Lecture Notes in Topology

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Note that these are rapidly taken and then even more swiftly typed notes, and as such errors might well occur. Be sure to check any oddities against the course literature [KF20]. Last updated March 1, 2016.

Throughout this document, □ signifies end proof, ▲ signifies end of example, and ⧫ signifies end of solution.

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1 Lecture I[1]

Note first of all that the lecture notes assumes familiarity with basic set theoretical concepts such as taking unions and intersections of sets, differences of sets, complements, et cetera. Furthermore the theory of metric spaces will be employed as a tool on which to base our intuition, whereby familiarity with this is beneficial. Having said that, some of the most basic properties of metric spaces will be introduced below.

1.1 Metric Spaces

Example 1.1.1. Let $X = \mathbb{R}$, the real line. Then $\rho(x, y) = |x - y|$ is a distance measure (the ordinary one), with the following properties:

(*i*) $\rho(x, y) \geq 0$,

(*ii*) $\rho(x, y) = 0$ if and only if $x = y$,

(*iii*) $\rho(x, y) = \rho(y, x)$,

(*iv*) The triangle inequality; for all x, y, z we have $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ (think of the sides of a triangle). ▲

We borrow these properties and generalise them to acquire a new theory; keep this in mind, as it is how we build the foundation of topology.

Definition 1.1.2 (Metric, Metric space)**.** Let X be a set and let $\rho : X \times X \to \mathbb{R}$ which has properties (*i*)–(*iv*) above. Then ρ is called a ***metric*** and (X, ρ) is a ***metric space***.

Example 1.1.3. Let $X = \mathbb{R}^m$, whence the elements are $\boldsymbol{x} = (x_1, x_2, \ldots, x_m)$. On this we may introduce many metrics. For example

$$\rho_1(\boldsymbol{x}, \boldsymbol{y}) = \sum_{j=1}^{m} |x_j - y_j| \quad \text{and} \quad \rho_2(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{\sum_{j=1}^{m} (x_j - y_j)^2}. \qquad ▲$$

1.2 Open Sets in Metric Spaces

Definition 1.2.1 (Ball)**.** In a metric space (X, ρ), we define the ***ball*** centred on x_0 with radius $r > 0$ as $B_r(x_0) = \{x \in X \mid \rho(x_0, x) < r\}$.

Definition 1.2.2 (Open set)**.** Let (X, ρ) be a metric space. The set $O \subset X$ is called ***open*** if for every $x_0 \in O$ there exists an $r > 0$ such that $B_r(x_0) \subset O$.

Example 1.2.3. Take $X = \mathbb{R}$. The interval $0 = (a, b)$ is an open set, since for any $c \in B$ we can construct a ball or radius $r = \min\{c - a, b - c\}$ which is contained in O.

The set $O = [a, b]$ on the other hand is not open, since taking $c = a$ or $c = b$, we can't construct *any* ball centred on c contained in O. ▲

Exercise 1.2.4. Let $X = \mathbb{R}^2$ (because it's easy to visualise). Draw (or describe) what the balls look like for ρ_1, ρ_2, and $\rho_\infty(\boldsymbol{x}, \boldsymbol{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

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Solutions. For ρ_2, the open ball of radius r around a is simply the interior of the circle centred on a with radius r, since ρ_2 is the Cartesian distance.

For ρ_1 the distance between a and x is the sum of the difference of their components. This sum must be less than r. The open ball then becomes a square with diagonals $2r$, the diagonals being parallel to the axes.

For ρ_∞ we take, for a given x, the biggest of the differences of the components of x and a. This will give us a square with sides $2r$, with sides parallel to the axes. ♦

Exercise 1.2.5. Let $X = C[a,b]$, the space of all continuous functions on $[a,b]$, with $\rho_\infty(\varphi,\psi) = \max\limits_{a\leq t\leq b}|\varphi(t)-\psi(t)|$ as the metric. Draw the ball $B_r(\varphi)$.

Solution. The ball of radius r around the function φ will be any function defined in the interval $[a,b]$ such that it is contained within an r-corridor around φ. ♦

1.3 Properties of Open Sets in Metric Spaces

In this subsection we will denote by $\mathscr{S}$ the set of all open subsets of a set metric space (X,ρ). In what follows we will establish four fundamental properties of the members of this set.

Let O_1 and O_2 belong to $\mathscr{S}$ (meaning that they are open). Then clearly their union is open as well, since all points in the union can have a sufficiently small ball constructed around them to be contained in (at least) one of O_1 or O_2, whence the ball would also be contained in their union.

It is clear that we can generalise this argument to any number of open sets, indeed even an uncountable number, giving us the first property.

(i) The union of any system of open sets is again open.

Having considered unions, it is sensible to consider intersections next. If we take $O_1, O_2 \in \mathscr{S}$, is $O_1 \cap O_2 \in \mathscr{S}$? Yes: if we take $x_0 \in O_1 \cap O_2$ there must exist an $r_1 > 0$ such that $B_{r_1}(x_0) \subset O_1$ and an $r_2 > 0$ such that $B_{r_2}(x_0) \subset O_2$. Taking $r = \min\{r_1, r_2\}$ we have a ball $B_r(x_0)$ which is contained in their intersection.

This time we cannot generalise quite as far: for a finite number of open sets it works fine, since the minimum of a finite set of positive numbers is again positive. However for infinite intersections we run into trouble with the method proposed above, since the minimum of an infinite set of numbers needn't exist (one might then suggest instead taking the infimum, but then we have the problem of this potentially being zero).

Of course showing that one method of proof doesn't work is not enough; we need a counter example.

Counterexample 1.3.1. Let $X = \mathbb{R}$ with the ordinary distance be our metric space, and consider the open sets

$$O_n = \left(-1-\frac{1}{n}, 1+\frac{1}{n}\right)$$

for $n = 1, 2, \ldots$.

Taking the intersection over all these we get

$$\bigcap_{n=1}^{\infty} O_n = [-1,1],$$

which we showed in Example 1.2.3 to not be open. ▲

Thus we have the second property.

(*ii*) The intersection of finitely many open sets is again open.

Finally the third and fourth properties are more of less obvious.

(*iii*) The set X itself is open.

(*iv*) The empty set is open.

To see the fourth property, study carefully the working in the definition of an open set. Since there is no $x_0 \in \varnothing$, it is true that for all those x_0, one can construct a ball centred on it contained in $\varnothing$.

Definition 1.3.2 (Closed set)**.** A set M is called ***closed*** if and only if it is a complement of some open set.

With this in hand we are again equipped to generalise things to form a new theory.

1.4 Topology

We now define the central concept of the entire course.

Definition 1.4.1 (Topology, topological space)**.** Let X be any set and let $\mathscr{S}$ be any system of subsets of X which satisfies properties

(*i*) The union of any system of sets in $\mathscr{S}$ is again in $\mathscr{S}$,

(*ii*) The intersection of finitely many sets in $\mathscr{S}$ is again in $\mathscr{S}$,

(*iii*) The entire set X itself is in $\mathscr{S}$,

(*iv*) The empty set is in $\mathscr{S}$.

Then the elements of $\mathscr{S}$ are called ***open sets*** and $\mathscr{S}$ is called a ***topology*** on X and $(X, \mathscr{S})$ is called a ***topological space***.

Example 1.4.2. As suggested by the way we constructed the four defining properties in the last subsection, all metric spaces are topological spaces. ▲

Example 1.4.3. Consider the set $X = \{a, b, c\}$, and the system of subsets $\mathscr{S} = \{\varnothing, X, O_1 = \{a\}, O_2 = \{b\}\}$. This system is not a topology on X since the union $O_1 \cup O_2$ isn't in $\mathscr{S}$. The system $\{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$ on the other hand is. ▲

Exercise 1.4.4. Construct all possible topologies on X as above. How many are there?

Solution. There are twenty-nine of them, and we will not spell them out below in the interest of not writing out something completely boring. Note however that most are similar; there are only nine fundamentally different looking ones. ⧫

We have the same definition of closed sets as in the case of metric spaces.

Definition 1.4.5 (Closed set)**.** Complements of open sets are called ***closed***.

To motivate the topological definition of continuity of a function we must first explore the case in metric spaces as, again, we are inspired by the properties therein. A function $f : X \to Y$, X having the metric ρ_X and Y having the metric ρ_Y, in such a setting is said to be continuous at a point x_0 if, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that having $\rho_X(x_0, x) < \delta$ implies that $\rho_Y(f(x_0), f(x)) < \varepsilon$.

Theorem 1.4.6. *The function $f : X \to Y$ is continuous if and only if for all $O \in \mathscr{S}_Y$ (being the set of all open subsets of Y) we have $f^{-1}(O) \in \mathscr{S}_X$ (this being defined analogously). That is to say, the preimage of any open set is open.*

Proof. ($\Rightarrow$) We take $y_0 = f(x_0)$ in what follows. Since $O \subset Y$ is open, there exists a ball $B_\varepsilon(y_0) \subset O$ for some $\varepsilon > 0$, for every $y_0 \in O$. Since f is continuous there therefore exists a $\delta > 0$ such that $\rho_X(x_0, x) < \delta$ such that $\rho_Y(f(x_0), f(x)) < \varepsilon$, whence $f(B_\delta(x_0)) \subset B_\varepsilon(y_0) \subset O$.

Taking the preimage the above we get $f^{-1}(O) \supset f^{-1}(f(B_\delta(x_0))) = B_\delta(x_0)$, which is open.

