Lecture Notes in Harmonic Analysis

Lectures by Dr Charles Moore

Throughout these notes, □ signifies end proof, ▲ signifies end of example, and ■ marks the end of exercise.

Table of Contents

Notes by Jakob Streipel. Last updated April 22, 2020.

Lecture 1 Fourier Analysis

1.1 Preliminaries and definitions

Definition 1.1.1 (Fourier coefficients, Fourier series)**.** Let $f \in L^1([-\pi,\pi))$, i.e.,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|\, dx < \infty.$$

For $n \in \mathbb{Z}$, we define the nth ***Fourier coefficient*** of f to be

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\, dt.$$

The *formal sum*

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{int}$$

is called the ***Fourier series*** of f.

When we write

$$f \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{int}$$

we mean *nothing more or less* than the coefficients in the series are the Fourier coefficients of f—we do not imply any sort of convergence (hence formal sum).

We will be interested, amongst other things, in

$$S_n f(x) = \sum_{k=-n}^{n} \hat{f}(k)e^{ikx},$$

called the nth ***partial sum*** of f. We want to study when $S_n f \to f$, and in what sense this converges.

For instance,

(*i*) does $S_n f(x) \to f(x)$ for all x, i.e., ***pointwise***?

(*ii*) does $S_n f(x) \to f(x)$ ***almost everywhere***, i.e., there exists a set E with $|E| = 0$ such that $S_n f(x) \to f(x)$ for all $x \notin E$?

(*iii*) does $S_n f \to f$ ***uniformly***? That is, given $\varepsilon > 0$ there exists some N such that for $n \geq N$, we have $|S_n f(x) - f(x)| < \varepsilon$ for all $x \in [-\pi,\pi)$.

(*iv*) does $S_n f \to f$ ***almost uniformly***? By this we mean, given $\varepsilon > 0$, there exists a set $E \subset [-\pi,\pi)$ such that $|E| < \varepsilon$ and $S_n f(x) \to f(x)$ uniformly on $[-\pi,\pi) \setminus E$.

(*v*) does $S_n f \to f$ in L^p norm? I.e.,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |S_n f(x) - f(x)|^p\, dx \to 0$$

as $n \to \infty$.

Date: January 21st, 2020.

(*vi*) does $S_n f \to f$ ***in measure***? I.e., given $\varepsilon > 0$, there exists some N such that for $n \geq N$,

$$|\{ x \in [-\pi, \pi) \mid |S_n f(x) - f(x)| > \varepsilon \}| < \varepsilon.$$

All by way of saying: there are *many* modes of convergence, and one can investigate any and all of them (though we likely will not).

Other questions one can ask of these objects are, for instance:

(*i*) Given $f \in L^1$, what can we say about the sequence $\{ \hat{f}(n) \}$? That is, what properties of f lead to properties of $\{ \hat{f}(n) \}$? For instance, if $f > 0$, what about $\hat{f}(n)$? What is f is continuous, or twice differentiable?

(*ii*) Conversely, given a sequence $\{ a_n \}$, does there exist a function f with $\hat{f}(n) = a_n$?

We can also ask these same questions in more general settings. For instance:

Definition 1.1.2 (Fourier coefficients of measure)**.** Suppose μ is a positive, finite measure on $[-\pi, \pi)$. Define

$$\hat{\mu}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-int} \, d\mu(t),$$

the ***Fourier coefficients of the measure*** μ.

What we have above is a special case of this, since if f is a function in L^1, then $f(t)\,dt$ is a measure, namely by defining

$$\mu(E) = \int_E f(t)\,dt.$$

It is possible to abstract this much further, too. Let G be a ***locally compact abelian group***, often abbreviated ***LCA group***. (In other words, G is a topological group, i.e., a group endowed with a topology, and the group operations $G \times G \to G$ by $(a, b) \mapsto a + b$ and $G \to G$ by $a \mapsto -a$ are continuous, and in particular this topology is locally compact.) Then there exists a (unique up to scalar) measure μ on G such that $\mu(E + x) = \mu(E)$ for all Borel sets E and $x \in G$ (i.e., the measure is translation invariant). This measure is called the ***Haar measure*** on G.

Definition 1.1.3 (Character)**.** A mapping $\gamma\colon G \to \mathbb{C}$ is called a ***character*** if $|\gamma(x)| = 1$ for all $x \in G$ and $\gamma(x + y) = \gamma(x)\gamma(y)$ (i.e., γ is a homomorphism).

Example 1.1.4. On $[-\pi, \pi)$, $\frac{dx}{2\pi}$ is a Haar measure, and the (continuous) characters are $\gamma(x) = e^{inx}$. ▲

Definition 1.1.5 (Dual group)**.** Let G be a locally compact abelian group. We define the ***dual group*** Γ of to be the group of all continuous characters on G. Given $f \in L^1(G)$ and $\gamma \in \Gamma$, set

$$\hat{f}(\gamma) = \int_G f(x)\gamma(-x)\,d\mu(x).$$

So $\hat{f}$ is a function on Γ.

Example 1.1.6. In the case where $G = [-\pi, \pi)$, we therefore have $\Gamma \cong \mathbb{Z}$, since we can identify each e^{inx} (uniquely) by n. ▲

1.2 Elementary facts

Proposition 1.2.1 (Riemann–Lebesgue lemma)**.** *If $f \in L^1([-\pi,\pi))$, then $|\hat{f}(n)| \to 0$ as $n \to \pm\infty$.*

Proof. We prove this mostly because it demonstrates a useful strategy: prove something for characteristic functions of intervals; show that it works also for linear combinations of such (i.e., step functions), and use the fact that step functions are (hopefully) dense in whatever space we are concerned with.

In particular: let $f(x) = \chi_{(a,b)}(x)$. Then

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \chi_{(a,b)}(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_a^b e^{-inx}\,dx \\ &= \frac{1}{2\pi}\frac{e^{-inx}}{-in}\bigg|_a^b = \frac{1}{2\pi}\frac{e^{-inb}-e^{-ina}}{-in},\end{aligned}$$

which since the denominator is bounded goes to 0 as $n \to \pm\infty$. Hence the proposition is true for characteristic functions of intervals, and since very step in this calculation is linear, it is true for linear combinations thereof, meaning step functions.

Since step functions are dense in L^1, if $f \in L^1([-\pi,\pi))$ there exists for any $\varepsilon > 0$ some step function s such that $\|f - s\|_1 < \varepsilon$. Since s is a step function and the proposition holds for those, there exists some N such that for $|n| \geq N$, $|\hat{s}(n)| < \varepsilon$.

Hence for $|n| \geq N$,

$$\begin{aligned}|\hat{f}(n)| &\leq |\hat{f}(n) - \hat{s}(n)| + |\hat{s}(n)| < \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x)-s(x))e^{-inx}\,dx\right| + \varepsilon \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)-s(x)|\,dx + \varepsilon = \|f-s\|_1 + \varepsilon < \varepsilon + \varepsilon = 2\varepsilon. \qquad \square\end{aligned}$$

Consider $f \in L^1([-\pi,\pi))$. We can extend this periodically to a function $f^\star$ on $\mathbb{R}$. That is, for $x \in \mathbb{R}$, there exists some $n \in \mathbb{Z}$ such that $x - 2n\pi \in [-\pi,\pi)$, and so we define $f^\star(x) = f(x + 2\pi n)$.

Generally speaking, we will abuse notation and just call f by $f^\star$.

In particular, this means that

$$\frac{1}{2\pi}\int_{c-\pi}^{c+\pi} f(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

for any c, since a function thus extended is 2π periodic.

This immediately gives us the following:

Proposition 1.2.2. *Let $f \in L^1([-\pi,\pi))$.*

(i) If $y \in \mathbb{R}$ and $g(x) = f(x-y)$, then $\hat{g}(n) = e^{-iyn}\hat{f}(n)$.

(ii) If $m \in \mathbb{Z}$ and $g(x) = e^{imx}f(x)$, then $\hat{g}(n) = \hat{f}(n-m)$.

Proof. Both of these are direct computations. $\square$

Definition 1.2.3 (Convolution)**.** Let $f, g \in L^1([-\pi, \pi))$. Define by

$$f * g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y) g(y)\, dy$$

the ***convolution*** of f and g.

Proposition 1.2.4. *If $f, g \in L^1([-\pi, \pi))$, then $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$.*

Lecture 2 L^2 Results

First let us remark that the Riemann–Lebesgue lemma is not true for measures: consider δ_0 on $[-\pi, \pi)$, i.e., $\delta_0(E) = 1$ if $0 \in E$, else $\delta_0(E) = 0$.

Then

$$\hat{\delta}_0(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-int}\, d\delta_0(t) = \frac{1}{2\pi}$$

uniformly for all n, which clearly does not go to 0.

2.1 Convolutions

Let us prove Proposition 1.2.4 from last time.

Proof. It is direct computation:

$$\begin{aligned} \widehat{f * g}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f * g(t) e^{-int}\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-y) g(y)\, dy e^{-int}\, dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-y) g(y) e^{-in(t-y)} e^{-iny}\, dy\, dt. \end{aligned}$$

Switching the order of integration (everything's finite, so no worries) we get

$$\begin{aligned} \widehat{f * g}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-y) g(y) e^{-in(t-y)} e^{-iny}\, dy\, dy \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(y) e^{-iny} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-y) e^{-in(t-y)}\, dt\, dy \\ &= \hat{f}(n) \frac{1}{2\pi} \int_{-\pi}^{\pi} g(y) e^{-iny}\, dy = \hat{f}(n)\hat{g}(n). \end{aligned}$$

□

Proposition 2.1.1 (Young's inequality)**.** *Let $1 \leq p \leq \infty$. If $f \in L^p([-\pi, \pi))$ and $g \in L^1([-\pi, \pi))$, then $\|f * g\|_p \leq \|f\|_p \|g\|_1$.*

Proof. First note that if $\|g\|_1 = 0$, then both sides of the inequality are trivially 0, so let us assume $\|g\|_1 \neq 0$.

First, let us take $1 \leq p < \infty$. Computing, we have

$$\begin{aligned} \|f * g\|_p^p &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f * g(x)|^p\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y) g(y)\, dy \right|^p dx \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left| f(x-y) \frac{g(y)\, dy}{\|g\|_1 2\pi} \right|^p \|g\|_1^p\, dx. \end{aligned}$$

Notice how $\frac{g(y)\, dy}{\|g\|_1 2\pi}$ is a measure of total mass 1, so by Jensen's inequality[1], since

Date: January 23rd, 2020.

[1] Saying, for $\mu(\Omega) = 1$ and φ convex, $\varphi(\int_\Omega f\, d\mu) \leq \int_\Omega \varphi(f)\, d\mu$.

$\varphi(s) = s^p$ is convex for $1 \leq p < \infty$, we have

$$\begin{aligned}
\|f * g\|_p^p &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(x-y)|^p \frac{|g(y)|\,dy}{\|g\|_1 2\pi}\,dx\|g\|_1^p \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x-y)|^p\,dx\frac{|g(y)|}{\|g\|_1}\,dy\|g\|_1^p \\
&= \|f\|_p^p\|g\|_1^p\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{g(y)\,dy}{\|g\|_1 2\pi} = \|f\|_p^p\|g\|_1^p.
\end{aligned}$$

Taking pth roots we are done.

This leaves $p = \infty$. We have $|f(x)| \leq \|f\|_\infty$ uniformly, so

$$\begin{aligned}
|f * g(x)| &= \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)g(y)\,dy\Big| \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x-y)||g(y)|\,dy \leq \|f\|_\infty\|g\|_1.
\end{aligned}$$

□

Proposition 2.1.2. *If $f \in L^1([-\pi,\pi))$ and $f' \in L^1([-\pi,\pi))$, then*

$$\hat{f}'(n) = in\hat{f}(n).$$

Proof. Again straightforward computation, this time with some integration by parts thrown in:

$$\begin{aligned}
\hat{f}'(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}\,dx = \frac{f(x)e^{-inx}}{2\pi}\Big|_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(-in)e^{-inx}\,dx \\
&= 0 + in\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = in\hat{f}(n),
\end{aligned}$$

where in the middle step we have used the periodic extension of f to see that $f(-\pi) = f(\pi)$. □

2.2 L^2 results

First note that $L^2([-\pi,\pi)) \subset L^1([-\pi,\pi))$.

We model much of this discussion from the intuition gained from finite dimensional inner product spaces:

Example 2.2.1. Let V be a finite dimensional inner product space, and let $e_1, e_2, \ldots, e_n$ be an orthonormal basis for V. In other words, we have an inner product $\langle\cdot,\cdot\rangle$ with the property that $\langle e_i, e_j\rangle = \delta_{ij}$ (where δ is the Kronecker delta). Like all inner products this induces a norm, $\|v\| = \langle v, v\rangle^{1/2}$.

