The Gibbs Phenomenon

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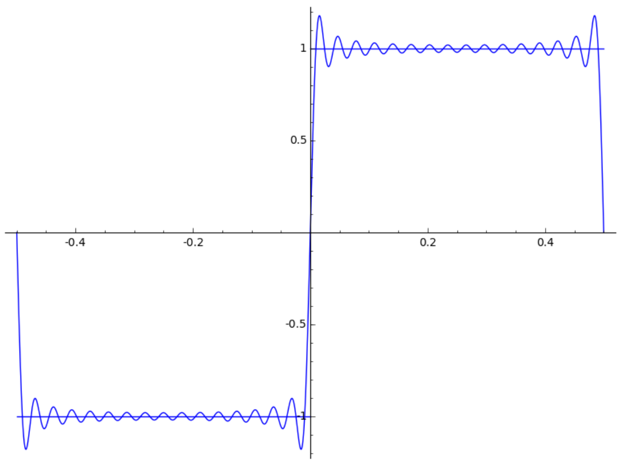
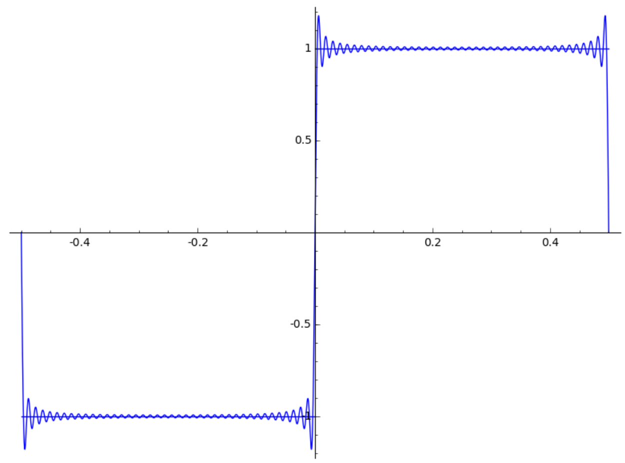
Example 1. Let f be the function of period 1 given by

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \frac{1}{2} \\ -1 & \text{if } -\frac{1}{2} < x < 0 \end{cases}$$

with a jump discontinuity at $x = 0$. According to Dirichlet's theorem, $S_N(f)(0)$ converges to the midpoint of the jump namely $\frac{f(0^+)+f(0^-)}{2} = \frac{1+(-1)}{2} = 0$. However, as you can see in the graphs below, for positive x near 0 the function $S_N(f)(x) = (D_N * f)(x)$ overshoots $f(0^+) = 1$, and the *height* of this bump (or overshoot) does not decay as $N \to \infty$. Despite of this phenomenon, the *width* of the bump tends to zero as $N \to \infty$.



(a) $S_{31}(f)$



(b) $S_{81}(f)$

Gibbs and Wilbraham discovered independently that this happens near the jump discontinuity of many other functions.

Gibbs Phenomenon. If f is a piecewise C^1 function, then near a jump discontinuity, the Fourier series of f overshoots (or undershoots) it by approximately 9% of the jump.[1]

Example 2. As before, let f be the function of period 1 given by

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \frac{1}{2} \\ -1 & \text{if } -\frac{1}{2} < x < 0 \end{cases}.$$

Find an approximation to the amount of overshoot of the Fourier series of f near $x = 0$.

Solution. The overshoot of the Fourier series of f near 0 is approximately

$$0.09 \underbrace{\left(f(0^+) - f(0^-)\right)}_{\text{size of the jump}} = 0.09 \times 2 = 0.18.$$

[1]More precisely, if $x = 0$ is a jump discontinuity, then

$$\max_{0<x\leq\frac{1}{2N}} S_N(f)(x) \simeq f(0^+) + c\underbrace{\left(f(0^+) - f(0^-)\right)}_{\text{jump}} \tag{1}$$

where $c = \int_0^1 \frac{\sin \pi t}{\pi t}\, dt - \frac{1}{2} \simeq 0.09$. The approximation (1) becomes an equality in the limit as $N \to \infty$.

Exercise 1. Let s be the sawtooth function of period 1 defined by $s(x) = 1/2 - x$ for $0 < x < 1$. Now assume that f is a function with a jump discontinuity at 0, and let $c = f(0^+) - f(0^-)$.

Check that for $g(x) = f(x) - cs(x)$,

$$g(0^+) = g(0^-).$$

Hint: $s(0^+) = 1/2$ and $s(0^-) = -1/2$.

Educational Note: The above exercise can be used to verify the Gibbs phenomenon for *any* piecewise C^1 function f with a jump discontinuity at 0, just by verifying it for the sawtooth function s; the latter is indeed what we did in class by explicit calculations.

Optional Reading: Avoiding the Gibbs Phenomenon

Theorem (No Gibbs Phenomenon)**.** Let $\Delta_N(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x}$ denote the Fejér kernel as usual, and let f be a piecewise continuous function of period 1. Then for any x and any N,

$$|(\Delta_N * f)(x)| \leq \max_{t\in[0,1]} |f(t)|.$$

Proof.

$$\begin{aligned}
|(\Delta_N * f)(x)| &= \left| \int_0^1 \Delta_N(x-t) f(t)\, dt \right| \\
&\leq \int_0^1 |\Delta_N(x-t)||f(t)|\, dt && \text{by triangle inequality for integrals} \\
&\leq \max_{t\in[0,1]} |f(t)| \int_0^1 \Delta_N(x-t)\, dt && \text{since } \Delta_N(x-t) \geq 0 \\
&= \max_{t\in[0,1]} |f(t)| && \text{because } \int_0^1 \Delta_N(x-t)\, dt = 1.
\end{aligned}$$

The reason that this argument fails to work for the Dirichlet kernel is that D_N takes both positive and negative values.

Exercise 2. Use the above theorem to argue that the Gibbs phenomenon does not occur if we use $\Delta_N * s$, rather than $D_N * s$, to approximate the sawtooth function s.