($\Leftarrow$) Now let $f^{-1}(O) \in \mathscr{S}_X$ for every $O \in \mathscr{S}_Y$. Take $O = B_\varepsilon(y_0)$, with $y_0 = f(x_0)$; $f^{-1}(O)$ is open in X meaning that there exists a $\delta > 0$ such that $B_\delta(x_0) \subset f^{-1}(O)$, whereby $f(B_\delta(x_0)) \subset B_\varepsilon(y_0)$, which by the argument above implies continuity. □

Thus our topological definition of continuity is clear.

Definition 1.4.7 (Continuity)**.** Let $(X, \mathscr{S}_X)$ and $(Y, \mathscr{S}_Y)$ be topological spaces, then $f : X \to Y$ is said to be ***continuous*** if and only if for all $O \in \mathscr{S}_Y$, we have $f^{-1}(O) \in \mathscr{S}_X$.

2 Lecture II[2]

2.1 More on Topology

We continue to construct and motivate the fundamentals of topology.

In the spirit of limits in metric spaces we define the notions of contact points and limit points in topologies.

Definition 2.1.1 (Neighbourhood)**.** Let $(X, \mathscr{S})$ be a topological space and take $x \in X$. Any open set $U \in \mathscr{S}$ containing x is called a ***neighbourhood*** of x, denoted $U(x)$.

Definition 2.1.2 (Contact point, limit point)**.** Let $(X, \mathscr{S})$ be a topological space and take $A \subset X$.

A point $x \in X$ is called a ***contact point*** of A if, for any neighbourhood $U(x)$ of x we have $U(x) \cap A \neq \varnothing$.

Moreover x is called a ***limit point*** of A if, for any neighbourhood $U(x)$ of x the intersection $U(x) \cap A$ contains infinitely many points belonging to A.

It is clear by definition that x being a limit point of a set A implies that it is also a contact point of A. The converse is not true, which we show by the following counterexample.

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Counterexample 2.1.3. Let $X = \mathbb{R}$ and take $A = \{x_n = 1/n\}$ for all $n = 1, 2, 3, \ldots$. Then every x_n is a contact point but not a limit point, since any ball of radius less than x_n around x_n will contain only finitely many points of A. Note on the other hand that $x = 0$ is a limit point of A. ▲

2.2 Comparison of Topologies

We can define a partial order on topologies as follows.

Definition 2.2.1 (Comparison of topologies)**.** Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be two topologies on the same set X. We say that $\mathscr{S}_1 \leq \mathscr{S}_2$ if $\mathscr{S}_1 \subset \mathscr{S}_2$, pronounced as "$\mathscr{S}_1$ being ***weaker*** than $\mathscr{S}_2$" or "$\mathscr{S}_2$ being ***stronger*** than $\mathscr{S}_1$."

We can introduce at least two topologies on any X; the trivial topology $\mathscr{S} = \{\varnothing, X\}$ and $\mathscr{S} = \mathscr{P}(X)$, the power set, being the set of all subsets of X. Naturally we will always have the former being weaker than the latter. Topologies are not always comparable, however.

Example 2.2.2. Let $X = \{a, b, c\}$. The topology $\mathscr{S}_1 = \{\varnothing, X, \{a\}\}$ is weaker than the topology $\mathscr{S}_2 = \{\varnothing, X, \{a\}, \{a, b\}\}$.

On the other hand, the topology $\mathscr{S}_3 = \{\varnothing, X, \{b\}\}$ is neither stronger than nor weaker than $\mathscr{S}_1$, since neither is a subset of the other. ▲

Definition 2.2.3 (Intersection of topologies)**.** Let $\mathscr{S}_\alpha$ be a collection of topologies on the same set X. We define the ***intersection*** of these topologies $\mathscr{S} = \bigcap_\alpha \mathscr{S}_\alpha$ as simply the set consisting of all open sets O common to all $\mathscr{S}_\alpha$.

Theorem 2.2.4. *Let $\mathscr{S}$ be the intersection of any number of topologies on some X, as defined above. Then $\mathscr{S}$ is again a topology.*

Proof. The only two axioms of a topology that we need check is the union of arbitrarily many sets and the intersection of finitely many sets.

Let $\{O_\beta\}$ be the set of all O_β in the intersection $\mathscr{S}$. That is to say, for all β, $O_\beta \in \mathscr{S}$. Thus since the intersection of topologies is just a set intersection, for all α we must have that for all β, $O_\beta \in \mathscr{S}_\alpha$, since if an O_β is in the intersection, it must have been in every original topology.

Now, since all $\mathscr{S}_\alpha$ are topologies, all possible unions of O_β belong to each $\mathscr{S}_\alpha$, whereby these unions also belong to $\mathscr{S}$. Similarly, since $\mathscr{S}_\alpha$ are topologies for all α, any intersection of finitely many O_β must belong to each $\mathscr{S}_\alpha$, and thus they belong to $\mathscr{S}$ as well. □

Definition 2.2.5 (Minimal topology, generator)**.** Let $\mathscr{A}$ be any collection of subsets of some set X. Then we denote by $\mathscr{S}(\mathscr{A})$ the ***minimal topology*** (in the sense of containing fewest elements) containing $\mathscr{A}$, where $\mathscr{S}(\mathscr{A})$ is said to be ***generated*** by $\mathscr{A}$.

It is worth spending a moment ensuring that this minimal topology always exist, which follows nicely from the last theorem: consider the intersection of all possible topologies on X that contain $\mathscr{A}$. By the theorem this is a topology, and it being minimal follows nature of set intersections.

Note also that there must always exist at least one topology containing $\mathscr{A}$—the topology $\mathscr{S} = \mathscr{P}(X)$, since it contains all possible subsets of X.

2.3 Continuous Functions

In real analysis, and indeed in functional analysis, compositions of continuous functions are again continuous. The same is true in topology, as shown by the following theorem.

Theorem 2.3.1. *Let $(X, \mathscr{S}_X)$, $(Y, \mathscr{S}_Y)$, and $(Z, \mathscr{S}_Z)$ be topological spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous functions, then $\varphi = g \circ f : X \to Z$ is a continuous function as well.*

Proof. Take any $O \in \mathscr{S}_Z$. Since g is continuous, we have by definition that $g^{-1}(O) \in \mathscr{S}_Y$. Similarly, since f is continuous, we must have $f^{-1}(g^{-1}(O)) \in \mathscr{S}_X$. Thus since $\varphi = g \circ f$, we have $\varphi^{-1} = f^{-1} \circ g^{-1}$, whence $\varphi^{-1}(O)$ is open and so by definition φ is continuous. □

Exercise 2.3.2. Consider once more $X = \{a, b, c\}$, equipped with the topology $\mathscr{S} = \{\varnothing, X, \{a\}, \{b, c\}\}$. Let the function $f : X \to X$ be defined by $f(a) = b$, $f(b) = c$, and $f(c) = a$. Is the function f continuous? What about $g : X \to X$ defined by $g(a) = a$, $g(b) = c$ and $g(c) = b$?

Solution. No, f is not continuous, because if we study the preimages of all open sets $O \in \mathscr{S}$, we find two that aren't in $\mathscr{S}$. More precisely, $f^{-1}(\varnothing) = \varnothing \in \mathscr{S}$, and $f^{-1}(X) = X \in \mathscr{S}$, which is good, but $f^{-1}(\{a\}) = \{c\} \notin \mathscr{S}$ and $f^{-1}(\{b, c\}) = \{a, b\} \notin \mathscr{S}$.

On the other hand g is, since $g^{-1}(\varnothing) = \varnothing$, $g^{-1}(X) = X$, $g^{-1}(\{a\}) = \{a\}$, and finally $g^{-1}(\{b, c\}) = \{b, c\}$. ♦

Exercise 2.3.3. Let $X = \{a_1, a_2, a_3, a_4\}$ equipped with the topology $\mathscr{S} = \{\varnothing, X, \{a_1, a_2, a_3\}, \{a_3\}, \{a_3, a_4\}\}$. Construct both a continuous function and a discontinuous function on X. (Try to avoid using the identity function.)

Solution. We first construct a continuous function $f : X \to X$. Let f be the constant function $f(a_i) = a_1$ for all $i = 1, 2, 3, 4$. It is easy verify that this is continuous: $f^{-1}(X) = f^{-1}(\{a_1, a_2, a_3\}) = X$, $f^{-1}(\varnothing) = f^{-1}(\{a_3\}) = f^{-1}(\{a_3, a_4\}) = \varnothing$, all of which belong to $\mathscr{S}$.

For a discontinuous function, consider $g : X \to X$ defined by $g(a_1) = g(a_2) = a_3$ and $g(a_3) = g(a_4) = a_1$. Here we have for example $f^{-1}(\{a_3\}) = \{a_1, a_2\}$ which is not in $\mathscr{S}$. ♦

2.4 Base of Topology

The motivation of creating bases of topologies again stem from metric spaces. If (X, ρ) is a metric space, we defined open sets in terms of very simple balls. Because of this we can consider every possible open set O as the union of all balls around every $x \in O$ small enough to be contained entirely in O:

$$O = \bigcup_{x \in O} B_r(x).$$

Example 2.4.1. Consider $X = \mathbb{R}$. Then the family of all balls $B_r(x) = (x - r, x + r)$ can be used to build all possible open sets. We can do better: if we instead limit ourselves to $\mathcal{B} = \{B_r(x)\}_{x \in \mathbb{Q}}$, the same conclusion holds, since any real number r can be approximated to any desired precision with rational numbers. ▲

Definition 2.4.2 (Base of topology)**.** Let $(X, \mathscr{S})$ be a topological space. A system of open sets $\mathcal{B}$, $\mathcal{B} \subset \mathscr{S}$, is called a ***base*** of the topology $\mathscr{S}$ if any $O \in \mathscr{S}$ can be represented as a union of sets belonging to $\mathcal{B}$.