For any $v \in V$ we can write $v = a_1e_1 + a_2e_2 + \ldots + a_ne_n$ for some $a_1, a_2, \ldots, a_n$, and we have

$$\langle v, e_j\rangle = a_1\langle e_1, e_j\rangle + a_2\langle e_2, e_j\rangle + \ldots + a_n\langle e_n, e_j\rangle = a_j,$$

so

$$v = \sum_{j=1}^{n}\langle v, e_j\rangle e_j.$$

▲

Now as it happens $L^2([-\pi,\pi))$ is an inner product space (though infinite dimensional), with the inner product

$$\langle f,g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}\,dt,$$

and the characters $\{\, e^{inx} \,\}_{n\in\mathbb{Z}}$ happen to be orthogonal, i.e., $\langle e^{inx}, e^{imx}\rangle = \delta_{mn}$.

Proposition 2.2.2. *If* $\{\, a_n \,\} \subset \mathbb{C}$, *then*

$$\Big\| \sum_{n=-N}^{N} a_n e^{inx} \Big\|_2^2 = \sum_{n=-N}^{N} |a_n|^2.$$

Proof. Straightforward computation—expand the square of the sum as a product of conjugates; mixed terms disappear. □

Proposition 2.2.3. *Let* $f \in L^2([-\pi,\pi))$. *Then for any* N,

$$\Big\| \sum_{n=-N}^{N} e^{inx}\hat{f}(n) - f \Big\|_2^2 = \|f\|_2^2 - \sum_{n=-N}^{N} |\hat{f}(n)|^2.$$

Proof. By writing the norm as the inner product and expanding, we get

$$\Big\| \sum_{n=-N}^{N} e^{inx}\hat{f}(n) - f \Big\|_2^2 = \Big\| \sum_{n=-N}^{N} e^{inx}\hat{f}(n) \Big\|_2^2 + \|f\|_2^2 - 2\,\mathrm{Re}\Big\langle \sum_{n=-N}^{N} e^{inx}\hat{f}(n), f \Big\rangle.$$

Now

$$\begin{aligned} 2\,\mathrm{Re}\Big\langle \sum_{n=-N}^{N} e^{inx}\hat{f}(n), f \Big\rangle &= 2\,\mathrm{Re}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=-N}^{N} e^{inx}\hat{f}(n)\overline{f(x)}\,dx \Big) \\ &= 2\,\mathrm{Re} \sum_{n=-N}^{N} \hat{f}(n)\overline{\hat{f}(n)} = 2 \sum_{n=-N}^{N} |\hat{f}(n)|^2. \end{aligned}$$

Combining these finishes the proof. □

Corollary 2.2.4 (Bessel's inequality)**.** *If* $f \in L^2([-\pi,\pi))$, *then for all* N

$$\sum_{n=-N}^{N} |\hat{f}(n)|^2 \le \|f\|_2^2,$$

and moreover

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \le \|f\|_2^2.$$

2.3 Cesàro mean

As discussed, we wish to discuss in what sense $S_n f(x) \to f(x)$. This, as it happens, is a hard question—it is true almost anywhere in the L^p sense for $p > 1$, but it is not at all a simple proof.

An easier question is to consider

$$\sigma_N f(x) = \frac{S_0 f(x) + S_1 f(x) + \ldots + S_N f(x)}{N+1},$$

i.e., the average.

The motivation for this is that the ***Cesàro mean*** of a sequence $\{ a_n \}$, i.e.,

$$s_1 = a_1, \quad s_2 = \frac{a_1 + a_2}{2}, \quad s_3 = \frac{a_1 + a_2 + a_3}{3}, \quad \ldots,$$

is quite well-behaved.

For instance:

Theorem 2.3.1. *If* $\lim_{n\to\infty} a_n = a$ *exists, then* $\lim_{n\to\infty} s_n = a$.

The converse is not true (for instance, take $a_n = (-1)^n$).

Lecture 3 Fejer Kernels

3.1 More about Cesàro means

Notice how, since $\sigma_n f(x)$ is the mean of $S_0 f(x), S_1 f(x), \ldots, S_n f(x)$, we can view $\sigma_n f(x)$ as

$$\sigma_n f(x) = \frac{1}{n+1}\hat{f}(-n)e^{-inx} + \ldots + \frac{n-1}{n+1}\hat{f}(-2)e^{-i2x} + \frac{n}{n+1}\hat{f}(-1)e^{-i1x} + \hat{f}(0)e^{i0x} + \\ + \frac{n}{n+1}\hat{f}(n)e^{inx} + \frac{n-1}{n+1}\hat{f}(2)e^{i2x} + \ldots + \frac{1}{n+1}\hat{f}(n)e^{inx}.$$

In other words, we can view $S_n f(x)$ as weighting each of the Fourier coefficients $\hat{f}(-n), \ldots, \hat{f}(n)$ with the uniform weight 1, whereas $\sigma_n f(x)$ weights the kth Fourier coefficients by $\frac{|k|}{n+1}$.

For these so-called ***Fejer means***, the natural question to ask is what happens as $n \to \infty$.

A similar question is to consider all Fourier coefficients, but weight them by something rapidly decaying, say for $0 < r < 1$, consider

$$a_r f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{inx},$$

called ***Abel means***. Here, of course, the natural question to ask is what happens as $r \to 1$.

Both of these are special cases of a more general theory of so-called ***Fourier multipliers***.

Date: January 28th, 2020.

With this goal in mind, particularly studying σ_n, let us formulate a more useful, closed form way of expressing this quantity.

First notice how

$$\begin{aligned} S_N f(x) &= \sum_{n=-N}^{N} \hat{f}(n) e^{inx} = \sum_{n=-N}^{N} e^{inx} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int}\, dt \\ &= \sum_{n=-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-in(t-x)}\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \sum_{n=-N}^{N} e^{in(x-t)}\, dt \\ &= f * D_N(x) \end{aligned}$$

where

$$D_N(t) = \sum_{n=-N}^{N} e^{int}$$

is called the ***Dirichlet kernel***. The bad news is that whilst this is a very pretty way of representing $S_N f(x)$, the Dirichlet kernel is a terrible kernel, in that it is in practice not easy to work with.

If instead we take the Cesàro mean of the Dirichlet kernels, i.e., $K_N \coloneqq \sigma_N D_N$, then we get the more well-behaved

$$\sigma_N f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) K_N(x-t)\, dt,$$

where $K_N(t)$ is called the ***Fejér kernel***. As it stands, thinking of the Fejér kernel as the Cesàro mean of the Dirichlet kernels is not very easy to work with either, so we will endeavour to rewrite it in a simple closed form.

First, consider the cleverly telescoping sum

$$\begin{aligned} e^{\frac{ix}{2}} \sum_{n=-N}^{N} e^{inx} - e^{-\frac{ix}{2}} \sum_{n=-N}^{N} e^{inx} &= \sum_{n=-N}^{N} e^{i(n+\frac{1}{2})x} - \sum_{n=-N}^{N} e^{i(n-\frac{1}{2})x} \\ &= e^{i(N+\frac{1}{2})x} - e^{-i(N+\frac{1}{2})x} = 2i \sin\Big(\Big(N + \frac{1}{2}\Big)x\Big). \end{aligned}$$

On the other hand, this says that

$$(e^{\frac{ix}{2}} - e^{-\frac{ix}{2}}) \sum_{n=-N}^{N} e^{inx} = 2i \sin\Big(\Big(N + \frac{1}{2}\Big)x\Big),$$

which means

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})}.$$

Hence the Fejér kernel is

$$\begin{aligned}
K_N(x) &= \frac{1+\frac{\sin(\frac{3}{2}x)}{\sin(\frac{x}{2})}+\frac{\sin(\frac{5}{2}x)}{\sin(\frac{x}{2})}+\ldots+\frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})}}{N+1} \\
&= \frac{1}{N+1}\frac{1}{\sin(\frac{x}{2})^2}\sum_{k=0}^{n}\sin\Big(\Big(k+\frac{1}{2}\Big)x\Big)\sin\frac{x}{2} \\
&= \frac{1}{N+1}\frac{1}{\sin(\frac{x}{2})^2}\sum_{k=0}^{n}\frac{1}{2}(\cos(kx)-\cos((k+1)x)) \\
&= \frac{1}{N+1}\frac{1}{\sin(\frac{x}{2})^2}\frac{-\cos((N+1)x)+1}{2},
\end{aligned}$$

where we have used the product to sum formula for sin in the third step and noticed that the sum is telescoping in the last step. Now the remaining cosine, by the half angle formula for sin, becomes

$$\frac{1}{N+1}\frac{1}{\sin(\frac{x}{2})^2}\sin\Big(\frac{N+1}{2}x\Big)^2,$$

so in all we have the closed form

$$K_N(x) = \frac{1}{N+1}\Big(\frac{\sin(\frac{N+1}{2}x)}{\sin\frac{x}{2}}\Big)^2$$

for the Fejér kernel.

Proposition 3.1.1. *The Fejér kernel K_N has the following properties:*

(i) $K_N(T) \geq 0$ *for all* N *and* t*;*

(ii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)\,dt = 1$ *for all* N*; and*

(iii) *if* I *is any open interval containing* 0*, then* $\lim_{N\to\infty}\|K_N\chi_{I^c}\|_\infty = 0$.

Proof. (i) This is obvious.

(ii) Viewing K_N again as $\sigma_N D_N$, notice how only the $n = 0$ terms from each part contributes when integrating from $-\pi$ to π, so we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)\,dt = \frac{1+1+\ldots+1}{N+1} = 1.$$

(iii) Suppose $(-\delta,\delta) \subset I$ for some $\delta > 0$. For $x \in [-\pi,\pi) \setminus (-\delta,\delta)$,

$$K_N(x) = \frac{1}{N+1}\Big(\frac{\sin(\frac{N+1}{2}x)}{\sin\frac{x}{2}}\Big)^2 \leq \frac{1}{N+1}\frac{\sin(\frac{N+1}{2}x)^2}{\sin(\frac{\delta}{2})^2}$$

since in the range $x \in [-\pi,\pi) \setminus (-\delta,\delta)$, $|\sin\frac{\delta}{2}| \leq |\sin\frac{x}{2}|$. The denominator is fixed and the numerator is bounded, so as $N \to \infty$ this vanishes. □

Remark 3.1.2. Any family of functions satisfying conditions (i), (ii), and (iii) is called an ***approximate identity***.

These approximate identities enjoy a couple of powerful features:

Theorem 3.1.3. *(i) Suppose $f \in L^p([-\pi,\pi))$ where $1 \le p < \infty$. Then $\sigma_n f \to f$ in L^p.*

(ii) If f is continuous and $f(\pi) = f(-\pi)$, then $\sigma_n f(x) \to f(x)$ uniformly.

(iii) If $f \in L^1([-\pi,\pi))$ and if x is a point of continuity of f, then $\sigma_n f(x) \to f(x)$.

Remark 3.1.4. As we shall see, all of these fail spectacularly for the Dirichlet kernel.

Remark 3.1.5. As hinted at, this theorem is not particular to σ_n coming from the Fejér kernel—indeed the theorem is true for any approximate identity.

Proof. (i) This is mostly a straight forward, but at one crucial step slightly careful, computation. We have

$$\begin{aligned}\|\sigma_N f - f\|_p^p &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sigma_N f(x) - f(x)|^p \, dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)K_N(x-t)\, dt - f(x)\Big|^p \, dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)\, dt - f(x)\Big|^p \, dx,\end{aligned}$$

where we have shifted the t integral. Since $K_N(t)$ has mass 1, we can factor this like

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\pi}^{\pi} \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x-t) - f(x))K_N(t)\, dt\Big|^p \, dx \\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-t) - f(x)|K_N(t)\, dt\Big)^p \, dx.\end{aligned}$$

Notice how $\frac{K_N(t)\,dt}{2\pi}$ is a measure of mass 1, so by Jensen's inequality this becomes

$$\begin{aligned}&\le \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-t) - f(x)|^p K_N(t)\, dt\, dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-t) - f(x)|^p \, dx K_N(t)\, dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \|f_t - f\|_p^p K_N(t)\, dt.\end{aligned}$$

Here we are letting $f_t(x) = f(x-t)$, a shift.

We know that

$$\lim_{t\to 0} \|f_t - f\|_p^p = 0,$$

so let us write the above as

$$\frac{1}{2\pi}\int_{|t|<\delta} \|f_t - f\|_p^p K_N(t)\, dt + \frac{1}{2\pi}\int_{|t|>\delta} \|f_t - f\|_p^p K_N(t)\, dt = I + II.$$

Choosing δ such that $|t| < \delta$ implies $\|f_t - f\|_p^p < \varepsilon$, we see immediately that $I < \varepsilon$.

On the other hand, with δ fixed, there exists some M such that for $N \geq M$, $K_N(t) \leq \varepsilon$ on $[-\pi, \pi) \setminus (-\delta, \delta)$. Hence if $N \geq M$ we have $II < (2\|f\|_p)^p \varepsilon$.

