

whereby we have $U_t^*U_t = U_tU_t^* = I$, meaning that $U_t^* = U_t^{-1}$. From this we also have that $\langle U_t\varphi_1, U_t\varphi_2\rangle = \langle \varphi_1, U_t^*U_t\varphi_2\rangle = \langle \varphi_1, \varphi_2\rangle$, showing that U_t preserves scalar products.

Definition 11.3.1 (Unitary operator)**.** Let H be a Hilber space and let $U : H \to H$ be an operator that has an inverse and preserves scalar products. Then U is called a ***unitary operator***.

This leads to quantum physicists saying things like "quantum evolution is unitary."

12 Lecture XII[13]

12.1 Spectral Decomposition of Self-adjoint Operators

Consider the finite dimensional case $\dim H = n$, and let $\mathscr{L}(H) \ni A = A^*$ be an Hermitian operator. Then as discussed Example 9.2.5, page 37 together

[13]Date: December 10, 2015.

with some basic linear algebra, the eigenvalues are real and the eigenvectors for distinct eigenvalues λ_j are all orthogonal. Moreover if we have $\lambda_1 < \lambda_2 < \ldots < \lambda_k$ with $\dim L_{\lambda_j} = m_j$, L_{λ_j} being the eigen subspace for the eigenvalue λ_j, then

$$\sum_{j=1}^{k} m_k = n.$$

Let P_λ denote the projector on the subspace L_λ, then

$$A = \sum_{j=1}^{k} \lambda_j P_{\lambda_j}, \tag{12.1.1}$$

in other words

$$A\varphi = \sum_{j=1}^{k} \lambda_j P_{\lambda_j} \varphi,$$

for all $\varphi \in H$, which we motivate as follows. From the right-hand side we have

$$\varphi = \sum_{j=1}^{k} P_{\lambda_j} \varphi,$$

since, because distinct eigenvalues are orthogonal, the eigen subspaces for distinct eigenvalues are orthogonal as well.

Let $\left(e_s^{(j)}\right)$ be an orthonormal basis of L_{λ_j}, then

$$P_{\lambda_j}\varphi = \sum_{s=1}^{k} \langle \varphi, e_s^{(j)} \rangle e_s^{(j)},$$

whereby

$$\varphi = \sum_{j=1}^{k} P_{\lambda_j}\varphi = \sum_{j=1}^{k}\sum_{s=1}^{k} \langle \varphi, e_s^{(j)} \rangle e_s^{(j)},$$

or equivalently $I = \sum\limits_{j=1}^{k} P_{\lambda_j}$. Thus by the linearity of the operator A,

$$A\varphi = A\sum_{j=1}^{k} P_{\lambda_j}\varphi = \sum_{j=1}^{k} A P_{\lambda_j}\varphi = \sum_{j=1}^{k} \lambda_j P_{\lambda_j}\varphi.$$

We would like to rewrite the expression in (12.1.1) in the form of an integral, since when we attempt to generalise it to the infinite dimensional case, the sum will naturally become an integral.

To do so, let

$$E^A(\lambda) = \sum_{\lambda_j < \lambda} P_{\lambda_j}, \tag{12.1.2}$$

which we claim is again a projector (i.e. by Theorem 11.2.7 last lecture, it is both Hermitian and idempotent). It being Hermitian is clear by the linearity of scalar products with respect to the first component and P_{λ_j} themselves being projectors and so Hermitian:

$$\begin{aligned}\big\langle E^A(\lambda)\varphi, \psi\big\rangle &= \Big\langle \sum_{\lambda_j<\lambda} P_{\lambda_j}\varphi, \psi\Big\rangle = \sum_{\lambda_j<\lambda} \langle P_{\lambda_j}\varphi, \psi\rangle = \sum_{\lambda_j<\lambda} \langle \varphi, P_{\lambda_j}\psi\rangle \\ &= \Big\langle \varphi, \sum_{\lambda_j<\lambda} P_{\lambda_j}\psi\Big\rangle = \big\langle \varphi, E^A(\lambda)\psi\big\rangle,\end{aligned}$$

whereas the idempotency follows from the next theorem.

Theorem 12.1.1. *Let H be a Hilbert space and L_1 and L_2 two subspaces of H. Then L_1 and L_2 are orthogonal if and only if the projectors P_{L_1} and P_{L_2} on the two subspaces commute.*

Proof. ($\Rightarrow$) We assume that L_1 and L_2 are orthogonal subspaces, and we look to prove that $P_{L_1}P_{L_2} = P_{L_2}P_{L_1}$.