Example 2.4.3. In any metric space, the set $\mathcal{B}$ of all balls is a base of the topology induced by these balls (as expected). ▲

In practice it is not convenient to check these unions manually, so instead we have the following necessary and sufficient condition for $\mathcal{B}$ being a base.

Theorem 2.4.4. *The system $\mathcal{B} \subset \mathscr{S}$ is a base of the topology $\mathscr{S}$ if and only if for each $O \in \mathscr{S}$ and for each $x \in O$ there exists a $B \in \mathcal{B}$ such that $x \in B \subset O$.*

Proof. ($\Rightarrow$) Suppose $\mathcal{B}$ is a base of the topology $\mathscr{S}$. Then each $O \in \mathscr{S}$ is the union of some $B \in \mathcal{B}$, by definition. For each $x \in O$ we pick one $B = B_x$ from this union, from which we see that there exists a $B = B_x$ such that $x \in B_x \subset O$.

($\Leftarrow$) Take any $O \in \mathscr{S}$. For each $x \in O$, there exists at least one $B = B_x \in \mathcal{B}$ such that $x \in B_x \subset O$, by assumption. Taking the union of all such B_x for all $x \in O$, we get O itself, so $\mathcal{B}$ is a base. □

We recall one more notion from metric spaces: separability. We say that the metric space (X, ρ) is separable if there exists a countable, everywhere dense subset M of X (everywhere dense meaning that for any $x \in X$, we can find a $y \in M$ such that y is as close to x as we like).

Example 2.4.5. The set $M = \mathbb{Q}$ is countable and everywhere dense in $\mathbb{R}$, meaning that $\mathbb{R}$ is separable. ▲

Again we mimic this property in the language of topology.

Definition 2.4.6 (Second axiom of countability)**.** Let $(X, \mathscr{S})$ be a topological space. The space is said to satisfy the ***second axiom of countability*** if there exists a base $\mathcal{B}$ of the topology that is countable.

Example 2.4.7. Take $X = \mathbb{R}$. Then $\mathcal{B} = \{(x - r, x + r)\}$, $r \in \mathbb{Q}$ is a base, as discussed earlier, however it is not countable, since there are uncountably many $x \in R$ on which to centre our intervals. If we instead modify it to be $\mathcal{B} = \{(x - r, x + r)\}$, $r, x \in \mathbb{Q}$, we do have a countable set, and it is everywhere dense since for any $x_0 \in \mathbb{R}$ and for all $O \in \mathscr{S}$ with $x_0 \in O$ there exists $x, r \in \mathbb{Q}$ such that $x_0 \in (x - r, x + r) \subset O$. ▲

That is to say:

Theorem 2.4.8. *The set of real numbers $\mathbb{R}$ satisfies the second axiom of countability.*

We can generalise this.

Theorem 2.4.9. *Any separable metric space satisfies the second axiom of countability.*

Proof. Let our metric space be (X, ρ). The proof is the same as for $\mathbb{R}$, except we use the countable, everywhere dense subset $M = \{x_1, x_2, \ldots, x_n, \ldots\}$ instead of the particular $\mathbb{Q}$. Take $\mathcal{B} = \{B_{1/m}(x_n)\}$ for $m, n \in \mathbb{Z}^+$. Then for any x in any open set O there exists an ball from $\mathcal{B}$ such that $x \in B_{1/m}(x_n) \subset O$.

Because M is everywhere dense, for every $\varepsilon > 0$ there exists an $x_n \in M$ such that $\rho(x, x_n) < \varepsilon$. Thus for every $O \in \mathscr{S}$ and every $x \in O$ there exists a $B_r(x) \subset O$.

We can always select $\varepsilon = 1/m$ for some appropriate m such that $x \in B_\varepsilon(x_n) \subset B_r(x) \subset O$. □

3 Lecture III[3]

3.1 Topology from a System of Neighbourhoods

In metric spaces, we construct open sets in terms of the union of neighbourhoods. So far we have constructed our topologies as a system of open sets, using which we defined neighbourhoods. We can go the other way around, as with metric spaces.

Consider the following: in a topological space $(X, \mathscr{S})$, a neighbourhood of some $x \in X$ is any $U_x \in \mathscr{S}$ such that $x \in U_x$. Now fix $x \in X$ and let $\mathscr{U}_x$ be the collection of all such neighbourhoods U_x of x. Two properties of the elements of $\mathscr{U}_x$ that are immediately obvious are the following:

(i) For all $U \in \mathscr{U}_x$, x belongs to U;

(ii) If U and V belong to $\mathscr{U}_x$, then $U \cap V$ belongs to $\mathscr{U}_x$ as well.

(The second one is a consequence of the intersection of two open sets again being open, and if x belongs to both, it must belong to the intersection.)

Theorem 3.1.1. *Suppose that for any $x \in X$ there is given a system of subsets of X denoted $\mathscr{U}_x$ which has the properties (i) and (ii) above. Now let $\mathscr{S}$ be the collection of all subsets of X such that $O \in \mathscr{S}$ if and only if, for all $x \in O$ there exists a $U \in \mathscr{U}_x$ such that $U \subset O$.*

The system $\mathscr{S}$ constructed this way, together with the empty set, defines a topology on X.

Proof. That $X \in \mathscr{S}$ is clear. It is also clear that if $\{O_\alpha\}$ is any collection of $O_\alpha \in \mathscr{S}$, then $\cup_\alpha O_\alpha \in \mathscr{S}$ as well since if x belongs to any one O_α, it must then also belong to the union of all O_α.

Let us next consider the intersection. Let O_1 and O_2 be in $\mathscr{S}$. Therefore for all $x \in O_1$ there exists some $U_1 \in \mathscr{U}_x$ such that $U_1 \subset O_1$, and for all $x \in O_2$ there exists some $U_2 \in \mathscr{U}_x$ such that $U_2 \subset O_2$.

Thus for any $x \in U = U_1 \cap U_2$ we $U \in \mathscr{U}_x$ by construction, and $x \in U \subset O_1 \cap O_2$ whence $O_1 \cap O_2 \in \mathscr{S}$. Thus $\mathscr{S} \cup \{\varnothing\}$ is a topology. □

Example 3.1.2. Consider metric spaces on countably infinite sequences of real numbers, such as

$$\ell_1 = \left\{ \boldsymbol{x} = (x_j) \,\middle|\, \sum_{j=1}^{\infty} |x_j| < \infty \right\}$$

with the metric $\rho_1(\boldsymbol{x}, \boldsymbol{y}) = \sum_{j=1}^{\infty} |x_j - y_j|$ or

$$\ell_\infty = \left\{ \boldsymbol{x} = (x_j) \,\middle|\, \sup_{1 \le j < \infty} |x_j| < \infty \right\}$$

[3]Date: February 4, 2016.

with the metric $\rho_\infty(\boldsymbol{x},\boldsymbol{y}) = \sup_{1\le j<\infty}|x_j - y_j|$.

If we consider instead the set of *any* sequences of real numbers

$$\mathbb{R}^{\mathbb{N}} = \{\boldsymbol{x} = (x_j),\, x_j \in \mathbb{R}\}$$

it is not so easy (but not impossible) to define a metric, however we can introduce a fairly natural topology. For all $\boldsymbol{x}$ we define $\mathscr{U}_{\boldsymbol{x}}$ to be the collection of all neighbourhoods $U^{\varepsilon_1\varepsilon_2\ldots\varepsilon_n}_{i_1 i_2\ldots i_n}(\boldsymbol{x}) = \{\boldsymbol{y} \mid |x_{i_k} - y_{i_k}| < \varepsilon_k,\, k = 1,2,\ldots,n\}$.

That is to say, $U^{\varepsilon_1\varepsilon_2\ldots\varepsilon_n}_{i_1 i_2\ldots i_n}(\boldsymbol{x})$ is the set of all sequences which, in the finite number of coordinates $i_1, i_2, \ldots, i_n$ differs from $\boldsymbol{x}$ with at most ε_i, and on the other coordinates there are no conditions. Clearly $\boldsymbol{x} \in \mathscr{U}_{\boldsymbol{x}}$ since it differs from $\boldsymbol{x}$ by precisely 0 in all coordinates, so condition (i) from above is satisfied. As is condition (ii), which we see attempting to take intersections between $U^{\varepsilon_1\varepsilon_2\ldots\varepsilon_n}_{i_1 i_2\ldots i_n}(\boldsymbol{x})$ and $U^{\delta_1\delta_2\ldots\delta_n}_{j_1 j_2\ldots j_n}(\boldsymbol{x})$.

Doing so is easy: for the indices s where i_k and j_l agree, we take the new ε_s to be the minimum of ε_{i_k} and δ_{j_l}, whereas for the remaining indices we leave the conditions as-is. Clearly this new set also belongs to $\mathscr{U}_{\boldsymbol{x}}$, whereby the collection of all $\mathscr{U}_{\boldsymbol{x}}$ does indeed define a topology, the so-called ***coordinate-wise topology***. ▲

We can do one better.