Finally, then, if δ small enough and $N \geq M$, then

$$\|\sigma_N f - f\|_p^p < \varepsilon + \varepsilon(2\|f\|_p)^p.$$

Lecture 4 Dirichlet Kernel

4.1 The Fejér kernels are nice

Proof continued. (b) Let $x \in [-\pi, \pi)$. Then

$$\begin{aligned} |\sigma_N f(x) - f(x)| &= \Big| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) K_N(t)\, dt - \frac{f(x)}{2\pi} \int_{-\pi}^{\pi} K_N(t)\, dt \Big| \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x-t) - f(x)| K_N(t)\, dt \\ &= \frac{1}{2\pi} \int_{|t|<\delta} + \frac{1}{2\pi} \int_{|t|>\delta} = I + II. \end{aligned}$$

By assumption f is continuous on $[-\pi, \pi]$, which is closed, so f is uniformly continuous. Hence given $\varepsilon > 0$ there exists $\delta > 0$ such that $|t| < \delta$ implies $|f(x-t) - f(x)| < \varepsilon$ for all x. Hence

$$I < \frac{1}{2\pi} \int_{|t|<\delta} \varepsilon K_N(t)\, dt < \varepsilon.$$

With δ fixed, choose M so large that if $n \geq M$, then

$$\|\chi_{(-\delta,\delta)} K_N\|_\infty < \varepsilon.$$

Therefore, since f is bounded because it is continuous,

$$II < \frac{1}{2\pi} \int_{|t|>\delta} 2\|f\|_\infty K_N(t)\, dt < 2\|f\|_\infty \varepsilon.$$

This means

$$|\sigma_N f(x) - f(x)| = I + II < \varepsilon(1 + 2\|f\|_\infty)$$

independent of x, so the convergence is uniform.

(c) We play a similar game, only with a slightly different approach at the very end. For the same I and II, we deal with I by noting that since x is a point of continuity of f, there exists for every $\varepsilon > 0$ some $\delta > 0$ so that $|f(x-t) - f(x)| < \varepsilon$ for $|t| < \delta$, and so $I < \varepsilon$ just as before.

Date: January 30th, 2020.

For II we again choose M large to ensure that $K_N(t)$ is small, but this time we bound

$$|f(x-t) - f(x)| \leq |f(x-t)| + |f(x)| \leq \|f\|_1 + |f(x)|,$$

where the L^1 norm is bounded by assumption and $|f(x)|$ is constant since x is fixed. The convergence follows. □

Definition 4.1.1 (Trigonometric polynomial)**.** Any finite sum of the form

$$\sum_{n=-M}^{N} a_n e^{inx}$$

is called a ***trigonometric polynomial***.

Corollary 4.1.2. *Trigonometric polynomials are dense in* $L^P([-\pi, \pi))$*,* $1 \leq p < \infty$*.*

Proof. Let $f \in L^p([-\pi, \pi))$. Then $\sigma_n f \to f$ in $L^p([-\pi, \pi))$ as $n \to \infty$. Since $\sigma_n f$ is a trigonometric polynomial, the corollary follows. □

4.2 Dirichlet kernels

Let $f \in L^1([-\pi, \pi))$. As we have previously discussed, $S_N f(x) = f * D_N(x)$, and we derived previously how

$$D_N(x) = \frac{\sin((N + \frac{1}{2})x)}{\sin \frac{x}{2}}.$$

We mentioned at the time that the Dirichlet kernel D_N is not a very good kernel. One large reason for this is this: D_N is not an approximate identity.

To see this, let us show that its L^1 norm is not bounded. In particular,

$$\begin{aligned} \|D_N\|_1 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big| \frac{\sin((N + \frac{1}{2})t)}{\sin \frac{t}{2}} \Big| \, dt \\ &\geq \frac{2}{\pi} \int_{-\pi}^{\pi} \frac{|\sin((N + \frac{1}{2})t)||t|}{} dt = \frac{2}{\pi} \int_{0}^{\pi} \frac{|\sin((N + \frac{1}{2})t)||t|}{} dt \end{aligned}$$

since $|\sin(x)| \leq |x|$. Changing variables $u = (N + \frac{1}{2})t$, this becomes

$$= \frac{2}{\pi} \int_0^{(N+\frac{1}{2})\pi} \frac{|\sin(u)|}{|u|} \, du = \frac{2}{\pi} \int_0^{(N+\frac{1}{2})\pi} \frac{|\sin(u)|}{u} \, du.$$

Splitting this integral up into integrals over intervals of length π (throwing away the last half) we get

$$\begin{aligned} \|D_N\|_1 &\geq \frac{2}{\pi} \sum_{k=0}^{N-1} \int_{k\pi}^{(k+1)\pi} \frac{|\sin(u)|}{u} \, du \geq \frac{2}{\pi} \sum_{k=0}^{N-1} \frac{1}{(k+1)\pi} \int_{k\pi}^{(k+1)\pi} |\sin(u)| \, du \\ &= \frac{4}{\pi^2} \sum_{k=0}^{N-1} \frac{1}{k+1} = \frac{4}{\pi^2} \sum_{k=1}^{N} \frac{1}{k} \geq \frac{4}{\pi^2} \log(N+1). \end{aligned}$$

Hence $\|D_N\|_1 \to \infty$ as $N \to \infty$.

4.3 Pointwise convergence of $S_N f(x)$

For $f \in L^1([-\pi,\pi))$, set $T_N f = S_N f(0)$. Then $T_N \colon L^1([-\pi,\pi)) \to \mathbb{C}$, and since the sum in S_N is linear, T_N is a linear functional (on the Banach space $L^1([-\pi,\pi))$, for those keeping count).

Consider in particular $f \in C([-\pi,\pi])$. Then

$$\begin{aligned} |T_N f| = |S_N f(0)| &= \Big| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(0-t) D_N(t)\, dt \Big| \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \|f\|_\infty |D_N(t)|\, dt \leq \|f\|_\infty \|D_N\|_1. \end{aligned}$$

Hence for every N, $T_N \colon C([-\pi,\pi]) \to \mathbb{C}$ is a linear bounded functional, so $\|T_N\| \leq \|D_N\|_1$.

Fixing an N, define $g(t) = \operatorname{sgn} D_N(t)$. Using this we create functions g_j on $[-\pi,\pi)$ such that g_j are continuous and $g_j \to g$ almost everywhere. Then

$$\lim_{j\to\infty} T_N g_j = \lim_{j\to\infty} S_N g_j(0) = \lim_{j\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} g_j(0-t) D_N(t)\, dt.$$

Now since g_j is bounded by construction, and since $D_N(t)$ is bounded because N is fixed, the integrand is bounded and so by Lebesgue's dominated convergence theorem we can bring the limit inside the integral. Hence

$$\begin{aligned} \lim_{j\to\infty} T_N g_j &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \lim_{j\to\infty} g_j(-t) D_N(t)\, dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) D_N(t)\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_N(t)|\, dt = \|D_N\|_1, \end{aligned}$$

where we have noted that $g(-t) = g(t)$ is even by construction.

Hence on $C([-\pi,\pi])$ we have a family $T_N \colon C([-\pi,\pi]) \to \mathbb{C}$ of bounded ($\|T_N\| \leq \|D_N\|_1$) linear functionals with $\|T_N\| \geq \frac{4}{\pi^2} \log(N+1)$.

By the Principle of uniform boundedness we must therefore have

$$\sup_N |T_N f| = \infty$$

for all f in a dense G_δ set of $C([-\pi,\pi])$. In other words, for all f in a dense set (that is the intersection of open sets) in $C([-\pi,\pi])$ we have $S_N f(0)$ growing without bound, and so for all those f we cannot have $S_n f(0) \to f(0)$.

Given any $x_0 \in [-\pi,\pi]$ we can repeat this argument, and so there exists a dense set of functions in $C([-\pi,\pi])$ such that $S_N f(x_0) \not\to f(x_0)$.

Lecture 5 The Principle of Uniform Boundedness

5.1 Some functional analysis

We wish to make more clear the statements at the end of last lecture, where we invoked the Principle of uniform boundedness.

Date: February 4th, 2020.

Let X be a normed vector space, say the norm is $\|\cdot\|$, meaning that we can give it a natural metric $d(x, y) = \|x - y\|$. Hence we have a metric space, and so on.

We say that $(X, \|\cdot\|)$ is a ***Banach space*** if it is a complete metric space (i.e., Cauchy sequences converse in the space).

For $T\colon X \to \mathbb{C}$, T linear, we define

$$\|T\| = \sup_{x \neq 0} \frac{|Tx|}{\|x\|}.$$

Equipped with such an operator norm, we can define X^*, the set of all such linear functionals T that are bounded, i.e., $\|T\| < \infty$.

This makes $(X^*, \|\cdot\|)$ a normed space too, in fact it, too, is a Banach space.

Theorem 5.1.1. *Let X be a Banach space and let $T\colon X \to \mathbb{C}$ be linear. The following are equivalent:*

(*i*) *T is bounded.*

(*ii*) *T is continuous.*

(*iii*) *T is continuous at a point $x \in X$.*

Proof. We show the implications in order. First, (*i*) implies (*ii*): We have

$$|Tx_1 - Tx_2| = |T(x_1 - x_2)| \leq \|T\|\|x_1 - x_2\|.$$

Since $\|T\| < \infty$ by assumption, this is a Lipschitz relation, whence T is continuous.

Next, (*ii*) implies (*iii*): This is trivial, since being continuous *everywhere* certainly includes being continuous at a point.

Finally, (*iii*) implies (*i*): Suppose T is continuous at $x_0 \in X$. In other words, there exists $\delta > 0$ such that $\|x_0 - x\| < \delta$ implies $|Tx - Tx_0| < 1$ (i.e., take $\varepsilon = 1$ in the continuity definition). Then for $x \in X$, $x \neq 0$, we have

$$\begin{aligned} |Tx| &= \Big|\frac{2\|x\|}{\delta} T\Big(\frac{\delta x}{2\|x\|}\Big)\Big| = \frac{2\|x\|}{\delta}\Big|T\Big(\frac{\delta x}{2\|x\|}\Big) + Tx_0 - Tx_0\Big| \\ &= \frac{2\|x\|}{\delta}\Big|T\Big(\frac{\delta x}{2\|x\|} + x_0\Big) - Tx_0\Big|. \end{aligned}$$

Since the two arguments of T are close—notice how $\frac{\delta x}{2\|x\|}$ by construction has norm less than δ—this means this is less than $\frac{2}{\delta}\|x\|$.

Hence $\|T\| < \frac{2}{\delta}$, do T is bounded. □

5.2 The Baire category theorem

Theorem 5.2.1 (Baire category theorem)**.** *Let X be a Banach space. Then the intersection of a countable number of dense open sets in X is dense in X (and by definition it is a G_δ set).*

Proof. Let $O_1, O_2, \ldots$ be open dense sets in X. Let $B(z, r)$ be an open ball in X or radios r around z. We need to find some

$$y \in B(z, r) \cap \Big(\bigcap_{n=1}^{\infty} O_n \Big).$$

The idea here is fairly simple: we want to somehow construct a good Cauchy sequence, since we know it must converge in X because X is complete. However, we also wish to remain inside all the sets we are intersecting, which at the moment is a problem: they're open, not closed.

Hence, shrink the $B(z, r)$ ball a bit, say halving its radius, so that the closure of the smaller ball is contained in $B(z, r)$. Since O_1 is dense in X, there must exist some point in this smaller ball with $x_1 \in O_1$.

In particular, since O_1 is open, there exists a whole open neighbourhood of x_1 contained in O_1. As with the original point, halve the radius here, take a closure, and there must exist some new point x_2 in this new ball with $x_2 \in O_2$.

Repeat this for all O_n, and we get a Cauchy sequence $\{ x_n \}$ with, say, $x_n \to y$. Since x_n lies in all the half radius balls which are closed, this limit lies inside the closed balls, and so we are done. □

Note how this proof works any locally compact space.

5.3 The Principle of uniform boundedness

We are now equipped to prove the titular result of this lecture, also known as the Banach-Steinhaus theorem.

Theorem 5.3.1 (Principle of uniform boundedness)**.** *Suppose X is a Banach space. Let T_α, $\alpha \in \Lambda$, be a collection of bounded linear functionals. Then either*

(*i*) *there exists some M such that $\|T_\alpha\| \leq M$ for all α, or*

(*ii*) $\sup_\alpha |T_\alpha x| = \infty$ *for all $x \in X$ belonging to a dense G_δ set.*

Proof. Let $\varphi(x) = \sup_\alpha |T_\alpha x|$. For $n = 1, 2, 3, \ldots$, set $V_n = \{ x \in X \mid \varphi(x) > n \}$. Note that if $x_0 \in V_n$, then $\varphi(x_0) > n$ so there exists some α_0 such that $|T_{\alpha_0} x_0| > n$.

Since T_{α_0} is continuous, there exists a neighbourhood U of x_0 such that $x \in U$ implies $|T_{\alpha_0} x| > n$. Hence V_n is open for all n.

Certainly it is true that either all V_n are dense in X, or at least one of them is not. Let us study each of these two possibilities in turn.

Suppose not all V_n are dense. In other words, there exists some V_N that is not dense, which in turn means there exists some x_0 and $r > 0$ such that $B(x_0, r) \subset V_N^c$.

If $\|x\| < r$, then $\varphi(x_0 + x) \leq N$, so for every α, $\|x\| < r$ implies $|T_\alpha(x_0 + x)| \leq N$.