Let (e_i) be an orthonormal basis of L_1 and let (f_j) be an orthonormal basis of L_2. Since L_1 and L_2 are orthonormal, $\langle e_i, f_j\rangle = 0$ for all i and j. Moreover we have

$$P_{L_1}\varphi = \sum_i \langle \varphi, e_i\rangle e_i \quad \text{and} \quad P_{L_2}\varphi = \sum_j \langle \varphi, f_j\rangle f_j,$$

from which by the linearity of scalar products we get

$$P_{L_2}P_{L_1}\varphi = \sum_j \Big\langle \sum_i \langle \varphi, e_i\rangle e_i, f_j\Big\rangle f_j = \sum_j\sum_i \langle \varphi, e_i\rangle\langle e_i, f_j\rangle f_j = 0$$

since $\langle e_i, f_j\rangle = 0$, and similarly

$$P_{L_1}P_{L_2}\varphi = \sum_i \Big\langle \sum_j \langle \varphi, f_j\rangle f_j, e_i\Big\rangle e_i = \sum_i\sum_j \langle \varphi, f_j\rangle\langle f_j, e_i\rangle e_i = 0,$$

for the same reason, whence $P_{L_1}P_{L_2} = P_{L_2}P_{L_1} = 0$.

($\Leftarrow$) If we now instead assume that $P_{L_1}P_{L_2} = P_{L_2}P_{L_1}$, meaning that we have equality between the two double sums above, for all φ, we must have $\langle e_i, f_j\rangle = \langle f_j, e_i\rangle = 0$ since the remaining terms in the sums can take on arbitrary values. ☐

Recalling (12.1.2), we now use it to express P_{λ_j} in a different way: $P_{\lambda_j} = E^A(\lambda_{j+1}) - E^A(\lambda_j)$, using which we reformulate (12.1.1) as

$$A = \sum_{j=1}^{k} \lambda_j(E^A(\lambda_{j+1}) - E^A(\lambda_j)),$$

which after an interlude on integration theory we will recognise as something meaningful.

12.1.1 Riemann–Stieltjes Integral

Recalling that we define the Riemann integral as

$$\int_a^b f(\lambda)\,d\lambda = \lim_{\Delta\to 0} \sum_{j=1}^{N-1} f(\lambda_j)(\lambda_{j+1} - \lambda_j),$$

where $\Delta = \max_{1\le j<N}\{\lambda_j + 1 - \lambda_j\}$, and $a = \lambda_1 < \lambda_2 < \ldots < \lambda_N = b$ is a partition of the interval $[a, b]$, assuming the limit exists.

The Riemann–Stieltjes integral is defined analogously, with a slight modification. Take $E(\lambda)$ to be a real-valued function (for now) that is monotonously increasing, i.e. that $E(\lambda') \ge E(\lambda)$ whenever $\lambda' \ge \lambda$. In other words, the so-called increment $E(\lambda') - E(\lambda)$ is nonnegative.

The Riemann-Stieltjes integral is then defined as

$$\int_a^b f(\lambda)\,dE(\lambda) = \lim_{\Delta\to 0}\sum_{j=1}^{N-1} f(\lambda_j)(E(\lambda_{j+1}) - E(\lambda_j)), \tag{12.1.3}$$

where now $\Delta = \max_{1\le j<N}\{E(\lambda_{j+1}) - E(\lambda_j)\}$, again provided the limit exists.

Note that if we take $E(\lambda) = \lambda$, this becomes the Riemann integral.

We now wish to generalise the concept of a Riemann–Stieltjes integral to have $E(\lambda)$ take its values in the space of linear, bounded operators on a Hilbert space: $E(\lambda) \in \mathscr{L}(H)$ for all λ.

The conditions on E are then the following:

(i) For all λ, $E(\lambda)$ is Hermitian (this to mimic E being real-valued above);

(ii) For all $\lambda' \ge \lambda$, we have $E(\lambda') \ge E(\lambda)$.

Of course for this to make sense we must define the notion of nonnegative operators.

Definition 12.1.2 (Nonnegative operator)**.** Let H be a Hilbert space and let $C \in \mathscr{L}(H)$. Then C is called ***nonnegative***, denoted $C \ge 0$, if $\langle C\varphi, \varphi\rangle \ge 0$ for all $\varphi \in H$.

We discuss the connection to linear algebra in the following example.

Example 12.1.3. Consider a finite dimensional Hilbert space H, say $\dim H = n$. Then we can represent any linear operator in this space as a matrix

$$C = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix},$$

which by Sylvester's criterion from linear algebra we know is positive semidefinite if and only if the determinants of the upper left $m \times m$ matrices are all nonnegative, for all $1 \le m \le n$. ▲

Example 12.1.4. Taking $H = L_2(\mathbb{R})$, the operator $C\varphi(x) = x^2\varphi(x)$ is nonnegative since

$$\langle C\varphi, \varphi\rangle = \int x^2|\varphi(x)|^2\,dx \ge 0.$$

Indeed, taking $C\varphi(x) = \gamma(x)\varphi(x)$ for any $\gamma(x) \ge 0$ is nonnegative. ▲

Thereby for $E(\lambda)$ satisfying conditions (i) and (ii) above and $f : \mathbb{R} \to \mathbb{R}$ we define the Riemann-Stieltjes integral as per (12.1.3), if this limit exists.

Remark 12.1.5. Note that if $E(\lambda)$ is discrete (i.e. it's a jump function), then the integral is just the same as the sum we produced earlier.

Thereby our now long studied expression (12.1.1) becomes

$$A = \int_{\lambda_1}^{\lambda_k} \lambda \, dE^A(\lambda),$$

where λ_1 and λ_k as before are the smallest and largest eigenvalues, respectively, since we wrote A as precisely the sort of sum seen in our definition of the Riemann–Stieltjes integral from before.

