THE TWISTED SECOND MOMENT OF L-FUNCTIONS ASSOCIATED TO HECKE–MAASS FORMS

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ABSTRACT. We establish an asymptotic formula for the twisted second moment of L-functions associated to Hecke–Maass forms, which can be used to deduce a zero-density estimate for these L-functions in the spectral aspect.

Keywords. moments of L-functions, zeros of L-functions, Hecke–Maass forms

1. Introduction

The study of moments of L-functions is one of the most important themes in analytic number theory. Many deep results, such as subconvexity bounds and nonvanishing of central L-values, can be obtained via the moments method. In recent works [7, 1, 8], it was proved that there exists a positive proportion of nonvanishing of central L-values for the family of Maass forms for $SL_2(\mathbb{Z})$. The proof relies on the asymptotic formulas of the mollified first and second moments of L-values at the central point $s = 1/2$. In this paper, we will establish an asymptotic formula for the twisted second moment of the L-values at $s = 1/2 + \sigma + i\tau$ with $\sigma > 0$. This formula has its own interests and could be further elaborated upon and utilized in the future. For example, Selberg [12] proved an analogous asymptotic formula for the twisted second moment of L-functions for the family of Dirichlet characters with large prime modulus q and used it to deduce a zero-density estimate for this family of Dirichlet L-functions.

To state our main result, we need some notation. Let $\{u_j\}$ be an orthonormal basis of $SL_2(\mathbb{Z})$ Hecke–Maass forms corresponding to Laplace eigenvalue $\frac{1}{4} + t_j^2$ with $t_j \geq 0$. In this case $\{u_j\}$ consists of even Maass forms and odd Maass forms according to $u_j(-\overline{z}) = u_j(z)$ or $u_j(-\overline{z}) = -u_j(z)$. Each $u_j(z)$ has the Fourier expansion

$$u_j(z) = \sum_{n \neq 0} \rho_j(n) \mathcal{W}_{s_j}(nz)$$

where $s_j = \frac{1}{2} + it_j$ and

$$\mathcal{W}_s(x + iy) = 2|y|^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|y|) e(x)$$

is the Whittaker function.

Let $\lambda_j(n)$ denote the n-th Hecke eigenvalue of $u_j(z)$. Then we have

$$\rho_j(\pm n) = \rho_j(\pm 1) \lambda_j(n) n^{-\frac{1}{2}}.$$

For each even Maass form u_j, the L-function associated to u_j is defined by

$$L(s, u_j) := \sum_{n=1}^{\infty} \frac{\lambda_j(n)}{n^s} \quad \text{for } \mathrm{Re}(s) > 1.$$

We also associate to u_j the symmetric square L-function

$$L(s,\mathrm{sym}^2 u_j) = \zeta(2s)\sum_{n\geq 1}\frac{\lambda_j(n^2)}{n^s} \quad \text{for } \mathrm{Re}(s)>1.$$

For $1<M<T$, we define

$$k(t) = e^{-\frac{(t-T)^2}{M^2}} + e^{-\frac{(t+T)^2}{M^2}}. \tag{1.1}$$

The main result of this paper is the following theorem.

Theorem 1.1. *Let T and M be large parameters with $T^{\varepsilon}\leq M\leq T^{1-\varepsilon}$, let $\sigma>0$ and $0<|\tau|<T^{\frac{1}{4}}$, and let $\ell<T^{\frac{1}{3}}$ be a square free integer. Let $\{u_j\}$ be an orthonormal basis of even Hecke–Maass cusp forms for $SL_2(\mathbb{Z})$ with Laplace eigenvalues $\frac{1}{4}+t_j^2$. Then we have*

$$\begin{aligned}
&{\sum_j}' \omega_j k(t_j)\lambda_j(\ell)|L(\tfrac{1}{2}+\sigma+i\tau,u_j)|^2\\
&\quad= \frac{\zeta(1+2\sigma)}{8\pi^2}\frac{\tau_{i\tau}(\ell)}{\ell^{1/2+\sigma}}\int_{-\infty}^{\infty}k(t)\tanh(\pi t)t\,dt + \frac{\zeta(1-2\sigma)}{8\pi^2}\frac{\tau_{i\tau}(\ell)}{\ell^{1/2-\sigma}}\int_{-\infty}^{\infty}k(t)\tanh(\pi t)t\left(\frac{t}{2\pi}\right)^{-4\sigma}dt\\
&\quad\quad+\frac{\zeta(1-2i\tau)}{8\pi^2}\frac{\tau_{\sigma}(\ell)}{\ell^{1/2-i\tau}}\int_{-\infty}^{\infty}k(t)\tanh(\pi t)t\left(\frac{t}{2\pi}\right)^{-2\sigma-2i\tau}dt\\
&\quad\quad+\frac{\zeta(1+2i\tau)}{8\pi^2}\frac{\tau_{\sigma}(\ell)}{\ell^{1/2+i\tau}}\int_{-\infty}^{\infty}k(t)\tanh(\pi t)t\left(\frac{t}{2\pi}\right)^{-2\sigma+2i\tau}dt\\
&\quad\quad+O(T^{\frac{13}{21}-\frac{26}{21}\sigma+\varepsilon}M+\ell^{1/2+\sigma}T^{1/2-2\sigma+\varepsilon}M^{-1/2}+\ell^{-1/2+\sigma}T^{1/4-2\sigma+\varepsilon}M)
\end{aligned} \tag{1.2}$$

where $\sum'$ restricts to even Maass forms, ω_j is given in (2.8).

Note that both sides in (1.2) are analytic in σ near 0. Letting $\sigma\to 0$, we have the following corollary.

Corollary 1.2. *Let T and M be large parameters with $T^{\varepsilon}\leq M\leq T^{1-\varepsilon}$. For $0<|\tau|<T^{\frac{1}{4}}$, and $\ell<T^{\frac{1}{3}}$ a square free integer, we have*

$$\begin{aligned}
&{\sum_j}' \omega_j k(t_j)\lambda_j(\ell)|L(\tfrac{1}{2}+i\tau,u_j)|^2\\
&\quad=\frac{1}{8\pi^2}\frac{\tau_{i\tau}(\ell)}{\ell^{1/2}}\int_{-\infty}^{\infty}k(t)\tanh(\pi t)t\left(2\log\left(\frac{t}{2\pi}\right)-\log\ell+2\gamma\right)dt\\
&\quad\quad+\frac{\zeta(1-2i\tau)}{8\pi^2}\frac{\tau(\ell)}{\ell^{1/2-i\tau}}\int_{-\infty}^{\infty}k(t)\tanh(\pi t)t\left(\frac{t}{2\pi}\right)^{-2i\tau}dt\\
&\quad\quad+\frac{\zeta(1+2i\tau)}{8\pi^2}\frac{\tau(\ell)}{\ell^{1/2+i\tau}}\int_{-\infty}^{\infty}k(t)\tanh(\pi t)t\left(\frac{t}{2\pi}\right)^{2i\tau}dt\\
&\quad\quad+O(T^{\frac{13}{21}+\varepsilon}M+\ell^{1/2}T^{1/2+\varepsilon}M^{-1/2}+\ell^{-1/2}T^{1/4+\varepsilon}M)
\end{aligned}$$

where γ is Euler's constant.