Hence for $\|x\| < r$, we have

$$|T_\alpha x| = |-T_\alpha(x_0 + x) + T_\alpha x_0| \leq 2N,$$

so that $\|T_\alpha\| \leq \frac{2N}{r}$, whence all T_α are bounded, and (*i*) holds.

Else, if all V_n are dense in X, then by the Baire category theorem,

$$V = \bigcap_{n=1}^{\infty} V_n$$

is a dense G_δ set in X, so for $x \in V$, $\varphi(x) = \infty$ (since it is greater than all $n \in \mathbb{N}$), so $\sup_\alpha |T_\alpha x| = \infty$, i.e., (ii) holds. ☐

Our short aside into functional analysis is now over.

5.4 A Tauberian theorem

Theorem 5.4.1 (Hardy's Tauberian theorem)**.** *Suppose $f \in L_1([-\pi, \pi))$ such that $\hat{f}(n) = O(\frac{1}{n})$ as $n \to \pm\infty$. Then $S_n f(x)$ and $\sigma_n f(x)$ converge for the same values of x and to the same limits.*

Also, if $\sigma_n f(x)$ converges uniformly on some set, so does $S_n f(x)$.

Remark 5.4.2. We know already that if $S_n f(x) \to L$, then $\sigma_n f(x) \to L$, even without the additional hypothesis on the growth of the Fourier coefficients (this is just saying that if the limit of a sequence exists, the limit of its Cesàro means is the same).

In this light, this theorem is a kind of ***Tauberian theorem***, giving a partial converse to the statement.

Recall moreover that we proved $\sigma_n f(x) \to f(x)$ at a point of continuity of f, giving immediately:

Corollary 5.4.3. *If $f \in L^1([-\pi, \pi))$ and $\hat{f}(n) = O(\frac{1}{n})$, then $S_n f(x) \to f(x)$ at any point of continuity of f.*

This tells us in particular that the functions f in the dense G_δ set from last time must not have $\hat{f}(n) = O(\frac{1}{n})$.

Lecture 6 Hardy's Tauberian Theorem

6.1 Proof of Hardy's Tauberian theorem

Proof. As per the remark we already know that $S_n f(x) \to L$ implies $\sigma_n f(x) \to L$, whence we focus our attention on the converse.

Let M and N be positive integers with $M > N$. We compute

$$\begin{aligned}
&\frac{M+1}{M-N}\sigma_M f(x) - \frac{N+1}{M-N}\sigma_N f(x) \\
&= \sum_{j=-M}^{M} \frac{M+1}{M-n}\Big(1 - \frac{|j|}{M+1}\Big)\hat{f}(j)e^{ijx} - \sum_{j=-N}^{N} \frac{N+1}{M-n}\Big(1 - \frac{|j|}{N+1}\Big)\hat{f}(j)e^{ijx} \\
&= \sum_{j=-N}^{N} \frac{M+1-|j|-(N+1-|j|)}{M-N}\hat{f}(j)e^{ijx} + \sum_{N<|j|\le M} \frac{M+1}{M-n}\Big(1 - \frac{|j|}{M+1}\Big)\hat{f}(j)e^{ijx} \\
&= S_N f(x) + \sum_{N<|j|\le M} \frac{M+1}{M-n}\Big(1 - \frac{|j|}{M+1}\Big)\hat{f}(j)e^{ijx}.
\end{aligned}$$

Date: February 6th, 2020.

We need to get a handle on the error at the end. Notice how

$$\Big|\sum_{N<|j|\leq M} \frac{M+1}{M-n}\Big(1-\frac{|j|}{M+1}\Big)\hat{f}(j)e^{ijx}\Big| \leq \frac{M+1}{M-N}\sum_{N<|j|\leq M}\Big(1-\frac{N+1}{M-N}\Big)\frac{c}{|j|}$$

by bounding $|j|$ below by $N+1$ and $\hat{f}(j) = O(\frac{1}{|j|})$ by assumption. This then is bounded by

$$c\sum_{N<|j|\leq M}\frac{1}{|j|} \leq C\log\frac{M}{N}.$$

Let $\varepsilon > 0$. Then for N large, take $M > N$ such that $\log\frac{M}{N} < \varepsilon$, i.e., $M \leq Ne^{\varepsilon}$. Then we have

$$\Big|\frac{M+1}{M-N}\sigma_M f(x) - \frac{N+1}{M-N}\sigma_N f(x) - S_N f(x)\Big| \leq C\varepsilon.$$

Now suppose $\sigma_N f(x) \to L$, meaning that we can choose N large enough so that for all $n \geq N$, $|\sigma_n f(x) - L| < \varepsilon^2$. In particular, $|\sigma_N f(x) - L| < \varepsilon^2$ and $|\sigma_M f(x) - L| < \varepsilon^2$.

This gives us

$$\begin{aligned}
|S_N f(x) - L| &= \Big|S_N f(x) - \Big(\frac{M+1}{M-N}L - \frac{N+1}{M-N}L\Big)\Big| \\
&= \Big|S_N f(x) - \Big(\frac{M+1}{M-N}(L - \Sigma_M f(x) + \sigma_M f(x)) - \frac{N+1}{M-N}(L - \sigma_N f(x) + \sigma_N f(x))\Big)\Big| \\
&< \Big|S_N f(x) - \Big(\frac{M+1}{M-N}\sigma_M f(x) - \frac{N+1}{M-N}\sigma_N f(x)\Big)\Big| + \Big(\frac{M+1}{M-N} + \frac{N+1}{M-N}\Big)\varepsilon^2 \\
&< C\varepsilon + \frac{M+N+2}{M-N}\varepsilon^2.
\end{aligned}$$

Since $N \leq M \leq e^{\varepsilon}N$, we have

$$\leq C\varepsilon + \frac{e^{\varepsilon}N + N + 2}{e^{\varepsilon}N - N}\varepsilon^2.$$

The 2 is negligible in context of N large, so we are left with estimating

$$\frac{e^{\varepsilon}+1}{e^{\varepsilon}-1}\varepsilon^2.$$

Now we can choose $\varepsilon > 0$ small enough that $\frac{\varepsilon^2}{e^{\varepsilon}-1} < C\varepsilon$ since this means $\varepsilon \leq C(e^{\varepsilon} - 1)$, which for $C = 1$ is just saying e^x lies above its tangent line.

This finishes the proof. □

Lemma 6.1.1. *Suppose* $f \in L^1([-\pi,\pi))$ *and assume*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big|\frac{f(t)}{t}\Big|\,dt < \infty.$$

Then $\lim_{n\to\infty} S_n f(0) = 0.$

Lecture 7 A Covering Lemma

7.1 The Principle of localisation

We start by proving the lemma at the end of last lecture.

Proof. We compute:

$$\begin{aligned} S_N f(0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin((n+\frac{1}{2})t)}{\sin\frac{t}{2}}\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(t)}{\sin\frac{t}{2}}\Big(\sin\frac{t}{2}\cos(nt)+\cos\frac{t}{2}\sin(nt)\Big)\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(t)}{\sin\frac{t}{2}}\cos\frac{t}{2}\sin(nt)\,dt. \end{aligned}$$

Notice how the first integral is *almost* $\hat{f}(n)$; in particular it is the real part of this Fourier coefficient. This means that since $f \in L^1([-\pi,\pi))$, and the Riemann–Lebesgue lemma says its Fourier coefficients must go to 0, the first integral also goes to zero.

In the same manner, since $\sin\frac{t}{2} \approx \frac{t}{2}$ close to 0, the second integral is also (essentially) the imaginary part of a Fourier coefficient of an $L^{(}[-\pi,\pi))$ function since $\frac{f(t)}{t}$ is assumed $L^1([-\pi,\pi))$, whence it, too, goes to 0. □

Theorem 7.1.1 (Principle of localisation)**.** *Let $f \in L^1([-\pi,\pi))$ and assume f vanishes on an open interval I. Then $S_N f(x) \to 0$ for all $x \in I$.*

Proof. Let $x \in I$. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big|\frac{f(t)}{x-t}\Big|\,dt < \infty$$

since I is open, and hence by Lemma 6.1.1 $S_N f(x) \to 0$. □

Theorem 7.1.2 (Dini's test)**.** *If $f \in L^1([-\pi,\pi))$ and if*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|f(t-t_0)-f(t_0)|}{|t|}\,dt < \infty,$$

then $S_N f(t_0) \to f(t_0)$.

Proof. Set $g(t) = f(t+t_0) - f(t_0)$, which then means that by construction g satisfies Lemma 6.1.1, whence $S_N g(0) \to 0$. Now let us translate this back to f. First, for $n \neq 0$ we have $\hat{g}(n) = e^{int_0}\hat{f}(n)$, and crucially for $n = 0$ we have $\hat{g}(0) = \hat{f}(0) - f(t_0)$, since only the 0th Fourier coefficient detects the constant.

Hence

$$\begin{aligned} S_N g(0) = \sum_{j=-N}^{N}\hat{g}(j)e^{ij0} &= \sum_{j=-N}^{N}\hat{f}(j)e^{ijt_0} - f(t_0) \\ &= S_N f(t_0) - f(t_0). \end{aligned}$$

Since this goes to 0, we see that $S_N f(t_0) \to f(t_0)$. □

Date: February 6th, 2020.

7.2 Almost everywhere convergence

A question we have been itching to answer for a while is this: does $S_N f(x) \to f(x)$ almost everywhere?

We know already that it's definitely not true pointwise—we showed that given any point there exists a dense set of continuous functions all of which do not converge at the point at hand.

It gets even worse: in 1925 Kolmogorov showed that there exists some $f \in L^1([-\pi, \pi))$ such that $S_N f(x)$ diverges almost everywhere, and later on went on to show that in fact there exists $f \in L^1([-\pi, \pi))$ such that $S_N f(x)$ diverges for all x.

But fear not! There is partial hope.

For any $p > 1$, we have $L^p([-\pi, \pi)) \subsetneq L^1([-\pi, \pi))$, and in such spaces we have better control. More on this in future.

Recall

Theorem 7.2.1 (Lebesgue differentiation theorem)**.** *If $f \in L^1(\mathbb{R}^n)$, then for almost every $x \in \mathbb{R}^n$,*

$$\lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy = f(x).$$

This famous result from measure theory can be rephrased in terms that look more familiar to us here and now.

Let

$$L_r(t) = \frac{\chi_{B(0,r)}(t)}{|B(0,r)|}.$$

Then the integral in the theorem can be rewritten as

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy = \int_{\mathbb{R}^n} f(y) L_r(x-y)\, dy.$$

In other words, the average we are interested in can be realised as the convolution of f against a kernel L_r.

As it happens, and this is not a coincidence, this is a pretty nice kernel. Notice how

$$\int_{\mathbb{R}^n} L_r(t)\, dt = 1,$$

how $L_r(t) \geq 0$, and how for any neigbhourhood U of 0, $L_r(t) \to 0$ on $\mathbb{R}^n \setminus U$. In other words, L_r, known as a ***box kernel*** is an approximate identity!

This suggests that we might be able to prove results similar to Lebesgue differentiation theorem for other approximate identities, and indeed we will do this in order to show that $\sigma_N f \to f$ almost everywhere for $f \in L^1([-\pi, \pi))$.

In order to accomplish this we need some tools.

Definition 7.2.2 (Maximal function)**.** For $f \in L^1([-\pi, \pi))$, we define the ***maximal function***

$$Mf(x) = \sup_{0<h<\pi} \frac{1}{2h} \int_{x-h}^{x+h} |f(t)|\, dt.$$

In other words, it is the supremum of the averages of f around x.

We also need the following covering lemma:

Lemma 7.2.3. *From any family of open intervals $\Omega = \{ I_\alpha \}$ in $[-\pi, \pi)$, we can extract a sequence of pairwise disjoint intervals $I_1, I_2, I_3, \ldots$, such that*

$$\Big|\bigcup_{n=1}^{\infty} I_n\Big| \geq \frac{1}{4}\Big|\bigcup_{\alpha} I_\alpha\Big|.$$

Proof. The proof of this is standard. Set

$$a_1 = \sup_{I \in \Omega} |I|,$$

which by virtue of living in $[-\pi, \pi)$ must be finite (and less than 2π). This means there must exist some $I_1 \subset \Omega$ such that $|I_1| > \frac{3}{4}a_1$. (We could have picked any constant between 0 and 1, of course, but this is enough for our purposes.)

Now let Ω_2 be the subfamily of Ω consisting of intervals disjoint from I_1, and play the same game: let

$$a_2 = \sup_{I \in \Omega_2} |I|,$$

choose an $I_2 \in \Omega_2$ such that $|I_2| > \frac{3}{4}a_2$, and so on, getting a sequence $I_1, I_2, I_3, \ldots, I_n$, all disjoint, and in general let Ω_{n+1} be the intervals from Ω disjoint from $I_1, I_2, \ldots, I_n$ and set

$$a_{n+1} = \sup_{I \in \Omega_{n+1}} |I|$$

and pick $I_{n+1} \in \Omega_{n+1}$ such that $|I_{n+1}| > \frac{3}{4}a_{n+1}$.