This is the simplest case of spectral decomposition of A.

12.2 The Infinite Dimensional Case

Let us now consider infinite dimensional Hilbert spaces H, again with $A \in \mathscr{L}(H)$ and A Hermitian. Select from H an orthonormal basis, call it (e_j). Moreover consider the subspace generated by the first K basis elements,

$$H_K = \Big\{ x = \sum_{j=1}^{K} x_j e_j \Big\},$$

with $\pi_K = P_{H_K}$ being the projector on this space. Let us now study

$$A_K = \pi_K A \pi_K,$$

an operator from H to H_K (although we can always restrict it to $A_K : H_K \to H_K$), called a finite dimensional dressing of A.

Now recalling that $(ABC)^* = C^* B^* A^*$ (from Lemma 7.2.1, page 27), we see immediately that A_K is Hermitian, whence we may call

$$A_K = \int_{S(A_K)} \lambda \, dE^{A_K}(\lambda),$$

since A_K is a finite dimensional operator.

This leads us nicely to one of the main results of functional analysis, on the spectral decomposition of arbitrary bounded Hermitian operators in Hilbert spaces, although presented without proof.

Theorem 12.2.1. *Let H be an infinite dimensional Hilbert space and let $A \in \mathscr{L}(H)$ be an Hermitian operator. Then the limit*

$$\lim_{K \to \infty} E^{A_K}(\lambda) = E^A(\lambda)$$

exists and moreover is

(*i*) *an orthogonal projection,*

(*ii*) *is monotonously increasing,*

and finally (*iii*) *is*

$$A = \int_{S(A)} \lambda \, dE^A(\lambda).$$

Note that the integral in the infinite dimensional case doesn't have the same elegant representation as a sum as the finite dimensional case.

12.3 Functions of Hermitian Operator

Recall how previously we discussed analytic functions of bounded operators. Using the previous result we are able to study more general functions of Hermitian operators.

Take $f : S(A) \to \mathbb{R}$, using which we define

$$f(A) = \int\limits_{S(A)} f(\lambda)\, dE^A(\lambda)$$

as the limit of the corresponding Riemann–Stieltjes integral.

Example 12.3.1. Take $g_\mu(\lambda)$ to be the step function

$$g_\mu(\lambda) = \begin{cases} 1, & \text{if } \lambda \le \mu, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$g_\mu(A) = \int\limits_{S(A)} g_\mu(A)\, dE^A(\lambda) = \int_{-\infty}^{\mu} dE^A(\lambda) = E^A(\mu) - E^A(-\infty) = E^A(\mu)$$

since $E^A(-\infty) = 0$ due to $A \in \mathscr{L}(H)$ implying that $S(A) \subset B_{\|A\|}(0)$ (see Theorem 10.3.1, page 43). ▲

This is one of the most useful formulae in function analysis, in particular once we during next lecture generalise it to the unbounded case.

References

[KF20] A. N. Kolmogorov, S. V. Fomin. *Introductory Real Analysis. Dover Publications*, 416 pages, New edition, 2000.

Notations

$\lvert\cdot\rvert$	Absolute value
$\{\cdot,\cdot\}$	Anti-commutator
$\bar{B}_r(a)$	Closed ball of radius r centred on a
$\boldsymbol{x}$	Vector
$\mathbb{C}$	Complex numbers
$[\cdot]$	Closure of a set
$[\cdot,\cdot]$	Commutator
$\overline{z}$	Conjugate of z
$\mathcal{D}$	Domain of definition
$\mathcal{D}_A$	Natural domain of definition
$\langle\cdot,\cdot\rangle$	Duality notation
$\langle A\rangle_\psi$	Average of A with respect to state ψ
ℓ_p	Space of sequences with finite p-norm
$\hbar$	Planck constant
$\mathcal{H}$	Generator of evolution
im	Image
ker	Kernel
$\leq$	Partial order relation
$\mathscr{L}(E_1, E_2)$	Space of linear operators between E_1 and E_2
$\mathbb{N}$	Natural numbers
$\lVert\cdot\rVert$	Norm
$\mathbb{Q}$	Rational numbers
$\mathbb{R}$	Real numbers
$\rho(x,y)$	Metric; distance between x and y
$\langle\cdot,\cdot\rangle$	Scalar product
σ^2	Dispersion
A^*	Adjoint operator of A
$B_r(a)$	Open ball of radius r centred on a
$C[a,b]$	Space of smooth functions on $[a,b]$

$C^{(n)}[a,b]$	Space of n times differentiable functions on $[a,b]$
c_0	Space of null sequences
E'	Dual space to E
$L^{\perp}$	Orthogonal complement of L
$L_2[a,b]$	Space of absolutely square integrable functions on $[a,b]$
$P[a,b]$	Space of polynomials on $[a,b]$
P_L	Orthogonal projection on L
$R(A)$	Resolvent set of A
$S(A)$	Spectrum of A
$S_r(a)$	Sphere of radius r centred on a
U_t	Evolution operator

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