The proof uses the standard method of approximate functional equation and Kuznetsov trace formula. Although many technical estimates we used were established in [8, 14], it is still worth providing a detailed proof of Theorem 1.1 to add to the literature for future reference. One of the main difficulties in the proof is extracting the off-diagonal main term, since it comes from the integrals involving both J-Bessel and K-Bessel functions. This relies on the uniform treatment of these integrals in [8].

Theorem 1.1 can be used to deduce a zero-density result in the spectral aspect which may have other applications. To describe the result, for $\sigma > 0$ and $H > 0$, we let

$$N(\sigma, H; u_j) := \#\{\rho = 1/2 + \beta + i\gamma \ : \ L(\rho, u_j) = 0, \ \beta > \sigma, \ |\gamma| < H\}.$$

and

$$N(\sigma, H) := \frac{1}{W} \sum_j{}' \omega_j k(t_j) N(\sigma, H; u_j)$$

where

$$W = \frac{1}{8\pi^2} \int_{-\infty}^{\infty} k(t) \tanh(\pi t) t dt. \tag{1.3}$$

We have the following zero-density estimate in the spectral aspect for the L-functions associated to Hecke–Maass forms.

Theorem 1.3. *Let $2/\log T < \sigma < 1/2$. For some sufficiently small $\delta, \ \theta > 0$, we have*

$$N(\sigma, H) \ll HT^{-\theta\sigma} \log T$$

uniformly in $3/\log T < H < T^{\delta}$.

Analogous results for the families of L-functions associated to modular forms were established by Kowalski and Michel [6] in the level aspect with great applications and by Hough [3] in the weight aspect. Our zero-density result in the spectral aspect can be used to remove the GRH assumption in Hejhal and Luo's work [4].

Remark 1.4. One can show by standard methods that $N(0, H; u_j) \asymp H \log T$ for $t_j \asymp T$ and hence $N(0, H) \asymp H \log T$. Theorem 1.3 shows that there are only very few L-functions with zeros of imaginary part less than T^{δ} and real part greater than $1/2 + C/\log T$.

To prove Theorem 1.3, one uses a similar argument as in [3, section 5] to derive from Theorem 1.1 the following mollified second moment.

Proposition 1.5. *There exist Dirichlet polynomials $\{M(s, u_j)\}$ with $\overline{M(s, u_j)} = M(\bar{s}, u_j)$ such that for $\delta, \theta > 0$ sufficiently small, we have*

$$\frac{1}{W} \sum_j{}' \omega_j k(t_j) |M(\tfrac{1}{2} + \sigma + i\tau, u_j) L(\tfrac{1}{2} + \sigma + i\tau, u_j)|^2 \le 1 + O\left(T^{-\theta\sigma}\right),$$

uniformly for $1/\log T \le \sigma \le 1$ and $|\tau| < T^{\delta}$. Moreover, for all τ

$$M(3/2 + i\tau, u_j) L(3/2 + i\tau, u_j) = 1 + O(T^{-\theta}).$$

Now Theorem 1.3 can be derived from Selberg's lemma [12, Lemma 14] and Proposition 1.5. See [6, section 4.2] and [12] for details or [3, section 2] for a outline of the proof. The rest of the paper is devoted to proving Theorem 1.1.

2. Preliminaries

2.1. **Approximate functional equation.** It is known that $L(s,u_j)$ has an analytic continuation to an entire function on $\mathbb{C}$ and satisfies the functional equation

$$\Lambda(s,u_j) := \gamma(s,t_j)L(s,u_j) = \Lambda(1-s,u_j).$$

where

$$\gamma(s,t) = \pi^{-s}\Gamma\left(\frac{s+it}{2}\right)\Gamma\left(\frac{s-it}{2}\right). \tag{2.1}$$

The proof of the following approximate functional equation for $|L(s,u_j)|^2$ can be found in [5, Theorem 5.3].

Lemma 2.1. *For $s = \frac{1}{2}+\sigma+i\tau$ with $0<\sigma\le 1$, we have*

$$|L(s,u_j)|^2 = \sum_{d=1}^{\infty}\frac{1}{d^{1+2\sigma}}\sum_{n=1}^{\infty}\frac{\lambda_j(n)\tau_{i\tau}(n)}{n^{\frac{1}{2}+\sigma}}V_s(nd^2,t_j) + \sum_{d=1}^{\infty}\frac{1}{d^{1-2\sigma}}\sum_{n=1}^{\infty}\frac{\lambda_j(n)\tau_{i\tau}(n)}{n^{\frac{1}{2}-\sigma}}\widetilde{V}_s(nd^2,t_j)$$

with

$$V_s(y,t) = \frac{1}{2\pi i}\int_{(3)} y^{-v}G_s(v,t)\frac{dv}{v}$$
$$\widetilde{V}_s(y,t) = \frac{1}{2\pi i}\int_{(3)} y^{-v}\widetilde{G}_s(v,t)\frac{dv}{v}.$$

where

$$G_s(v,t) = \frac{\gamma(s+v,t)\gamma(\overline{s}+v,t)}{\gamma(s,t)\gamma(\overline{s},t)}e^{v^2}$$
$$\widetilde{G}_s(v,t) = \frac{\gamma(1-s+v,t)\gamma(1-\overline{s}+v,t)}{\gamma(s,t)\gamma(\overline{s},t)}e^{v^2}.$$

Here

$$\tau_\nu(n) = \sum_{ab=|n|}\left(\frac{a}{b}\right)^{\nu}$$

is the generalized divisor function.

Likewise, for $s=\frac{1}{2}+\sigma+i\tau$, we have

$$|\zeta(s+it)\zeta(s-it)|^2 = \sum_{d=1}^{\infty}\frac{1}{d^{1+2\sigma}}\sum_{n=1}^{\infty}\frac{\tau_{it}(n)\tau_{i\tau}(n)}{n^{\frac{1}{2}+\sigma}}V_s(nd^2,t) + \sum_{d=1}^{\infty}\frac{1}{d^{1-2\sigma}}\sum_{n=1}^{\infty}\frac{\tau_{it}(n)\tau_{i\tau}(n)}{n^{\frac{1}{2}-\sigma}}\widetilde{V}_s(nd^2,t). \tag{2.2}$$

The functions V_s and $\widetilde{V}_s$ satisfy the following properties.