This process might eventually stop, if we run out of intervals, but if it does just let $I_n = \varnothing$ for n large, so that we can still talk about a countably infinite family below without splitting into cases.

Notice how

$$\infty > \Big|\bigcup_{k=1}^{\infty} I_k\Big| = \sum_{k=1}^{\infty} |I_k|,$$

since they are disjoint, and hence $|I_k| \to 0$ as $k \to \infty$.

This means that

$$\bigcap_{k=1}^{\infty} \Omega_k = \varnothing,$$

for if there is some interval J in this intersection, then $|J| \leq a_k$ for all k, but the a_k must go to zero by the above argument.

Because of this, consider $I_{\alpha_0} \in \Omega$, and let k be the first index such that $I_{\alpha_0} \notin \Omega_k$. Then $I_{\alpha_0} \cap I_{k-1} \neq \varnothing$, and since

$$|I_{k-1}| > \frac{3}{4}a_{k-1} = \frac{3}{4} \sup_{I \in \Omega_{k-1}} |I|$$

we have in particular $|I_{k-1}| > \frac{3}{4}|I_{\alpha_0}|$, since $I_{\alpha_0} \in \Omega_{k-1}$.

Now consider $4I_{k-1}$, by which we mean an interval with the same centre (which we'll cenote c) as I_{k-1}, but four times the length. Then $4I_{k-1} \supseteq I_{\alpha_0}$. This is easy to see: pick $z \in I_{k-1} \cap I_{\alpha_0}$. Then for any $y \in I_{\alpha_0}$, we have

$$d(c, y) \leq d(y, z) + d(z, c) \leq |I_{\alpha_0}| + |I_{k-1}| \leq \frac{8}{3}|I_{k-1}|,$$

which very loosely bounded by $4|I_{k-1}|$.

Then finally

$$\Big|\bigcup_{\alpha} I_\alpha\Big| \le \Big|\bigcup_{k=1}^{\infty} 4I_k\Big| \le \sum_{k=1}^{\infty}|4I_k| = 4\sum_{k=1}^{\infty}|I_k| = 4\Big|\bigcup_{k=1}^{\infty} I_k\Big|$$

since the I_k are disjoint. □

Lecture 8 Almost Everywhere Convergence

8.1 Weak type results

Theorem 8.1.1. *For $f \in L^1([-\pi,\pi))$, we have*

$$|\{\, t \mid Mf(t) > \lambda \,\}| \le \frac{8\pi}{\lambda}\|f\|_1$$

for every $\lambda > 0$.

Proof. For each $sinT_\lambda = \{\, t \mid Mf(t) > \lambda \,\}$, we can choose an interval I_s centred as s such that

$$\frac{1}{|I_s|}\int_{I_s} |f(t)|\, dt > \lambda.$$

In doing so we create a collection $\{\, I_s \mid s \in T_\lambda \,\}$, and from such a family we can, by Lemma 7.2.3, select a subfamily of disjoint $I_1, I_2, \dots$ such that

$$\Big|\bigcup_{n=1}^{\infty} I_n\Big| \ge \frac{1}{4}\Big|\bigcup_{s\in T_\lambda} I_s\Big|.$$

Then

$$\begin{aligned}|T_\lambda| = |\{\, t \mid Mf(t) > \lambda \,\}| &\le \Big|\bigcup_{s\in T_\lambda} I_s\Big| \le 4\Big|\bigcup_{n=1}^{\infty} I_n\Big| \\ &= 4\sum_{n=1}^{\infty}|I_n| < \frac{4}{\lambda}\sum_{n=1}^{\infty}\int_{I_n}|f(t)|\, dt \le \frac{8\pi}{\lambda}\|f\|_1.\end{aligned}$$

□

Remark 8.1.2. It is *not* true that $\|Mf\|_1 \le c\|f\|_1$. Notice how, if it were, we would have

$$|\{\, t \mid Mf(t) > \lambda \,\}| \le \frac{\|Mf\|_1}{\lambda} \le \frac{c\|f\|_1}{\lambda}$$

by the Chebyshev inequality. This, then, is not true, but only in the tricky sense that the middle step is false—we just showed that in spite of this, the first and last expressions are indeed ordered thusly.

This is known as a ***weak type*** $(1,1)$ ***inequality***, because it does not *quite* give an inequality of 1-norms, but it does give such an inequality in measure, in some sense.

Notice how in the proof of the above theorem, when we pass from the integrals over I_n to $\|f\|_1$, we might, in principle, lose a fair amount of precision, in the event that the I_n don't cover much of $[-\pi,\pi)$. We can make this a little more precise:

Date: February 13rd, 2020.

Lemma 8.1.3. *For $f \in L^1([-\pi, \pi))$, we have*

$$|\{\, t \mid Mf(t) > 2\lambda \,\}| \leq \frac{4}{\lambda} \int_{\{\, t \mid |f(t) > \lambda| \,\}} |f(t)|\, dt$$

for all $\lambda > 0$.

Proof. Fix $\lambda > 0$ and write $f = g + b$ where

$$g(x) = \begin{cases} f(x), & \text{if } |f(x)| \leq \lambda, \\ 0, & \text{otherwise} \end{cases}$$

is the good function, and

$$b(x) = f(x) - g(x) = \begin{cases} f(x), & \text{if } |f(x)| > \lambda, \\ 0, & \text{otherwise} \end{cases}$$

is the bad function. Then for any interval I centred at x, we have

$$\begin{aligned} \frac{1}{|I|} \int_I |f(t)|\, dt &\leq \frac{1}{|I|} \int_I |g(t)|\, dt + \frac{1}{|I|} \int_I |b(t)|\, dt \\ &\leq Mg(x) + Mb(x) \leq \lambda + Mb(x). \end{aligned}$$

Taking supremum over all such I, we consequently have

$$Mf(x) \leq \lambda + Mb(x).$$

Hence

$$\begin{aligned} |\{\, t \mid Mf(t) > 2\lambda \,\}| &\leq |\{\, t \mid Mg(t) > \lambda \,\}| + |\{\, t \mid Mb(t) > \lambda \,\}| \\ &= |\{\, t \mid Mb(t) > \lambda \,\}| \leq \frac{8\pi \|b\|_1}{\lambda} \\ &= \frac{4}{\lambda} \int_{-\pi}^{\pi} |b(t)|\, dt = \frac{4}{\lambda} \int_{\{\, t \mid |f(t) > \lambda| \,\}} |f(t)|\, dt. \end{aligned}$$ □

As discussed, for 1-norms we only have these weak type results, however for p-norms, $p > 1$, we have stronger results:

Corollary 8.1.4. *For $1 < p \leq \infty$, there exists a constant c_p such that*

$$\|Mf\|_p \leq c_p \|f\|_p.$$

Proof. First let us handle $p = \infty$, for it is much quicker. We have

$$\frac{1}{|I|} \int_I |f(t)|\, dt \leq \|f\|_\infty,$$

and so $Mf(x) \leq \|f\|_\infty$, and we are done.

Now let $1 < p < \infty$. We start with a classic trick, consequence of Fubini's theorem:

$$\|Mf\|_p^p = \frac{1}{2\pi} \int_{-\pi}^{\pi} Mf(t)^p\, dt = \frac{1}{2\pi} \int_0^\infty p\lambda^{p-1} |\{\, t \mid Mf(t) > \lambda \,\}|\, d\lambda.$$

It then becomes an exercise in using our Lemma 8.1.3 and switching order of integration with Fubini's theorem strategically. In particular, setting $T_\lambda = \{\, y \mid |f(t)| > \frac{\lambda}{2} \,\}$,

$$\begin{aligned}
\|Mf\|_p^p &\le \frac{1}{2\pi}\int_0^\infty p\lambda^{p-1}\frac{8}{\lambda}\int_{T_\lambda}|f(t)|\,dt\,d\lambda \\
&= \frac{4}{\pi}\int_0^\infty p\lambda^{p-2}\int_{-\pi}^{\pi}\chi_{T_\lambda}(t)|f(t)|\,dt\,d\lambda \\
&= \frac{4}{\pi}\int_{-\pi}^{\pi}\int_0^\infty p\lambda^{p-2}\chi_{T_\lambda}(t)|f(t)|\,d\lambda\,dt \\
&= \frac{4}{\lambda}\int_{-\pi}^{\pi}\int_0^{2|f(t)|} p\lambda^{p-2}|f(t)|\,d\lambda\,dt \\
&= \frac{4}{\pi}\int_{-\pi}^{\pi}|f(t)|\frac{p}{p-1}\lambda^{p-1}\Big|_{\lambda=0}^{\lambda=2|f(t)|}\,dt \\
&= \frac{4}{\pi}\int_{-\pi}^{\pi}|f(t)|\frac{p}{p-1}(2|f(t)|)^{p-1}\,dt \\
&= \frac{82^{p-1}p}{p-1}\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)|^p\,dt = c_p^p\|f\|_p^p.
\end{aligned}$$

Hence, taking pth roots, $\|Mf\|_p \le c_p\|f\|_p$. □

Remark 8.1.5. Notice how the constant depending on p blows up as $p \to 1$—this illustrates exactly why we cannot have the strong type $(1,1)$ inequality discussed above.

Corollary 8.1.6. *Let* $\{\, L_N(t) \,\}_{N\in\mathbb{N}}$ *be a sequence of approximate identities that are all even. For* $f \in L^1([-\pi,\pi))$*, set*

$$f^*(x) = \sup_N |L_N * f(x)|.$$

Then $f^*(x) \le Mf(x)$*.*

Proof. The idea is the standard ***wedding cake decomposition***, by which we mean approximating a given L_N with a finite set of characteristic functions of nested intervals B_k.

Taking, without loss of generality, $x = 0$, something like

$$\begin{aligned}
|L_N * f(0)| = \Big|\frac{1}{2\pi}\int_{-\pi}^{\pi}L_N(t)f(t)\,dt\Big| &\le \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=1}^m a_k\chi_{B_k}(t)|f(t)|\,dt \\
&= \frac{1}{2\pi}\sum_{k=1}^m a_k\int_{B_k}|f(t)|\,dt = \frac{1}{2\pi}\sum_{k=1}^m a_k|B_k|\frac{1}{|B_k|}\int_{B_k}|f(t)|\,dt \\
&\le \frac{Mf(0)}{2\pi}\sum_{k=1}^m a_k|B_k| \le \frac{1}{2\pi}Mf(0)\|L_N\|_1\gamma,
\end{aligned}$$

where $\gamma > 1$ measures how far off the wedding cake is from L_N. Taking supremum over N we get $f^*(0) \le Mf(0)\gamma$ for all $\gamma > 1$, and taking infimum over γ we get $f^*(0) \le Mf(0)$ as desired. □

Corollary 8.1.7. *If $f \in L^1([-\pi,\pi))$, then for almost every $x \in [-\pi,\pi)$,*

$$\lim_{h\to 0} \frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt = f(x).$$

Proof. Note first of all that the result is definitely true for continuous f. Define

$$Tf(x) = \limsup_{h\to 0^+} \frac{1}{2h}\int_{x-h}^{x+h} |f(t)-f(x)|\,dt.$$

We want to show that $Tf(x) = 0$ almost everywhere, because then the $\liminf$ is also zero, what with the integrand being nonnegative, and so the limit is 0.

Let $\varepsilon > 0$. Choose $k \in \mathbb{N}$ and a continuous g such that $\|f-g\|_1 < \frac{1}{k}$, which is doable since continuous functions are dense in L^1. Then for every $h > 0$,

$$\begin{aligned}\frac{1}{2h}\int_{x-h}^{x+h} |f(t)-g(t)-(f(x)-g(x))|\,dt &\le \frac{1}{2h}\int_{x-h}^{x+h} |f(t)-g(t)|\,dt + |f(x)-g(x)| \\ &\le M(f-g)(x) + |f(x)-g(x)|.\end{aligned}$$

Notice how

$$\begin{aligned}&\frac{1}{2h}\int_{x-h}^{x+h} |f(t)-f(x)|\,dt \\ &\quad\le \frac{1}{2h}\int_{x-h}^{x+h} |f(t)-g(t)-(f(x)-g(x))|\,dt + \frac{1}{2h}\int_{x-h}^{x+h} |g(t)-g(x)|\,dt.\end{aligned}$$

Taking $\limsup$, this tells us that

$$Tf(x) \le M(f-g)(x) + |f(x)-g(x)|.$$

Then by Chebyshev's inequality we have

$$\begin{aligned}|\{\, x \mid Tf(x) > \varepsilon \,\}| &\le |\{\, x \in M(f-g)(x) > \varepsilon \,\}| + |\{\, x \mid |f(x)-g(x)| > \varepsilon \,\}| \\ &\le \frac{C}{\varepsilon}\|f-g\|_1 + \frac{1}{\varepsilon}\|f-g\|_1 \le \frac{c}{k\varepsilon}.\end{aligned}$$

Letting $k \to \infty$, this means $|\{\, x \mid Tf(x) > \varepsilon \,\}| = 0$, and so, taking a sequence of $\varepsilon \to 0$, this gives us $Tf(x) = 0$ almost everywhere. ☐

Lecture 9 More Almost Everywhere Convergence

9.1 Generalising a theme

The key steps in the proof at the end of last lecture was to a) know the result for a dense class and b) have a maximal function that gave us control in a weak or strong type sense. We can play this game in many more situations.