Example 3.1.3. Consider now $X = \mathbb{R}^A$, where A is any set. Then the elements of X are $\boldsymbol{x} = (x_\alpha)_{\alpha\in A}$. Intuitively, $\boldsymbol{x}$ is an object which to every $\alpha \in A$ maps an element of $\mathbb{R}$, that is to say, $\boldsymbol{x} : A \to \mathbb{R}$, so $\mathbb{R}^A$ is the space of all functions from A to $\mathbb{R}$. Here similarly $\mathscr{U}_{\boldsymbol{x}}$ is the set of all neighbourhoods $U^{\varepsilon_1\varepsilon_2\ldots\varepsilon_n}_{i_1 i_2\ldots i_n}(\boldsymbol{x}) = \{\boldsymbol{y} = \boldsymbol{y}(t) \mid |\boldsymbol{x}(t_k) - \boldsymbol{y}(t_k)| < \varepsilon_k,\, k = 1,2,\ldots,n\}$, which defines the ***pointwise topology of functions***. ▲

3.2 Axioms of Separability

Using these neighbourhoods we formulate two axioms.

Definition 3.2.1 (Axioms of separability)**.** Let $(X, \mathscr{S})$ be a topological space.

It is said to satisfy the ***first axiom of separability***, T_1, if for any two points $x, y \in X$, $x \neq y$, there exists a neighbourhood U_x of x and a neighbourhood V_y of y such that $y \notin U_x$ and $x \notin V_y$.

The topological space is said to satisfy the ***second axiom of separability***, T_2, otherwise known as the Hausdorff axiom if for any two points $x, y \in X$, $x \neq y$, there exists a neighbourhood U_x of x and a neighbourhood V_y of y such that $U_x \cap V_y = \varnothing$.

It is clear that T_2 implies T_1, whereas intuition suggests that the converse isn't true. We would like to show this by a simple counterexample, but this cannot be done for topologies on finite sets $X = \{a_1, a_2, \ldots, a_n\}$!

Theorem 3.2.2. *If a finite topological space $(X, \mathscr{S})$, $X = \{a_1, a_2, \ldots, a_n\}$, is T_1, then $\mathscr{S}$ must be the set of all subsets of X, and thus Hausdorff.*

Proof. Taking any two distinct a_i and a_j, a_j fixed, since the space is T_1 there exists an open set $O \in \mathscr{S}$ such that $a_i \in O$ but $a_j \notin O$, whereby a_i cannot be a contact point of $\{a_j\}$. Since this is true for all $a_i \neq a_j$, the set $\{a_j\}$ has no limit points other than a_j, whence it is its own closure, meaning that is is closed.[4] Thus all points are closed.

Furthermore, since in a topology any finite intersection of open sets is open, by duality any finite union of closed sets must be closed. In particular, in our finite topology, the complement of any point is a finite union of the remaining point, which we showed to be closed, and so the union is closed, and therefore the original point must be open.

Since all points are open, they belong to the topology, and therefore all possible unions of points do as well, whereby the topology must be the one of all subsets of X. Finally this space is Hausdorff since, in the definition, we take the neighbourhoods of the two points to be precisely the singletons containing the two points. □

Therefore we must use a slightly more complicated counterexample.

Counterexample 3.2.3. We define the so-called ***cofinite topology*** on an infinite set X. Co- in cofinite here refers to 'complementary', motivating the definition: the cofinite topology of an infinite set X is the set of all complements to finite subsets of X (as well as the empty set). Thus O is in $\mathscr{S}$ if there exists a finite set $A \subset X$ suchh that $O = X \setminus A$.

Let us use the cofinite topology of $X = \{a_1, a_2, \ldots, a_n, \ldots\}$. Clearly this topology is T_1, since taking any two points a_i and a_j, we can take the neighbourhoods U_{a_i} and U_{a_j} to be $X \setminus \{a_j\}$ and $X \setminus \{a_i\}$ respectively.

However it is not Hausdorff, because if we take any two (nonempty) open sets O_1 and O_2, there exists finite subsets A and B of X such that $O_1 = X \setminus A$ and $O_2 = X \setminus B$, whence

$$O_1 \cap O_2 = (X \setminus A) \cap (X \setminus B) = X \setminus (A \cup B),$$

whereby the intersection of two open sets cannot be empty unless $A \cup B = \varnothing$, which is not the case by assumption. Since this is true for any nonempty open sets, it is true in particular for neighbourhoods. ▲

Theorem 3.2.4. *All metric spaces are Hausdorff.*

Proof. Since any two points $x \neq y$ from the metric space have a nonzero distance, one can construct balls of half this distance around either point which therefore do not intersect. □

3.3 Closure of a Set

Definition 3.3.1 (Closure)**.** Let $(X, \mathscr{S})$ be a topological space and let $M \subset X$. Then $[M]$, called the ***closure*** of M, is the set of all contact points of M.

Example 3.3.2. In $X = \mathbb{R}$, with $M = (a, b)$, the closure of M is $[M] = [(a, b)] = [a, b]$. ▲

It is clear that M is always a subset of its closure, since any point of a set is a contact point to this set. Moreover, as suggested by the name, the closure of a set is closed.

[4] Strictly speaking we haven't yet shown these last two implications, but they're obvious enough and we will very shortly.

Theorem 3.3.3. *Let $(X, \mathscr{S})$ be a topological set and let $M \subset X$. Then $[M]$ is closed.*

Proof. Take any $x \notin [M]$. Then there exists a neighbourhood U_x of x such that $U_x \cap M = \varnothing$. Suppose we have a $z \in [M]$, with $z \in U_x$ as well, whence U_x is an open set containing both x and z. Thus U_x can be considered instead as a neighbourhood V_z of z, with $V_z \cap M = \varnothing$. Thus we have found a neighbourhood V_z of $z \in [M]$ without any point $y \in M$, so $z \notin [M]$, a contradiction. Thus for all $z \in X \setminus [M]$ there exists neighbourhoods $U_x \cap [M] = \varnothing$, whence $X \setminus [M]$ is open, so $[M]$ is closed. □

3.4 Compact Topological Spaces

The sets we call compact in $\mathbb{R}$ are ones that are closed and bounded. Generalising this to topology is not quite as straight forward, since we need not have any concept of distance for the boundedness.

Definition 3.4.1 (Cover)**.** Let $(X, \mathscr{S})$ be a topological space and let A be a subset of X. A collection of subsets $\{O_\alpha\}$ is called a ***cover*** of A if and only if $A \subset \bigcup\limits_{\alpha} O_\alpha$.

Definition 3.4.2 (Compact set)**.** Let $(X, \mathscr{S})$ be a topological space. The set $K \subset X$ is called ***compact*** if each of its open covers has a finite subcover. In other words, for each cover $\{O_\alpha\}$, $O_\alpha \in \mathscr{S}$, such that $K \subset \bigcup\limits_{\alpha} O_\alpha$, there exists $O_1, O_2, \ldots, O_n \in \{O_\alpha\}$ such that $K \subset \bigcup\limits_{j=1}^{n} O_j$.

Theorem 3.4.3. *Continuous maps preserve compactness.*

Proof. Let $(X, \mathscr{S}_X)$ and $(Y, \mathscr{S}_Y)$ be two topological spaces. Recall that a function $f : X \to Y$ is continuous if and only if the pre-image of any open set in Y is open in X.

We now take any open cover in Y, and we find the pre-image of this in X. Since f is continuous, the pre-image of this open cover must be open. Then since X is assumed to be compact, we can find a finite subcover of this pre-image. We now map this finite subcover back to Y by means of f, which is again an open cover since we started with a cover in Y, and since the we mapped a finite subcover—a collection of finitely many open sets—the result must also be a collection of finitely many open sets, so we have acquired a finite subcovering in Y. □

3.5 Centred Systems

Definition 3.5.1 (Centred system)**.** A system of sets $\{F_\alpha\}$ is called ***centred*** if all of its finite subsystems have non-empty intersections. That is to say, for all possible combinations of $\alpha_1, \alpha_2, \ldots, \alpha_n$, we have $\bigcap\limits_{j=1}^{n} F_{\alpha_j} \neq \varnothing$.

Theorem 3.5.2. *Let $(X, \mathscr{S})$ be a topological space. A set X is compact if and only if each centred system $\{F_\alpha\}$ of closed subsets has non-empty intersections.*

Proof. ($\Rightarrow$) Let X be compact, and define $G_\alpha = X \setminus F_\alpha$ for all α. Then since by assumption F_α is closed, G_α is open. Moreover since

$$\bigcap_{j=1}^{n} F_{\alpha_j} \neq \varnothing,$$

by duality we must have

$$X \setminus \bigcap_{j=1}^{n} F_{\alpha_j} = \bigcup_{j=1}^{n} G_{\alpha_j} \neq X.$$

Thus $\{G_\alpha\}$ cannot cover X, whereby again by duality $\{F_\alpha\}$ must have non-empty intersection. □

4 Lecture IV[5]

4.1 Relative Topology

Suppose that we have some topological space $(X, \mathscr{S})$, and we have a subset $Y \subset X$. We would like to equip this Y with a topology as well. The obvious way to do this is the following.

Definition 4.1.1 (Relative topology)**.** Let $(X, \mathscr{S}_X)$ be a topological space and let $Y \subset X$. We call $\mathscr{S}_Y = \{O_Y = O_X \cap Y, O_X \in \mathscr{S}_X\}$ the ***relative topology*** on Y.

Theorem 4.1.2. *Let X be a compact topological space and let Y be a* closed *subset of X. Then Y is also compact in its relative topology.*

Proof. Let $\{F_\alpha\}$ be a centred system of closed subsets in Y, recalling that Y is closed in X. Then $F_\alpha = F_\alpha \cap Y$ is closed in X as well. Thus $\{F_\alpha\}$ is a centred system of closed subsets in X, which is compact, meaning that $\bigcap_\alpha F_\alpha \neq \varnothing$ in X, but since $F_\alpha \subset Y$ it cannot be empty in Y either. Thus Y is compact. □

The converse holds as well, assuming the topology on X is Hausdorff.