Lemma 2.2. *Let $s=\frac{1}{2}+\sigma+i\tau$ with $0<\sigma\le 1$. Let $A>0$, $T>1$ and $\varepsilon>0$. For $1\le |t| < T^{1+\varepsilon}$ and $|\tau|<|t|^{\frac{1}{4}}$, we have*

$$V_s(y,t) \ll_A \left(1+\frac{y}{t^2}\right)^{-A}, \qquad \widetilde{V}_s(y,t) \ll_A \left(1+\frac{y}{t^2}\right)^{-A}$$

and

$$V_s(y,t) = \frac{1}{2\pi i}\int_{\varepsilon - i\log T}^{\varepsilon+i\log T} y^{-v}G_s(v,t)\frac{dv}{v} + O_{\varepsilon,A}\left(y^{\varepsilon}T^{-A}\right) \tag{2.3}$$

$$\widetilde{V}_s(y,t) = \frac{1}{2\pi i}\int_{\varepsilon - i\log T}^{\varepsilon + i\log T} y^{-v}\widetilde{G}_s(v,t)\frac{dv}{v} + O_{\varepsilon,A}\left(y^{\varepsilon}T^{-A}\right). \tag{2.4}$$

$$V_s(y,t) = 1 + O_A\Big(\Big(\frac{y}{t^2}\Big)^A\Big), \tag{2.5}$$

$$\widetilde{V}_s(y,t) = \frac{\gamma(1-s,t)\gamma(1-\overline{s},t)}{\gamma(s,t)\gamma(\overline{s},t)} + O\Big(\Big(\frac{y}{t^2}\Big)^A\Big). \tag{2.6}$$

Proof. This is analogous to [8, Lemma 4.1]. It follows from the Stirling formula and suitable shifts of the contour integral as in [5, Proposition 5.4]. The main term in (2.5) and (2.5) are the residues from the pole at $v = 0$ while the residues at the other poles v are exponentially small in view of $|\mathrm{Im}(v)| \asymp t$. □

2.2. **Kuznetsov trace formula.**

Definition 2.3 (Space of test functions)**.** Let $S > \frac{1}{2}$. We set $\mathscr{H}(S)$ to be the space of functions $h(t)$ which extend to even holomorphic functions on the strip $\{t + i\sigma : |\sigma| \leqslant S\}$ such that

$$h(t + i\sigma) \ll e^{-\pi|t|}(1+|t|)^{-N},$$

holds uniformly for some $N > 6$.

Definition 2.4 (Bessel kernel)**.** Let $s \in \mathbb{C}$. For $x \in \mathbb{R}_+$ we define

$$B_s(x) = \frac{\pi}{\sin(\pi s)}\big(J_{-2s}(4\pi\sqrt{x}) - J_{2s}(4\pi\sqrt{x})\big),$$
$$B_s(-x) = 4\cos(\pi s)K_{2s}(4\pi\sqrt{x}).$$

Let $h(t)$ be a test function in $\mathscr{H}(S)$. Define

$$\mathscr{H} = \int_{-\infty}^{\infty} h(t)\tanh(\pi t)t\,dt, \quad \text{and} \quad \mathscr{H}(x) = \int_{-\infty}^{\infty} h(t)B_{it}(x)\tanh(\pi t)t\,dt$$

for $x \in \mathbb{R}$.

For $m, n \geq 1$ we have the Kuznetsov trace formula

$$\begin{aligned}&\sum_j{}' \omega_j h(t_j)\lambda_j(m)\lambda_j(n) + \frac{1}{4\pi}\int_{-\infty}^{\infty}\omega(t)h(t)\tau_{it}(m)\tau_{it}(n)\,dt \\ &= \frac{1}{8\pi^2}\delta(m,n)\mathscr{H} + \frac{1}{8\pi^2}\sum_{c=1}^{\infty}\Big(\frac{S(m,n;c)}{c}\mathscr{H}\Big(\frac{mn}{c^2}\Big) + \frac{S(m,-n;c)}{c}\mathscr{H}\Big(\frac{-mn}{c^2}\Big)\Big),\end{aligned} \tag{2.7}$$

where $\sum'$ restricts to even Maass forms and

$$\omega_j = \frac{1}{2L(1,\mathrm{sym}^2 u_j)}, \qquad \omega(t) = \frac{1}{|\zeta(1+2it)|^2}. \tag{2.8}$$

It is known that $|\zeta(1+it)| \gg 1/\log(3+|t|)$ and hence

$$\omega(t) \ll \log^2(3+|t|). \tag{2.9}$$

2.3. **Mellin transform of Bessel kernel.** The Mellin transform of the Bessel kernel $B_{it}(x)$ is defined by

$$\widetilde{B}_{it}(s) = \int_{\mathbb{R}^\times} B_{it}(x)|x|^{s-1}dx. \tag{2.10}$$

The explicit formula for $\widetilde{B}_{it}(s)$ is given in the following lemma.

Lemma 2.5. [8, Lemma 9.1] *For $|\mathrm{Im}(t)| < \mathrm{Re}(s) < \frac{1}{4}$, the integral (2.10) is absolutely convergent and*

$$\widetilde{B}_{it}(s) = \frac{\gamma(s,t)}{\gamma(1-s,t)}, \tag{2.11}$$

with $\gamma(s,t)$ as defined in (2.1).

2.4. **Voronoi summation formula.** We have the following Voronoi summation formula for the divisor function (see [5, Theorem 4.10] or [10]).

Lemma 2.6. *Let $w(x) \in C_c^\infty(\mathbb{R}^\times)$. Let $c \geq 1$ and $(a,c) = 1$ with $a\bar{a} \equiv 1 \pmod{c}$. Then*

$$\sum_{n\neq 0} \tau_{i\tau}(n) e\Big(\frac{an}{c}\Big) w(n) = c^{2i\tau-1}\zeta(1-2i\tau)\widetilde{w}_{-i\tau}(0) + c^{-2i\tau-1}\zeta(1+2i\tau)\widetilde{w}_{i\tau}(0)$$
$$+ \frac{1}{c}\sum_{n\neq 0} \tau_{i\tau}(n) e\Big(\frac{-\bar{a}n}{c}\Big)\widetilde{w}_{i\tau}\left(\frac{n}{c^2}\right),$$

where

$$\widetilde{w}_s(0) = \int_{\mathbb{R}^\times} w(x)|x|^s dx,$$
$$\widetilde{w}_s(y) = \int_{\mathbb{R}^\times} w(x) B_s(xy)\, dx \quad \textit{for} \quad y \neq 0.$$

3. Properties of Bessel integrals

For $1 \ll T^\varepsilon \leq M \leq T^{1-\varepsilon}$ and $k(t)$ as in (1.1), we define

$$h(t;v) = k(t)G_s(v,t) \qquad \text{and} \qquad h'(t;v) = k(t)\widetilde{G}_s(v,t)$$

with $\mathrm{Re}(v) = \varepsilon$ and $|\mathrm{Im}(v)| \ll \log T$.

Let

$$\mathscr{H}(x) = \int_{-\infty}^{\infty} h(t;v) B_{it}(x) \tanh(\pi t) t dt$$
$$\mathscr{H}'(x) = \int_{-\infty}^{\infty} h'(t;v) B_{it}(x) \tanh(\pi t) t dt.$$

The following results are essentially established in [14, Lemmas 7.1 and 7.2] (also see [8, Lemmas 7.1 and 7.3]).