Theorem 9.1.1. *Let $f \in L^1([-\pi,\pi))$. Then $\sigma_N f(x) \to f(x)$ almost everywhere.*

Date: February 18th, 2020.

Recall how we have

$$\sigma_N f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)K_N(x-t)\,dt,$$

where

$$K_N(x) = \frac{1}{N+1}\Big(\frac{\sin((N+\frac{1}{2})x)}{\sin\frac{x}{2}}\Big)^2$$

is the Fejér kernel.

To prove this we will make use of the following lemmata.

Lemma 9.1.2. *For all N we have*

$$|K_N(t)| \le L_N(t) = \begin{cases} N+1, & \text{if } -\frac{\pi}{N+1} < t < \frac{\pi}{N+1}, \\ \frac{\pi^2}{(N+1)t^2}, & \text{if } \frac{\pi}{N+1} \le |t| \le \pi. \end{cases}$$

Proof. As it happens, both bounds are true for all t; we've just taken the best one where applicable. Hence we compute

$$|K_N(t)| = \Big|\sum_{j=-N}^{N}\Big(1-\frac{|j|}{N+1}\Big)e^{ijt}\Big| \le 1 + \sum_{j=1}^{N}\Big|\Big(1-\frac{|j|}{N+1}\Big)2i\sin(jt)\Big|$$
$$= 1 + 2\sum_{j=1}^{N}\Big(1-\frac{j}{N+1}\Big) = 1 + 2\Big(N - \frac{1}{N+1}\frac{N(N+1)}{2}\Big) = N+1,$$

whence the first part holds. Next, notice how for $x\in[-\frac{\pi}{2},\frac{\pi}{2}]$, $|\sin x| \ge \frac{2}{\pi}x$, and therefore we have $|\sin\frac{x}{2}| \ge \frac{2}{\pi}\frac{|x|}{2} = \frac{|x|}{\pi}$ for $x\in[-\pi,\pi]$. Therefore

$$K_N(x) = \frac{1}{N+1}\Big(\frac{\sin((N+\frac{1}{2})x)}{\sin\frac{x}{2}}\Big)^2 \le \frac{1}{N+1}\frac{1}{(\frac{|x|}{\pi})^2} = \frac{\pi^2}{(N+1)x^2}. \qquad \square$$

This $L_N(t)$ that bounds $K_N(t)$ is almost, but not quite, an approximate identity of the kind we studied in the wedding cake argument last time.

Lemma 9.1.3. *$L_N(t)$ is even, $\|L_N\|_1 \le 2$ for all N, and for $\delta > 0$, $L_n(t)\to 0$ uniformly on $[-\pi,\pi)\setminus(-\delta,\delta)$.*

Proof. That L_N is even and that it goes to 0 uniformly away from the origin is clear. For the L^1 bound, we compute

$$\|L_N\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} L_N(t)\,dt = \frac{1}{2\pi}\Big(\frac{2\pi}{N+1}(N+1) + 2\int_{\frac{\pi}{N+1}}^{\pi}\frac{\pi^2}{(N+1)t^2}\,dt\Big)$$
$$= 1 + \frac{1}{\pi}\frac{-\pi^2}{(N+1)t}\Big|_{\frac{\pi}{N+1}}^{\pi} = 1 + \frac{1}{\pi}\frac{-\pi^2}{(N+1)\pi} + \frac{1}{\pi}\frac{\pi^2}{(N+1)\frac{\pi}{N+1}}$$
$$= 2 - \frac{1}{N+1} \le 2. \qquad \square$$

Based on the same argument as last time, we therefore have

Corollary 9.1.4. *Define* $f^*(x) = \sup_N |L_N * f(x)|$. *Then* $\|f^*\|_p \leq 2c_p\|f\|_p$ *and*

$$|\{\, x \mid f*(x) > \lambda \,\}| \leq \frac{c\|f\|_1}{\lambda}$$

for every $\lambda > 0$.

Correspondingly, we get

Corollary 9.1.5. *Define*

$$\mathcal{M}f(x) = \sup_N |\sigma_N f(x)| = \sup_N \Big| \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(x-t) f(t)\, dt \Big|.$$

Then $\|\mathcal{M}f\|_p \leq c_p \|f\|_p$ *and*

$$|\{\, x \mid \mathcal{M}f(x) > \lambda \,\}| \leq \frac{c\|f\|_1}{\lambda}$$

for all $\lambda > 0$.

Proof. This follows since

$$\Big| \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(x-t) f(t)\, dt \Big| \leq |L_N * f(x)|$$

since K_N is dominated by a nice even and wedding-cake sort of kernel. ☐

With this under our belt, we are equipped to prove Theorem 9.1.1.

Proof of Theorem 9.1.1. The idea is identical to the proof last time. Let $\varepsilon > 0$ and $k \in \mathbb{N}$. Pick a trigonometric polynomial g so that $\|f - g\|_1 < \frac{1}{k}$, since the trigonometric polynomials are dense in $L^1([-\pi, \pi))$. Set

$$Tf(x) = \limsup_{N \to \infty} |f(x) - \sigma_N f(x)|.$$

As before, we want to show that $Tf(x) = 0$ almost everywhere.

Of course we have

$$|f(x) - \sigma_N f(x)| \leq |f(x) - g(x)| + |g(x) - \sigma_N g(x)| + |\sigma_N g(x) - \sigma_n f(x)|,$$

where the middle term goes to 0 as $N \to \infty$ since $\sigma_N g(x)$ is a trigonometric polynomial approximating the trigonometric polynomial $g(x)$.

Taking lim sup, we get

$$Tf(x) \leq |f(x) - g(x)| + 0 + \mathcal{M}(g - f)(x),$$

and so by Chebyshev's inequality

$$\begin{aligned} |\{\, x \mid Tf(x) > 2\varepsilon \,\}| &\leq |\{\, x \mid |f(x) - g(x)| > \varepsilon \,\}| + |\{\, x \mid \mathcal{M}(g-f)(x) > \varepsilon \,\}| \\ &\leq \frac{\|f-g\|_1}{\varepsilon} + \frac{c\|f-g\|_1}{\varepsilon} = \frac{C\|f-g\|_1}{\varepsilon} < \frac{C}{k\varepsilon}. \end{aligned}$$

This is true for every k, and the left-hand side is independent of k, so let k go to infinity and we have $|\{\, x \mid Tf(x) > 2\varepsilon \,\}| = 0$, Now taking a sequence of epsilons going to 0, this tells us $|\{\, x \mid Tf(x) > 0 \,\}| = 0$, so $Tf(x) = 0$ almost everywhere. ☐

This is the same strategy Carleson used in 1966 to settle Lusin's conjecture, showing

Theorem 9.1.6 (Carleson's theorem). *If $f \in L^2([-\pi, \pi))$, then $S_N f(x) \to f(x)$ almost everywhere.*

Using this, Hunt showed a year later that if $f \in L^p([-\pi, \pi))$ for $1 < p < \infty$, then $S_N f(x) \to f(x)$ almost everywhere as well.

This is much, much harder since, as we have discussed, the Dirichlet kernel D_N giving rise to $S_N f(x)$ is not an approximate identity. In fact, its norm grows like $\log N$ without bound.

Hence the main result of Carleson and Hunt to accomplish this was

Lemma 9.1.7. *For $f \in L^1([-\pi, \pi))$, set*

$$\mathcal{M} f(x) = \sup_N \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(x - t) f(t)\, dt \right|.$$

Then for $1 < p < \infty$ there exists some c_p depending only on p such that $\|\mathcal{M} f\|_p \le c_p \|f\|_p$.

Carleson did $p = 2$, and Hunt adopted Carleson's argument and did $1 < p < \infty$. Carleson's own proof of this is very terse, but to give a sense of the delicacy of the arugments involved, when it was later written up in a book the proof occupied pages 24 through 121.

All this said: assuming this lemma, Carleson's theorem is straight forward—it follows in exactly the same way as our two previous almost everywhere results, for exactly the same reason.

Lecture 10 Herglotz's Theorem

10.1 Making Fourier coefficients out of a sequence

A natural question to ask is whether, given some sequence $\{ a_n \}_{n \in \mathbb{Z}}$, we can find a function $f \in L^1([-\pi, \pi))$ such that $a_n = \hat{f}(n)$. Or, if $a_n = \hat{\mu}(n)$ for some measure μ.

Definition 10.1.1. A sequence $\{ a_n \}$ is called ***positive definite*** if for any sequence of complex numbers $\{ z_j \}$ with only a finite number of nonzero elements we have

$$\sum_{m,n} a_{n-m} z_n \overline{z_m} \ge 0.$$

Theorem 10.1.2 (Herglotz's theorem). *Let $\{ a_n \} \subset \mathbb{C}$. Then $\{ a_n \}$ is positive definite if and only if there exists a positive measure μ on $[-\pi, \pi)$ such that $a_n = \hat{\mu}(n)$ for all $n \in \mathbb{Z}$.*

Date: February 25th, 2020.

Proof. Let us first prove the converse direction, because that is the easy part. Suppose $a_n = \hat{\mu}(n)$ for some positive measure μ. Consider $\{ z_j \}$ with only finitely many nonzero elements. Then

$$0 \le \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big| \sum_{m \in \mathbb{Z}} z_m e^{-imt} \Big|^2 \, d\mu(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{m,n} z_n \overline{z_m} e^{imt} e^{-int} \, d\mu(t)$$
$$= \sum_{m,n} z_n \overline{z_m} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(n-m)t} \, d\mu(t) = \sum_{m,n} z_n \overline{z_m} \hat{\mu}(n-m) = \sum_{m,n} z_n \overline{z_m} a_{n-m},$$

where switching the order of summation is motivated by only finitely many z_j being nonzero, so the sum is finite. □

For the forward direction we need much more machinery, and we will get back to it once the machinery is in hand.

Lemma 10.1.3. *Given $\{ a_n \}$, then $a_n = \hat{\mu}(n)$ for some positive measure μ if*

$$\sum_{j=-n}^{n} \Big(1 - \frac{|j|}{n+1} \Big) a_j e^{ijt} \ge 0$$

for all n and t.

In other words, $a_n = \hat{\mu}(n)$ for some positive measure μ if

$$K_n * \sum_{j=-n}^{n} a_j e^{ijt} \ge 0$$

for all n and t.

To prove this, let us go on a brief functional analysis tangent. Suppose X is a normed space, and consider a linear mapping $T : X \to \mathbb{C}$. Set

$$\|T\| = \sup_{x \neq 0} \frac{|Tx|}{\|x\|} = \sup_{\|x\|=1} |Tx|.$$

If $\|T\| < \infty$, then T is a bounded linear functional, which is equivalent to T being continuous.

Let X^* be the space of all bounded linear functionals on X. Then $(X^*, \|\cdot\|)$ is in fact a normed space, called the ***dual space*** of X.

There is no need to stop there: X was a normed space and from it we made X^*, but now X^* is a normed space, so we can make the dual space X^{**} of X^* of bounded linear functionals from X^* to $\mathbb{C}$.

Now consider $(X, \|\cdot\|)$ again, and the family of bounded linear functionals $T \in X^*$. Then we can also equip X with the ***weak topology*** from the family, i.e., the weakest (or coarsest) topology on X such that all $T \in X^*$ are continuous. That is to say, the weak topology is generated by $\{ T^{-1}(U) \mid T \in X^* \text{ and } U \subset \mathbb{C} \text{ open} \}$.

This topology has fewer open sets than $(X, \|\cdot\|)$, hence *weak*.

Now X^* also has two topologies; $(X, \|\cdot\|)$ and the weak topology from X^{**}. Given $x \in X$, define $x^* \in X^{**}$ by

$$x^*(T) = Tx$$

for $T \in X^*$, i.e., the evaluation functional. This gives us get another kind of topology: on X^*, the ***weak-∗ topology*** is the weakest topology which makes all x^* continuous.

This topology is then generated by $\{ {x^*}^{-1}(U) \mid x \in X \text{ and } U \subset \mathbb{C} \text{ open} \}$, which is a subset of the generators of the weak topology on X^*, so is indeed weaker (since it makes potentially fewer maps continuous).

With this we can make sense of the following tool we need to prove the lemma:

Theorem 10.1.4 (Banach–Alaoglu theorem)**.** *The closed unit ball of X^*, i.e., $\{ T \mid \|T\| \leq 1 \}$, is compact in the weak-∗ topology.*

Proof of Lemma 10.1.3. Set

$$\sigma_n(t) = \sum_{j=-n}^{n} \Big(1 - \frac{|j|}{n+1}\Big) a_j e^{ijt}.$$

By assumption $\sigma_n(t) \geq 0$. Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\sigma_n(t)| \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sigma_n(t) \, dt = a_0,$$

so $\|\sigma_n\|_1 = a_0$ for all n. Hence $\sigma_n \in B(0, a_0) \subset L^1([-\pi, \pi))$ for all n. But $L^1([-\pi, \pi))$ is a proper subset of the space of finite measures on $[-\pi, \pi)$, which is equal to $C([-\pi, \pi))^*$.