Theorem 4.1.3. *Let X be a Hausdorff space and let Y be a compact subset. Then Y is closed.*

Proof. By assumption, $Y \subset X$ and Y is compact. Thus any open cover of Y has a finite subcover. We want to prove that $X \setminus Y$ is open, and so we pick some $b \in X \setminus Y$. Now take any $z \in Y$. Since X is Hausdorff, there exists some neighbourhood $U_b^{(z)}$ of b (depending on z) and some neighbourhood V_z of z such that $U_b^{(z)} \cap V_z = \varnothing$. This construction we can perform for all $z \in Y$.

If we now for all $z \in Y$ take the union of these, we have $\bigcup_z V_z \supset Y$, an open cover of Y. Thus since Y is compact there exists a finite collection $V_{z_1}, V_{z_2}, \dots, V_{z_n}$ such that

$$\bigcup_{j=1}^{n} V_{z_j} \supset Y$$

[5] Date: February 8, 2016.

as well. This union is open, since we are in a topology. If we now take the finitely many corresponding $U_b^{(z)}$, and take their intersection (which is open, since it's a finite amount), we get

$$U_b = \bigcap_{j=1}^{n} U_b^{(z_j)} \in \mathscr{S},$$

so for $b \in X \backslash Y$ we have found a neighbourhood U_b of b such that $U_b \cap \big(\bigcap_{j=1}^{n} V_{z_j}\big) = \varnothing$, where the union is a cover of Y, so we also have $U_b \cap Y = \varnothing$. This construction can be done for any $b \in X \setminus Y$, as b was arbitrary, so for all $b \in X \setminus Y$ there exists a neighbourhood U_b of b such that $U_b \cap Y = \varnothing$, whence $X \setminus Y$ is open, and therefore Y is closed. □

5 Lecture V[6]

5.1 More on Open and Closed Sets

Recall how in Theorem 3.2.2, page 9 we showed that in any T_1 topology over a finite set, points are both open and closed. Let us consider the same for T_1 topologies over any set.

Theorem 5.1.1. *Let $(X, \mathscr{S})$ be any T_1 topological space. Then for every $x \in X$, $M = \{x\}$ is closed.*

Proof. The proof is similar to that of the aforementioned theorem: we show that M is its own closure, and therefore closed.

Take any $y \neq x$, then for y to be a contact point of M (and thus in the closure of M) every neighbourhood of y must contain a point of M, i.e. x. But $\mathscr{S}$ is T_1, whereby there exists at least one neighbourhood of y which does not contain x, and so y cannot be a contact point of M.

Therefore the only contact point of M is x itself, and so $[M] = M$. □

Remark 5.1.2. An immediate consequence of this is that in any T_1 topology, any finite subset is closed, since it is the union of a finite number of points, which are closed.

We might now ask ourselves if the other part of the aforementioned theorem is true as well; that in any T_1 topology, all points are open as well. This is not the case, as demonstrated by the following counterexample.

Counterexample 5.1.3. The cofinite topology on $X = \{a_1, a_2, \dots, x_n, \dots\}$ is, as demonstrated in Counterexample 3.2.3, T_1. However in it no points are open. This is easy to see: the open sets in the cofinite topology are complements of finite sets, and so for singletons to be open their complement would have to be finite, which cannot be the case since our universe X is infinite. ▲

[6]Date: February 9, 2016.

5.2 Compact Sets and Countability

Example 5.2.1. Let $X = \mathbb{R}$. It is not compact, since if we take the open covering $(-1/2, 1/2)$, $(0, 1)$, $(1/2, 3/2)$, et cetera, we can find no finite subcovering, since if we stop at finitely many intervals, we have covered only a finite subset of the real line. ▲

Theorem 5.2.2. *Let X be a compact space. Then any* infinite *subset has a limit point.*

Proof. Suppose there is an infinite subset A without limit points. Then there must exist a countable subset $\{x_1, x_2, \ldots, x_n, \ldots\} \subset A$, and since A has no limit points, this countable subset doesn't either.

Consider now $F_n = \{x_n, x_{n+1}, \ldots\}$, the set of all but the $n-1$ points of this countable subset. Then clearly $F_1 \supset F_2 \supset \ldots \supset F_n \supset \ldots$, and taking finitely many intersections we have

$$\bigcap_{n=1}^{m} F_n = F_m \neq \varnothing,$$

whence $\{F_n\}$ is a centred system. Moreover each F_n is closed, since it contains all of its limit points (there are none), whence we have a centred system of closed subsets in a compact set X. Thus by completeness, the intersection of all infinitely many of them must be nonempty as well.

However, in how we constructed them, the intersection is empty, since we exhaust the list if we let m tend to infinity, and therefore we have encountered a contradiction, and our original assumption of A not having any limit points must be false. □

This previous theorem directly motivates the next definition.

Definition 5.2.3 (Countably compact)**.** A topological space is called ***countably compact*** is all infinite subsets have limit points.

We proved in the previous theorem that compactness implies countable compactness. The converse does not hold.

Theorem 5.2.4. *A topological space $(X, \mathscr{S})$ is countably compact if and only if one of the following two equivalent statements holds:*

- (i) *Every* countable *open cover of X has a finite subcover,*
- (ii) *Every* countable *centred system of closed subsets of X has nonempty intersection.*

Proof. First of all (i) and (ii) being equivalent is clear: it is just Theorem 3.5.2 repeated with countable sets instead of any set.

Let us then consider whether (ii) implies countable compactness. This is also clear: suppose any centred system of countably many closed subsets has nonempty intersection, then by previous theorem works as well; $\{F_n\}$ was specifically constructed to be countable, and so works here as well.

Finally suppose our space is countably compact. Then there exist limits points of any infinite subset, and we would like to show that this implies nonempty intersections of centred systems of countably many closed subsets.

Let $\{F_n\}$, $n \in \mathbb{N}$, be a centred system of closed sets. We construct using these

$$\Phi_m = \bigcap_{n=1}^{m} F_n \neq \varnothing,$$

which is nonempty because $\{F_n\}$ is centred and closed because F_n are. We also have $\Phi_1 \supset \Phi_2 \supset \ldots \supset \Phi_n \supset \ldots$ by construction. There are now two possibilities: either we reach stability, meaning that there exists some M, such that for all $N \geq M$ we have $\Phi_N = \Phi_{N+1} = \ldots$, in which case we trivially have nonempty intersections of the infinitely many F_n since this intersection is just equal to Φ_N.

The other possibility is that we don't have stability, in which case we have some $\Phi_1 \supset \Phi_{n_1} \supset \Phi_{n_2} \supset \ldots$, where $\Phi_{n_i} \neq \Phi_{n_j}$ whenever $i \neq j$.

Now if for each consecutive pair Φ_{n_j} and $\Phi_{n_{j+1}}$ we pick an $x_j \in Phi_{n_j} \backslash \Phi_{n_{j+1}}$, by construction $\{x_j\}$ is an infinite set, and it must have a limit point, say x_0, because of countable compactness. Now take Φ_{m_j}, then all $\{x_j, x_{j+1}, \ldots\} \supset \Phi_{m_j}$, but Φ_{m_j} is closed, so $x_0 \in \Phi_{m_j}$, for any Φ_{m_j}, so

$$x_0 \in \bigcap_{j=1}^{\infty} \Phi_{m_j} = \bigcap_{m=1}^{\infty} \Phi_m \neq \varnothing. \qquad \square$$

Recall from the end of Lecture II the discussion of bases of topologies, along with the second axiom of countability.

Theorem 5.2.5. *Let $(X, \mathscr{S})$ satisfy the second axiom of countability. Then from any open covering of X it is possible to select a countable subcovering.*

Proof. We have that

$$X \subset \bigcup_s O_s,$$

$O_s \in \mathscr{S}$, and we would like to show that there exists $O_{s_1}, O_{s_2}, \ldots, O_{s_n}, \ldots$, such that

$$X \subset \bigcup_{j=1}^{\infty} O_{s_j}.$$

Now since X satisfies the second axiom of countability, there exists a countable base $\mathcal{B} = \{G_1, G_2, \ldots, G_n, \ldots\}$ such that for all $O \in \mathscr{S}$ there exists some G_k such that $O = \bigcup G_k$.

Now let O_s be an open covering of X, whence due to us having a basis we must have that for all O_s, and for all $x \in O_s$, there exists some $G_k \in \mathcal{B}$ (where $k = k(s,x)$ depends on s and x) such that $x \in G_k \subset O_s$.

Of course

$$\bigcup_s O_s = \bigcup_{\forall s,x} G_{k(s,x)} \supset X.$$

Now for each k we take one of $O_s \supset G_k$, whence we have that $\{O_{s_1}, O_{s_2}, \ldots, O_{s_n}, \ldots\}$ is a countable subcovering of X. $\square$

Now combining the various things we know we have the following theorem.

Theorem 5.2.6. *For spaces satisfying the second axiom of countability, the notions of compactness and countable compactness coincide.*

Proof. We have showed that countable compactness means three equivalent things:

(i) By definition, any infinite set has a limit point;

(ii) Any countable open covering has a finite subcovering;

(iii) Countably centred systems of closed subsets have nonempty intersections.

Clearly compactness implies (ii); how about the converse? Yes, the last theorem gives us a countable subcover, and countable compactness means that a countable subcover has a finite subcover, so in all we have compactness. □

Thus in particular, in any separable metric space, compactness is equivalent to countable compactness, however both of these things are often hard to test in practice. Thus in applications, the first equivalent statement (i) is often the easiest to check.

Example 5.2.7. Consider a closed interval $[a, b]$ on the real number line. This interval is compact, since any infinite sequence has a limit point.