Lemma 3.1. *There exists a Schwartz function $g(r)$ satisfying $g^{(j)}(r) \ll_{j,A,\varepsilon} (1+|r|)^{-A}$ for any $j, A \geq 0$, and such that $\mathscr{H}(x) = \mathscr{H}_+(x) + \mathscr{H}_-(x) + O(T^{-A})$ and $\mathscr{H}'(x) = \mathscr{H}'_+(x) + \mathscr{H}'_-(x) + O(T^{-A})$ for $|x| > 1$ with*

$$\mathscr{H}_\pm(x^2) = MT^{1+\varepsilon}\int_{-M^\varepsilon/M}^{M^\varepsilon/M} g(Mr)e(Tr/\pi \mp 2x\cosh r)dr$$

$$\mathscr{H}_\pm(-x^2) = MT^{1+\varepsilon}\int_{-M^\varepsilon/M}^{M^\varepsilon/M} g(Mr)e(Tr/\pi \pm 2x\sinh r)dr$$

$$\mathscr{H}'_\pm(x^2) = MT^{1-4\sigma+\varepsilon}\int_{-M^\varepsilon/M}^{M^\varepsilon/M} g(Mr)e(Tr/\pi \mp 2x\cosh r)dr$$

$$\mathscr{H}'_\pm(-x^2) = MT^{1-4\sigma+\varepsilon}\int_{-M^\varepsilon/M}^{M^\varepsilon/M} g(Mr)e(Tr/\pi \pm 2x\sinh r)dr.$$

Moreover, $\mathscr{H}(x) = O(T^{-A})$ *and* $\mathscr{H}'(x) = O(T^{-A})$ *for* $1 < |x| \ll T^2$*. For* $|x| \le 1$*, we have* $\mathscr{H}(x) \ll_{A,\varepsilon} MT^{1-2A+\varepsilon}\sqrt{|x|}$ *and* $\mathscr{H}'(x) \ll_{A,\varepsilon} MT^{1-4\sigma-2A+\varepsilon}\sqrt{|x|}$*. We also have* $\mathscr{H}_\pm(x) = O(T^{-A})$ *for* $1 < x \le M^{2-\varepsilon}T^2$*.*

4. Analysis of Hankel transformations

We collect the results for the Hankel transform arising in the Voronoi summation formula in this section. Most of the results are given in [8] with slight modifications here. In view of Definition 2.4 and [13, 3.61 (1) (2)], we have

$$B_s(x) = \frac{\pi}{2\sin(\pi s)}\left((e^{2\pi is}-1)H^{(1)}_{2s}(x) + (e^{-2\pi is}-1)H^{(2)}_{2s}(x)\right)$$

where $H^{(1)}_\nu(x)$ and $H^{(2)}_\nu(x)$ are the Hankel functions. By the asymptotic expansions in [13, 7.2 (1, 2) and 7.23 (1)], there exists smooth function $W_{2i\tau}(x)$ such that for $x > 1$ and $|\tau| < x^{1/4}$,

$$B_{i\tau}(x) = \sum_\pm \frac{e(\pm 2\sqrt{x} - 1/8)}{\sqrt[4]{x}} W_{2i\tau}(\pm\sqrt{x}) + O\left(\frac{1+|\tau|^{2K}}{x^{(2K+1)/4}}\right) \tag{4.1}$$

$$B_{i\tau}(-x) = O\left(\frac{\exp(-3\pi\sqrt{x})}{\sqrt[4]{x}}\right). \tag{4.2}$$

with

$$x^j W^{(j)}_{2i\tau}(x) \ll 1.$$

Let $w(x) \in C_c^\infty([1,2])$ satisfying $w^j(x) \ll_j (\log T)^j$ for all $j \ge 0$. For $|\Lambda| \gg T^2$, define

$$w(x,\Lambda) = w(|x|)\mathscr{H}(\Lambda x), \qquad w'(x,\Lambda) = w(|x|)\mathscr{H}'(\Lambda x).$$

Let $\widetilde{w}_{i\tau}(y,\Lambda)$ be the Hankel transform defined by

$$\widetilde{w}_{i\tau}(y,\Lambda) = \int_{\mathbb{R}^\times} w(x,\Lambda)B_{i\tau}(xy)dx$$

and

$$\widetilde{w'}_{i\tau}(y,\Lambda) = \int_{\mathbb{R}^\times} w'(x,\Lambda)B_{i\tau}(xy)dx.$$

We may assume that $\Lambda > 0$ since $\widetilde{w}_{i\tau}(y,\Lambda) = \widetilde{w}_{i\tau}(-y,-\Lambda)$ and $\widetilde{w'}_{i\tau}(y,\Lambda) = \widetilde{w'}_{i\tau}(-y,-\Lambda)$.

Lemma 4.1. *Suppose $\Lambda \gg T^2$, $|\tau| < T^{1/4}$ and $y \geq T$. We have*

$$\widetilde{w}_{i\tau}(\pm y, \Lambda) = \frac{MT^{1+\varepsilon}}{\sqrt[4]{y}} \Psi^{\pm}(\sqrt{y/\Lambda}, \sqrt{\Lambda}) + O(T^{-A})$$

and

$$\widetilde{w'}_{i\tau}(\pm y, \Lambda) = \frac{MT^{1-4\sigma+\varepsilon}}{\sqrt[4]{y}} \Psi^{\pm}(\sqrt{y/\Lambda}, \sqrt{\Lambda}) + O(T^{-A})$$

with

$$\Psi^{+}(x, \Delta) = \int_{-M^{\varepsilon}/M}^{M^{\varepsilon}/M} e(Tr/\pi) g(Mr) \widehat{V}(\Delta(x - \cosh r)) dr$$

for $\Delta > M^{1-\varepsilon}T$ and $\Psi^{+}(x, \Delta) = 0$ for $\Delta \leq M^{1-\varepsilon}T$, and

$$\Psi^{-}(x, \Delta) = \int_{-M^{\varepsilon}/M}^{M^{\varepsilon}/M} e(Tr/\pi) g(Mr) (\widehat{V}(\Delta(x + \sinh r)) + \widehat{V}(\Delta(x - \sinh r))) dr$$

where $\widehat{V}(x)$ is a Schwartz function satisfying

$$\widehat{V}^{(j)}(x) \ll_{j,A} \left(1 + \frac{|x|}{\log T}\right)^{-A}$$

for any $j, A \geq 0$.

Proof. Using the (4.1) and (4.2) for $B_{i\tau}(x)$ and Lemma 3.1 for $\mathscr{H}(x)$ and $\mathscr{H}'(x)$. See [8, Lemma 8.1] for details. □

Proposition 4.2. *The function $\Psi^{-}(x, \Delta)$ is negligibly small unless $x < \frac{M^{\varepsilon}}{M}$, in which case $\Psi^{-}(x, \Delta) \ll \frac{T^{\varepsilon}}{\Delta}$. Similarly, $\Psi^{+}(x, \Delta)$ is negligibly small unless $|x - 1| < \frac{M^{\varepsilon}}{M^2}$, in which case $\Psi^{+}(x, \Delta) \ll \frac{T^{\varepsilon}}{\Delta\sqrt{|x-1|}}$. Moreover, $\Psi^{+}(1, \Delta) \ll T^{-A}$ if $\Delta \leq T^{2-\varepsilon}$.*

Proof. See [8, Propositions 8.6 and 8.7]. □

5. The twisted second moment

We devote this section to the proof of Theorem 1.1. Using Lemma 2.1 we can rewrite the twisted second moment as

$$\begin{aligned}
\mathcal{M}_2(\ell) &:= \sum_j{}' \omega_j k(t_j) \lambda_j(\ell) |L(s, u_j)|^2 \\
&= \sum_{d=1}^{\infty} \frac{1}{d^{1+2\sigma}} \sum_{n=1}^{\infty} \frac{\tau_{i\tau}(n)}{n^{\frac{1}{2}+\sigma}} \sum_j{}' \omega_j k(t_j) \lambda_j(\ell) \lambda_j(n) V_s(nd^2, t_j) \\
&\quad + \sum_{d=1}^{\infty} \frac{1}{d^{1-2\sigma}} \sum_{n=1}^{\infty} \frac{\tau_{i\tau}(n)}{n^{\frac{1}{2}-\sigma}} \sum_j{}' \omega_j k(t_j) \lambda_j(\ell) \lambda_j(n) \widetilde{V}_s(nd^2, t_j).
\end{aligned}$$

In view of Lemma 2.2, we may truncate the summations over n and d to $nd^2 \leq T^{2+\varepsilon}$ with a negligible error term.