Recall how $T\colon C([-\pi, \pi)) \to \mathbb{C}$ being bounded and linear means there exists some μ such that

$$Tf = \frac{1}{2\pi} \int_{-\pi}^{\pi} f \, d\mu.$$

By Banach–Alaoglu theorem there exists a subsequence $\{ \sigma_{n_j} \}$ and a measure μ such that $\sigma_{n_j} \to \mu$ in the weak-∗ sense, so

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \sigma_{n_j}(t) g(t) \, dt \to \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \, d\mu(t)$$

for all $g \in C([-\pi, \pi))$. Note that if $g \geq 0$, then so is

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \sigma_{n_j}(t) g(t) \, dt \geq 0$$

for all n_j. Hence for all $g \geq 0$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \, d\mu(t) \geq 0$$

so μ is a positive measure.

For each $n \in \mathbb{Z}$, $e^{inx} \in C([-\pi, \pi))$, so

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \sigma_{n_j}(t) e^{-ikt} \, dt \to \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt} \, d\mu(t).$$

But for n_j large, with k fixed,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sigma_{n_j}(t)e^{-ikt}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{\ell=-n_j}^{n_j}\Big(1-\frac{|\ell|}{n_j+1}\Big)a_\ell e^{i\ell t}e^{-ikt}\,dt = \Big(1-\frac{|k|}{n_j+1}\Big)a_k,$$

because only the $\ell = k$ term survives. Let $n_j \to \infty$, and this reads $\hat{\mu}(k) = a_k$, since the right-hand side is equal to the definition of $\hat{\mu}(k)$. □

Lecture 11 Harmonic Functions

11.1 Finishing Herglot's theorem

With Lemma 10.1.3 in place we are equipped to finally prove the forward direction of Herglotz's theorem.

Proof. Assume $\{ a_n \}_{n\in\mathbb{Z}}$ is positive definite. We need to show that a_n satisfies the lemma, i.e.,

$$\sum_{j=-n}^{n}\Big(1-\frac{|j|}{n+1}\Big)a_j e^{ijt} \geq 0$$

for all t and n, whence there exists a positive measure μ with $\hat{\mu}(n) = a_n$.

Fix N and t and consider the sequence

$$e^{-iNt}, e^{-i(N-1)t}, \ldots, e^{-it}, 1, e^{it}, \ldots, e^{iNt}.$$

Then by definition of a_n being positive definite,

$$0 \leq \sum_{-N\leq n,m\leq N} a_{n-m}e^{int}e^{-imt} = \sum_j a_j c_{j,N} e^{ijt},$$

where $c_{j,N}$ counts the number of ways to write $j = n - m$ with $|n| \leq N$ and $|m| \leq N$. In other words,

$$c_{j,N} = \begin{cases} 2N-1-|j|, & \text{if } |j| \leq 2N, \\ 0, & \text{otherwise.} \end{cases}$$

This means that for all N and t,

$$0 \leq \sum_{j=-2N}^{2N}((2N+1)-|j|)a_j e^{ijt},$$

which when divided by $2N + 1$ becomes

$$0 \leq \sum_{j=-2N}^{2N}\Big(1-\frac{|j|}{2N+1}\Big)a_j e^{ijt}.$$

Note how no part of this argument requires N be an integer—taking half integers we recover the required inequality. □

Date: February 27th, 2020.

11.2 Harmonic functions and kernels

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be analytic on $\overline{D}$ (by which we mean it is analytic on some open neighbourhood of $\overline{D}$). Let $0 < r < 1$, and set

$$f_r(\theta) = f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}.$$

These are of course C^∞ functions—the power series are absolutely convergent, so we can differentiate termwise.

By Cauchy's theorem,

$$\begin{aligned} f_r(\theta) &= \frac{1}{2\pi i} \int_{|\xi|=1} \frac{f(\xi)}{\xi - re^{i\theta}}\, d\xi = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{f(e^{it})}{e^{it} - re^{i\theta}} ie^{it}\, dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(e^{it})}{1 - re^{i(\theta - t)}}\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) C_r(\theta - t)\, dt, \end{aligned}$$

where

$$C_r(s) = \frac{1}{1 - re^{is}} = \sum_{n=0}^{\infty} (re^{is})^n = \sum_{n=0}^{\infty} r^n e^{isn}$$

is the ***Cauchy kernel***. In a slight abuse of notation we will write things like

$$f(\theta) \coloneqq f(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta},$$

and so viewed in this way $\hat{f}(n) = a_n$, and (per above), $\hat{C}_r(n) = r^n$. In this view things work out very nicely:

$$\hat{f}_r(n) = \widehat{f * C_r}(n) = \hat{f}(n)\hat{C}_r(n) = a_n r^n,$$

as expected.

Suppose u is ***harmonic*** on $\overline{D}$, meaning

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

From complex variables we know that $u = \operatorname{Re} f$ for some analytic function $f = u + iv$, with u and v satisfying the ***Cauchy–Riemann equations*** $u_x = v_y$ and $u_y = -v_x$.

Hence

$$u(z) = \frac{1}{2}(f(z) + \overline{f(z)}).$$

Now if, as above, we have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

we get

$$u(z) = \frac{1}{2}\Big(\sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{a_n z}^n + 2\operatorname{Re} a_0\Big).$$

Evaluating this at $z = re^{i\theta}$, we write

$$u(re^{i\theta}) = \frac{1}{2}\Big(\sum_{n=1}^{\infty} a_n r^n e^{in\theta} + \sum_{n=1}^{\infty} \overline{a_n} r^n e^{in\theta} + 2\operatorname{Re} a_0\Big) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\theta},$$

where

$$c_n = \begin{cases} \frac{a_n}{2}, & \text{if } n = 1, 2, 3, \ldots, \\ \frac{\overline{a_{-n}}}{2}, & \text{if } n = -1, -2, -3, \ldots, \\ \operatorname{Re} a_0 & \text{if } n = 0. \end{cases}$$

Since we are viewing u as the real part of a complex analytic function, consider u real-valued. The above is true also for $r = 1$, so we have

$$u(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta},$$

ans so $\hat{u}(n) = c_n$. Hence $u(re^{i\theta}) = u_r(\theta) = u * P_r(\theta)$ where

$$P_r(\theta) = \sum_{n=1}^{\infty} r^n e^{in\theta} + \sum_{n=-\infty}^{-1} r^{-n} e^{in\theta} + 1$$

is the so-called ***Poisson kernel***. Let us compute a nicer form of this:

$$\begin{aligned} P_r(\theta) &= C_r(\theta) - 1 + \overline{C_r(\theta) - 1} + 1 = C_r(\theta) + \overline{C_r(\theta)} - 1 \\ &= \operatorname{Re}(2C_r(\theta) - 1) = \operatorname{Re}\Big(\frac{2}{1 - re^{i\theta}} - 1\Big) = \operatorname{Re}\Big(\frac{2}{1 - re^{i\theta}} - \frac{1 - re^{i\theta}}{1 - re^{i\theta}}\Big) \\ &= \operatorname{Re}\Big(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\Big) = \operatorname{Re}\Big(\frac{1 + re^{i\theta}}{1 - re^{i\theta}} \cdot \frac{1 - re^{-i\theta}}{1 - re^{-i\theta}}\Big) \\ &= \operatorname{Re}\Big(\frac{1 + re^{i\theta} - re^{-i\theta} - r^2}{1 - re^{-i\theta} - re^{i\theta} + r^2}\Big) = \operatorname{Re}\Big(\frac{1 - r^2 + r(2i\sin\theta)}{1 - 2r\cos\theta + r^2}\Big) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}. \end{aligned}$$

Hence, for u harmonic in $\overline{D}$,

$$u_r(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{it}) P_r(\theta - t)\, dt = u * P_r(\theta).$$

Lecture 12 Harmonic Functions, continued

12.1 Harmonic conjugates and such

Proposition 12.1.1. *If $f = u + iv$ is analytic on $\overline{D}$, and if $f(0)$ is real, then*

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t) H_r(\theta - t)\, dt$$

Date: March 3rd, 2020.

where

$$H_r(s) = 2C_r(s) - 1$$

is the **Herglotz kernel**.

Proof. We have $u = \frac{1}{2}(f + \overline{f})$, so

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)H_r(\theta - t)\,dt &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}(f(t) + \overline{f(t)})(2C_r(\theta - t) - 1)\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)C_r(\theta - t)\,dt + \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(t)}C_r(\theta - t)\,dt - \frac{1}{4\pi}\int_{-\pi}^{\pi} f(t) + \overline{f(t)}\,dt.\end{aligned}$$

The first integral is simply $f(re^{i\theta})$ by previous calculations, and the last integral is $\operatorname{Re} f(0)$ by Cauchy's mean-value theorem.

The middle integral is the trickier part: let

$$g(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(t)}C_r(\theta - t)\,dt.$$

Then $\hat{g}(n) = \hat{\overline{f}}(n)\hat{C}_r(n)$, but

$$\overline{f(\theta)} = \sum_{n=0}^{\infty} \overline{\hat{f}(n)}e^{-in\theta},$$

so this has only nonpositive indices, whereas

$$C_r(\theta) = \sum_{n=0}^{\infty} r^n e^{in\theta}$$

has only nonnegative indices. Hence $\hat{g}(n) = 0$ except when $n = 0$, in which case $\hat{g}(0) = \hat{\overline{f}}(0)r^0 = \overline{f(0)}$.

Hence when $f(0)$ is real, we get the desired result. □

Notice how

$$H_r(\theta) = 2C_r(\theta) - 1 = 2\frac{1}{1 - re^{i\theta}} - 1 = \frac{2}{1 - re^{i\theta}} - \frac{1 - re^{i\theta}}{1 - re^{i\theta}} = \frac{1 + re^{i\theta}}{1 - re^{i\theta}}.$$

Hence

$$H_r(\theta - t) = \frac{1 + re^{i(\theta - t)}}{1 - re^{i(\theta - t)}} = \frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}}.$$

Hence if $z = re^{i\theta}$, we have the real and imaginary parts

$$\begin{aligned}f(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)\frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)\frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}} \cdot \frac{e^{-it} - re^{-i\theta}}{e^{-it} - re^{-i\theta}}\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)\frac{1 - r^2}{1 - 2r\cos\theta + r^2}\,dt + \frac{i}{2\pi}\int_{-\pi}^{\pi} u(t)\frac{2r\sin(\theta - t)}{1 - 2r\cos\theta + r^2}\,dt.\end{aligned}$$

In other words,

$$v(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)\frac{2r\sin(\theta - t)}{1 - 2r\cos\theta + r^2}\,dt = Q_r * u(\theta)$$

where

$$Q_r(s) = \frac{2r\sin(\theta - t)}{1 - 2r\cos\theta + r^2}$$

is the ***conjugate kernel***.

Hence in summary, if $f(z) = u(z) + iv(z)$ is analytic on $\overline{D}$, then

(*a*) $f(re^{i\theta}) = C_r * f(e^{i\theta})$ where $C_r(s) = \dfrac{1}{1 - re^{is}}$;

(*b*) $u(re^{i\theta}) = P_r * u(e^{i\theta})$ where $p_r(s) = \dfrac{1 - r^2}{1 - 2r\cos s + r^2}$;

(*c*) $f(re^{i\theta}) = H_r * u(e^{i\theta})$ if $f(0)$ is real, where $H_r(s) = 2C_r(s) - 1$; and

(*d*) $v(re^{i\theta}) = Q_r * u(e^{i\theta})$ where $Q_r(s) = \dfrac{2r\sin s}{1 - 2r\cos s + r^2}$.

In other words, knowing the values of the real part of an analytic function on the boundary of $\overline{D}$ is enough to determine everything about the function in the interior (maybe requiring $f(0)$ real).

12.2 Boundary values

Now for sort of the converse problem; we know from above that if we have an analytic function we can understand it by studying it on the unit circle.

Now we ask, given a function defined on the unit circle, can we extend it in some sensible way to a function on the interior?

Let f be defined on $|z| = 1$ or, equivalently, on $[-\pi, \pi)$.

Given such an f, can we extend it to a harmonic function f in D such that $f(re^{i\theta}) \to f(e^{i\theta})$ in some sense? Can you make such an extended f continuous on $\overline{D}$? (Probably not in general—the boundary function itself needn't be continuous to start with).

Can you make

$$\lim_{r\to 1} f(re^{i\theta}) = f(e^{i\theta})$$

for every θ? For almost all θ?

Or, perhaps, if we consider $f_r(\theta) = f(re^{i\theta})$ a family of functions, can we make $f_r \to f$ in L^p?

All of these are variants of so-called ***Dirichlet problems***. We will study several of them.