In fact, this is where the topological notion of compactness stems from. The more abstract notion of cover becomes necessary when one generalises away the properties of metric spaces. ▲

6 Lecture VI[7]

6.1 Compactness and Related Notions

We recall the definitions of various notions along with certain important theorems, and attempt to relate these to other notions under relevant assumptions.

Recall that a topological space $(X, \mathscr{S})$ is compact if all open covers have finite subcovers, and that this is equivalent to all centred systems of closed subsets have nonempty intersections. With this we showed three important things: first, that continuous maps preserve compactness. Secondly, that if X is compact and $Y \subset X$ is closed, then Y is also compact. Thirdly we shoed that if X is a Hausdorff space and $Y \subset X$ is compact, then Y is closed.

Problem 6.1.1. Show that if X is not Hausdorff, then a compact subset might not be closed.

Solution. Consider the set $X = \{a, b, c\}$ with the topology $\mathscr{S} = \{\varnothing, X, \{a\}, \{a, b\}\}$. Clearly this is not Hausdorff since it isn't even T_1 (there are for example no neighbourhoods of b not also containing a).

Clearly everything is compact; since it is a finite topology, any cover is automatically a finite subcover. Moreover, for example $\{a\}$ is not closed, since $X \setminus \{a\} = \{b, c\} \notin \mathscr{S}$. ♦

We then proved a theorem saying that if a topological space $(X, \mathscr{S})$ is compact, then every infinite subset of X has a limit point.

Problem 6.1.2. Prove that the converse isn't true. That is to say, find a topological space where every infinite subset has limit points, yet the space isn't compact.

[7]Date: February 11, 2016.

Solution. Consider ω_1, the set of all countable ordinal numbers, on which we can create a topology similar to that of the open intervals in the real numbers, since the ordinals are totally ordered, so the open sets are ω_1 itself, $\varnothing$, and $(\alpha, \beta) = \{\gamma \in \omega_1 \mid \alpha < \gamma < \beta\}$, along with any possible unions and finite intersections of such.

This space is not compact, since for example the set $\{U_\alpha \mid \alpha < \omega_1\}$, where $U_\alpha = \{\beta \in \omega_1 \mid \beta < \alpha\}$, is a cover without any finite subcover.

However it is countably compact, because if O is any infinite subset of ω_1, we can always find an increasing sequence, and any infinite sequence of ordinals converges to its supremum, which is then a limit point. To show that we can always find an increasing sequence, let $o_1 = \min O$ (the ordinals are well-ordered; this always exists). We then let o_2 be the smallest $o \in O$ that is strictly greater than o_1, and so on, with o_{n+1} being defined as the smallest $o \in O$ strictly greater than o_n. ♦

We are interested in the class of spaces where these two notions are equivalent, which prompted the definition of countably compact spaces: a topological space is called countably compact if every finite subset has limit points.

We then showed that if a topological space satisfies the second axiom of countability (i.e. there exists a countable base), then the notions of countable compactness and compactness coincide.

The reason we care so deeply about compactness and its variations stems largely from integration theory. As in introductory analysis, where it is easy to integrate over closed intervals (the prototype of compact sets on the real line) it is similarly easy to integrate in general over compact sets.

In this spirit we introduce a new notion of compactness.

Definition 6.1.3 (Local compactness)**.** We call a space ***locally compact*** if any point has a base of neighbourhoods, the closures of which are compact.

Example 6.1.4. The real line $\mathbb{R}$ is locally compact (but not compact) since around any point we can build neighbourhoods (that is, open intervals), the closures of which are closed intervals, which are of course compact. ▲

6.2 Compactness in Metric Spaces

Let us now venture back into the jungle of metric spaces and explore these notions in this setting.

We recall that a metric space (X, ρ) satisfies the second axiom of countability if and only if it is separable (i.e. there exists a countable set $M = \{x_1, x_2, \ldots, x_n, \ldots\}$ such that $[M] = X$).

Combining this with the previous result we therefore conclude that in a separable metric space, compactness and countable compactness coincide. Therefore in a separable metric space we can check whether it is compact by just checking whether every infinite subset has a limit point.

Example 6.2.1. Consider the metric space ℓ_2 of infinite sequences $\boldsymbol{x} = (x_j)$ which satisfy

$$\sum_{j=1}^{\infty} x_j^2 < \infty,$$

with the distance

$$\rho(\boldsymbol{x},\boldsymbol{y}) = \Big(\sum_{j=1}^{\infty}|x_j - y_j|\Big)^{1/2} < \infty.$$

This space is separable, whereby it satisfies the condition above.

Here the closed unit ball $\overline{B}_1(\mathbf{0}) = \{\boldsymbol{x} \in \ell_2 \mid \rho(\boldsymbol{x},\mathbf{0}) \leq 1\}$ is closed but not compact. To see this, consider the sequence $\boldsymbol{e}_1 = (1,0,\ldots)$, $\boldsymbol{e}_2 = (0,1,0,\ldots)$, et cetera, where $\boldsymbol{e}_n$ is zero everywhere except at coordinate n where it is 1. Clearly this sequence belongs to $\overline{B}_1(\mathbf{0})$ since all of the distances to $\mathbf{0}$ are exactly 1, yet it has no limit point, since $\rho(\boldsymbol{e}_n,\boldsymbol{e}_m) = \sqrt{2}$ for all $n \neq m$.

The same is true in ℓ_1. ▲

Problem 6.2.2. Consider the metric space ℓ_∞ of bounded sequences. Contrary to ℓ_1 and ℓ_2 it is not separable, so to decide compactness it is enough to consider whether infinite subsets have limit points. Even so, decide whether the closed unit ball is compact or not.

Solution. Recall first of all that the closed unit ball is

$$B = \Big\{\boldsymbol{x} = (x_j)\ \Big|\ \sup_{1\leq j<\infty}|x_j| \leq 1\Big\}.$$

We suspect this is not compact. Note that B is closed, and since ℓ_∞ is a metric space it must automatically be Hausdorff, and we know that any closed subset of a compact Hausdorff space is compact. Thus if the closed ball B was compact, then its closed subset $\{\boldsymbol{e}_n\}$, $\boldsymbol{e}_n = (0,\ldots,0,1,0,\ldots)$ (the 1 being in the nth coordinate) would have to be compact as well.

That $\{\boldsymbol{e}_n\}$ is closed is not hard to see: clearly in its complement we can create around any sequence $\boldsymbol{x}$ a ball of some small radius such that this ball doesn't include any of $\boldsymbol{e}_n$.

Finally it remains to verify that $\{\boldsymbol{e}_n\}$ is indeed not compact. This is also easy to see; if we construct the set $\{B_{1/2}(\boldsymbol{e}_n\}$ of open balls with radii 1/2 centred around each $\boldsymbol{e}_n$, then this is definitely an open cover of $\{\boldsymbol{e}_n\}$, but it certainly doesn't have a finite subcover, if we had only finitely many such balls, then we could cover only finitely many $\boldsymbol{e}_n$ since each ball is constructed specifically to cover only one $\boldsymbol{e}_n$. ♦

We now define a few new, closely related notions.

Definition 6.2.3 (Net)**.** Let (X,ρ) be a metric space and let A be a finite subset of X. Then A is called an ***ε-net*** of X if, for any $x \in X$, there exists an $a \in A$ such that $\rho(x,a) < \varepsilon$.

Definition 6.2.4 (Total boundedness)**.** Let (X,ρ) be a metric space. Suppose there for every $\varepsilon > 0$ exists an ε-net. Then it is called ***totally bounded***.

Since we introduce the notion of total boundedness, we should also have a notion of boundedness.

Definition 6.2.5 (Boundedness)**.** A subset B of a metric space (X,ρ) is called ***bounded*** if there exists an $R > 0$ such that $\rho(x,0) < R$ for all $x \in B$.

Note that the definitions of net and total boundedness are formulated for the entire metric space X. We can formulate the same for subsets of a metric space X.

Definition 6.2.6 (Net)**.** Let (X, ρ) be a metric space and let B be a subset of X. If there exists a finite subset A of X such that, for any $b \in B$, there exists an $a \in A$ such that $\rho(b, a) < \varepsilon$, then A is called an ***ε-net*** of B.

Definition 6.2.7 (Total boundedness)**.** Let (X, ρ) be a metric space. A subset B of X is called ***totally bounded*** if there for every $\varepsilon > 0$ exists an ε-net of B.

Example 6.2.8. Consider again the metric space ℓ_2, and in it the closed unit ball $B = \{\boldsymbol{x} \mid \rho(\boldsymbol{0}, \boldsymbol{x}) \leq 1\}$. Clearly B is bounded, but it is not totally bounded.

Suppose there exists a finite ε-net for every $\varepsilon > 0$. Then there should exist a set $A = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\}$ (n depending on ε) such that this finite set is sufficiently close to, for example, all $\boldsymbol{e}_j$ from before. But clearly this cannot be, because if we take, say, $\varepsilon = \sqrt{2}/2$, $\boldsymbol{a}_i$ can be close to only one $\boldsymbol{e}_j$, and so A is close to only finitely many $\boldsymbol{e}_j$. ▲

On the other hand, total boundedness implies boundedness.

Theorem 6.2.9. *Total boundedness implies boundedness.*

Proof. If a set B is totally bounded, then for every $\varepsilon > 0$ there exists a finite set $A = \{a_1, a_2, \ldots, a_n\}$ (where n depends on ε) such that

$$\bigcup_{j=1}^{n} B_\varepsilon(a_j) \supset B.$$

Thus taking $R = \varepsilon + \max_{1 \leq j \leq n} \rho(a_j, 0)$, we clearly have $\rho(x, 0) < R$ for every $x \in B$. □

Sometimes, these coincide.