Next we use the expressions of V_s and $\widetilde{V}_s$ as in (2.3) and (2.4). Then we apply the Kuznetsov trace formula (2.7) inside the v-integral with the test functions:

$$h(t; v) = k(t) G_s(v, t) \quad \text{and} \quad h'(t; v) = k(t) \widetilde{G}_s(v, t).$$

Moreover, for the diagonal terms and continuous spectrum we revert the v-integral to V_s and $\widetilde{V}_s$ with negligible errors.

It follows that

$$\mathcal{M}_2(\ell) = \mathcal{D} + \mathcal{O} - \mathcal{C} + \mathcal{D}' + \mathcal{O}' - \mathcal{C}' + O(T^{-A})$$

where $\mathcal{D}$ and $\mathcal{D}'$ are the diagonal terms

$$\mathcal{D} = \frac{1}{8\pi^2}\frac{\tau_{i\tau}(\ell)}{\ell^{1/2+\sigma}}\sum_{d=1}^{\infty}\frac{1}{d^{1+2\sigma}}\int_{-\infty}^{\infty} k(t)V_s(\ell d^2, t)\tanh(\pi t)tdt$$
$$\mathcal{D}' = \frac{1}{8\pi^2}\frac{\tau_{i\tau}(\ell)}{\ell^{1/2-\sigma}}\sum_{d=1}^{\infty}\frac{1}{d^{1-2\sigma}}\int_{-\infty}^{\infty} k(t)\widetilde{V}_s(\ell d^2, t)\tanh(\pi t)tdt,$$

and the terms coming from the continuous spectrum are (in view of (2.2))

$$\mathcal{C} + \mathcal{C}' = \frac{1}{4\pi}\int_{-\infty}^{\infty}\omega(t)\Big|\zeta(\frac{1}{2}+\sigma+i\tau+it)\zeta(\frac{1}{2}+\sigma+i\tau-it)\Big|^2\tau_{it}(\ell)k(t)dt.$$

The off-diagonal terms $\mathcal{O}$ and $\mathcal{O}'$ are

$$\mathcal{O} = \frac{1}{2\pi i}\int_{\varepsilon-i\log T}^{\varepsilon+i\log T}\frac{1}{8\pi^2}\mathcal{O}(v)\frac{dv}{v}$$

$$\mathcal{O}' = \frac{1}{2\pi i}\int_{\varepsilon-i\log T}^{\varepsilon+i\log T}\frac{1}{8\pi^2}\mathcal{O}'(v)\frac{dv}{v}$$

with

$$\mathcal{O}(v) = \sum_{\substack{n\neq 0\\ |n|d^2\leq T^{2+\varepsilon}}}\frac{\tau_{i\tau}(n)}{|n|^{\frac{1}{2}+\sigma+v}d^{1+2\sigma+2v}}\sum_{c=1}^{\infty}\frac{S(\ell,n;c)}{c}\mathscr{H}\left(\frac{n\ell}{c^2};v\right)$$

$$\mathcal{O}'(v) = \sum_{\substack{n\neq 0\\ |n|d^2\leq T^{2+\varepsilon}}}\frac{\tau_{i\tau}(n)}{|n|^{\frac{1}{2}-\sigma+v}d^{1-2\sigma+2v}}\sum_{c=1}^{\infty}\frac{S(\ell,n;c)}{c}\mathscr{H}'\left(\frac{n\ell}{c^2};v\right)$$

and

$$\mathscr{H}(x;v) = \int_{-\infty}^{\infty} h(t;v)B_{it}(x)\tanh(\pi t)tdt$$
$$\mathscr{H}'(x;v) = \int_{-\infty}^{\infty} h'(t;v)B_{it}(x)\tanh(\pi t)tdt.$$

5.1. **Continuous spectrum.** Using the bound $\zeta(1/2+it) \ll (1+|t|)^{13/84+\varepsilon}$ (due to Bourgain [2]) and the Phragmén–Lindelöf principle, we have

(5.1) $$\zeta(1/2+\sigma+it) \ll (1+|t|)^{\frac{13}{84}-\frac{13}{42}\sigma+\varepsilon}$$

for $\sigma > 0$. Using (2.9) and (5.1), we deduce that

$$\mathcal{C} + \mathcal{C}' \ll T^{\frac{13}{21}-\frac{26}{21}\sigma+\varepsilon}M.$$

5.2. **Diagonal terms.** Inserting the definition of $V_s(y,t)$ and $\widetilde{V}_s(y,t)$, we have $\mathcal{D}+\mathcal{D}'$ equals

$$\frac{1}{8\pi^2}\frac{\tau_{i\tau}(\ell)}{\ell^{1/2+\sigma}}\int_{-\infty}^{\infty}k(t)\tanh(\pi t)t\frac{1}{2\pi i}\int_{(3)}\ell^{-v}\zeta(1+2\sigma+2v)\frac{\gamma(s+v,t)\gamma(\overline{s}+v,t)}{\gamma(s,t)\gamma(\overline{s},t)}e^{v^2}\frac{dv}{v}dt$$
$$+\frac{1}{8\pi^2}\frac{\tau_{i\tau}(\ell)}{\ell^{1/2-\sigma}}\int_{-\infty}^{\infty}k(t)\tanh(\pi t)t\frac{1}{2\pi i}\int_{(3)}\ell^{-v}\zeta(1-2\sigma+2v)\frac{\gamma(1-s+v,t)\gamma(1-\overline{s}+v,t)}{\gamma(s,t)\gamma(\overline{s},t)}e^{v^2}\frac{dv}{v}dt.$$

Next we shift the v-integral to $\mathrm{Re}(v)=-A$ in both integrals for a large $A>0$. The main contribution is coming from the residue at $v=0$. The contribution from the residue at $v=-\sigma$ from the first integral and the residue at $v=\sigma$ from the second integral are canceled. The contribution from the residues at the other places and the integral at $\mathrm{Re}(v)=-A$ are negligible.