Theorem 12.2.1. *Suppose* $f \in L^1([-\pi, \pi))$. *Define*

$$f(re^{i\theta}) = f * P_r(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)P_r(\theta - t)\,dt.$$

Then

(*i*) $f(re^{i\theta})$ *is harmonic in* D;

(*ii*) *if* $1 \le p < \infty$ *and if* $f \in L^p([-\pi, \pi))$, *then* $f_r \to f$ *in* L^p;

(*iii*) *if* θ *is a point of continuity of* f, *then*

$$\lim_{r\to 1} f(re^{i\theta}) = f(\theta);$$

(*iv*) *if f is continuous, then $f_r(\theta) \to f(\theta)$ uniformly.*

Most of this is a consequence of the following fact:

Lemma 12.2.2. *The Poisson kernel $P_r(t)$ is an approximate identity.*

Proof. First,

$$P_r(t) = \frac{1 - r^2}{1 - 2r\cos t + r^2} \geq 0$$

since $2r\cos t \leq r^2 + \cos(t)^2 \leq r^2 + 1$ by the arithmetic and geometric mean inequality.

Secondly,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\,dt = P_r * 1(0) = 1$$

per our previous calculation.

Finally, if $0 < \delta < \pi$, then for $\delta \leq |\theta| \leq \pi$,

$$P_r(\theta) \leq \frac{1 - r^2}{1 - 2r\cos\theta + r^2},$$

so as $r \to 1$, $P_r(\theta) \to 0$ for $\delta \leq |\theta| \leq \pi$

Lecture 13 The Dirichlet Problem

13.1 The Classical Dirichlet problem

Proof continued. For the second part, let $\varepsilon > 0$. We have by Jensen's inequality that

$$|f(re^{i\theta}) - f(e^{i\theta})|^p \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i(\theta-t)}) - f(e^{i\theta})|^p P_r(t)\,dt$$

$$= \frac{1}{2\pi}\int_{|t|<\delta} |f(e^{i(\theta-t)}) - f(e^{i\theta})|^p P_r(t)\,dt + \frac{1}{2\pi}\int_{\delta<|t|<\pi} |f(e^{i(\theta-t)}) - f(e^{i\theta})|^p P_r(t)\,dt$$

with $\delta > 0$ to be determined.

Taking the notation $f_{(t)}(e^{i\theta}) = f(e^{i(\theta-t)})$ to be a shift, this means

$$\|f_r - f\|_p^p \leq \frac{1}{2\pi}\int_{|t|<\delta} \|f_{(t)} - f\|_p^p P_r(t)\,dt + \frac{1}{2\pi}\int_{\delta<|t|<\pi} \|f_{(t)} - f\|_p^p P_r(t)\,dt = I + II.$$

It is a fact that for $f \in L^p([-\pi,\pi))$, $f_{(t)} \to f$ in L^p (it is obviously true for continuous functions, and they're dense in L^p), so to estimate I, we can choose $\delta < 0$ small enough that $\|f_{(t)} - f\|_p^p < \varepsilon$, so

$$I < \frac{1}{2\pi}\int_{|t|<\delta} \varepsilon P_r(t)\,dt \leq \varepsilon.$$

For II,

$$II \leq \frac{1}{2\pi}\int_{\delta<|t|\leq\pi} (2\|f\|_p)^p P_r(t)\,dt = \frac{2^p\|f\|_p^p}{2\pi}\int_{\delta<|t|\leq\pi} P_r(t)\,dt.$$

Date: March 10th, 2020.

Now the last integral we can make as small as we like by choosing r sufficiently close to 1 since $P_r(t)$ is an approximate identity, and so we are done.

For the third part, if θ is a point of continuity of f, let $\varepsilon > 0$ and choose $\delta > 0$ so that $|t| < \delta$ implies $|f(e^{i(\theta-t)}) - f(e^{i\theta})| < \varepsilon$. Then

$$\begin{aligned}|f(re^{i\theta}) - f(e^{i\theta})| &\le \frac{1}{2\pi}\int_{|t|<\delta} |f(e^{i(\theta-t)}) - f(e^{i\theta})| P_r(t)\,dt \\ &\qquad + \frac{1}{2\pi}\int_{\delta<|t|\le\pi} |f(e^{i(\theta-t)}) - f(e^{i\theta})| P_r(t)\,dt \\ &\le \frac{1}{2\pi}\int_{|t|<\delta} \varepsilon P_r(t)\,dt + \frac{1}{2\pi}\int_{\delta<|t|\le\pi} |f(e^{i(\theta-t)}) - f(e^{i\theta})|\varepsilon\,dt\end{aligned}$$

by choosing r close enough to 1 for the bound in the second term. Hence

$$|f(re^{i\theta}) - f(e^{i\theta})| \le \varepsilon + \Big(\|f\|_1 + \frac{|f(e^{i\theta})|}{2\pi}\Big)\varepsilon$$

which we can make arbitrarily small.

Finally, for the fourth part: if f is continuous then it is uniformly continuous, and so f is bounded and we can bound $|f(e^{i\theta})| \le \|f\|_\infty$ above, and take the same δ for every θ. □

Remark 13.1.1. The classical Dirichlet problem on D is: assume f is continuous on $[-\pi, \pi)$ and periodic, and assume $\Delta u = 0$ on D and $u\Big|_{\partial D} = f$. Then

$$\lim_{r\to 1} u(re^{i\theta}) = f(e^{i\theta}),$$

which we proved: the Poisson kernel to the rescue!

Now we want to try to do sort of the opposite: if we have a harmonic function inside D, then can we say that it is the Poisson integral of something on the boundary?

Theorem 13.1.2. *Suppose f is a complex-valued harmonic function on D. Set $f_r(\theta) = f(re^{i\theta})$. Then*

(*i*) *for $1 < p \le \infty$, f is the Poisson integral of an L^p function if and only if $\sup_{0<r<1} \|f_r\|_p < \infty$.*

(*ii*) *if $p = 1$, then f is the Poisson integral of a measure if and only if $\sup_{0<r<1} \|f_r\|_1 < \infty$.*

(*iii*) *f is the Poisson integral of a finite positive measure if and only if f is non-negative.*

Remark 13.1.3. The last theorem already gives one of the directions of (*i*) and (*ii*), and the forward direction of (*iii*) is obvious.

Remark 13.1.4. To see that the measure consideration in (*ii*) is strictly necessary, notice how

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}$$

is Harmonic (easy way: it's the harmonic extension of 1 as per the last theorem). Now we have

$$\lim_{r\to 1} P_r(\theta) = 0$$

for all $\theta \neq 0$, where we get the limit ∞. Hence

$$P_r(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\, d\delta(t)$$

is the Poisson integral of the Dirac measure.

Lecture 14 Converse Problem

14.1 Converse to this problem

Proof. First let us handle the easy half. If $f(re^{i\theta}) = P_r * f(e^{i\theta})$ and $f_r \to f$ in L^p, then by Theorem 12.2.1, $\sup_r \|f_r\|_p < \infty$.

Similarly, if $f(re^{i\theta}) = P_r * \mu(e^{i\theta})$, then

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|\, d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\, d\mu(t)\right| d\theta \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\, d\theta\, d|\mu|(t) = \frac{1}{2\pi}|\mu|([-\pi,\pi)),\end{aligned}$$

where by $|\mu|([-\pi,\pi))$ is the ***total variation*** of μ.

Finally, if μ is a positive measure, then

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\, d\mu(t) \geq 0$$

since the integrand and the measure are nonnegative.

Now for the hard half.

First, for $1 < p \leq \infty$, without loss of generality we can assume (by normalising appropriately) that $\sup_r \|f_r\|_p < 1$. So $\{ f_r \}_r$ is in the unit ball in L^p.

Now since $L^p = (L^q)^*$, where $\frac{1}{p} + \frac{1}{q} = 1$ (by which we mean for $T\colon L^p \to \mathbb{C}$ there exists $g \in L^q$ such that

$$Tf = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\overline{g}\, d\theta$$

or similar representation theorems).

Hence the f_r are also in the unit ball in $(L^q)^*$. By the Banach–Alaoglu theorem there exists a subsequence f_{r_j} and $g \in (L^q)^* = L^p$ such that $f_{r_j} \to g$ weak-$*$. So

$$\int_{-\pi}^{\pi} f_{r_j}(\theta)h(\theta)\, d\theta \to \int_{-\pi}^{\pi} g(\theta)h(\theta)\, d\theta$$

Date: March 12th, 2020.

for all $h \in L^q$. In particular, fixing $0 < r < 1$ and θ, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f_{r_j}(t)\, dt \to \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(\theta)\, d\theta$$

since $P_r(\theta - t)$ is a perfectly innocent L^q function for r and θ fixed. In other words, on the one hand

$$f(rr_j e^{i\theta}) \to \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(\theta)\, d\theta$$

as $j \to \infty$, and on the other hand

$$f(rr_j e^{i\theta}) \to f(re^{i\theta})$$

as $r_j \to 1$, finishing the first part.

For the second part, the $p = 1$ case, again normalise to $\sup_r \|f_r\|_1 < 1$. Then f_r is in the unit ball of L^1, which is contained in the unit ball of finite measures, which is equal to the unit ball of $C([-\pi, \pi])^*$.

So there exists f_{r_j} and μ such that $f_{r_j} \to \mu$ weak-$*$, i.e.,

$$\int_{-\pi}^{\pi} f_{r_j}(\theta) h(\theta)\, d\theta \to \int_{-\pi}^{\pi} h(\theta)\, d\mu(\theta)$$

for all $h \in C([-\pi, \pi])$.

Again fixing r and θ, we then have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f_{r_j}(t)\, dt = f(rr_j e^{i\theta}) \to \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\, d\mu(t)$$

which, as $r_j \to 1$, is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\, d\mu(t)$$

and as $j \to \infty$ goes to $f(re^{i\theta})$, finishing the second part.

Finally if $f \geq 0$ on D, then the measure is positive. To see this, notice how from the second part we know that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f_{r_j}(e^{i\theta}) h(\theta)\, d\theta \to \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)\, d\mu(\theta)$$

for all $h \in C([-\pi, \pi])$. In particular this is true for $h \geq 0$, and even more particularly, it is true for $h \geq 0$ approximating the characteristic functions of open intervals, and so combinations of same, and so generally we see that μ is nonnegative on any set, so $\mu \geq 0$. $\square$

Playing the same classic $\varepsilon/3$ game we have played many times before, we then easily get:

Theorem 14.1.1. *If $f(re^{i\theta}) = P_r * h(e^{i\theta})$, $h \in L^1([-\pi, \pi))$, then*

$$\lim_{r \to 1} f(re^{i\theta}) = h(e^{i\theta})$$

almost everywhere.

Proof. Let $\varepsilon > 0$, $k \in \mathbb{N}$. Choose g continuous such that $\|h - g\|_1 < \frac{1}{k}$. Then

$$\begin{aligned}|f(re^{i\theta}) - h(e^{i\theta})| &= |P_r * h(e^{i\theta}) - h(e^{i\theta})| \\ &\leq |P_r * (h-g)(e^{i\theta})| + |P_r * g(e^{i\theta}) - g(e^{i\theta})| + |g(e^{i\theta}) - h(e^{i\theta})|.\end{aligned}$$

Recall how we know, because $P_r(\theta)$ is an approximate identity and using a wedding cake argument, that

$$\sup_r \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t)\varphi(t)\, dt \leq CM\varphi(\theta)$$

where M is the Hardy–Littlewood maximal function. Then we have

$$Tf(\theta) \coloneqq \limsup_{r \to 1} |P_r * h(e^{i\theta}) - h(e^{i\theta})| \leq CM(h-g)(e^{i\theta}).$$

Hence

$$|\{\, \theta \mid Tf(\theta) > \varepsilon \,\}| \leq |\{\, \theta \mid M(h-g)(e^{i\theta}) > \frac{\varepsilon}{C} \,\}| \leq \frac{C\|h-g\|_1}{\varepsilon / c} \leq \frac{C}{\varepsilon k}$$

using the type of weak type-1 inequalities we have established in the past. Now let $k \to \infty$ and we see that

$$|\{\, \theta \mid T(\theta) > \varepsilon \,\}| = 0$$

so $Tf(\theta) = 0$ almost everywhere. □

This theorem is not true for measures: the limit does not recover the function, but instead recovers the ***absolutely continuous*** of the measure, as was proved by Fatou in 1904 (though in less generality):

Theorem 14.1.2. *Let μ be a finite complex-valued measure in the unit circle. Suppose $d\mu = g(\theta)\, d\theta + d\nu$ where $g \in L^1([-\pi, \pi))$, $d\nu$ and $d\theta$ have disjoint support, and $d\nu$ is* **singular** *meaning it is supported on a set of measure* 0.

If

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t)\, d\mu(t),$$

then

$$\lim_{r \to 1} f(re^{i\theta}) = g(\theta)$$

almost everywhere.

We have already explored a particular example of this:

Example 14.1.3. We have

$$P_r(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t)\, d\delta(t)$$

where the limit as $r \to 1$ is 0 if $\theta \neq 0$ and infinity of $\theta = 0$—in other words, we do not recover the delta measure in the limit. ▲

Index