Theorem 6.2.10. *Let $X = \mathbb{R}^n$, for example with the Euclidean distance. Then boundedness and total boundedness coincide.*

Proof. We describe the proof as a geometric construction in $\mathbb{R}^2$, but it is not hard to see that the same works in any finite dimensional space $\mathbb{R}^n$.

We know that total boundedness implies boundedness in all cases, so it remains to verify that in this particular case the opposite holds. Thus we have some set B which is bounded, meaning that we can enclose it in a sufficiently large ball of radius $R > 0$.

Now imagine enclosing this ball (which in $\mathbb{R}^2$ is just a circle) in a square of sides $\sqrt{R}$, and then partitioning this square into smaller squares of sides $\varepsilon > 0$. This partitions the square into finitely many smaller squares.

If we then let A be the set of all vertices of these smaller ε squares, the worst case for any $b \in B$ is that it is $\sqrt{2}\varepsilon$ away from some the closest vertex in A. Thus if we take $\varepsilon' = \sqrt{2}\varepsilon$, then for every $\varepsilon' > 0$ there exists a finite ε'-net of B. □

It might then seem natural to assume that since in finite dimensional spaces, total boundedness and boundedness are the same, we would have something similar in infinite dimensional spaces. That is, in infinite dimensional spaces this is not the case. However it is not so: there exists infinite dimensional spaces that are totally bounded. We will present an example of this later.

Our ultimate aim of this discussion is to relate total boundedness to countable compactness.

Theorem 6.2.11. *Let (X, ρ) be a metric space. Then if X is countably compact, then it is totally bounded.*

Proof. Let X be countably compact but *not* totally bounded. Then there must exist some $\varepsilon_0 > 0$ such that there is no finite ε_0-net for X. Therefore if we take any $x_1 \in X$, there must exist another point x_2 such that $\rho(x_1, x_2) > \varepsilon_0$ (otherwise there would exist a ε_0-net for x_1). We continue this iteratively; now that we have $\{x_1, x_2\}$, there must exist another point x_3 such that its distance to the other two is greater than ε_0, et cetera.

Continuing this we get a set $\{x_1, x_2, \ldots, x_n, \ldots\}$ with $\rho(x_n, x_{n+1}) > \varepsilon_0$ for all n, whence the set cannot have a limit point. □

In the same spirit we attempt to relate total boundedness and separability.

Theorem 6.2.12. *Let (X, ρ) be a totally bounded metric space. Then X is separable.*

Proof. Take any $\varepsilon_n = 1/n$. Then by the total boundedness there exists a finite ε_n-net $A_n = \{a_1, a_2, \ldots, a_n\}$. Consider now

$$A = \bigcup_{n=1}^{\infty} A_n,$$

which must be countable since it is a countable union of finite sets. Because each of A_n are ε_n-nets, then $[A] = X$. □

Combining all of the above knowledge we get the following chain of implications.

Theorem 6.2.13. *Let (X, ρ) be a metric space. If X is countably compact, then it is totally bounded. If X is totally bounded, then it is separable. If X is separable, then it satisfies the second axiom of countability. In such a space, compactness and countable compactness are equivalent, and so all of the above are equivalent.*

Therefore we have not only what we found last lecture, that in a separable metric space, countable compactness and compactness coincide, but we can in fact drop the separability condition.

Theorem 6.2.14. *In the class of metric spaces, countable compactness is the same as compactness.*

Exercise 6.2.15. Let $X = \mathbb{R}^{\mathbb{N}}$. Determine whether it (i) is Hausdorff, (ii) is T_1, (iii) satisfies the second axiom of countability, (iv) is compact, and finally (v) if it is countably compact.

Solution. Recall first of all that the neighbourhoods of a point $x \in X$ are the sets of any other $y \in X$ that are sufficiently close to x in finitely many coordinates, and otherwise unrestricted. This space is T_1, since if we take any two distinct x and y in X, there must exist some coordinate i where the two differ, say by ε. We then let the neighbourhood U_x of x be the set of all elements that are less than $\varepsilon/2$ away from x in coordinate i, and similarly the neighbourhood U_y we take to be all elements that are less than $\varepsilon/2$ away from y in coordinate i. Thus $x \notin U_y$ and $y \notin U_x$.

For the same reason it is a Hausdorff space: since the possible values of the ith coordinate of elements in U_x don't agree with the possible values of the ith coordinate of elements in U_y, elements of the two can never agree in the ith coordinate and so there can be no common elements between the two; their intersection is empty.

This space satisfies the second axiom of countability since $\mathbb{R}$ does (open intervals with rational endpoints form a countable base, for example), and $\mathbb{R}^{\mathbb{N}}$ is a countable number of copies of $\mathbb{R}$, so taking $|\mathbb{N}|$ copies of such rational bases gives us a countable base.

Since $\mathbb{R}^{\mathbb{N}}$ is countably compact, compactness and countable compactness are equivalent, so we need only show one of them. It is easy to see that the space is not countably compact: take the infinite subset S of $\mathbb{R}^{\mathbb{N}}$ containing all sequences where the first coordinate is arbitrary and the rest are 0; the first coordinate is the entirety of $\mathbb{R}$, but of course $\mathbb{R}$ isn't countably compact: the integers, for example, are an infinite set without a limit point. ⧫

Exercise 6.2.16. Let $X = \mathbb{R}^{\mathbb{R}}$. Determine whether it ($i$) is Hausdorff, ($ii$) is T_1, (iii) satisfies the second axiom of countability, (iv) is compact, and finally (v) if it is countably compact.

Solution. Recall that the elements of $\mathbb{R}^{\mathbb{R}}$ can be thought of as arbitrary functions f from $\mathbb{R}$ to $\mathbb{R}$.

By the same argument as in the previous problem this space is both T_1 and Hausdorff.

Certainly for the second axiom of countability the same construction as above does not work, since this time around the space is an *uncountable* copies of $\mathbb{R}$. Beyond that we are not certain: we do not have any results at hand for disproving the existence of a countable base (other than contradiction), however it certainly seems intuitively obvious that $\mathbb{R}^{\mathbb{R}}$ shouldn't have one; a base is made up of open sets, and here open sets are arbitrary except for finitely many points, and a countable number of finite points is still countable, we we couldn't even cover every coordinate with a countable base.

For countable compactness, the same argument as in the previous problem applies: take the set of all functions that are zero everywhere except at one point, and at this point let them take arbitrary values. Then this is an infinite set, and clearly it isn't countably compact.

By similar reasoning it is not compact: one can cover $\mathbb{R}$ with infinitely many intervals of the form $(n - 1/2, n + 1/2)$ for all $n \in \mathbb{Z}$, and so by taking these neighbourhoods uncountably many times, for each point in $\mathbb{R}$, we have an infinite covering or $\mathbb{R}^{\mathbb{R}}$. But certainly this covering lacks a finite subcovering, since even one single copy of $\mathbb{R}$ covered thus lacks a finite subcovering, let alone $|\mathbb{R}|$ copies of $\mathbb{R}$. ⧫

Exercise 6.2.17. Let X be an infinite set equipped with its cofinite topology. Determine whether it (i) satisfies the second axiom of countability, (ii) is compact, and finally (iii) if it is countably compact.

Solution. If X itself is countable, then certainly it contains a countable base, because then X has only countably many finite subsets and therefore the cofinite topology on $\mathbb{R}$ has only countably many open sets, so then it satisfies the second axiom of countability.

If X is uncountable, it is slightly trickier. Suppose that we do have a countable base. Then if there's a countable base for the whole space, there must also exist a countable set of neighbourhoods $N_1, N_2, \ldots$ around every point $x \in X$ such that any neighbourhood U around x contains N_i for some $i \in \mathbb{N}$.

Now for each i, let $F_i = X \setminus N_i$, which by definition must be finite since N_i is open and consider

$$F = \{x\} \cup \Big(\bigcup_{i \in \mathbb{N}} F_i \Big),$$

which is a union of countably many finite sets, so it is itself countable. But X is uncountable, so there must exist some $y \in X \setminus F$, which is then the set of all points not equal to x that are contained in all N_i, and we now take $V = X \setminus \{y\}$ for one such y.

Since $\{y\}$ is finite, V is open, and indeed it is an open neighbourhood of x, since it contains x. However we cannot have $x \in N_i \subset V$ for any n, since y was constructed to belong to all N_i, and V specifically does not contain y. So in the case of an uncountable X, the cofinite topology on X does not satisfy the second axiom of countability.

Certainly the cofinite topology is compact regardless of X, since only finitely many points are outside any one open set. We show it more carefully: let $\mathbb{C} = \{C_\alpha\}$ be any open cover, and take any $C \in \mathbb{C}$. Then $X \setminus C$ contains finitely many points, call them $\{x_1, x_2, \ldots, x_n\}$.

Now, since $\mathbb{C}$ is a cover, there must for every x_i exist some $C_i \in \mathbb{C}$ such that $x_i \in C_i$. Then $\{C_1, C_2, \ldots, C_n\}$ constructed thusly is a finite subcover, making the space compact, and therefore also countably compact. ♦

7 Lecture VII[8]

7.1 Topological Concepts in Metric Spaces

Recall a few things. We call a metric space (X, ρ) ***totally bounded*** if for every $\varepsilon > 0$ there exists a finite ε-net, which is a set A_ε such that for every $x \in X$, there exists an $a \in A_\varepsilon$ such that $\rho(x, a) < \varepsilon$.

We showed that if a metric space is countably compact, then it is also totally bounded. Moreover we showed that in any topological space, metric or not, compactness implies countable compactness.