The contribution from the residue at $v=0$ gives

$$\frac{1}{8\pi^2}\frac{\tau_{i\tau}(\ell)}{\ell^{1/2+\sigma}}\zeta(1+2\sigma)\int_{-\infty}^{\infty}k(t)\tanh(\pi t)tdt$$
$$+\frac{1}{8\pi^2}\frac{\tau_{i\tau}(\ell)}{\ell^{1/2-\sigma}}\zeta(1-2\sigma)\int_{-\infty}^{\infty}k(t)\tanh(\pi t)t\frac{\gamma(1-s,t)\gamma(1-\overline{s},t)}{\gamma(s,t)\gamma(\overline{s},t)}dt.$$

By Stirling formula, we have

$$\frac{\gamma(1-s,t)\gamma(1-\overline{s},t)}{\gamma(s,t)\gamma(\overline{s},t)}=\left(\frac{t}{2\pi}\right)^{-4\sigma}+O(|t|^{-4\sigma-3/4}).$$

Thus the main term is

$$\frac{1}{8\pi^2}\frac{\tau_{i\tau}(\ell)}{\ell^{1/2+\sigma}}\zeta(1+2\sigma)\int_{-\infty}^{\infty}k(t)\tanh(\pi t)tdt$$
$$+\frac{1}{8\pi^2}\frac{\tau_{i\tau}(\ell)}{\ell^{1/2-\sigma}}\zeta(1-2\sigma)\int_{-\infty}^{\infty}k(t)\tanh(\pi t)t\left(\frac{t}{2\pi}\right)^{-4\sigma}dt+O(\ell^{-1/2+\sigma}T^{-4\sigma+1/4+\varepsilon}M).$$

6. Off-diagonal terms

We will study the off-diagonal terms $\mathcal{O}$ and $\mathcal{O}'$ in this section. By Lemma 3.1, we may impose the condition $|\frac{n\ell}{c^2}|\gg T^2$ with a negligible error. Let $\sum_R \nu(|x|/R)$ be a dyadic partition of unity for $\mathbb{R}\setminus\{0\}$ with $R=2^{j/2}$ and $\nu(r)\in C_c^\infty[1,2]$. It suffices to consider $\mathcal{O}_d(R,v)$ and $\mathcal{O}'_d(R,v)$ (see (6.1) and (6.4)) via inserting the partition of unity into $\mathcal{O}(v)$ and $\mathcal{O}'(v)$ for the n-sums.

6.1. **The off-diagonal term $\mathcal{O}$.** Let

$$\mathcal{O}_d(R,v):=\frac{1}{R^{1/2+\sigma+v}}\sum_{1\le c\ll\frac{\sqrt{\ell R}}{T}}\frac{1}{c}\sum_{n\neq0}\tau_{i\tau}(n)S(\ell,n;c)w\left(\frac{n}{R},\frac{\ell R}{c^2};v\right) \tag{6.1}$$

for $R\le T^{2+\varepsilon}/d^2$, where

$$w(x,\Lambda;v)=w(|x|;v)\mathscr{H}(\Lambda x;v)\qquad w(r;v)=\frac{\nu(r)}{r^{1/2+\sigma+v}}.$$

First we open the Kloosterman sum $S(\ell,n;c)$ and apply the Voronoi formula (Lemma 2.6) to the sum over n. The computations for the first term and the second term from the Voronoi formula are similar, so we will just deal with the first term here.

We reverse the procedures of partition of unity and extend the sum of c, d with a negligible error. So the contribution from this to $\mathcal{O}$ is

$$\mathcal{F}_1 = \frac{1}{8\pi^2}\zeta(1-2i\tau)\int_{-\infty}^{\infty} k(t)\tanh(\pi t)t\frac{1}{2\pi i}\int_{(\varepsilon)} G_s(v,t)\mathcal{Z}(v,t)\frac{dv}{v}dt$$

where

$$\mathcal{Z}(v,t) = \sum_{d=1}^{\infty}\sum_{c=1}^{\infty}\frac{S(\ell,0;c)}{c^{2-2i\tau}d^{1+2\sigma+2v}}\widetilde{B}_{it}\left(\frac{\ell}{c^2};\frac{1}{2}-\sigma-i\tau-v\right)$$

and

$$\widetilde{B}_{it}(y;s) = \int_{\mathbb{R}^\times} B_{it}(xy)|x|^{s-1}dx = |y|^{-s}\widetilde{B}_{it}(s)$$

where $\widetilde{B}_{it}(s)$ is defined in (2.10). Thus

$$\mathcal{Z}(v,t) = \ell^{-1/2+\sigma+i\tau+v}\zeta(1+2\sigma+2v)\sum_{c=1}^{\infty}\frac{S(\ell,0;c)}{c^{1+2\sigma+2v}}\widetilde{B}_{it}(\frac{1}{2}-\sigma-i\tau-v).$$

Note that since ℓ is square free,

$$\sum_{c=1}^{\infty}\frac{S(\ell,0;c)}{c^{1+2\sigma+2v}} = \frac{\tau_{\sigma+v}(\ell)}{\ell^{\sigma+v}\zeta(1+2\sigma+2v)}.$$

We have

$$\mathcal{F}_1 = \frac{\zeta(1-2i\tau)}{8\pi^2}\ell^{-1/2+i\tau}\int_{-\infty}^{\infty} k(t)\tanh(\pi t)t\frac{1}{2\pi i}\int_{(\varepsilon)} G_s(v,t)\tau_{\sigma+v}(\ell)\widetilde{B}_{it}(\frac{1}{2}-\sigma-i\tau-v)\frac{dv}{v}dt.$$

By the Mellin transform (2.11) of the Bessel kernel (shift the v-integral if necessary), we have

(6.2)
$$\mathcal{F}_1 = \frac{\zeta(1-2i\tau)}{8\pi^2}\ell^{-1/2+i\tau}\int_{-\infty}^{\infty} k(t)\tanh(\pi t)t\frac{1}{2\pi i}\int_{(\varepsilon)} \tau_{\sigma+v}(\ell)\frac{\gamma(\overline{s}+v,t)\gamma(1-s-v,t)}{\gamma(s,t)\gamma(\overline{s},t)}e^{v^2}\frac{dv}{v}dt.$$

By a similar calculation, the contribution to $\mathcal{O}$ from the second term in the Voronoi formula is

(6.3)
$$\mathcal{F}_2 = \frac{\zeta(1+2i\tau)}{8\pi^2}\ell^{-1/2-i\tau}\int_{-\infty}^{\infty} k(t)\tanh(\pi t)t\frac{1}{2\pi i}\int_{(\varepsilon)} \tau_{\sigma+v}(\ell)\frac{\gamma(s+v,t)\gamma(1-\overline{s}-v,t)}{\gamma(s,t)\gamma(\overline{s},t)}e^{v^2}\frac{dv}{v}dt.$$

We will combine $\mathcal{F}_1$ and $\mathcal{F}_2$ with similar terms from $\mathcal{O}'$ to give the off-diagonal main terms.

For the dual sum, define (we have suppressed v from the notation)

$$\widetilde{\mathcal{O}_d}(R) = \sum_{1\leq c\ll\frac{\sqrt{\ell R}}{T}}\frac{1}{c^2}\sum_{n\neq 0}\tau_{i\tau}(n)S(\ell-n,0;c)\widetilde{w}_{i\tau}\left(\frac{nR}{c^2},\frac{\ell R}{c^2}\right)$$

with

$$\widetilde{w}_{i\tau}(y,\Lambda) = \int_{\mathbb{R}^\times} w(x,\Lambda;v)B_{i\tau}(xy)dx.$$

The contribution from the third term in the Voronoi formula to $\mathcal{O}_d(R,v)$ is $R^{1/2-\sigma-v}\widetilde{\mathcal{O}_d}(R)$. Note that for $\ell \le T^{1/3}$ and $c^2 \ll \ell R/T^2$, we have $nR/c^2 \gg T^{5/3}$. By Lemma 4.1,

$$\widetilde{w}_{i\tau}\left(\pm\frac{nR}{c^2},\frac{\ell R}{c^2}\right) = \frac{MT^{1+\varepsilon}\sqrt{c}}{(nR)^{1/4}}\Psi^{\pm}\left(\sqrt{\frac{n}{\ell}},\frac{\sqrt{\ell R}}{c}\right) + O(T^{-A}).$$