Therefore, in metric spaces, compactness implies total boundedness. Moreover we showed that in a metric space, countable compactness and compactness coincide.

Theorem 7.1.1. *In a metric space, the notions of compactness and countable compactness coincide.*

Proof. First of all, if a topological space has a countable base (meaning that it satisfies the second axiom of countability), then countable compactness implies compactness.

Thus we need to show that in a metric space, countable compactness implies the second axiom of countability (which, we recall, in a metric space is equivalent to being separable).

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Thus we need to show that countable compactness implies separability. However countable compactness implies total boundedness, which implies separability (just take $\varepsilon = 1/n$ and take the union of all ε-nets). □

Thus we know that in a metric space countable compactness (which is the same as compactness here) implies total boundedness. One might then ask whether the converse holds.

Intuition suggests that this is not the case, for we would need the space to also be complete. This is indeed true.

Counterexample 7.1.2. Take $X = \mathbb{Q} \cap [0, 1]$. It is certainly totally bounded; take as finite $\varepsilon = 1/n$ nets $A_{1/n} = \{0.\alpha_1\alpha_2 \ldots \alpha_n\}$, that is the set of all decimals n digits long.

However not every infinite set has a limit point (in X); consider any irrational number in $[0, 1]$, for example $\sqrt{2} - 1$. The sequence of rational approximations of this is a Cauchy sequence, but its limit does not belong to X. Therefore X is not countably compact. ▲

Theorem 7.1.3. *A metric space is countably compact if and only if it is totally bounded and complete.*

Proof. ($\Rightarrow$) Let X be a compact metric space. We know that it is totally bounded from before, so we need to show that it is also complete. To see this, let $\{x_j\}$ be a Cauchy sequence in X. Now recall that compactness implies countable compactness, whereby any infinite set has a limit point.

But our Cauchy sequence is an infinite set, so it to must have a limit point x in X. It remains to show that this x must be the limit of the sequence, but this is clear, since it being a limit point means that every neighbourhood of x contains infinitely many points of the sequence.

($\Leftarrow$) Now let X be totally bounded and complete, and take any infinite subset $A \subset X$. It has to have a limit point, and for any $\varepsilon = 1/n$, there exists a finite ε-net N_n.

We start with N_1, and take around each $x \in N_1$ the ball of radius 1. By the pigeon hole principle, at least one of these balls, say $B_1(x(1))$, contains infinitely many points $x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}, \ldots \subset A$. This ball is also totally bounded, since A itself is totally bounded, so there exists a ball $B_{1/2}(x^{(1/2)})$ containing an infinite subsequence of the above, such that $x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)}, \ldots \in B_{1/2}(x^{(1/2)})$.

We continue this, and in general we take $y_k = x_k^{(k)}$ as the kth term of the kth subsequence. Now $\{y_k\}$ must be a Cauchy sequence, since they're contained in smaller and smaller balls, and since the space is complete there exists a $y \in X$ such that

$$\lim_{k \to \infty} y_k = y,$$

and certainly y is a limit point of A, so X is countably compact, since A was any infinite subset. Thus since we are in a metric space X is also compact. □

We recall another result from earlier. If (X, ρ) is a complete metric space and M is a closed subset, then (M, ρ) is a compact metric space.

We also showed that in $\mathbb{R}^n$¡ total boundedness is equivalent to boundedness. Thus, combining these two results, in $\mathbb{R}^n$ any closed and bounded set is compact.

7.2 Relative Compactness

Thinking back to $\mathbb{R}$, we have that a set like $[a, b]$ is compact, whereas a set (a, b) is not, but it *almost* is, in some sense; we need only add its endpoints. This notion has a name.

Definition 7.2.1 (Relative compactness)**.** A set is called ***relatively compact*** if its closure is compact.

It is immediately obvious from the previous discussion that the following holds.

Theorem 7.2.2. *In a complete metric space, a set is relatively compact if it is totally bounded.*

Example 7.2.3. In $\mathbb{R}^n$, relative compactness and boundedness are equivalent. ▲

Let us now consider notions like this in the metric space of continuous functions $X = C[a, b]$. We know from knowledge of functional analysis that this space is complete, and the metric we use is

$$\rho(\varphi_1, \varphi_2) = \max_{a \le x \le b} |\varphi_1(x) - \varphi_2(x)|,$$

for any two functions $\varphi_1, \varphi_2 \in C[a, b]$. Now consider a family of functions $\Phi \subset C[a, b]$. We are interested in some criterion on it being relatively compact (much like how in $\mathbb{R}^n$, boundedness implies relative compactness).

To do this, we require two new (possibly, depending on the reader's familiarity with analysis) notions.

Definition 7.2.4 (Uniformly bounded)**.** A system of functions Φ from $C[a, b]$ is ***uniformly bounded*** if there exists a K such that for all $x \in [a, b]$ we have $|\varphi(x)| < K$ for all $\varphi \in \Phi$.

Definition 7.2.5 (Equicontinuous)**.** A system of functions Φ is called ***equicontinuous*** if for all $\varepsilon > 0$ there exists a δ such that $|x - x'| < \delta$ implies that $|\varphi(x) - \varphi(x')| < \varepsilon$ for all $\varphi \in \Phi$.

Note that for just one function, the above is the definition of uniform continuity. Note also that the distinguishing feature here is that δ doesn't depend on x.

Theorem 7.2.6. *A system of functions Φ from $C[a, b]$ is relatively compact if and only if it is uniformly bounded and equicontinuous.*

Proof. First of all we know that since $C[a, b]$ is a complete metric space, relative compactness and total boundedness are equivalent. So we wish to prove that the above to notions imply total boundedness.

($\Leftarrow$) Suppose Φ is totally bounded. Then for every $\varepsilon > 0$ there exists a finite ε-net $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$.

For each such functions we let

$$\max_{a \le x \le b} |\varphi_j(x)| = K_j < \infty,$$

and certainly since there are finitely many, there exists $K = \max_j K_j$. Now since it is an ε-net, for every $\varphi \in \Phi$ there exists a φ_j such that

$$\rho(\varphi, \varphi_j) = \max_{a \le x \le b} |\varphi(x) - \varphi_j(x)| < \varepsilon.$$

Now we use a standard trick; certainly we have $|\varphi(x)| \leq |\varphi(x)-\varphi_j(x)|+|\varphi_j(x)|$. Taking maximums of both sides, we get

$$\max_{a\leq x\leq b}|\varphi(x)| \leq \max_{a\leq x\leq b}|\varphi(x)-\varphi_j(x)| + \max_{a\leq x\leq b}|\varphi_j(x)| < \varepsilon + K,$$

which does not depend on φ, and thus Φ is uniformly bounded.

Now for the equicontinuity, since $[a,b]$ is closed, our functions φ are not only continuous but also uniformly continuous. Thus for every $\varepsilon > 0$ there exists a $\delta_j > 0$ such that $|x-x'| < \delta_j$ implies that $|\varphi_j(x)-\varphi_j(x')| < \varepsilon$.

Now, again since there are finitely many elements in the net, take $\delta = \min_j \delta_j > 0$. Then for every $\varepsilon > 0$ we must have that $|x-x'| < \delta$ imply that $|\varphi(x)-\varphi(x')| < \varepsilon$, for every φ_j in the net.

Now take any $\varphi \in \Phi$. Then we have

$$\begin{aligned}|\varphi(x)-\varphi(x')| &\leq |\varphi(x)-\varphi_j(x)| + |\varphi_j(x)-\varphi_j(x')| + |\varphi_j(x')-\varphi(x')| \\ &\leq \varepsilon+\varepsilon+\varepsilon = 3\varepsilon,\end{aligned}$$

which of course goes to 0 as ε does. Therefore Φ being relatively compact implies it being both uniformly bounded and equicontinuous.

($\Leftarrow$) Let Φ be both uniformly bounded and equicontinuous. Then there exists a K such that $|\varphi(x)| \leq K$ for every $x \in [a,b]$ and every $\varphi \in \Phi$, and there also exists for every $\varepsilon > 0$ some $\delta > 0$ such that $|\varphi(x)-\varphi(x')| < \varepsilon$ if $|x-x'| < \delta$.

Now partition $[a,b]$ into parts of width δ, and partition $[-K,K]$ into parts of lengths ε (which is of course splitting the relevant region of the plane into a finite net).

Now approximate any $\varphi \in \Phi$ with a piecewise linear function $\tilde{\varphi}$ that connects vertices of this net, such that $|\varphi(x)-\tilde{\varphi}(x)| < \varepsilon$ for all x.

Then since the total number of such $\tilde{\varphi}$ is finite, this is a finite ε-net, and therefore we have total boundedness, and therefore also relative compactness. □

A result related to this is Peano's theorem, which says that for an ordinary differential equation with initial condition,

$$\frac{dx(t)}{dt} = f(x(t)),$$

with $x(t_0) = x_0$, if $f \in C(\mathbb{R})$, then there exists a solution (though it says nothing about uniqueness).

References

[KF20] A. N. Kolmogorov, S. V. Fomin. *Introductory Real Analysis. Dover Publications*, 416 pages, New edition, 2000.

Notations

$\lvert\cdot\rvert$	Absolute value
$[M]$	Closure of the set M
$\mathscr{P}(X)$	The power set of X
$\mathbb{R}$	Real numbers
$\mathscr{S}(\mathscr{A})$	The minimal topology containing $\mathscr{A}$
$B_r(x)$	Ball of radius r centred on x
$U(x)$	A neighbourhood of x
T_1	First axiom of separability
T_2	Second axiom of separability

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