Recall that $\Psi^+(x,\Delta)=0$ for $\Delta \le M^{1-\varepsilon}T$ (by Lemma 4.1). Using the formula for the Ramanujan sum $S(n,0;c) = \sum_{m|(n,c)} m\mu\left(\frac{c}{m}\right)$ and Proposition 4.2 for $\Psi^{\pm}$, we have that $\widetilde{\mathcal{O}_d}(R)$ is bounded by the sum of

$$\widetilde{\mathcal{O}_d}^{-}(R) = \frac{MT^{1+\varepsilon}}{\sqrt{\ell}R^{3/4}} \sum_{0<n<\ell/M^{2-\varepsilon}} \frac{\tau(n)}{n^{1/4}} \sum_{m|\ell+n} \sqrt{m} \sum_{mc \ll \frac{\sqrt{\ell R}}{T}} \frac{|\mu(c)|}{c^{1/2}}$$

and

$$\widetilde{\mathcal{O}_d}^{+}(R) = \frac{MT^{1+\varepsilon}}{\ell^{1/4}R^{3/4}} \sum_{0<|k|<\ell/M^{2-\varepsilon}} \frac{\tau(\ell+k)}{\sqrt{|k|}} \sum_{m|k} \sqrt{m} \sum_{mc \le \frac{\sqrt{\ell R}}{M^{1-\varepsilon}T}} \frac{|\mu(c)|}{c^{1/2}}$$

with $k = n-\ell$. Note that the $n=\ell$ term is removed by Proposition 4.2. We may assume $R \ge T^{2-\varepsilon}/\ell$ because otherwise the c-sums have no terms.

If $\ell \le M^{2-\varepsilon}$, then $\widetilde{\mathcal{O}_d}^{-}(R)$ and $\widetilde{\mathcal{O}_d}^{+}(R)$ vanish since the n-sum and k-sum have no terms. Otherwise, we have

$$\widetilde{\mathcal{O}_d}^{-}(R) \ll \frac{MT^{1/2+\varepsilon}}{\ell^{1/4}R^{1/2}} \sum_{0<n<\ell/M^{2-\varepsilon}} \frac{\tau(n)\tau(\ell+n)}{n^{1/4}} \ll \frac{\ell^{1/2}T^{1/2+\varepsilon}}{R^{1/2}M^{1/2}},$$

and

$$\widetilde{\mathcal{O}_d}^{+}(R) \ll \frac{M^{1/2}T^{1/2+\varepsilon}}{R^{1/2}} \sum_{0<|k|<\ell/M^{2-\varepsilon}} \frac{\tau(k)\tau(\ell+k)}{|k|^{1/2}} \ll \frac{\ell^{1/2}T^{1/2+\varepsilon}}{R^{1/2}M^{1/2}}.$$

Thus the contribution from them to $\mathcal{O}_d(R,v)$ is bounded by

$$\frac{R^{-\sigma}\ell^{1/2}T^{1/2+\varepsilon}}{M^{1/2}} \ll \ell^{1/2+\sigma}T^{1/2-2\sigma+\varepsilon}M^{-1/2}.$$

Hence its contribution to $\mathcal{O}(v)$ is

$$O\left(\sum_{d\le T^{1+\varepsilon}} \frac{1}{d^{1+2\sigma+2v}}\ell^{1/2+\sigma}T^{1/2-2\sigma+\varepsilon}M^{-1/2}\right) = O(\ell^{1/2+\sigma}T^{1/2-2\sigma+\varepsilon}M^{-1/2}).$$

6.2. **The off-diagonal term $\mathcal{O}'$.** Let

$$\mathcal{O}'_d(R,v) := \frac{1}{R^{1/2-\sigma+v}} \sum_{1\le c \ll \frac{\sqrt{\ell R}}{T}} \frac{1}{c} \sum_{n\ne 0} \tau_{i\tau}(n) S(\ell,n;c) w'\left(\frac{n}{R},\frac{\ell R}{c^2};v\right) \tag{6.4}$$

for $R \le T^{2+\varepsilon}/d^2$, where

$$w'(x,\Lambda;v) = w'(|x|;v)\mathscr{H}'(\Lambda x;v) \qquad w'(r;v) = \frac{\nu(r)}{r^{1/2-\sigma+v}}.$$

This term is handled the same as in $\mathcal{O}$. We open the Kloosterman sum $S(\ell, n; c)$ and apply the Voronoi formula to the n-sum. The contribution from the first term in the Voronoi formula to $\mathcal{O}'$ is

$$\mathcal{F}_1' = \frac{1}{8\pi^2}\zeta(1-2i\tau)\int_{-\infty}^{\infty} k(t)\tanh(\pi t)t\frac{1}{2\pi i}\int_{(\varepsilon)} \widetilde{G}_s(v,t)\mathcal{Z}'(v,t)\frac{dv}{v}dt$$

where

$$\mathcal{Z}'(v,t) = \sum_{d=1}^{\infty}\sum_{c=1}^{\infty}\frac{S(\ell,0;c)}{c^{2-2i\tau}d^{1-2\sigma+2v}}\widetilde{B}_{it}\left(\frac{\ell}{c^2};\frac{1}{2}+\sigma-i\tau-v\right).$$

By change of variables,

$$\mathcal{Z}'(v,t) = \ell^{-1/2-\sigma+i\tau+v}\zeta(1-2\sigma+2v)\sum_{c=1}^{\infty}\frac{S(\ell,0;c)}{c^{1-2\sigma+2v}}\widetilde{B}_{it}(\frac{1}{2}+\sigma-i\tau-v).$$

Note that since ℓ is square free,

$$\sum_{c=1}^{\infty}\frac{S(\ell,0;c)}{c^{1-2\sigma+2v}} = \frac{\tau_{-\sigma+v}(\ell)}{\ell^{\sigma+v}\zeta(1-2\sigma+2v)} = \frac{\tau_{\sigma-v}(\ell)}{\ell^{\sigma+v}\zeta(1-2\sigma+2v)}.$$

Thus we have

(6.5)
$$\mathcal{F}_1' = \frac{\zeta(1-2i\tau)}{8\pi^2}\ell^{-1/2+i\tau}\int_{-\infty}^{\infty} k(t)\tanh(\pi t)t\frac{1}{2\pi i}\int_{(\varepsilon)}\tau_{\sigma-v}(\ell)\frac{\gamma(\overline{s}-v,t)\gamma(1-s+v,t)}{\gamma(s,t)\gamma(\overline{s},t)}e^{v^2}\frac{dv}{v}dt.$$

A similar calculation for the second term in the Voronoi formula gives

(6.6)
$$\mathcal{F}_2' = \frac{\zeta(1+2i\tau)}{8\pi^2}\ell^{-1/2-i\tau}\int_{-\infty}^{\infty} k(t)\tanh(\pi t)t\frac{1}{2\pi i}\int_{(\varepsilon)}\tau_{\sigma-v}(\ell)\frac{\gamma(s-v,t)\gamma(1-\overline{s}+v,t)}{\gamma(s,t)\gamma(\overline{s},t)}e^{v^2}\frac{dv}{v}dt.$$

For the dual sum, define (we have suppressed v from the notation)

$$\widetilde{\mathcal{O}_d'}(R) = \sum_{1\le c\ll\frac{\sqrt{\ell R}}{T}}\frac{1}{c^2}\sum_{n\neq 0}\tau_{i\tau}(n)S(\ell-n,0;c)\widetilde{w'}_{i\tau}\left(\frac{nR}{c^2},\frac{\ell R}{c^2}\right)$$

with

$$\widetilde{w'}_{i\tau}(y,\Lambda) = \int_{\mathbb{R}^\times} w'(x,\Lambda;v)B_{i\tau}(xy)dx.$$

The contribution from the third term in the Voronoi formula to $\mathcal{O}_d'(R,v)$ is $R^{1/2+\sigma-v}\widetilde{\mathcal{O}_d'}(R)$.

Note that for $\ell\le T^{1/3}$ and $c^2\ll \ell R/T^2$, we have $nR/c^2\gg T^{5/3}$. By Lemma 4.1,

$$\widetilde{w'}_{i\tau}\left(\pm\frac{nR}{c^2},\frac{\ell R}{c^2}\right) = \frac{MT^{1-4\sigma+\varepsilon}\sqrt{c}}{(nR)^{1/4}}\Psi^{\pm}\left(\sqrt{\frac{n}{\ell}},\frac{\sqrt{\ell R}}{c}\right)+O(T^{-A}).$$

Recall that $\Psi^+(x,\Delta)=0$ for $\Delta\le M^{1-\varepsilon}T$ (by Lemma 4.1). Using the formula for the Ramanujan sum $S(n,0;c) = \sum_{m|(n,c)} m\mu\left(\frac{c}{m}\right)$ and Proposition 4.2 for $\Psi^{\pm}$, we have that $\widetilde{\mathcal{O}_d'}(R)$ is bounded by the sum of

$$\widetilde{\mathcal{O}_d'}^{-}(R) = \frac{MT^{1-4\sigma+\varepsilon}}{\sqrt{\ell}R^{3/4}}\sum_{0<n<\ell/M^{2-\varepsilon}}\frac{\tau(n)}{n^{1/4}}\sum_{m|\ell+n}\sqrt{m}\sum_{mc\ll\frac{\sqrt{\ell R}}{T}}\frac{|\mu(c)|}{c^{1/2}}$$

and

$$\widetilde{\mathcal{O}'_d}^{+}(R) = \frac{MT^{1-4\sigma+\varepsilon}}{\ell^{1/4}R^{3/4}} \sum_{0<|k|<\ell/M^{2-\varepsilon}} \frac{\tau(\ell+k)}{\sqrt{|k|}} \sum_{m|k} \sqrt{m} \sum_{mc\leq \frac{\sqrt{\ell R}}{M^{1-\varepsilon}T}} \frac{|\mu(c)|}{c^{1/2}}$$

with $k = n - \ell$. Note that the $n = \ell$ term is removed by Proposition 4.2. We may assume $R \geq T^{2-\varepsilon}/\ell$ because otherwise the c-sums have no terms.

If $\ell \leq M^{2-\varepsilon}$, then $\widetilde{\mathcal{O}'_d}^{-}(R)$ and $\widetilde{\mathcal{O}'_d}^{+}(R)$ vanish since the n-sum and k-sum have no terms. Otherwise, we have

$$\widetilde{\mathcal{O}'_d}^{-}(R) \ll \frac{MT^{1/2-4\sigma+\varepsilon}}{\ell^{1/4}R^{1/2}} \sum_{0<n<\ell/M^{2-\varepsilon}} \frac{\tau(n)\tau(\ell+n)}{n^{1/4}} \ll \frac{\ell^{1/2}T^{1/2-4\sigma+\varepsilon}}{R^{1/2}M^{1/2}},$$

and

$$\widetilde{\mathcal{O}'_d}^{+}(R) \ll \frac{M^{1/2}T^{1/2-4\sigma+\varepsilon}}{R^{1/2}} \sum_{0<|k|<\ell/M^{2-\varepsilon}} \frac{\tau(k)\tau(\ell+k)}{|k|^{1/2}} \ll \frac{\ell^{1/2}T^{1/2-4\sigma+\varepsilon}}{R^{1/2}M^{1/2}}.$$

Thus the contribution from them to $\mathcal{O}'_d(R, v)$ is bounded by

$$\frac{R^{\sigma}\ell^{1/2}T^{1/2-4\sigma+\varepsilon}}{M^{1/2}} \ll d^{-2\sigma}\ell^{1/2}T^{1/2-2\sigma+\varepsilon}M^{-1/2}.$$

Hence its contribution to $\mathcal{O}'(v)$ is

$$O\left(\sum_{d\leq T^{1+\varepsilon}} \frac{1}{d^{1+2v}}\ell^{1/2}T^{1/2-2\sigma+\varepsilon}M^{-1/2}\right) = O(\ell^{1/2}T^{1/2-2\sigma+\varepsilon}M^{-1/2}).$$

6.3. **The off-diagonal main term.** Making the change of variables $v \to -v$ in (6.2), the sum of the inner integrals in $\mathcal{F}_1$ and $\mathcal{F}'_1$ equals the reside at $v = 0$. Thus we have

$$\mathcal{F}_1 + \mathcal{F}'_1 = \frac{\zeta(1-2i\tau)}{8\pi^2}\ell^{-1/2+i\tau}\tau_\sigma(\ell)\int_{-\infty}^{\infty} k(t)\tanh(\pi t)t\frac{\gamma(1-s,t)}{\gamma(s,t)}dt$$

By Stirling's formula

$$\frac{\gamma(1-s,t)}{\gamma(s,t)} = \left(\frac{t}{2\pi}\right)^{-2\sigma-2i\tau} + O(t^{-2\sigma-3/4}).$$

The O-term contributes $O(\ell^{-1/2+\sigma}T^{-2\sigma+1/4+\varepsilon}M)$. The main term is

$$\begin{aligned}\mathcal{F}_1 + \mathcal{F}'_1 = &\frac{\zeta(1-2i\tau)}{8\pi^2}\ell^{-1/2+i\tau}\tau_\sigma(\ell)\int_{-\infty}^{\infty} k(t)\tanh(\pi t)t\left(\frac{t}{2\pi}\right)^{-2\sigma-2i\tau}dt\\ &+ O(\ell^{-1/2+\sigma}T^{-2\sigma+1/4+\varepsilon}M).\end{aligned}$$

A similar calculation gives

$$\begin{aligned}\mathcal{F}_2 + \mathcal{F}'_2 &= \frac{\zeta(1+2i\tau)}{8\pi^2}\ell^{-1/2-i\tau}\tau_\sigma(\ell)\int_{-\infty}^{\infty} k(t)\tanh(\pi t)t\frac{\gamma(1-\overline{s},t)}{\gamma(\overline{s},t)}dt\\ &= \frac{\zeta(1+2i\tau)}{8\pi^2}\ell^{-1/2-i\tau}\tau_\sigma(\ell)\int_{-\infty}^{\infty} k(t)\tanh(\pi t)t\left(\frac{t}{2\pi}\right)^{-2\sigma+2i\tau}dt\\ &\quad+ O(\ell^{-1/2+\sigma}T^{-2\sigma+1/4+\varepsilon}M).\end{aligned}$$

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